mathematics

MDPI

Article

Shortage Policies for a Jump Process with Positive and Negative Batch Arrivals in a Random Environment

Yonit Barron

Industrial Engineering and Management, Ariel University, Ariel 40700, Israel; ybarron@ariel.ac.il;
Tel.: +972-50-7749774

Abstract: We study a continuous-review stock management of a retailer for a single item in a limited storage (buffer) in a random environment. The stock level fluctuates according to two independent compound Poisson processes with discrete amounts of items (batches) that enter and leave the storage facility. The storage facility is controlled by a three-parameter base-stock replenishment policy. All items exceeding the storage capacity are transferred to an unlimited foreign facility. In addition, a restricted backlogging possibility is permitted; additional demands for items are lost sales. We further assume a random shelf life, the possibility of total inventory collapse, and a random lead time. Applying Markov theory, we derive the optimal control parameters minimizing the long-run expected total cost. A sensitivity analysis is conducted focusing on the comparison between the pure lost-sales policy and a partial backordering policy. Accordingly, we identify cases where one policy is cost effective compared to the other, particularly with respect to the batch patterns (sign, rate, average, and variability), and the associated costs.

Keywords: inventory; batch arrival; base-stock policy; backlog; lost sales

MSC: 90B05; 90B06; 60J28; 60G51

Citation: Barron, Y. Shortage Policies for a Jump Process with Positive and Negative Batch Arrivals in a Random Environment. *Mathematics* **2024**, *12*, 1341. https://doi.org/10.3390/math12091341

Academic Editor: Xiang Song

Received: 2 April 2024
Revised: 18 April 2024
Accepted: 22 April 2024
Published: 28 April 2024

1. Introduction

The planning of a supply chain subject to market uncertainty is challenging and requires decisions on when and how many items should be ordered. This challenge is increased due to rapid changes in the economic environment, increasing both customer intolerance and market competitiveness. As a result, retailers have to anticipate increasing changes in customer consumption and returned items as e-commerce sales soar and, hence, must carefully manage their stock in order to reap the benefits of these changes. Despite the implementation of advanced inventory-management tools and advanced forecasting of customer demands and returns, shortages are still inevitable. With regard to managing stock, retailers must maintain a high stock level since shortages reduce customer satisfaction and service levels. On the other hand, maintaining a high stock level that is further increased by returned items reduces the profit. Faced with this trade-off, retailers must decide whether to allow a (perhaps limited) backlogged level or to direct the customer to a competing company, thereby losing his/her loyalty. In addition, the complexity of managing stock is exacerbated by other factors, such as random delivery times, random amount sizes, random shelf lives, and unexpected events.

Motivated by the above problem, we consider the control management of a retailer with a single-item two-sided bounded storage (buffer) that faces jump-pattern arrivals. The storage facility is continuously controlled according to a triple-parameter base-stock (S, s, B) policy for $0 \leq s < S$ and $B \geq 0$, which is an extension of the conventional (S, s)-type. Under the (S, s) policy, when the on-hand stock level drops to level s or below, an order is issued to bring the stock up to level $S > s$, which the supplier will carry out after an exponentially distributed lead time. We extend the (S, s) policy by assuming a combination of backorders and lost sales. That is, in the case of zero on-hand stock, limited B backlogged items are allowed and additional demands for items are lost. Thus, S is the replenishment level, s is the reorder point, and B is the maximum backlogged items permitted during a stockout. The stock level process belongs to the class of jump models: it jumps upward and downward at random times, and stays constant in between jumps. When a positive (negative) batch arrives, the stock increases (decreases). A positive jump is caused by the entry of returned or loaded items into the system, or by the receipt of produced items from the manufacturer. A negative jump is caused by the demand or consumption of items. The batch arrivals are governed by two independent compound Poisson processes with positive batches, and each batch size follows a given discrete probability distribution. Since the storage facility has a limited capacity, any arriving amount exceeding it is transferred immediately to a second storage facility. We further assume two prototypes of shelf lives for the items held in the storage facility: each item has an i.i.d. exponentially distributed shelf life, after which it is useless and out of stock, and a total loss of all items in stock occurs at exponential times. The random shelf life of an item can be the result of spoilage, perishability, or failure. The possibility of losing all items may occur due to a malfunction of the storage facility, the bankruptcy of the retailer, an obsolescence event, an external disaster, or changing customer needs. In such cases, all on-hand items (if any) lose their value at once, and the stock collapses instantaneously to zero, after which the items are unusable.

The customer–retailer–supplier flow, operations processes, and outputs/inputs from/to the retailer are illustrated in Figure 1. We use blue and red arrows to indicate positive batches' inputs and negative batches' outputs of items, respectively.

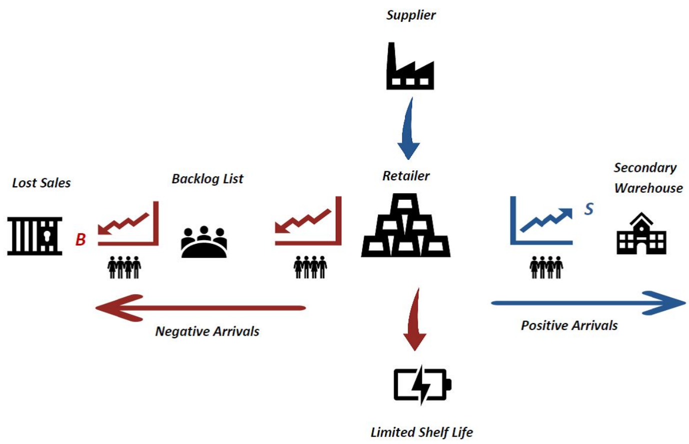

Figure 1. A general supply chain structure and operations process.

Our study assumes negative and positive jumps of random sizes occurring at random times. Positive jumps may be caused, e.g., by returned items from customers, returned items for recycling, used product collection, production packets received from the manufacturing plant [1], and canceled orders, such as booked rooms in hotels. With regard to returned items, customers increasingly opt to return products, and with the higher share of e-commerce channel sales, product return rates are soaring upwards. Therefore, it is no longer feasible for retailers to ignore returned items when determining the optimal inventory policy since it could result in overstocking and overestimation of the profit. Moreover, retail e-commerce has been growing rapidly and, with it, the number of returned items [2]. For example, the National Retail Federation found that, in the U.S., USD one billion in merchandise was returned, which is a 66% increase from five years earlier. The lack of power of touch at online stores further increases the number of returns; brick-and-mortar stores have a return rate as high as 10% of total sales [3]; U.S. consumers returned goods worth USD 261 billion, and the return rates for online sales sometimes exceeded 30% [4]; a California-based online clothing retailer disclosed that it made just under USD 400 million in net sales a year ago, but paid out USD 385 million for returned items. Buyers return a huge number of packages they buy from Amazon and other e-commerce sites, so much that retailers are sometimes left with little choice but to get rid of large swaths of inventory at a cost https://www.cnbc.com/2018/12/13/returned-goods-are-a-problem-for-retailers-resellers-are-cashing-in.html, accessed on 1 January 2024. A recent data on sales, returns, and return rates can be found in [2,4,5].

Our model is motivated by several practical applications. Take, for example, Israeli organic vegetable marketing farm Aley Bodek (https://aleybodek.co.il, accessed on 1 January 2024). The farm, established in the early 2000s in Kibbutz Be'erot Yitzhak in Israel, uses hydroponic methods of growing vegetables and fruits, using substrates transplanted from the ground to greenhouses that ensure fresh, high-quality agricultural produce. Aley Bodek markets its produce in packed boxes to specific retailers, who, in turn, market the produce to industrial plants, hotels, and other institutions. Clearly, the supply is random and depends on weather and other conditions. When the stock at the retailer falls below a certain level, an order is placed and the product arrives from the farm after some random time. Due to the uniqueness of the product, the retailer maintains a limited waiting list for the product, while customers not waitlisted are directed to competing companies. Another common application can be found in health systems, especially the management of blood units in emergency and operating rooms. The blood units are necessary for all medical activities, and usually arrive in batches from a central blood bank. Here, positive jumps represent the arrival of units, and negative jumps represent patients' needs. Of course, a safe stock of blood units is required. Since the capacity of the storage facility is limited, the surplus blood units are disposed of or used for other needs, such as academic or laboratory research. The negative stock expresses the number of blood units required for the non-urgent patients [6]. Another example comes from reliability and maintenance services. Consider the case of parts delivered to maintenance sites in isolated locales (where the delivery requires some lead time). Repair activities increase the stock level, and using the parts lowers the stock level. Due to the geographical distance, the maintenance site allows backlogged parts when stock is out; when the next replenishment arrives, these backlogged parts are satisfied first, and then, the stock is filled up to its maximum capacity (see, e.g., [7]).

We note that, although our model is described in the context of the stock management of a retailer, it can be generalized to a wide variety of other contexts. For example, in financial- and cash-management models, a positive jump may represent a bailout from the government, and a negative jump may represent the effect of a financial crisis [8]. Take, for example, the Israeli contracting company Sharbat Brothers. Sharbat Brothers invests in the construction of buildings around the world. The company's cash flow includes rental fees (positive jumps) and maintenance costs and other transactions for the public sector (negative jumps). The bank requires the company to maintain a certain level of financial cash flow, below which the required amount of money is transferred from other sources with some overhead time. The negative cash level represents a deficit; any financial withdrawal beyond the maximum permitted level is not allowed and forces the company to turn to more expensive sources. For additional examples, see, e.g., [9,10].

This paper further assumes that the positive and negative jumps arrive according to independent compound Poisson processes. The Poisson assumption is a common one in inventory models; statistical analysis has shown that it models customers' needs to retailers in service systems, queueing systems, and inventory systems well. One of the specific advantages of the Poisson process is that it exactly captures a similar effect as the short-term non-stationarity encountered in some arrival processes [11–13].

We further consider a random batch size. Clearly, a random batch size complicates the analysis since each inventory level faces two-sided jumps from several levels and some levels are not necessarily occupied between two replenishments. In practice, retailers usually deal with batch pattern arrivals. Examples include spare parts and drugs [14] and supplier-constrained or discounted items. Other examples come from chemical processing, water purification systems [15], medical equipment, and out-of-fashion products such as newspapers or personal computers [16]. As an excellent application for a batch type, we mention the tourism companies managed by travel agents. Travel agents reserve several hotel rooms in advance or allocate seats on planes (negative jumps) and may cancel the entire reservation due to a security situation or other unexpected event (positive jumps). Similarly, there are activities for teens that are given at clubs or at academic institutions and are open only through centralized registration at the schools.

Our model employs a triple-parameter (S, s, B) control policy, which extends the base-stock (S, s) replenishment policy by allowing limited shortages of up to B items. In general, the base-stock class includes two common policies: the constant-order policy (Q, r) and the variable-order up-to-level policy (S, s). In this paper, we assume the latter. As the above examples demonstrate, a variable-order size is a practical control policy in numerous scenarios [17].

As a result of the random arrivals (in terms of time and size) and uncertain shelf lives, shortages are common in most customer–retailer–supplier flows. Especially currently, with the quick flow of open information, any shortage of inventory causes dissatisfaction, lowers the service level, redirects customers to competing retailers, and increases unsupported reviews. There are two common policies for tackling shortages: the backordering policy, where unsatisfied demand is put on a waiting list until the next replenishment, and the lost-sales policy, where unsatisfied demand is redirected to other providers and is considered as lost sales. Both policies have a strong anchor in the literature and in practice. The backordering policy is commonly employed in the case of famous brand products, fashionable commodities, or critical products. For example, backordering is necessary to ensure the reliability and availability of critical products like military systems and medical supplies [18]. In monopolies, the absence of competitors may lead to a situation where customers may prefer to wait while their items are backordered if a shortage occurs [19]. On the other hand, for more common items, only 15% of customers will wait if a shortage occurs, while the remaining 85% will visit another store [20]; this percentage may even increase with the volatility and predictability of the customers' needs [21]. Examples include the apparel industry, the grocery industry, the steel industry, and the fast-moving consumer goods industry; more examples are given in [22].

From the retailer's perspective, the backordering policy improves the service level and customer satisfaction; however, it requires the payment of overhead costs and the building of appropriate infrastructure. On the other hand, the lost-sales policy results in a loss of customers, and less profitability. In should be noted that many retailers allow some limited backordering, so as not to lose customers and to take advantage of the fluctuations in customers' consumption habits. To address this trade-off, we propose a combination of the two policies whereby some of the items (up to B) are backordered and additional demands for items are lost. An excellent example of such a combination of backordering and lost-sales policies is found in the field of healthcare. Here, customers are waitlisted for treatment by a specialist and, thus, can be considered backlogged items. However, the waiting list is limited in the number of waits. Thus, when the specialist's calendar fills up, the customers are referred to other specialists, and are considered as lost sales. For more examples, we refer the reader to [23,24] and the references therein.

We seek to determine the optimal control levels s^*, S^*, and B^* that minimize the overall average cost of managing the stock. To that end, we assume the following cost structure: (i) a fixed cost per order and a purchasing cost per item, (ii) a cost for handling a positive batch, (iii) a linear cost for holding the on-hand stock and a backordering cost for not satisfying the customer on time (both costs are charged per item per unit time), (iv) a transfer cost for each item exceeding level S, (v) a lost cost for unsatisfied demand beyond level $-B$, and (vi) a loss cost for each useless item due to end of life or an unexpected event.

Using a Markovian formulation, we derive the steady-state probabilities of having i $(-B \leq i \leq S)$ items in the storage facility. These probabilities enable us to derive, by simulation-based optimization, the optimal control parameters and other operating characteristics of the system to be used. Based on an extensive set of experiments, we can obtain valuable insights, such as the interplay between the positive and negative batches (rate, mean, and variance) and the impact of this interplay on the system's performance, the economic benefit of backordering compared to lost sales, and how the best policy responds to changes in the system's characteristics (costs, lead times, and arrivals). It should be noted that, from a theoretical point of view, using the exponential distribution for the shelf life and lead time simplifies the analysis due to the Markovian property. If, however, a general or constant distribution variable is introduced for the lead time or for the shelf life, the system becomes significantly complicated as its performance is a function of many factors. In such cases, strong assumptions are required in order to obtain explicit results, and deriving the optimal or near-optimal control parameters is analytically complex. However, when either the shelf life or the lead time is not exponentially distributed, our results may still be used as an approximation.

The contribution of this paper is fourfold. First, we develop an easy-to-implement mathematical framework for studying processes, and formulate closed-form (albeit cumbersome) expressions for stationary distributions and costs. These expressions are then used to obtain, numerically, the optimal control parameters S^*, s^*, and B^*, so as to minimize the average total cost per time unit. The scope of the problem covers a wide range of real-world problems, such as the uncertainty of lead times, random arrivals (rate, mean, and variance), shelf lives, and unexpected disasters. To the best of our knowledge, no mathematical framework studying this combination of backordering and lost-sales policies has been explored in the literature; thus, our model improves the understanding of stock management systems.

Second, we coordinate positive and negative random jumps to the arrival processes; in doing so, we capture the reality of the changing patterns of customers/manufacturers. Studying the impact of the positive jumps on the system's performance reveals that the optimal control parameters are significantly impacted by these jumps, and integrating them into the derivation may lead to substantial cost savings. Our results further imply that the optimal parameters are more sensitive to the average batch size than to its variability. Interestingly, when the jumps are fixed to k, the optimal parameters and cost can be approximated by k times the corresponding values obtained for the unit-size system (the Poisson process), especially when more negative batches arrive and at high lost cost. In addition, our numerical analysis demonstrates the similarities between systems with the same average total arrival patterns (positive and negative).

Third, we introduce a replenishment policy that is a combination of backordering and lost-sales strategies, and focus on determining the optimal backlogged level that minimizes the total cost. From a practical point of view, a mixture of backordering and lost sales emphasizes the interplay between the loss cost due to stockout events and the overhead cost due to a high stock level maintained to retain customers. Although several papers have considered a mixture of backordering and lost-sales policies, to the best of our knowledge, none of them have studied the framework addressed here; thus, our model significantly expands the perspective on dealing with shortage and has a wide range of applications. For example, our study shows that allowing backordering may yield substantial cost savings. Surprisingly, the optimal backlogged level B^* is decreasing (increasing) in outflow (inflow). That is, the fewer the negative batches there are, the more economically profitable the backordering policy is. It is further shown that the benefit of backordering increases with the variability of the outflows; however, the impact of inflows' variability is more complicated.

Finally, using numerical examples, we compare the backordering policy ($B > 0$) and the pure lost-sales policy (corresponding to $B = 0$) in a variety of scenarios, and provide retailers with managerial insights and practical applications when considering changing their replenishment policy during a stockout. By that, the current study is adapted to the dynamic reality of rapid changes in order to maintain a high service level and to stay competitive. For example, our comparison shows that the maximum backlogged cost, beyond which the policy is not worthwhile, decreases as more outflows arrive, but as the variability increases, the outflows' arrival rate is less significant.

The remainder of the paper is organized as follows. In Section 2, we review the relevant literature. In Section 3, we present the basic features, assumptions, and notation. In Section 4, we formulate the mathematical model. Using numerical examples and a sensitivity analysis, Section 5 investigates the impact of the parameters on the system's performance under the two policies; a comparison, conclusions, and managerial insights derived from the results are presented. Finally, Section 6 presents concluding remarks and some directions for future research.

2. Literature Review

The main innovative focus of this paper is on two major topics: (1) systems with positive and negative jumps under the base stock policy and (2) different attitudes to stockout: a full backordering policy, a pure lost-sales policy, and a combination of both. In this section, we discuss previous research on these topics.

The literature on inventory systems under base-stock policies with inflows and outflows can be broadly separated into periodic- and continuous-review inventory systems. In this section, we address only continuous-review inventory systems. Schrady [25] was the first to introduce the concept of reusable inventory and study the classical economic order quantity problem (known as the EOQ model) with returned items. Gajdalo [26] extended Schrady's work by assuming independent compound Poisson processes for the inflows and outflows. Since those pioneering works, several policies have been suggested for a single-item inventory system with bilateral movements, e.g., an optimal (a, b) policy with no backorders or lost sales [27], an (s, Q)-type policy with backorders and fixed or zero lead times [28], differently sized lot sizes over a planning horizon [29], band policies with lost sales [30], and band policies with emergency supply [21]. For periodic-review systems with returned items, see [31] and the references therein.

Studies on inventory systems under the backordering policy include linear demand rate models [32], exponential demand rate models [33], EOQ and EPQ models [7], discrete probability processes [34], lot-sizing systems and remanufacturing models [35], power demand models [23], and stock-dependent demand models [36,37].

Base-stock replenishment systems with lost sales are more challenging, especially when dealing with a short shelf life. Here, the inventory level is constant during a shortage, and thus, the treatment of mathematical formulae for lost sales is more difficult than for backorders [11]. Hence, only various heuristic policies have been developed for continuous-review systems with lost sales. Studies on perishable systems with lost sales include [38,39]. We further refer the reader to [22,40] for excellent reviews of base-stock inventory systems with shortages.

In this paper, we address a policy that is a mixture of backordering and lost sales. A considerable body of literature has been written on this mixture. Montgomery et al. [41] were the first to study such a policy where α is the fraction of the excess demand backordered. Rosenberg [42] and Moinzadeh [43] reformulated Montgomery et al.'s model by assuming deterministic demand so that the optimal solution can be easily obtained. Several papers have studied the (Q, r) inventory models with partial backorders and lost sales (e.g., [44] and, more recently, [18,45]). Chang and Lo [46] proposed an approach to overcome the drawback of traditional methods for improving the continuous and discrete lead time with a mixture of backorders and lost sales. Sicilia et al. [33] analyzed an inventory system with a mixture of backorders and lost sales, where the backordered demand rate is an exponential function of the customer's waiting time. Applying a Markov decision model, Wang and Tang [47] obtained the optimal dynamic rationing levels for multiple demand classes' priorities. Taleizadeh and Zamani-Dehkordi [48] presented an inventory system with partial backordering where some of the backlogged items become lost sales. Li et al. [49] presented a production–inventory (M, m) control model in which a reflected Brownian motion governs the inventory level variation and the total amount of stockout is a mixture of backordering and lost sales. Taleizadeh et al. [24] developed four independent EPQ profit-maximization problems for four different shortage situations. More recently, Wang et al. [50] built a computational model using the response surface methodology to determine the levels of factors, including the quantity of backlogged items. San-José et al. [23] derived the economic order quantity for a time-dependent power demand rate where only a fixed proportion of the demand during a stockout is satisfied.

In our study, we consider an exponentially distributed lead time and shelf life [14]. A stochastic lead time is a factor that introduces uncertainty and challenges the determination of the optimal policy. When an inventory system is characterized by random inflows and outflows, any change in lead time has an immediate impact on every step of the supply chain, as well as on the safety stock, the out-of-stock loss, and the service level [46]. In practice, a stochastic lead time fits the case where the lead time depends on different logistics factors [51]. For example, consider a retailer that has several independent suppliers, each behaving as an M/M/1 system. The lead time for each supplier can be interpreted as the total time for handling and delivering the batch. Queueing theory implies that the total (sojourn) time in such a system is an exponentially distributed random variable. We further assume a random shelf life. In particular for models with a variable lead time, the life time of items from the time they reach the retailer until the time they perish may be uncertain even for items with a known expiration date [17]. Therefore, our assumption of an exponential life time is more applicable than it seems. Here, Markovian models with random arrivals, exponential shelf lives, and total losses are discussed in [11] and the references therein.

To outline our position, an overview of the most relevant literature studies concerning the continuous-review base-stock policy is given in Table A1 in Appendix A. It seems that the discussed studies are not as comprehensive as the one we present in this paper. As far as we know, the combination of a continuous-review base-stock policy for jump processes with a mixture of backorderings and lost sales has not been explored in the literature; hence, the model developed below significantly contributes to the existing literature.

3. Description, Notation, and Assumptions

3.1. Problem Statement

We considered a continuous-review stock-level process $\mathbb{I} = \{I(t), t \geq 0\}$ fluctuating due to two independent arrival processes of negative and positive batches of items. The stock level $I(t)$ has a double-sided bound: an upper bound S and a lower bound -B, where -$B \leq I(t) \leq S$. The stock is managed according to a triple-parameter base-stock policy $(S, s, B), 0 \leq s < S, B \geq 0$, with random lead times and random shelf lives. Specifically, we assume the following:

(i) *Arrival processes*. We assume that the on-hand stock process is fluctuating due to negative and positive batches of items (downward and upward jumps, respectively). A negative batch represents items on demand or items stored for specific needs. A positive batch represents returned items, refunds, or arriving items from the manufacturer. The negative batches arrive according to a compound Poisson process with an exponentially distributed inter-arrival time at rate λ. The positive batches arrive according to another independent compound Poisson process with an exponentially distributed inter-arrival time at rate η (there are no restrictions on the ratio between η and λ). The (absolute) size of a negative batch is a random variable D, where D is independent of the arrival process, and has a discrete probability distribution function (PDF) $P_D(d), d = 1, 2, \ldots$, a cumulative distribution function (CDF) $F_D(\cdot)$, and a coefficient of variation cv_D. Similarly, the size of a positive batch is denoted by R, where R is a random variable independent of the arrival process, and has a PDF $P_R(r), r = 1, 2, \ldots$, CDF $F_R(\cdot)$, and cv_R. We note that, although we assume fixed rates λ and η, it is easy to generalize the model to support state-dependent rates $\lambda(i)$ and $\eta(i)$, respectively, as a function of i items in stock (i may be positive or negative; see Part 2 of Remark 1).

(ii) *Upper bound*. We assume that the on-hand stock process is limited from above by level S. This upper bound can be considered as the storage capacity, or the safety or sanitary constraints on the storage capacity. Thus, all items in the batch exceeding capacity S are immediately transferred to an unlimited secondary storage or warehouse, at some cost.

(iii) *Lower bound*. The stock level is bounded from below by level -B $(B \geq 0)$. The negative stock level indicates the number of permitted backlogged items.

(iv) *Control policy of the stock*. We introduce the triple-parameter base-stock (S, s, B) control policy, which generalizes the well-known (S, s) policy. Under the (S, s) policy, whenever the stock level decreases to or below s (either due to negative batches, short shelf lives, or a total loss of all on-hand items in stock), a replenishment is ordered to bring the stock level up to its maximum capacity S. The replenishment is carried out after an exponential lead time. During the lead time, new orders are not allowed; thus, at most one oncoming order exists at any given time. Our policy implements a mixture of backorders and lost sales, i.e., in the case of zero on-hand stock, B backlogged items are allowed. Each backlogged item is incurred with some financial compensation per time unit until it is satisfied. A negative batch (or portion thereof) that arrives when $I(t) =$ -B is considered as lost sales, at the price of lost cost and the decrease in the service level. Note that the special case $B = 0$ means no backorders are allowed, meaning that a negative batch (or portion thereof) that arrives when there is a zero stock is considered as lost sales. This policy is known as a *pure* lost-sales policy. Thus, under the (S, s, B) policy, the quantity replenished at time t is random, depending on $I(t)$, and equals $max(S - I(t), 0)$.

(v) *Lead time*. We assume an *exponential lead time with parameter* μ.

(vi) *Limited shelf life*. We assume that each stored item lasts a random amount of time before it becomes unusable. The shelf life of an item is an i.i.d. r.v. with an exponential distribution at rate θ. We further consider a limited shelf life of all items in stock due to a disaster, an unexpected event, or obsolescence. Here, all on-hand items lose their value at once after an exponential time at rate Ξ, whereupon the stock collapses instantaneously to zero and all items become unusable.

Based on the assumptions outlined above, we distinguish between two shortage policies for different domains of B, namely $B = 0$ and $B > 0$:

(i) **Pure lost-sales policy**. Here, we assume that $B = 0$; i.e., no backlogged items are allowed, and $I(t)$ lies in the interval $[0, S]$. Any negative batch (or portion thereof) that faces zero stock is lost. A typical sample path of $I(t)$ under the lost-sales policy is depicted in Figure 2. The black (blue) arrows present negative (positive) batches. The number of transferred items and of lost items are indicated by the dashed blue and red lines, respectively. When $I(t)$ drops below level s, a replenishment is ordered and arrives after some lead time (*LT*); the lead time duration is indicated by the dotted gray line. Finally, the actual amount replenished is represented by the yellow line. Figure 2 illustrates two scenarios. During the first lead time (*LT* 1), we see that $I(t) > 0$; i.e., all batches are satisfied. By contrast, during the second lead time (*LT* 2), there are two incidents of lost sales. In the first lost-sales event, just a portion of the batch is lost (the first red segment). In the second lost-sales event, the entire batch is lost (the second red segment). Note that, when the order arrives, $\mathbb{I}$ starts over with S items; thus, due to the zero bound, the actual quantity replenished is always less than or equal to S items.

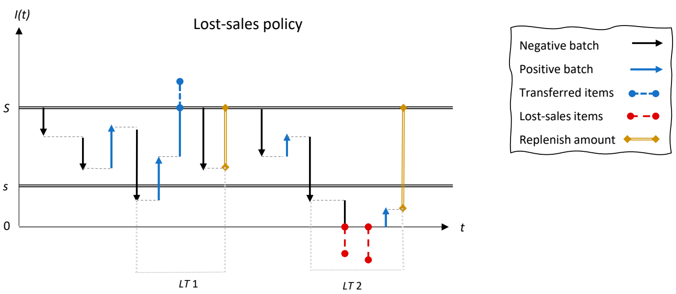

Figure 2. A typical sample path of $I(t)$ under the lost-sales policy.

(ii) **Mixture of backorders and lost-sales policy**. Here, we assume that $B > 0$; i.e., a maximum of B backlogged items are permitted, and all additional demand for items is considered as lost sales. Thus, $I(t)$ lies in the interval $[-B, S]$. For brevity, we will refer to this policy as a *backordering* policy. A typical sample path of $I(t)$ under the backordering policy is depicted on the right-hand side of Figure 3. We emphasize that, as long as $I(t) > 0$, the stock-level processes under the pure lost-sales and backordering polices have the same distribution. They are distributed differently when a negative batch arrives, causing the stock level to drop below level zero, until the replenishment order is carried out, whereupon $I(t)$ starts over at level S. This process is graphically represented under the magnifying glass in Figure 3. We see that, under the lost-sales policy, there are two incidents of lost sales, part of the first negative batch and the subsequent batch. Under the backordering policy, due to level $-B < 0$, the part that was lost in the first negative batch is backlogged and, regarding the subsequent batch, partly is backlogged and or partly is lost (the red segment on the right-hand side of Figure 3). Here, the actual quantity replenished can be even more than S items, as illustrated in Figure 3.

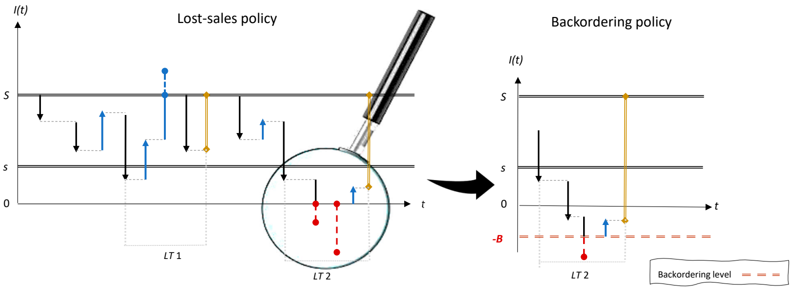

Figure 3. A typical sample path of $I(t)$ under the backordering policy.

3.2. Parameters and Costs

In this section, our notation and analysis are presented as a function of a general level B, $B \geq 0$. Substituting $B = 0$ leads to the pure lost-sales policy, while substituting $B > 0$ leads to the backordering policy. We start by summarizing the model's variables, parameters, and costs.

Decision variables

S	upper bound (the storage capacity and the replenishment level); $S > 0$ (items)
s	reorder level; $0 \leq s < S$ (items)
$-B$	lower bound (B is the maximum number of backlogged items); $B \geq 0$ (items)

Parameters and functions

μ	lead time rate (per time unit)
λ	arrival rate of negative batches (per time unit)
D	random (absolute) size of a negative batch (items)
$P_D(d), F_D(d)$	$P_D(d) = P(D = d)$ (PDF), $F_D(d) = P(D \leq d)$ (CDF)
$\widehat{F}_D(d)$	$\widehat{F}_D(d) = 1 - F_D(d) = P(D > d)$
cv_D	coefficient of variation of D
η	arrival rate of positive batches (per time unit)
R	random size of a positive batch (items)
$P_R(r), F_R(r)$	$P_R(r) = P(R = r)$ (PDF), $F_R(r) = P(R \leq r)$ (CDF)
$\widehat{F}_R(r)$	$\widehat{F}_R(r) = 1 - F_R(r) = P(R > r)$
cv_R	coefficient of variation of R
θ	expiration rate per item in stock, end of life (per time unit)
Ξ	loss rate of all items in stock (per time unit)

The costs

K_o, c_o	fixed and item purchasing cost per replenishment, respectively (USD/order)
c_θ	cost of each expired item (USD/item)
c_ϵ	cost of each lost item in case of total loss (USD/item)
c_l	penalty cost of an unsatisfied item (beyond the B backlogged items) (USD/item)
$\gamma(i)$	transfer cost of j items to a secondary storage facility (when the stock exceeds level S) (USD for j items)
c_r	overhead cost for each item of a positive batch (USD/item)
$h(i)$	holding cost for i on-hand items in stock per unit time ($0 < i \leq S$) (USD for i items)
$\beta(i)$	backordering cost for (-i) backlogged items per unit time (-$B \leq i < 0$) (USD for $-i$ items)
TC	long-run average total cost per unit time (USD/time unit)

Assumptions on the costs

(1) When the on-hand stock exceeds level S, a cost $\gamma(j)$ is charged for transferring the i excess items to a secondary storage facility. It is reasonable to assume that $\gamma(j)$ is an increasing and non-convex function of j. Formally, $\gamma'(j) > 0, \gamma''(j) \leq 0$; i.e., the growth rate of $\gamma(j)$ is decreasing (or constant) in j. That is, the greater the number of items that are transferred together in the same shipment, the lower the transfer cost of each item. This assumption is practical, since, usually, most of the transfer cost is for shipping and handling and is less affected by the number of items. Thus, we assume that $\gamma(j) = \Upsilon + c_\gamma \cdot (j)^\gamma, 0 < \gamma \leq 1$, where $\Upsilon, c_\gamma \geq 0$ are constant, and γ is the concavity coefficient. Note that a linear cost function is included, when $\gamma = 1$. It is worth pointing that our analysis is not limited to the above assumption, and also fits a general cost function.

(2) The holding cost $h(i)$ is linear in the number of the i on-hand items, and has the structure $h(i) = h \cdot i$, $0 < i \leq S$. This assumption is reasonable due to the fixed capacity of the storage, and therefore, the significant increase in the holding cost is due to the cost of each item in stock.

(3) The backordering cost $\beta(i)$ is an increasing and non-concave function of -i, for -$B \leq i < 0$; i.e., β'(-i) > 0, β''(-i) ≥ 0. That is, the growth rate of $\beta(i)$ is increasing (or constant) in -i. The convexity (non-concave) assumption means that, the more items that are backlogged (up to B items), the higher the compensation payment that the retailer pays. Thus, the backordering cost includes the additional cost due to the decrease in the service level.

(4) As in practice, we assume that $c_r < c_o$; i.e., the overhead cost of an arriving item is lower than the cost of ordering a new item.

(5) Finally, the cost of a lost-sales item is higher than the marginal cost of a backlogged item, i.e., $c_l > \beta'$(-i) for -$B \leq i < 0$. This assumption is well established because, in today's competitive reality, lost-sales items result in the loss of customers. Conversely, in the case of backordering, customers remain loyal to the retailer, albeit with increasing reluctance.

As mentioned above, it may happen that, due to a low stock level, part of a negative batch is satisfied (decreasing the stock to level 0), part of it is backlogged (up to B items), and the remaining part is lost. Similarly, it may happen that a positive batch exceeds level S, in which case, all items above level S are transferred to a secondary storage facility. Specifically, assume that $i \geq 0$ items are held in stock. When a positive batch of D items arrives, we distinguish between three cases: (1) if $D \leq i$, then the whole batch is satisfied and the stock level continues with $(i - D)$ items (0 items possible); (2) if $i < D \leq i + B$, then i items are satisfied, and the rest of the $(D - i)$ items are backlogged at a cost of $\beta(i - D)$ (note that $D - i \leq B$); (3) if $D > i + B$, then B items are backlogged at a cost of β(-B), and the remaining $(D - (i + B))$ items are lost, each at a cost of c_l. To summarize, when $D > i$, then i items are immediately satisfied, $\min(D - i, B)$ items will be satisfied later, and the remaining $\max(D - i - B, 0)$ items are lost (when $B = 0$, then i items are satisfied and $D - i$ items are lost). Accordingly, when a positive batch of R items arrives, if $R \leq S - i$, then the stock level increases by $i + R$ items; otherwise, if $R > S - i$, the stock increases to level S, and the remaining $R - (S - i)$ items are transferred to a secondary storage facility at a cost of $\gamma(R - (S - i))$.

4. Mathematical Description

4.1. The State Diagram of the Markov Chain

Let $\mathbb{I} = \{I(t) : t \geq 0\}$ denote the stock level at time t and state space Ω. Based on the exponential assumptions and the compound Poisson processes, it is clear that the stock process $\mathbb{I}$ is a continuous-time Markov chain (CTMC). Assume that $I(t) = i$ items are held in stock. When $i \in \{s+1, \ldots, S\}$, two different scenarios should be considered according to the replenishment placement: (1) with oncoming replenishment, when the stock increases from level s or below to level i by a positive batch, or (2) without oncoming replenishment, when the stock decreases from levels $i+1, \ldots, S$ to level i by a negative batch, or increases from levels $s+1 \ldots i-1$ to level i by a positive batch. By contrast, when $i \in \{\text{-}B, \ldots, s\}$, only the case with oncoming replenishment holds. Thus, systems with i items are split into two types, according to their replenishment placement. To distinguish between the two types, we mark states with oncoming replenishment by a superscript plus, and states without replenishment by a superscript minus. That is, the state space Ω is composed of two sets: *(i)* the set $\Omega^+ = \{\text{-}B, \ldots, S\}$ that includes states during lead time; *(ii)* the set marked by a minus $\Omega^- = \{(s+1)^-, (s+2)^-, \ldots, S^-\}$ that includes states without replenishment. The state space is, thus, Ω=$\Omega^+ \cup \Omega^-$. Figure 4 presents the state diagram of the Markov chain $\mathbb{I}$. For clarification, the states with oncoming replenishment are shaded in yellow (without a shortage) or red (with a shortage), and states without replenishment are shaded in in gray; the transition rates are marked by arrows.

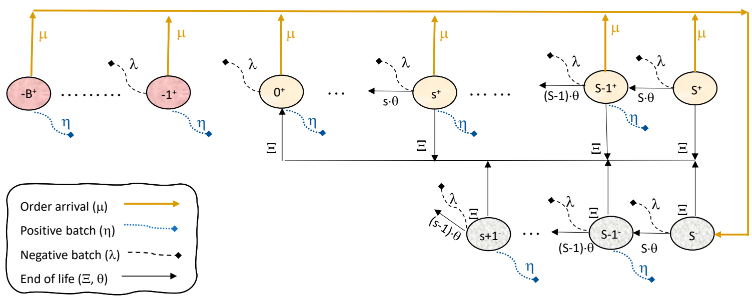

Figure 4. The state diagram of the Markov chain.

Let

$$P^+(i,j,t) = P(I(t) = j \mid I(0) = i), i \in \Omega, j \in \Omega^+,$$
$$P^-(i,j,t) = P(I(t) = j \mid I(0) = i), i \in \Omega, j \in \Omega^-. \tag{1}$$

Let $\pi_j^+ = \lim_{t\to\infty} P(i,j,t)$, and $\pi_j^- = \lim_{t\to\infty} P^-(i,j,t)$. Then, $\pi_j^+ \left(\pi_j^-\right)$ denotes the steady-state probability of having $j, j \in \Omega^+$ $(j \in \Omega^-)$ items in stock with (without) oncoming replenishment. Applying Markov theory, we obtain that the steady-state distribution of the stock level exists as the state space is finite, and the Markov chain is irreducible. Denote by $\lambda_j = \lambda \cdot p_D(j), j = 1, 2, \ldots, \infty$ the arrival rate of a positive batch with j items, and by $\eta_j = \eta \cdot p_R(j), j = 1, 2, \ldots, \infty$ the arrival rate of a negative batch with j items. By the law of total probability, we have:

$$\lambda = \sum_{j=1}^{\infty} \lambda_j, \quad \eta = \sum_{j=1}^{\infty} \eta_j. \tag{2}$$

In addition, denote by $\widehat{F}_D(k) = 1 - F_D(k) = p(D > k)$ the probability of a negative batch being larger than k items, and by $\widehat{F}_R(k) = 1 - F_R(k) = 1 - P(R > k)$ the probability of a negative batch being larger than k items.

Claim 1. *Assume that $s > 0$ and $B > 0$. It is easy to verify that the steady-state probabilities $\{\pi_i^+, \pi_{i'}^-\}_{i\in\Omega^+, i'\in\Omega^-}$ for the stock process $\mathbb{I}$ satisfy the following balance equations.*
States during lead time:

$$(i)\ \pi_S^+(\lambda + S\cdot\theta + \Xi + \mu) = \eta \sum_{k=-B}^{S-1} \pi_k^+ \cdot \widehat{F}_R(S-k-1), \qquad i = S,$$

$$(ii)\ \pi_i^+(\lambda + i\cdot\theta + \Xi + \eta + \mu) = (i+1)\cdot\theta\pi_{i+1}^+ + \sum_{k=i+1}^{S} \pi_k^+ \cdot \lambda_{k-i} + \sum_{k=-B}^{i-1} \pi_k^+ \cdot \eta_{i-k}, \qquad s+1 \le i \le S-1,$$

$$(iii)\ \pi_s^+(\lambda + s\cdot\theta + \Xi + \eta + \mu) = \left(\pi_{s+1}^+ + \pi_{s+1}^-\right)(s+1)\cdot\theta + \sum_{k=s+1}^{S} \left(\pi_k^+ + \pi_k^-\right)\cdot\lambda_{k-s} + \sum_{k=-B}^{s-1} \pi_k^+ \cdot \eta_{s-k}, \qquad i = s,$$

$$(iv)\ \pi_i^+(\lambda + i\cdot\theta + \Xi + \eta + \mu) = (i+1)\cdot\theta\pi_{i+1}^+ + \sum_{k=i+1}^{S} \pi_k^+ \cdot \lambda_{k-i} + \sum_{k=s+1}^{S} \pi_k^- \cdot \lambda_{k-i} + \sum_{k=-B}^{i-1} \pi_k^+ \cdot \eta_{i-k}, \qquad 0 < i < s,$$

$$(v)\ \pi_0^+(\lambda + \eta + \mu) = \pi_1^+ \cdot\theta + \sum_{k=1}^{S} \pi_k^+ \cdot \lambda_k + \sum_{k=s+1}^{S} \pi_k^- \cdot \lambda_k + \sum_{k=-B}^{-1} \pi_k^+ \cdot \eta_{-k} + \Xi\cdot\left(\sum_{k=1}^{S} \pi_k^+ + \sum_{k=s+1}^{S} \pi_k^-\right), \qquad i = 0,$$

$$(vi)\ \pi_i^+(\lambda + \eta + \mu) = \sum_{k=i+1}^{S} \pi_k^+ \cdot \lambda_{k-i} + \sum_{k=s+1}^{S} \pi_k^- \cdot \lambda_{k-i} + \sum_{k=-B}^{i-1} \pi_k^+ \cdot \eta_{i-k}, \qquad -B < i < 0,$$

$$(vii)\ \pi_{-B}^+(\eta + \mu) = \lambda\left[\sum_{k=-B+1}^{S} \pi_k^+ \cdot \widehat{F}_D(k+B-1) + \sum_{k=s+1}^{S} \pi_k^- \cdot \widehat{F}_D(k+B-1)\right], \qquad i = -B.$$

States without replenishment:

$$(viii)\ \pi_S^-(\lambda + S\cdot\theta + \Xi) = \eta \sum_{k=s+1}^{S-1} \pi_k^- \cdot \widehat{F}_R(S-k-1) + \mu \sum_{k=-B}^{S} \pi_k^+, \qquad i = S,$$

$$(ix)\ \pi_i^-(\lambda + i\cdot\theta + \Xi + \eta) = (i+1)\cdot\theta\cdot\pi_{i+1}^- + \sum_{k=i+1}^{S} \pi_k^- \cdot \lambda_{k-i} + \sum_{k=s+1}^{i-1} \pi_k^- \cdot \eta_{i-k}, \qquad s+1 < i \le S-1, \tag{3}$$

$$(x)\ \pi_{s+1}^-(\lambda + (s+1)\cdot\theta + \Xi + \eta) = (s+2)\cdot\theta\cdot\pi_{s+2}^- + \sum_{k=s+2}^{S} \pi_k^- \cdot \lambda_{k-s-1} \qquad i = s+1.$$

When $s = 0$ and $B > 0$, equations (iii) and (iv) of Claim 1 are omitted. When $s > 0$ and $B = 0$ (i.e., the pure lost-sales policy), then $\Omega^+ = \{0, \ldots, S\}$, $\Omega^- = \{(s+1)^-, \ldots, S^-\}$ and Equations (v)–(vii) of Claim 1 are replaced by

$$\begin{aligned}\pi_0^+(\eta + \mu) =& \lambda\left[\sum_{k=1}^{S} \pi_k^+ \cdot \widehat{F}_D(k-1) + \sum_{k=s+1}^{S} \pi_k^- \cdot \widehat{F}_D(k-1)\right] \\ &+ \pi_1^+\theta + \Xi\cdot\left(\sum_{k=1}^{S} \pi_k^+ + \sum_{k=s+1}^{S} \pi_k^-\right), \quad i = B = 0.\end{aligned} \tag{4}$$

Otherwise, if $s = B = 0$, then $\Omega^+ = \{0, \ldots, S\}$, $\Omega^- = \{1^-, \ldots, S^-\}$, and Equations (iii)–(vii) of Claim 1 are replaced by Equation (4).

4.2. The Expected Total Cost

The cost structure includes a fixed cost and a variable cost for each replenishment, a cost for handling each arriving item of a positive batch, a cost for transferring each item exceeding level S, a lost cost for each unsatisfied item, a cost for each expired item, a cost for a total collapse of the stock, and a cost for maintaining the stock (i.e., a holding cost for each on-hand item and a backordering cost for each backlogged item, if allowed). Our aim is to minimize the average total cost TC per time unit. To that end, denote by I^+ the average number of on-hand (positive) items in stock per time unit, and by I^- the average number of backlogged items per time unit. Formally,

$$I^+ = \sum_{i=1}^{S} i \cdot \pi_i^+ + \sum_{i=s+1}^{S} i \cdot \pi_i^-, \quad I^- = -\sum_{i=-B}^{-1} i \cdot \pi_i^+. \tag{5}$$

(Recall that, when $B = 0$, backordering is not allowed, and thus, $I^- = 0$.)

Applying Markov theory, the long-run expected cost components can be obtained in terms of the steady-state probabilities, and are given by

$$\begin{aligned} TC = & \underbrace{\sum_{i=-B}^{S} \mu\pi_i^+ \cdot (K_o + c_o(S-i))}_{\text{Replenishment cost}} + \underbrace{c_r\eta \cdot E(R)}_{\text{Overhead cost}} + \underbrace{\sum_{i=1}^{S} h(i) \cdot \pi_i^+ + \sum_{i=s+1}^{S} h(i) \cdot \pi_i^-}_{\text{Holding cost}} + \underbrace{\sum_{i=-B}^{-1} \beta(i) \cdot \pi_i^+}_{\text{Backordering cost}} \\ & + \underbrace{\sum_{i=-B}^{S} \sum_{k=S+1-i}^{\infty} \eta_k \cdot \gamma(i+k-S) \cdot \pi_i^+ + \sum_{i=s+1}^{S} \sum_{k=S+1-i}^{\infty} \eta_k \cdot \gamma(i+k-S) \cdot \pi_i^-}_{\text{Transfer cost}} \\ & + \underbrace{(c_\theta\theta + c_\epsilon\Xi) \cdot I^+}_{\text{End-of-life costs}} + \underbrace{c_l \left(\sum_{i=-B}^{S} \sum_{k=i+B+1}^{\infty} \lambda_k \cdot (k-i-B) \cdot \pi_i^+ + \sum_{i=s+1}^{S} \sum_{k=i+B+1}^{\infty} \lambda_k \cdot (k-i-B) \cdot \pi_i^- \right)}_{\text{Lost-sales cost}}. \end{aligned} \tag{6}$$

We emphasize that, although the costs incurred by random shelf life and total loss have similar expressions, they are derived in different paths. To be specific, when $i > 0$ items are held in stock during the lead time, each of these items has an exponentially distributed shelf life at rate θ. By the exponential properties, the time until the first item expires is exponential at rate $i \cdot \theta$. The cost incurred by random shelf life is, thus, $\left(c_\theta \cdot (i \cdot \theta) \cdot \pi_i^+\right)$ (here, the multiplication by i is due to the *rate* of the minimum of exponentially distributed random variables). By contrast, when a total loss event occurs at rate Ξ, all i items in stock become useless, each at cost c_ϵ. Thus, the loss cost is $\left((i \cdot c_\epsilon) \cdot \Xi \cdot \pi_i^+\right)$ (here, the multiplication by i is due to the *total* cost of the i items). Similar calculations lead to the case without oncoming replenishment.

Remark 1. ***1. Constant costs.*** *We assume that* $h(i) = c_h \cdot i, i > 0$*. It is easy to verify that the holding cost can be written as* $\sum_{i=1}^{S} h(i) \cdot \pi_i^+ + \sum_{i=s+1}^{S} h(i) \cdot \pi_i^- = c_h \cdot I^+$*. Similarly, if we assume that the backordering cost has the structure* $\beta(i) = -c_\beta \cdot i, i < 0$*, i.e., the backordering cost is constant per item, then it can be written as* $\sum_{i=-B}^{-1} \beta(i) \cdot \pi_i^+ = c_\beta \cdot I^-$*.*

2. State-dependent rates. *Although we assume fixed rates* λ *and* η*, it is easy to generalize the model to support state-dependent rates* $\lambda(i)$ *and* $\eta(i)$*. In this case, we make the following adjustments:*

(a) *Replace* λ *by* $\lambda(i)$ *on the left-hand side of Claim 1* (1–6, 8–10)*. Similarly, insert* λ *into the sum in Claim 1* (7)*, and replace* λ *by* $\lambda(k)$*.*

(b) *Replace* $\pi_i^\pm \cdot \lambda_k$ *by* $\pi_i^\pm \cdot \lambda_k(i)$ *and* $\pi_i^\pm \eta_k$ *by* $\pi_i^\pm \cdot \eta_k(i)$ *in Claim 1* (1–6, 8–10) *and in Equation* (6)*.*

(c) *The overhead cost* $c_r \cdot \eta \cdot E(R)$ *in Equation* (6) *becomes* $c_r \cdot \overline{\eta} \cdot E(R)$*, where* $\overline{\eta}$ *is the average size of a positive batch per time unit, i.e.,* $\overline{\eta} = \left(\sum_{n=-B}^{S} \eta(i) \cdot \pi_i^+ + \sum_{n=s+1}^{S} \eta(i) \cdot \pi_i^- \right)$*.*

5. Numerical Examples

In this section, we study, numerically, the sensitivity of the optimal controllers to the system's parameters and costs; we will focus mainly on the effect of the random arrivals (rate, mean, and variance) and the lost vs. backlogged costs on the pure lost-sales and backordering policies. We start with the pure lost-sales policy (Section 5.1). Here, we perform a hierarchical two-step sensitivity analysis. In the first step, we assume a fixed batch size (for all arrivals); then, in the second step, we extend the analysis to include batch size variability and, similarly, for the corresponding backordering policy (Section 5.2). The analysis is completed by comparing the policies and identifying situations where one policy is more cost effective than another. We note that, although our mathematical analysis does not assume any restrictions on the ratio between η and λ, we nevertheless see that, typically, the output rate exceeds the input rate. Thus, we focus on the cases where $\lambda \geq \eta$.

5.1. Pure Lost-Sales Policy

Under the pure lost-sales policy, any unsatisfied item is lost. Setting $B = 0$ in (3)–(6) leads to the steady-state probabilities and the corresponding costs. We start with constant positive and negative batch sizes.

Fixed batch size:

In order to study the effect of the system's parameters on the optimal control parameters, we start with constant batch sizes $D, R = \{1, 2, 3\}$; i.e., $p_D(D) = 1$ and 0 otherwise, and $p_R(R) = 1$ and 0 otherwise. Here, $\lambda = \lambda_D$, and $\eta = \eta_R$. The rate λ varies in $\{5, 7.5, 10\}$, and we fix $\eta = 5$; thus, the percentage of positive arrivals is $\eta/(\eta + \lambda) = \{50\%, 40\%, 33.3\%\}$, respectively. The total loss rate is set to $\Xi = 0.025$; the perishability rate is $\theta = 0.1$; the lead time μ varies in $\{0.05, 0.1\}$. We assume that

$$K_o = 50,\ c_o = 2.5,\ c_\theta = 1,\ c_\epsilon = 1,\ c_r = 0.5,\ c_h = 1. \tag{7}$$

We further let $c_l \in \{10, 25, 50\}$, and $\gamma(i) = 10 + i$ (i.e., the transfer cost includes a fixed component $\Upsilon = 10$ and a cost per each transferred item $c_\gamma = 1$). Our aim is to derive the optimal S^* and s^* minimizing the total cost as given in Equation (6). Clearly, the optimal control parameters depend on all parameters and costs. However, we focus mainly on the impact of the arrival rate, lead time, batch size, and penalty cost. Accordingly, Table 1 presents the optimal S^*, s^* and the total cost $TC^*(S^*, s^*, B = 0)$ as functions of λ, μ, and c_l for the batch size combinations $(D, R) = \{(1,1), (2,1), (2,2), (3,1), (3,2), \text{and } (3,3)\}$. We note that we assume $D \geq R$ since, when dealing with stock management, it only makes sense that the average amount of outflows exceeds the average amount of inflows.

Table 1. (S^*, s^*) and TC^* as functions of $\lambda,\ \mu,\ c_l$ for different combinations of (D, R).

B = 0		D, R											
S^*, s^*		D = 1, R = 1		D = 2, R = 1		D = 2, R = 2		D = 3, R = 1		D = 3, R = 2		D = 3, R = 3	
TC^*		μ = 0.05	μ = 0.1	μ = 0.05	μ = 0.1	μ = 0.05	μ = 0.1	μ = 0.05	μ = 0.1	μ = 0.05	μ = 0.1	μ = 0.05	μ = 0.1
λ = 5	c_l = 10	15.0 15.91	15.0 17.80	27.0 53.26	27.0 53.08	25.0 28.43	24.0 30.68	49.14 98.97	47.12 96.17	34.0 60.66	34.0 61.08	34.0 40.53	33.0 42.98
	c_l = 25	18.0 23.39	18.0 24.2	63.32 113.55	56.26 102.39	33.0 43.89	32.0 44.27	117.76 216.2	103.64 193.35	71.34 125.72	64.28 114.99	47.0 64.20	45.0 64.06
	c_l = 50	23.2 35.17	22.2 34.03	108.74 200.66	90.65 167.32	43.9 67.44	40.8 63.96	206.160 385.57	167.124 316.61	119.79 219.77	100.62 185.82	62.17 99.53	58.15 93.53
λ = 7.5	c_l = 10	17.0 30.42	17.0 31.41	48.14 97.97	46.13 94.98	31.0 58.28	31.0 58.47	80.38 167.54	77.33 160.49	62.20 124.67	59.18 120.77	46.3 86.09	45.2 85.42
	c_l = 25	35.12 62.21	32.10 57.66	117.77 214.53	102.63 189.45	68.33 121.86	62.27 110.95	201.145 370.13	173.130 324.36	148.101 272.19	130.84 240.63	101.56 181.43	91.47 164.04
	c_l = 50	59.33 108.22	50.25 92.28	204.135 383.02	165.122 313.79	115.77 213.72	97.60 179.92	353.280 662.84	284.228 540.03	258.200 485.34	210.159 398.37	172.123 319.16	143.97 267.39
λ = 10	c_l = 10	26.0 51.66	26.0 51.25	70.30 143.79	66.27 137.98	50.13 100.89	49.11 98.21	114.63 236.77	197.57 225.51	94.45 192.53	88.40 184.32	74.27 150.05	71.24 145.00
	c_l = 25	60. 32 110.94	54.26 99.23	172.12 317.32	149.103 278.29	120.78 219.36	105.64 194.55	286.219 525.14	250.190 458.48	231.174 424.95	199.144 372.40	179.126 327.73	156.104 289.43
	c_l = 50	105.72 196.59	87.55 163.03	298.247 568.06	243.192 462.98	209.160 390.52	170.130 321.33	392.367 962.26	350.319 767.24	360.316 762.18	326.264 619.97	312.252 584.41	253.196 479.47

We see that, as expected, S^*, s^* and TC^* are increasing in λ, c_l, and D and decreasing in μ; clearly, as demand increases (rate and size) and as the costs of the shortage increase (i.e., c_l increases), the control parameters and the total cost increase. In addition, the effect of μ is significant as a function of c_l; that is, for high values of c_l, shortening the lead time causes a more significant decrease in S^* and s^* (e.g., for $c_l = 10$, the results for $\mu = 0.05$ and $\mu = 0.1$ are almost the same, in contrast to the results for $c_l = 25, 50$). Table 1 further shows that the impact of the arriving batches on the system's performance is sharply pronounced. To emphasize this impact, we distinguish between batch size (D, R), where $D = R$, and batch size (D, R), where $D > R$.

We start with batch size (D, R), where $D = R$. Figure 5a,b show the curves S^* and TC^*, respectively, as functions of λ and c_l, where $(D, R) = \{(1, 1), (2, 2), (3, 3)\}$. Similar behavior is observed for s^*. The black, blue, and red lines refer to $\lambda = \{5, 7.5, 10\}$, respectively. The solid, dashed, and dotted lines refer to $c_l = \{10, 25, 50\}$, respectively.

Figure 5 (and the results for s^*) imply that S^*, s^*, and TC^* have a similar pattern: the growth rate is increasing in λ and c_l; we see that the slopes of the red ($\lambda = 10$) and dotted ($c_l = 50$) lines increase more sharply than the other lines. Somewhat surprisingly, it further seems that S^* and TC^* are almost linear as a function of $D = R$. That is, when $D = R = k$, $k = 2, 3 \ldots$, the control parameter S^* and the total cost TC^* may be approximated by k times the results for the unit batch size; i.e.,

$$\begin{aligned} S^*(D = R = k) &\approx k \cdot S^*(D = R = 1), \\ TC^*(D = R = k) &\approx k \cdot TC^*(D = R = 1). \end{aligned} \tag{8}$$

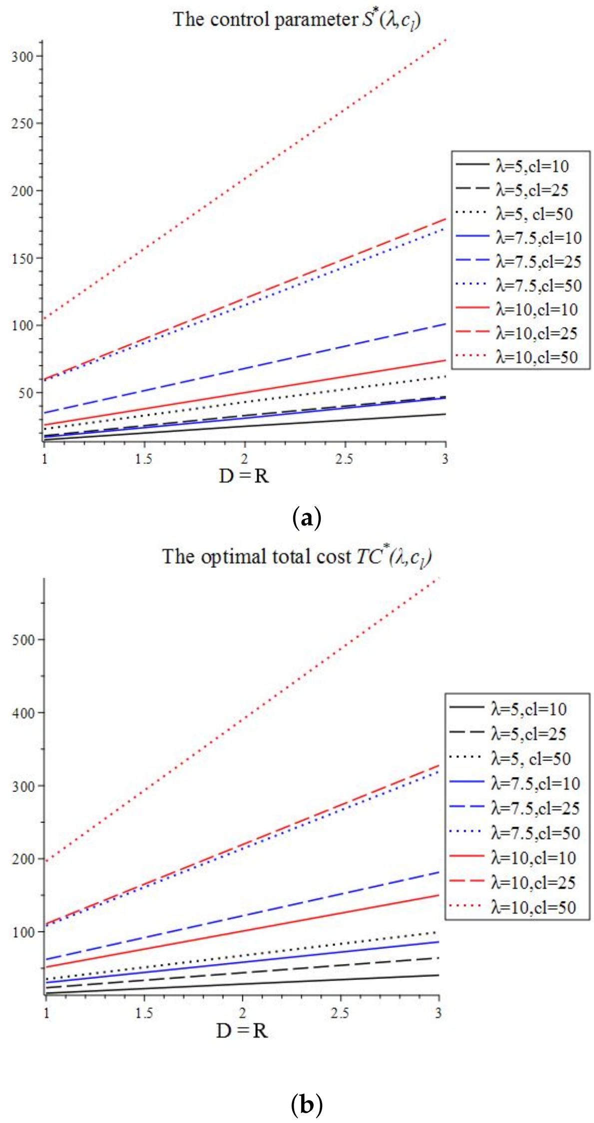

Figure 5. The optimal S^* (**a**) and TC^* (**b**) as functions of λ, c_l for $D = R = \{1, 2, 3\}$.

(This approximation barely works for s^*.) To explore the implications of this result, let $\%error$ be the percentage of the error obtained by using (8), i.e.,

$$\%error_S = \frac{S^*(D=R=k) - k{\cdot}S^*(D=R=1)}{S^*(D=R=k)},$$
$$\%error_{TC} = \frac{TC(D=R=k) - k{\cdot}TC^*(D=R=)}{TC^*(D=R=k)}, k=2,3. \quad (9)$$

Table 2 tabulates $\%error_S, \%error_{TC}$ for $\mu = 0.05, \lambda = \{5, 7.5, 10\}$ and $c_l = \{10, 25, 50\}$. We see that, approximating S^* and TC^* by using (8) is most effective for high λ and c_l. It should be noted that a similar approximation for s^* performs significantly worse and yields lower values. Thus, we recommend that (8) be applied as an upper bound for S^* and TC^*, especially for high λ and c_l and low k.

Table 2. The percentage $\%error_S, \%error_{TC}$ for $\lambda = \{5, 7.5, 10\}$ and $c_l = \{10, 25, 50\}$.

		$k=2$		$k=3$	
		$\%error_S$	$\%error_{TC}$	$\%error_S$	$\%error_{TC}$
$\lambda = 5$	$c_l = 10$	−20%	−11.9%	−32.3%	−17.7%
	$c_l = 25$	−9.0%	−6.58%	−14.89%	−9.29%
	$c_l = 50$	−6.97%	−4.30%	−11.29%	−6.0%
$\lambda = 7.5$	$c_l = 10$	−9.67%	−4.39%	−10.86%	−6.0%
	$c_l = 25$	−2.94%	−2.12%	−3.96%	−2.86%
	$c_l = 50$	−2.6%	−1.27%	−2.9%	−1.7%
$\lambda = 10$	$c_l = 10$	−4%	−2.4%	−5.4%	−3.28%
	$c_l = 25$	0%	−1.14%	−0.55%	−1.55%
	$c_l = 50$	−0.47%	−0.68%	−0.96%	−0.91%

We now proceed to batch size (D, R), where $D > R$. Here, increasing R while keeping D fixed decreases the control parameters; probably, since the stock level increases only with positive batches, the risk of a shortage decreases, and thus, S^* and s^* are set lower. In order to delve deeper into the impact of the inflows, we fix $\lambda = 7.5$ and $D = 3$. Figure 6 represents S^* (big circles) and s^* (small circles) for $R \in \{1, 2, 3\}$ and $c_l \in \{10, 25, 50\}$. The figure clearly shows the effect of R on lowering S^* and s^*, and the effect of c_l on increasing S^* and s^* (with a slight increase in the difference $(S^* - s^*)$). This means that, as the cost of the shortage increases, it is profitable to order earlier and in larger quantities. To complete the study, Figure 7 shows the curves of TC^* as a function of R, for $D = \{2, 3\}$ and $c_l = \{10, 25, 50\}$. Accordingly, we see that TC^* is increasing in c_l, and decreasing in R; the slope of that decrease seems to be independent of D (the solid, dashed, and dotted lines seem to be parallel). Figure 7 further shows that the slope of the decrease of TC^* as a function of R becomes sharper as c_l increases, which emphasizes the impact of inflows on the total cost and the need to integrate them when deriving the optimal control parameters, especially when shortages are costly.

To complete our study of the effect of the system's parameters on the optimal controllers when the batch size is fixed, we compare systems that have the same average number of arriving batches. Specifically, we focus on systems with the same λD and ηR. Our numerical study includes three such combinations of (D, R):

(*i*) Systems with $(\lambda = 5, \eta = 5, D = 2, R = 1)$ and $(\lambda = 10, \eta = 5, D = 1, R = 1)$, where $\lambda D = 10$ and $\eta R = 5$ (the submatrices shaded in blue in Table 1);

(*ii*) Systems with $(\lambda = 7.5, \eta = 5, D = 2, R = 1)$ and $(\lambda = 5, \eta = 5, D = 3, R = 1)$, where $\lambda D = 15$ and $\eta R = 5$ (the submatrices shaded in red in Table 1);

(*iii*) Systems with $(\lambda = 7.5, \eta = 5, D = 2, R = 2)$ and $(\lambda = 5, \eta = 5, D = 3, R = 2)$, where $\lambda D = 15$ and $\eta R = 10$ (the submatrices shaded in olive in Table 1).

The results show that, overall, systems with the same λD and ηR have similar optimal control parameters and total cost. However, it should be noted that the values are slightly more sensitive to changes in the batch size than to changes in the arrival rate. For example, comparing the systems (i) shows that (recall that increasing λ and D increases the controllers and cost), although both have $\lambda D = 10$, the system with $\lambda = 5$, $D = 2$ yields slightly higher values ($S^* = 27, s^* = 0, TC^* = 53.26$) compared to the system with $\lambda = 10, D = 1$ (here, we obtain $S^* = 26, s^* = 0, TC^* = 51.66$). Thus, decreasing D is more impactful than increasing λ.

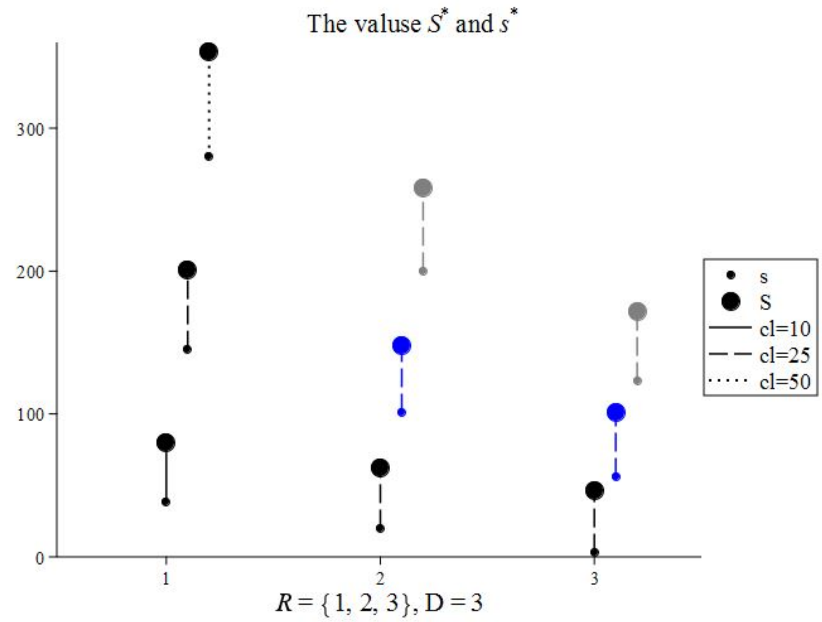

Figure 6. S^*, s^* for $D = 3$, $R = \{1, 2, 3\}$, $c_l = \{10, 25, 50\}$, $\lambda = 7.5$.

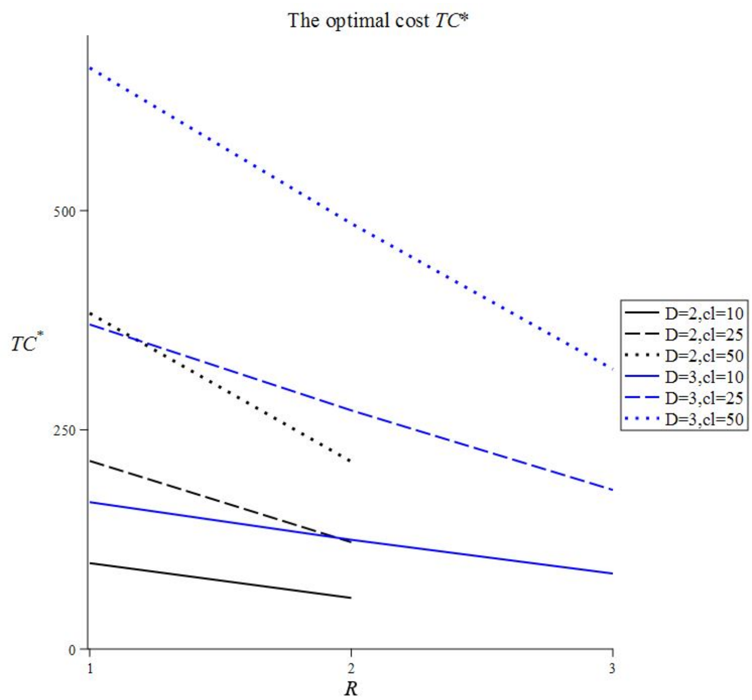

Figure 7. TC^* as a function of R, for $D = \{2, 3\}$, $c_l = \{10, 25, 50\}$.

Variable batch size:

Next, we allow D and R to vary. We fix $E(D) = 3$ and $E(R) = 2$, and assume the following four distributions of D with coefficient of variation $cv_D = \{0, 0.66, 1.15, 2.21\}$ (where $E(D) = 3$):

$$(i)\ \begin{array}{l} D = 3 \\ (cv = 0) \end{array} \quad (ii)\ \begin{array}{l} R = \left\{ \begin{array}{cc} 1 & 0.5 \\ 5 & 0.5 \end{array} \right\} \\ (cv = 0.66) \end{array} \quad (iii)\ \begin{array}{l} R = \left\{ \begin{array}{cc} 1 & 0.75 \\ 9 & 02.5 \end{array} \right\} \\ (cv = 1.15) \end{array} \quad (iv)\ \begin{array}{l} D = \left\{ \begin{array}{cc} 1 & 8/9 \\ 25 & 1/9 \end{array} \right\} \\ (cv = 2.21) \end{array} \tag{10}$$

and five distributions of R with coefficient of variation $cv_R = \{0, 0.5, 0.86, 1.41, 2.39\}$ (where $E(R) = 2$):

$$(i)\ \begin{array}{l} R = 2 \\ (cv = 0) \end{array} \quad (ii)\ \begin{array}{l} R = \left\{ \begin{array}{cc} 1 & 0.5 \\ 3 & 0.5 \end{array} \right\} \\ (cv = 0.5) \end{array} \quad (iii)\ \begin{array}{l} R = \left\{ \begin{array}{cc} 1 & 0.75 \\ 5 & 02.5 \end{array} \right\} \\ (cv = 0.86) \end{array} \quad (iv)\ \begin{array}{l} R = \left\{ \begin{array}{cc} 1 & 8/9 \\ 10 & 1/9 \end{array} \right\} \\ (cv = 1.41) \end{array}$$
$$(v)\ \begin{array}{l} R = \left\{ \begin{array}{cc} 1 & 23/24 \\ 25 & 1/24 \end{array} \right\}. \\ (cv = 2.39) \end{array} \tag{11}$$

Table 3 tabulates (S^*, s^*) and TC^* for each pair of distributions $D \in \{i, \ldots, iv\}$ and $R \in \{i, \ldots, v\}$ (for a total of 20 pairs). The table includes four submatrices, each corresponding to a different combination of $\lambda \in \{5, 7.5\}$ and $c_l \in \{10, 20\}$. Here, we fix $\mu = 0.05$; other values are given in (7). Note that the shaded cells with $cv_D = cv_R = 0$ correspond to the case of $D = 3, R = 2$, as given in Table 1.

Table 3. (S^*, s^*) and TC^* as functions of cv_D, cv_R (where $E(D) = 3,\ E(R) = 2$), for different values of $\lambda,\ c_l$.

	S^*, s^* TC^*			E(R) = 2		
	E(D) = 3	***(i)*** cv_R **= 0**	***(ii)*** cv_R **= 0.5**	***(iii)*** cv_R **= 0.86**	***(iv)*** cv_R **= 1.41**	***(v)*** cv_R **= 2.39**
λ **= 5** c_l **= 10**	***(i)*** cv_D **= 0**	34.0 60.66	34.0 61.35	36.0 63.04	40.0 68.95	51.0 74.61
	(ii) cv_D **= 0.66**	36.0 63.29	37.0 63.93	38.0 65.21	42.0 70.85	52.0 76.21
	(iii) cv_D **= 1.15**	40.0 67.82	40.0 68.35	42.0 69.40	44.0 74.10	54.0 78.97
	(iv) cv_D **= 2.21**	51.0 81.14	51.0 81.50	52.0 82.17	52.0 85.30	57.0 87.45
λ **= 5** c_l **= 25**	***(i)*** cv_D **= 0**	71.34 125.72	71.34 126.91	73.35 129.64	80.40 142.46	85.39 148.63
	(ii) cv_D **= 0.66**	73.34 130.03	74.35 131.15	75.35 133.31	82.40 145.65	88.40 151.34
	(iii) cv_D **= 1.15**	78.35 137.50	79.36 138.79	80.36 140.03	85.42 151.23	94.41 156.12
	(iv) cv_D **= 2.21**	90.40 159.81	90.40 160.56	91.40 161.88	96.45 170.41	99.43 171.32
λ **= 7.5** c_l **= 10**	***(i)*** cv_D **= 0**	62.20 124.67	62.20 125.00	63.20 125.99	66.21 131.93	70.19 133.55
	(ii) cv_D **= 0.66**	63.19 126.25	63.19 126.60	63.19 127.29	66.21 133.11	71.18 134.65
	(iii) cv_D **= 1.15**	64.17 128.94	64.17 129.29	65.17 129.94	67.19 135.14	72.17 136.54
	(iv) cv_D **= 2.21**	66.16 136.92	66.15 137.36	67.16 138.11	68.16 142.63	74.14 142.85
λ **= 7.5** c_l **= 25**	***(i)*** cv_D **= 0**	148.101 272.19	149.102 272.78	150.102 274.4	157.106 287.26	158.104 287.04
	(ii) cv_D **= 0.66**	150.102 274.83	150.102 275.45	151.102 276.66	158.108 289.3	159.105 288.95
	(iii) cv_D **= 1.15**	152.103 279.42	153.103 280.05	154.103 281.22	160.109 292.92	161.107 292.31
	(iv) cv_D **= 2.21**	159.107 293.57	160.107 294.35	165.11 295.7	166.112 306.27	166.11 303.77

Table 3 shows that letting D and R vary has a similar effect on S^*, s^*, and TC^*. In most cases, increasing the variability increases the controllers; in fact, it seems that S^* is more sensitive to variability and increases faster than s^*. Surprisingly, we see that, for high cv_R, the control parameter s^* even decreases; see Column (v) of Table 3. Apparently, letting a positive batch vary has a similar effect as increasing its average size: in both cases, s^* is decreasing (as we already inferred from Table 1). This is probably to keep the stock level balanced and low and, thus, reduce costs.

Conclusion 1. *(pure lost-sales policy):*

1. *The control parameters S^*, s^* and the total cost TC^* are increasing in λ, c_l, and D, and decreasing in R and μ.*
2. *The increase of S^* and TC^* in D is relatively constant. Accordingly, when $D = R = k$, $k = 2, 3 \ldots,$ approximating S^* and TC^* by k times the corresponding values of the Poisson process (i.e., the unit batch size) performs well, especially when λ and c_l are high.*
3. *Comparing systems with the same average inflow and outflow shows similarities between the controllers and cost, where the batch size has a slightly more significant effect than the arrival rate.*
4. *Increasing the variability of the batch size increases S^*, s^*, and TC^*, especially S^*. Quite surprisingly, we see that the impact of the variability of D and R on the optimal control parameters is similar.*
5. *Investigating the effect of the batch size distribution shows that, for both D and R, the average batch size has a more significant impact on the system's performance than the variability.*
6. *Integrating positive batches (with the focus on the average size) into the analysis significantly impacts the optimal control parameters, improves the system's performance, and may reduce costs.*

5.2. Backordering Policy

We next consider the policy in which the retailer allows backordering up to B items ($B > 0$) in order to improve the service level and retain customers. Each backlogged item incurs some financial compensation per time unit until it is satisfied in the next replenishment. A negative batch (or portion thereof) that exceeds level -B is lost. Here, the level -B is a lower bound for the stock level (clearly, $B = 0$ corresponds to the pure lost-sales policy). Setting $B > 0$ in (3)–(6) leads to the steady-state probabilities and costs under the backordering policy.

Clearly, the decision variable B is impacted by all parameters of the system. Accordingly, and further in Section 5.1, we focus on the impact of λ, c_l and the distribution of D and R on the level B. To do so, we assume a backordering cost $\beta(i) = 1.5\cdot(\text{-}i)$, $i < 0$; all other values and costs are equal to those given in Table 1 and in Equation (7).

Fixed batch size:

We start with fixed batch sizes and consider the combinations $(D, R) = \{(1,1), (2,1), (2,2), (3,1), (3,2), (3,3)\}$. Let λ vary in $\{5, 7.5, 10\}$ and c_l vary in $\{10, 25, 50\}$. Here, we assume that the storage capacity is fixed to the optimal capacity of the pure lost-sales policy, i.e., $S = S^*(B = 0)$ (see Table 1). This assumption follows the reality that changing the shortage policy (from pure lost sales to backordering) usually changes the reorder point (s^*) without changing the storage capacity. For each set (λ, c_l, D, R), Table 4 presents the optimal order point s^*, the optimal backlogged level -B^*, and the cost $TC^*(S, s^*, \text{-}B^*)$ (we present the results for $\mu = 0.05$; the results for $\mu = 0.1$ vary accordingly). In the table (and similar to Table 1), systems that have the same average arriving size are shaded in the same color.

The parameters s^* and B^*. Similar to Table 1, Table 4 shows that s^* is increasing in λ, c_l, and D (for fixed R), and decreasing in R (for fixed D). We further see that B^* is increasing in c_l, and that it eventually approaches a pure backordering policy when c_l is extremely high (i.e., when $B = \infty$). However, B^* changes unexpectedly and even somewhat surprisingly. In contrast to s^*, B^* is decreasing in D and λ and increasing in R. The result is that the difference s^*-(-B^*) is increasing in D and λ. A possible explanation lies in the simultaneous vision of s^* and B^*. When D and λ are increasing and R is fixed, the stock level must be replenished fast in order to avoid shortage (i.e., s^* is increasing), so backorders become less frequent and less profitable. Thus, B^* is increasing and the number backlogged items is decreasing. By contrast, when R is increasing and D is fixed, the stock level increases, and the risk of a shortage is already decreasing. Thus, B^* increases to allow a relatively large number of backorders while still maintaining profitability. It is interesting to see that increasing D and R simultaneously increases both s^* and B^* (see the pairs $(D, R) = \{(1, 1), (2, 2), (3, 3)\}$). Similar to Equation (8), Table 4 shows that, when $D = R = k$, $k = 2, 3$, then B^* can be approximated by k times the value of B^* for the unit batch size, i.e., $B^*(k) \approx k{\cdot}B^*(1)$.

Table 4. (S, s^*, B^*) and TC^* as functions of λ, c_l, for different combinations of (D, R).

$S^*, s^*, -B^*$ TC^* $(\%E_B)$		**D = 1, R = 1**	**D = 2, R = 1**	**D = 2, R = 2**	**D = 3, R = 1**	**D = 3, R = 2**	**D = 3, R = 3**
λ = 5	c_l = 10	15.0. −7 13.39 (15.83%)	27.0. −2 52.14 (2.10%)	25.0. −13 23.12 (18.67%)	49.13. −1 98.22 (0.75%)	34.0. −5 57.37 (5.42%)	34.0 −19 32.33 (20.23%)
	c_l = 25	18.0. −18 15.74 (32.7%)	63.30. −5 109.54 (3.53%)	33.0. −36 28.42 (35.24%)	117.76. −4 213.08 (1.44%)	71.30. −13 114.68 (8.78%)	47.0. −54 40.74 (36.40%)
	c_l = 50	23.0. −∞ 16.77 (52.30%)	108.51. −∞ 158.05 (21.23%)	43.0. −∞ 30.75 (54.40%)	206.132. −∞ 336.32 (12.77%)	119.48. −∞ 149.33 (32.05%)	62.0. −∞ 45.26 (59.09%)
λ = 7.5	c_l = 10	17.0. −2 29.17 (4.10%)	48.14. −1 97.47 (0.51%)	31.0. −4 55.75 (4.34%)	80.37. −1 167.15 (0.23%)	62.19. −2 123.34 (1.06%)	46.0. −6 82.29 (4.41%)
	c_l = 25	35.10. −5 57.78 (7.12%)	117. 76. −3 212.36 (1.01%)	68.28. −11 113.03 (7.24%)	201.148. −3 368.29 (0.33%)	148.98. −7 266.71 (2.01%)	101.49. −16 168.21 (7.28%)
	c_l = 50	59.21. −∞ 75.36 (30.36%)	204.132. −∞ 335.34 (12.44%)	115.47. −∞ 146.95 (31.24%)	353. 272. −∞ 621.3 (6.26%)	258.176. −∞ 433.04 (10.77%)	172.88. −∞ 218.4 (31.57%)
λ = 10	c_l = 10	26.0 −1 51.06 (1.16%)	70.30. −1 143.56 (0.16%)	50.12. −2 99.69 (1.18%)	114.63. −1 236.61 (0.06%)	94.45. −2 191.84 (0.35%)	74.26. −3 148.26 (1.19%)
	c_l = 25	60.31. −3 108.43 (2.26%)	172.124. −2 315.86 (0.47%)	120.75. −6 214.36 (2.27%)	286.223. −2 523.83 (0.24%)	231.172. −5 421.47 (0.81%)	179.122. −9 320.23 (2.28%)
	c_l = 50	105.50. −∞ 156.67 (20.40%)	298.221. −∞ 518.18 (8.78%)	209.130. −∞ 308.35 21.05%)	392.315. −∞ 936.05 (2.72%)	360.295. −∞ 713.24 (6.42%)	253. 145. −∞ 458.46 (21.55%)

The optimal total cost TC^*. As expected, TC^* is increasing in c_l, λ, and D (for fixed R), and decreasing in R (for fixed D). Here, we see that TC^* is more sensitive to D than to R, and thus, TC^* increases also when both D and R increase. Comparing Tables 1 and 4 reveals that $TC^*(B = 0) > TC^*(B)$ in many cases; i.e., backordering is economically viable. In order to better our understanding of these cases, let $\%E_B$ be the percentage of cost reduction obtained by applying the backordering policy instead of the lost-sales policy, i.e.,

$$\%E_B = \frac{TC^*(B = 0) - TC^*(B)}{TC^*(B = 0)}. \tag{12}$$

The percentage $\%E_B$ is indicated by the italic numbers in Table 4. We see that the backordering policy yields a significant cost reduction, in particular for high c_l and low λ (in fact, in most cases, the reduction is very significant). Clearly, the greater the cost of the shortage, the greater the economic benefit of the backordering policy. By contrast, the greater the outflow of items (by increasing $\lambda D - \eta R$), the level B^* is achieved quicker, which makes the backordering policy less efficient. For example, for $c_l = 10$, $\lambda = 5$, $D = 3$, and $R = 1$, the cost reduction is 0.75%, and when $\lambda = 10$, the cost reduction is less than 1.2%. We further see that systems that have the same average arriving size (i.e., the same λD and ηR) yield similar -B^*, s^*, and TC^* (see, respectively, the blue, red, and olive submatrices in Table 4).

Variable batch size:

Next, we allow the batch sizes to vary. To do so, we apply the distributions given in (10) for D, and those given in (11) for R (where $E(D) = 3, E(R) = 2$). For each pair (cv_D, cv_R), we set the storage capacity to the corresponding value for the lost-sales policy; i.e., we set $S = S^*_{B=0}(cv_D, cv_R)$, and derive the optimal control parameters s^*, -B^*, and total cost $TC^*(S, s^*, \text{-}B^*)$. Table 5 presents the suboptimal policy for the combinations $(\lambda, c_l) = \{(5, 10), (5, 25), (7.5, 10), (7.5, 25)\}$ as functions of cv_D and cv_R. Note that the shaded cells in Table 5 refer to $D = 3, R = 2$ (i.e., where $cv = 0$; see also Table 4). We further let $\%\widetilde{E}_B$ be the percentage of cost reduction obtained by applying the backordering policy instead of the lost-sales policy, i.e.,

$$\%\widetilde{E}_B = \frac{TC^*(B = 0, cv_D, cv_R) - TC^*(B, cv_D, cv_R)}{TC^*(B = 0, cv_D, cv_R)}. \tag{13}$$

The percentage $\%\widetilde{E}_B$ is indicated by the italic numbers in Table 5.

The optimal control parameters s^* and -B^*. Table 5 shows that s^* is less sensitive to the batch size variability; the changes are minimal (if they exist) and are not monotonic. By contrast, in most cases, B^* is significantly affected by both inflow variability (cv_R) and outflow variability (cv_D). Increasing cv (for both inflow and outflow) usually increases B^*; i.e., more backlogged items are permitted. It further seems that B^* is more sensitive to cv_D, probably due to the fact that the average outflow $\lambda E(D) = \{15, 22.5\}$ is higher than the average inflow $\eta E(R) = 10$. To better understand the effect of cv_D and cv_R on B^*, Figure 8 shows the curves B^* as a function of cv_R for two combinations: (1) lower outflow and higher lost cost, $\lambda = 5, c_l = 25$ (black lines), and (2) higher outflow and lower lost cost, $\lambda = 7.5, c_l = 10$ (blue lines). For each combination, the parameter B, for $cv_D = \{0, 0.66, 1.15, 2.21\}$, is indicated by the solid, dashed, dotted, and dashed–dotted lines, respectively. It is clearly shown that, when λ is relatively high and c_l is low, B^* is weakly affected by cv_R and by cv_D (see Figure 8, the blue lines). The low values of B^* can be explained by the fact that, when outflows are frequent, and the lost cost (i.e., the cost of losing customers) is relatively low, the economic benefit of backordering decreases. Thus, the variability of the inflows contributes to the prevention of a shortage, and further reduces the need for backordering (i.e., B^* decreases). However, when outflows are less frequent, and the lost cost is high, the economic benefit of backordering increases and B^* increases (see Figure 8, the black lines); this benefit is further emphasized by the increase in the variance of cv_R and cv_D. Note that, although only the combination $\lambda = 5, c_l = 25$ is presented, B^* varies accordingly also for other values.

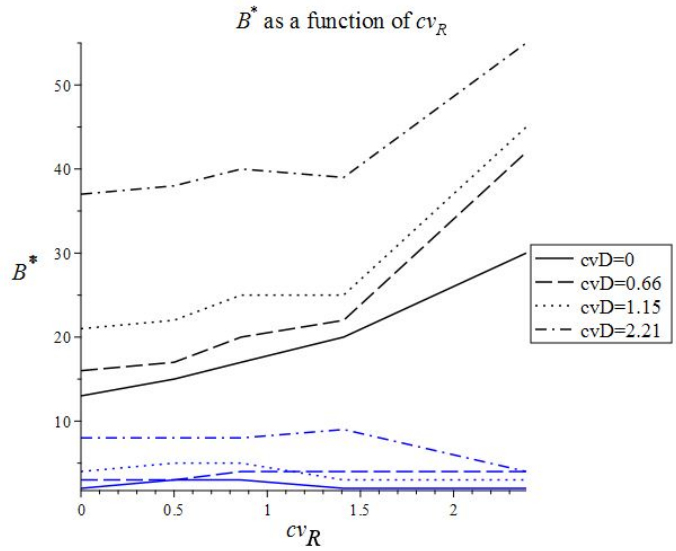

Figure 8. $B^*(cv_R)$ for (1) $(\lambda, c_l) = (5, 25)$ (black lines) and (2) $(\lambda, c_l) = (7.5, 10)$ (blue lines).

Table 5. $(S, s^*, \text{-}B^*)$ and TC^* as functions of cv_D, cv_R (where $E(D) = 3,\ E(R) = 2$), for different values of $\lambda,\ c_l$.

$S^*, s^*, \text{-}B^*$ TC^* $\%\widetilde{E}_B$		**E(R) = 2**				
E(D) = 3		**(i) $cv_R = 0$**	**(ii) $cv_R = 0.5$**	**(iii) $cv_R = 0.86$**	**(iv) $cv_R = 1.41$**	**(v) $cv_R = 2.39$**
$\lambda = 5$ $c_l = 10$	**(i) $cv_D = 0$**	34.0, −5 57.37 (5.42%)	34.0, −6 58.03 (5.41%)	36.0, −6 59.5 (5.61%)	40.0, −7 65.98 (4.30%)	51.0, −10 71.89 (3.78%)
	(ii) $cv_D = 0.66$	36.0, −6 59.17 (6.50%)	37.0, −7 59.81 (6.44%)	38.0, −7 61.1 (6.30%)	42.0, −8 67.37 (4.91%)	52.0, −11 73.03 (4.17%)
	(iii) $cv_D = 1.15$	40.0, −9 62.4 (7.99%)	40.0, −9 62.93 (7.92%)	42.0, −10 64.13 (7.59%)	44.0, −10 69.77 (5.84%)	54.0, −12 75.19 (4.78%)
	(iv) $cv_D = 2.21$	51.0, −15 72.52 (10.62%)	51.0, −15 72.91 (10.53%)	52.0, −15 73.8 (10.18%)	52.0, −14 78.26 (8.25%)	57.0, −21 82.66 (5.47%)
$\lambda = 5$ $c_l = 25$	**(i) $cv_D = 0$**	71.30, −13 114.68 (8.78%)	71.27, −15 115.31 (9.14%)	73.27, −17 116.7 (9.98%)	80.33, −20 129.29 (9.24%)	85.27, −30 129.09 (13.14%)
	(ii) $cv_D = 0.66$	73.26, −16 116.65 (10.28%)	74.27, −17 117.26 (10.59%)	75.26, −20 118.55 (11.07%)	82.33, −22 130.91 (10.12%)	88.26, −42 130.22 (13.95%)
	(iii) $cv_D = 1.15$	78.25, −21 120.36 (12.46%)	79.25, −22 120.96 (12.84%)	80.25, −25 122.13 (12.78%)	85.31, −25 133.98 (11.40%)	94,26, −45 133.29 (14.62%)
	(iv) $cv_D = 2.21$	90.20, −37 132.84 (16.87%)	90.20, −38 133.32 (16.96%)	91.20, −40 134.33 (17.01%)	96.27, −39 145.01 (14.90%)	99.22, −55 143.48 (16.25%)
$\lambda = 7.5$ $c_l = 10$	**(i) $cv_D = 0$**	62.19, −2 123.34 (1.06%)	62.19, −3 123.74 (1.00%)	63,19, −3 124.77 (0.96%)	66.21, −2 130.95 (0.74%)	70.18, −2 132.59 (0.71%)
	(ii) $cv_D = 0.66$	63.18, −3 124.41 (1.45%)	63.18, −3 124.81 (1.41%)	63.18, −4 125.77 (1.19%)	66.20, −3 131.87 (0.93%)	71.17, −3 133.47 (0.87%)
	(iii) $cv_D = 1.15$	64.15, −4 126.19 (2.13%)	64.16, −5 126.65 (2.04%)	65.16, −5 127.5 (1.87%)	67.18, −3 133.67 (1.08%)	72.16, −3 135.07 (1.07%)
	(iv) $cv_D = 2.21$	66.8, −8 131.87 (3.68%)	66.8, −8 132.36 (3.64%)	67.9, −8 133.33 (3.46%)	68.12, −9 136.06 (4.60%)	74.11, −4 141.08 (1.23%)
$\lambda = 7.5$ $c_l = 25$	**(i) $cv_D = 0$**	148.98, −7 266.71 (2.01%)	149.99, −8 267.09 (2.08%)	150.98, −9 267.9 (2.36%)	157.104, −11 280.43 (2.37%)	158.98, −25 275.65 (3.96%)
	(ii) $cv_D = 0.66$	150.98, −8 267.86 (2.53%)	150.98, −9 268.23 (2.62%)	151.98, −11 269.01 (2.76%)	158.104, −13 281.49 (2.69%)	159.98, −26 276.67 (4.24%)
	(iii) $cv_D = 1.15$	152.98, −11 269.98 (3.37%)	153.98, −12 270.38 (3.45%)	154.98, −14 271.18 (3.57%)	160.04, −15 283.51 (3.21%)	161.98, −27 278.65 (4.67%)
	(iv) $cv_D = 2.21$	159.98, −19 277.58 (5.44%)	160.98, −19 278.04 (5.54%)	165.97, −21 278.98 (5.65%)	166.103, −22 290.79 (5.05%)	166.98, −31 286.33 (5.74%)

The total cost TC^*. Table 5 shows that TC^* is increasing in both cv_D and cv_R, with a more notable growth rate in cv_D. Comparing the costs $TC^*(B > 0)$ and $TC^*(B = 0)$ (as measured by $\%\widetilde{E}_B$) shows that backordering yields a significant cost reduction, especially for low λ and high c_l. To better understand the impact of randomness on this cost reduction, Figure 9 shows the curves $\%\widetilde{E}_B$ as a function of cv_R for the combinations $(\lambda, c_l) = \{(5, 25), (5, 10), (7.5, 10)\}$ as indicated by the black, red, and blue lines, respectively. Here, also, for each combination, the results for $cv_D = \{0,\ 0.66,\ 1.15,\ 2.21\}$ are indicated by the solid,dashed, dotted, and dashed–dotted lines, respectively. Similar to above, Table 5 and Figure 9 show that, when the lost cost is low and the benefit of backordering is in doubt, adding variability to inflows only strengthens this doubt, and therefore, the economic benefit decreases, i.e., $\%\widetilde{E}_B$ decreases (see, e.g., the red and blue lines for $c_l = 10$).

By contrast, when the lost cost is high, the financial benefit of backordering increases, and this increase is boosted by the increase in the variability of the inflows (see the black lines for $c_l = 25$). To summarize, the measure $\%\widetilde{E}_B$ is decreasing in λ, and increasing in c_l and cv_D. The impact of cv_R is more challenging as the randomness of the inflows intensifies the economic benefit of the existing policy, i.e., when $\%\widetilde{E}_B$ is low (high), increasing vc_R decreases (increases) it even more.

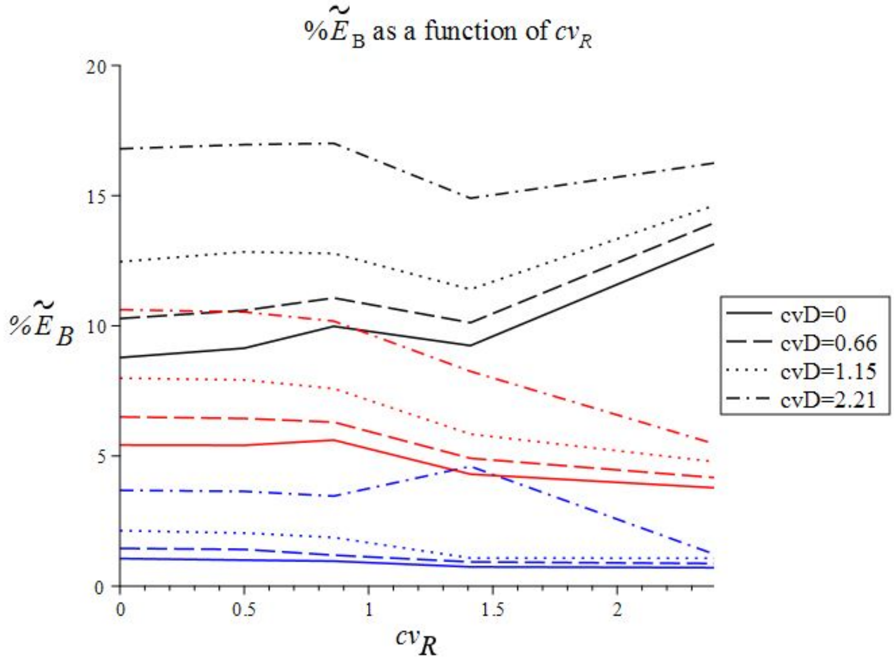

Figure 9. $\%\widetilde{E}_B(cv_R)$ for $(\lambda, c_l) = \{(5,25), (5,10), (7.5,10)\}$ as indicated by the black, red, and blue lines, respectively.

The effect of c_β. In the above discussion, we assumed that the retailer considers a backordering policy instead of a lost-sales policy. We show that backordering yields a significant cost reduction (in addition to raising the service level and maintaining customer satisfaction). To complete the discussion, it would be interesting to identify situations where the lost-sales policy is preferable. Clearly, the financial benefit of one policy or the other depends on the costs and, in particular, the interplay between the backordering cost and the lost cost, c_β and c_l, respectively. Increasing c_β (or, alternatively, decreasing c_l) decreases the economic benefit of backordering. To better understand this interplay, we assume that a backordering policy is being implemented, and that the retailer examines the policy's profitability as a function of the cost c_β. First, we fix $cv_R = 0.86$, and calculate for each set (λ, c_l, cv_D) the total cost $TC^*(S, s^*, B^*)$ as a function of c_β (the values S, s^*, and B^* are given in Table 5). For example, for $\lambda = 5, c_l = 10, cv_D = 0$, and $cv_R = 0.86$, we obtain $TC^*(S = 36, s^* = 0, \text{-}B^* = -6) = 105.420502 + 7.519728 \cdot c_\beta$. Figure 10 indicates the variable component of $TC^*(c_\beta)$ for $(\lambda, c_l) = \{(5,10), (5,25), (7.5,10), (7.5,25)\}$ by the gray, black, blue, and red lines, respectively. For each pair, the solid (dashed) line refers to $cv_D = 0$ ($cv_D = 1.15$). For example, when $(\lambda, c_l, cv_D) = (5, 10, 0)$, we obtain $7.519728 \cdot c_\beta$ (as indicated by the gray line). We see that the growth rate of $TC^*(c_\beta)$ is sharper when c_l and cv_D are high and λ is low. That is, when the lost cost is high, and fewer (but more variable) outflows arrive, the backlogged cost has a greater impact on the total cost. In particular, $TC^*(c_\beta)$ is more sensitive to changes in c_β when c_l is high (as indicated by the black and red lines for $c_l = 25$ in Figure 10).

Furthermore, let $c_{\beta\_}max$ be the maximum cost for which the backordering policy is more profitable compared to the lost-sales policy. In order to derive $c_{\beta\_}max$, we compare the total cost $TC^*(c_\beta)$ for $B > 0$ (given in Table 5) with the corresponding $TC^*(B = 0)$ of the lost-sales policy (given in Table 3). Solving $TC^*(c_\beta) = TC^*(B = 0)$ yields $c_{\beta\_}max$.

Let $cv_R = 0.86$, and let cv_D vary in $\{0, 0.66, 1.15, 2.21\}$. Figure 11 points out $c_{\beta\_}max$ for the pairs $(\lambda, c_l) = \{(5,10), (5,25), (7.5,10), (7.5,25)\}$ as indicated by the gray and black circles and the blue and red squares, respectively (i.e., each pair is assigned four values of $c_{\beta\_}max$ corresponding to cv_D). Note that, for $c_\beta > c_{\beta\_}max$, the backordering policy is less profitable. Figure 11 implies that the backordering policy has the highest economic benefit when λ is low and c_l is high. Similar to Table 5, Figure 11 shows that increasing cv_D increases the economic benefit of backordering, and eventually, the values $c_{\beta\_}max$ become closer.

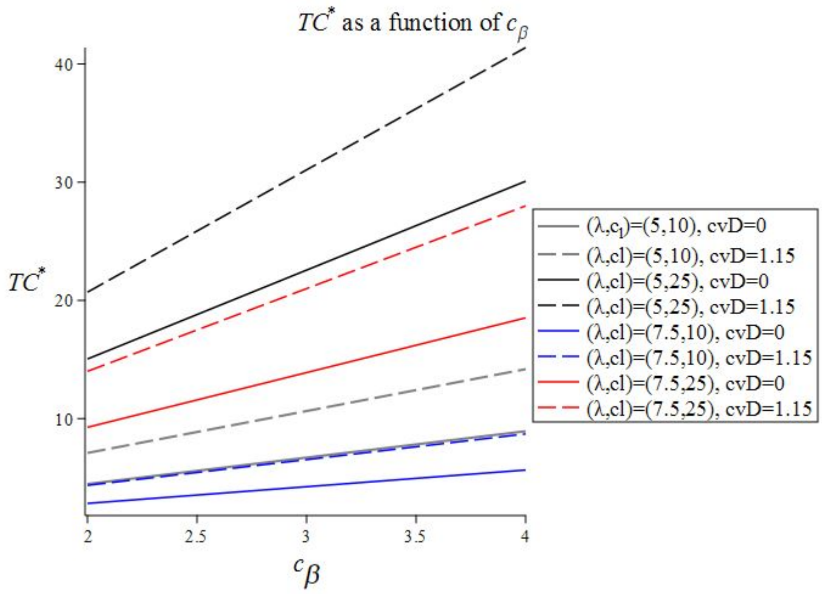

Figure 10. $TC^*(c_\beta)$ for $(\lambda, c_l) = \{(5,10), (5,25), (7.5,10), (7.5,25)\}$ as indicated by the gray, black, blue, red lines, respectively.

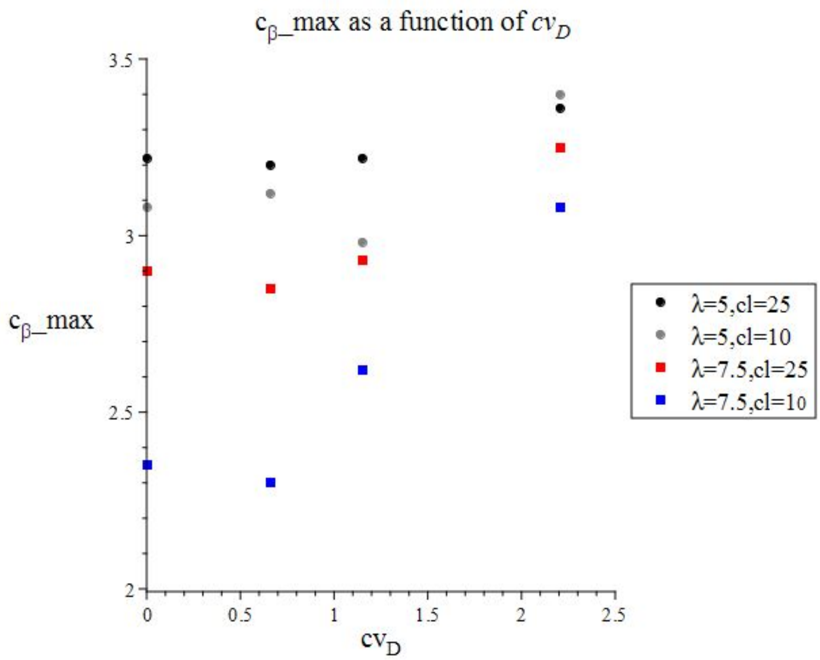

Figure 11. $c_{\beta\_}max$ for $(\lambda, c_l) = \{(5,10), (5,25), (7.5,10), (7.5,25)\}$ as indicated by the gray, black, blue, and red lines, respectively.

Conclusion 2. *(Backordering policy).*

1. *Increasing c_l and (in most cases) cv_R increases both s^* and B^*. However, the negative batch D (average and variance), the rate λ, and the average positive batch $E(R)$ have an inverse effect on s^* and B^*. The total cost TC^* is increasing in $\lambda E(D)$ and decreasing in $E(R)$; letting the batch size vary increases TC^* (except for the high cv_R).*
2. *Systems that have the same average batch size yield similar optimal control parameters, s^* and B^*, and a similar total cost TC^*.*
3. *When the batch size satisfies $D = R = k$, then B^* can be approximated by using the unit Poisson system, i.e., $B^*(k) \approx k \cdot B^*(1)$.*
4. *Comparing the backordering policy and the pure lost-sales policy shows that the former yields a significant cost reduction, particularly for high c_l and low λ. Thus, increasing the randomness of outflows increases the economic benefit of the backordering policy; however, increasing the randomness of inflows intensifies the economic benefit of the existing policy.*
5. *Accordingly, the maximum backlogged cost c_{β}_max, at which the backordering policy is economically preferable, is decreasing in λ and is increasing in c_l.*

Managerial insights:

We summarize the main managerial insights and rules of thumb of our study:

- The total cost increases with the demands (arrival rate and size) and decreases with the returns.
- The batch size distribution has a significant impact on the efficiency of the backordering policy. As a rule of thumb, we can point out that the fewer the demands (rate and quantity) and the higher the variance, the backordering policy is more recommended.
- Our study shows that integrating returns (with the focus on the average size) into the analysis significantly impacts the optimal control parameters, improves the system's performance, and may reduce costs. Specifically, when more returns arrive, we recommend increasing the backordering level (i.e., more backordered items are permitted).
- The mean batch size has a more significant impact on the system's performance than the variability. Additionally, systems that have the same average batch size yield similar optimal control parameters.

6. Concluding Remarks and Further Research

This paper studies a continuous-review triple-parameter (S, s, B) stock replenishment model for a jump process with positive and negative batch arrivals that combines backordering and lost-sales policies during a stockout. The model is characterized by random lead times, random life times (for each and all items in stock), and random arrivals (rate, mean, and variance). Using a Markovian framework, we derived the steady-state probabilities of the stock level, and constructed closed-form expressions for the average cost functions. Using a numerical analysis, we studied the impact of the different parameters, such as arrival rates, jump size distributions (mean and variance), lead times, and the costs, on the system's performance. Specifically, we focused on the impact of the arrival rates and batch sizes on the economic benefit of a backordering versus a lost-sales policy, and identified cases where one policy is more cost effective than the other. It is shown that the fewer the outflows (rate and quantity) and the higher the variance, the more economically beneficial the backordering policy. Surprisingly, more inflows increase the optimal number of backordered items and, thus, increase the economic benefit of the backordering policy.

There are several avenues for future research. In this paper, we assumed that the item's life time is an exponentially distributed random variable. A natural expansion is to assume an m-Erlang (or phase-type)-distributed life time. Under an m-Erlang distribution, the life time of each item consists of m phases, each of which is an exponentially distributed random variable. The family of m-Erlang distributions provides flexibility in model design that enables covering more cases in practice (note that as m approaches ∞, the life time becomes deterministic). Another interesting direction would be to consider an emergency supply during a stockout. This supply can be an appropriate, albeit costly, response to a stockout under both policies, $B = 0$ and $B > 0$. Here, it would be interesting to derive the optimal control parameters under the discounted cost criterion. As in many real-world relationships, the fixed order cost is paid when the order is placed, whereas the purchasing cost is paid when the stock is replenished. This difference in the timing of the payments and the cost of the backorders clearly has a significant effect on the profitability of one policy versus the other and, thus, is worth further investigation. Finally, investigating other replenishment policies, such as the (Q, r) policy with a mixture of backordering and lost sales, is an obvious extension. Under the (Q, r) policy, when the stock drops to level r, a fixed order of size Q is placed. Thus, the stock level is no longer known at the replenishment time, which makes the analysis more challenging, but worthwhile.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Acknowledgments: We sincerely thank Michael Dreyfuss for his help in designing parts of the simulation program.

Conflicts of Interest: The author declares no conflicts of interest.

Appendix A

In Table A1, we summarize recent and relevant literature studies concerning the continuous-review base-stock policy.

Table A1. Relevant literature studies concerning the continuous-review base-stock policy.

Continuous-Review Base-Stock Models		Model Features						Optimization Approach
-Returns -Backorder+ Lost Sales -Limited Shelf Life -Total Loss	Paper	Demand/Return	Shelf Life/Loss	Lead Time	Base-Stock Inventory Policy	Stockout Policy	Additional Features	
	[1]	Linear, Comp. Pois.	Infinite	Zero	Multi-targets(s, S)	Not allowed	• Disc. Criteria • Cash management	■ Hamilton-Jacobi-Bellman equation
	[9]	Brownian	Decay rate	Det., Stoch.	(Q, r)	Backorder, lost sales	• Ave. criteria	■ Kuhn-Tucker conditions ■ numerical
	[25]	Pois.	General/ no loss	General	$(S, S-1)$	Lost sales	• Extension to backorder • Ave. criteria	■ Queueing modelling ■ Algorithmic approach
	[21]	Fixed rates	Fixed rate	Fixed	(Q, r)	Backorder	• Ave. criteria • Correlated demands/returns	■ Derivation
	[13]	Linear	Decay rate	Det., Stoch.	(Q, r)	Backorder	• Ave. criteria	■ Kuhn-Tucker conditions ■ numerical
	[48]	Pois.	exp	exp	(s, Q)	Lost sales	• Ave. criteria • Two types of customers	■ Markov equations
	[6]	State-dep. Pois.	exp/ exp	exp	(S, s)	Lost sales	• Ave. criteria • Cancelations	■ Recursive formulas ■ numerical
	[11]	General	Three-period	Zero	$(S, s, S(x))$	Lost sales or backorder	• Emergency option • Disc. criteria	■ Closed-form optimum ■ Structural properties
	[4]	Linear + MAP	no	exp	(M, S, s)	Lost sales	• Disc. criteria	■ Martingales ■ First passage times analysis
	This study	**Comp. Pois.**	**exp /exp**	**exp**	**(S, s, B)**	**Lost sales, Partial Backorder**	• **Ave. criteria** • **Limited storage capacity**	■ Markov chain

References

1. Huang, S.; Lu, H.; Lin, J.; Ponte, B. On the dynamics of return collection in closed-loop supply chains. *Int. J. Prod. Res.* **2023**, *62*, 909–932. [CrossRef]
2. Jeganathan, K.; Selvakumar, S.; Saravanan, S.; Anbazhagan, N.; Amutha, S.; Cho, W.; Joshi, G.P.; Ryoo, J. Performance of stochastic inventory system with a fresh item, returned Item, refurbished item, and multi-class customers. *Mathematics* **2022**, *10*, 1137. [CrossRef]
3. Smith, E.G. How online retailers predict your perfect outfit. *Wall Str. J.* **2015**, *5*.
4. Cui, H.; Rajagopalan, S.; Ward, A.R. Predicting product return volume using machine learning methods. *Eur. J. Oper. Res.* **2020**, *281*, 612–627. [CrossRef]
5. Ambilkar, P.; Dohale, V.; Gunasekaran, A.; Bilolikar, V. Product returns management: A comprehensive review and future research agenda. *Int. J. Prod.* **2022**, *60*, 3920–3944. [CrossRef]
6. Meneses, M.; Marques, I.; Cárdenas-Barrón, A. Blood inventory management: Ordering policies for hospital blood banks under uncertainty. *Int. Oper. Res.* **2023**, *30*, 273–301. [CrossRef]
7. Gharaei, A.; Hoseini Shekarabi, S.A.; Karimi, M. Optimal lot-sizing of an integrated EPQ model with partial backorders and re-workable products: An outer approximation. *Int. J. Syst. Sci. Oper. Logist.* **2023**, *10*, 2015007. [CrossRef]
8. Li, R.; Liu, Y.; Teng, J.T.; Tsao, Y.C. Optimal pricing, lot-sizing and backordering decisions when a seller demands an advance-cash-credit payment scheme. *Eur. J. Oper. Res.* **2019**, *278*, 283–295. [CrossRef]
9. Azcue, P.; Muler, N. Optimal cash management problem for compound Poisson processes with two-sided jumps. *Appl. Math. Optim.* **2019**, *80*, 331–368. [CrossRef]
10. Barron, Y. A stochastic card balance management problem with continuous and batch-type bilateral transactions. *Oper. Res. Perspect.* **2023**, *10*, 100274. [CrossRef]
11. Kouki, C.; Legros, B.; Babai, M.Z.; Jouini, O. Analysis of base-stock perishable inventory systems with general lifetime and lead-time. *Eur. J. Oper. Res.* **2020**, *287*, 901–915. [CrossRef]
12. Presman, E.; Sethi, S.P. Inventory models with continuous and Poisson demands and discounted and average costs. *Prod. Oper. Manag.* **2006**, *15*, 279–293. [CrossRef]
13. Sahin, I. On the continuous-review (s, S) inventory model under compound renewal demand and random lead times. *J. Appl. Probab.* **1983**, *20*, 213–219. [CrossRef]
14. Pacheco-Velázquez, E.A.; Robles-Cárdenas, M.; Juárez Ordóñez, S.; Damy Solís, A.E.; Cárdenas-Barrón, L.E. A heuristic model for spare parts stocking based on Markov chains. *Mathematics* **2023**, *11*, 3550. [CrossRef]
15. Barron, Y. An order-revenue inventory model with returns and sudden obsolescence. *Oper. Lett.* **2017**, *46*, 88–92. [CrossRef]
16. Barron, Y. Integrating replenishment policy and maintenance services in a stochastic inventory system with bilateral movements. *Mathematics* **2023**, *11*, 864. [CrossRef]
17. Chen, K.; Song, J.S.; Shang, J.; Xiao, T. Managing hospital platelet inventory with mid-cycle expedited replenishments and returns. *Prod. Oper. Manag.* **2022**, *31*, 2015–2037. [CrossRef]
18. Sermkulpat, V.; Buddhakulsomsiri, J. An (R, Q) Inventory Policy for Medicine under Limited Storage with a Backlog. Doctoral Dissertation, Thammasat University, Bangkok, Thailand, 2022.
19. Rabinowitz, G.; Mehrez, A.; Chu, C.W.; Patuwo, B.E. A partial backorder control for continuous review (r, Q) inventory system with Poisson demand and constant lead time. *Comput. Oper. Res.* **1995**, *22*, 689–700. [CrossRef]
20. Barron, Y.; Hermel, D. Shortage decision policies for a fluid production model with MAP arrivals. *Int. J. Prod. Res.* **2017**, *55*, 3946–3969. [CrossRef]
21. Poormoaied, S.; Atan, Z.; de Kok, T.; Van Woensel, T. Optimal inventory and timing decisions for emergency shipments. *IISE Trans.* **2020**, *52*, 904–925. [CrossRef]
22. Bijvank, M.; Huh, W.T.; Janakiraman, G. 1. Lost-sales inventory systems. In *Research Handbook on Inventory Management*; Edward Elgar Publishing: Cheltenham Glos, UK, 2023; pp. 2–26.
23. San-José, L.A.; Sicilia, J.; Pando, V.; Alcaide-López-de-Pablo, D. An inventory system with time-dependent demand and partial backordering under return on inventory investme. *Comput. Oper. Res.* **2022**, *145*, 105861. [CrossRef]
24. Taleizadeh, A.A.; Soleymanfar, V.R.; Govindan, K. Sustainable economic production quantity models for inventory systems with shortage. *J. Clean. Prod.* **2018**, *174*, 1011–1020. [CrossRef]
25. Schrady, D.A. A deterministic inventory model for reparable items. *Nav. Res. Logist. Q.* **1967**, *14*, 391–398. [CrossRef]
26. Gajdalo, S. *Heuristics for Computing Variable Safety Levels/Economic Order Quantities for Repairable Items*; AMC Inventory Research Office, Institute of Logistics Research, US Army Logistics Management Center: Fort Lee, VA, USA, 1973.
27. Heyman, D.P. Return policies for an inventory system with positive and negative demands. *Nav. Res. Q.* **1978**, *25*, 581–596. [CrossRef]
28. Yuan, X.M.; Cheung, K.L. Modeling returns of merchandise in an inventory system. *Oper. Res. Spektrum* **1998**, *20*, 147–154. [CrossRef]
29. Schulz, T.; Voigt, G. A flexibly structured lot sizing heuristic for a static remanufacturing system. *Omega* **2014**, *44*, 21–31. [CrossRef]
30. Barron, Y.; Dreyfuss, M. A triple (S, s, ℓ)-thresholds base-stock policy subject to uncertainty environment, returns and order cancellations. *Comput. Oper. Res.* **2021**, *134*, 105320. [CrossRef]

31. Fleischmann, M.; Kuik, R. On optimal inventory control with independent stochastic item returns. *Eur. J. Oper. Res.* **2003**, *151*, 25–37. [CrossRef]
32. Choné, P.; Linnemer, L. Linear demand systems for differentiated goods: Overview and user's guide. *Int. J. Ind. Organ.* **2020**, *73*, 102663. [CrossRef]
33. Sicilia, J.; San-José, L.A.; García-Laguna, J. An inventory model where backordered demand ratio is exponentially decreasing with the waiting time. *Ann. Oper. Res.* **2012**, *199*, 137–155. [CrossRef]
34. Goedhart, J.; Haijema, R.; Akkerman, R. Modelling the influence of returns for an omni-channel retailer. *Eur. J. Oper. Res.* **2023**, *306*, 1248–1263. [CrossRef]
35. Ho, T.F.; Lin, C.C.; Lin, C.L. Determining the optimal inventory and number of shipments for a two-resource supply chain with correlated demands and remanufacturing products allowing backorder. *Mathematics* **2020**, *8*, 548. [CrossRef]
36. Jani, M.Y.; Patel, H.A.; Bhadoriya, A.; Chaudhari, U.; Abbas, M.; Alqahtani, M.S. Deterioration control decision support system for the retailer during availability of trade credit and shortages. *Mathematics* **2023**, *11*, 580. [CrossRef]
37. Thilagavathi, R.; Viswanath, J.; Cepova, L.; Schindlerova, V. Effect of inflation and permitted three-slot payment on two-warehouse inventory system with stock-dependent demand and partial backlogging. *Mathematics* **2022**, *10*, 3943. [CrossRef]
38. Chen, K.; Xiao, T.; Wang, S.; Lei, D. Inventory strategies for perishable products with two-period shelf-life and lost sales. *Int. J. Prod.* **2021**, *59*, 5301–5320. [CrossRef]
39. Kouki, C.; Arts, J.; Babai, M.Z. Performance evaluation of a two-echelon inventory system with network lost sales. *Eur. J. Oper. Res.* **2023**, *314*, 647–664. [CrossRef]
40. Perera, S.C.; Sethi, S.P. A survey of stochastic inventory models with fixed costs: Optimality of (s,S) and (s,S)-type policies-Continuous-time case. *Prod. Oper. Manag.* **2023**, *32*, 154–169. [CrossRef]
41. Montgomery, D.C.; Bazaraa, M.S.; Keswani, A.K. Inventory models with a mixture of backorders and lost sales. *Nav. Res. Logist. Q.* **1973**, *20*, 255–263. [CrossRef]
42. Rosenberg, D. A new analysis of a lot-size model with partial backlogging. *Nav. Res. Logist. Q.* **1979**, *26*, 349–353. [CrossRef]
43. Moinzadeh, K. Operating characteristics of the $(S-1,S)$ inventory system with partial backorders and constant resupply times. *Manag. Sci.* **1989**, *35*, 472–477. [CrossRef]
44. Braglia, M.; Castellano, D.; Marrazzini, L.; Song, D. A continuous review (Q,r) inventory model for a deteriorating item with random demand and positive lead time. *Comput. Oper. Res.* **2019**, *109*, 102–121. [CrossRef]
45. Guo, Z.; Chen, H. Cost evaluation of a (Q,r,K) inventory model with two demand classes of lost sales and backorders. In Proceedings of the IFIP International Conference on Advances in Production Management Systems, Trondheim, Norway, 17–21 September 2023; Springer Nature: Cham, Switzerland, 2023; pp. 257–270.
46. Chang, C.T.; Lo, T.Y. On the inventory model with continuous and discrete lead time, backorders and lost sales. *Appl. Math. Model.* **2009**, *33*, 2196–2206. [CrossRef]
47. Wang, D.; Tang, O. Dynamic inventory rationing with mixed backorders and lost sales. *Int. J. Prod. Econ.* **2014**, *149*, 56–67. [CrossRef]
48. Taleizadeh, A.A.; Zamani-Dehkordi, N. A stochastic lot sizing model with partial backordering and imperfect production processes. *Int. J. Inventory Res.* **2017**, *4*, 75–96. [CrossRef]
49. Lin, H.J. Two-critical-number control policy for a stochastic production inventory system with partial backlogging. *Int. J. Prod. Res.* **2017**, *55*, 4123–4135. [CrossRef]
50. Wang, C.N.; Dang, T.T.; Nguyen, N.A.T. A computational model for determining levels of factors in inventory management using response surface methodology. *Mathematics* **2020**, *8*, 1210. [CrossRef]
51. Uzunoglu Kocer, U.; Yalcin, B. Continuous review (s,Q) inventory system with random lifetime and two demand classes. *Opsearch* **2020**, *57*, 104–118. [CrossRef]