mathematics

*Article*

# A New Robust Iterative Scheme Applied in Solving a Fractional Diffusion Model for Oxygen Delivery via a Capillary of Tissues

**Godwin Amechi Okeke [1,*], Akanimo Victor Udo [1], Nadiyah Hussain Alharthi [2] and Rubayyi T. Alqahtani [2]**

1 Functional Analysis and Optimization Research Group Laboratory (FANORG), Department of Mathematics, School of Physical Sciences, Federal University of Technology, Owerri, P.M.B. 1526 Owerri, Imo State, Nigeria; akanimo17@gmail.com

2 Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University, Riyadh P.O. Box 90950, Saudi Arabia; nhalharthi@imamu.edu.sa (N.H.A.); rtalqahtani@imamu.edu.sa (R.T.A.)

* Correspondence: godwin.okeke@futo.edu.ng

**Abstract:** In this paper, we constructed a new and robust fixed point iterative scheme called the UO iterative scheme for the approximation of a contraction mapping. The scheme converges strongly to the fixed point of a contraction mapping. A rate of convergence result is shown with an example, and our scheme, when compared, converges faster than some existing iterative schemes in the literature. Furthermore, the stability and data dependence results are shown. Our new scheme is applied in the approximation of the solution to the oxygen diffusion model. Finally, our results are applied in the approximation of the solution to the boundary value problems using Green's functions with an example.

**Keywords:** UO iterative scheme; boundary value problem; strong convergence; $\mathcal{T}$-stability; almost $\mathcal{T}$-stability; oxygen diffusion model; Green's function; data dependence; rate of convergence

**MSC:** 47H09; 47H10; 47J26; 35R11



**Citation:** Okeke, G.A.; Udo, A.V.; Alharthi, N.H.; Alqahtani, R.T. A New Robust Iterative Scheme Applied in Solving a Fractional Diffusion Model for Oxygen Delivery via a Capillary of Tissues. *Mathematics* **2024**, *12*, 1339. https://doi.org/10.3390/math12091339

Academic Editors: Godwin Chdid Ugwunnadi, Hammed Anuoluwapo Abass and Maggie Aphane

Received: 29 March 2024
Revised: 23 April 2024
Accepted: 25 April 2024
Published: 28 April 2024



## 1. Introduction

Let $\mathcal{B}$ be a nonempty closed convex subset of a Banach space $\mathbb{E}$. A self-mapping $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ is called a contraction if for any $\delta \in [0, 1)$, $x, y \in \mathbb{E}$,

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \delta\|x - y\|, \tag{1}$$

holds. An element $\tau^* \in \mathcal{B}$ is called a fixed point of $\mathcal{T}$ if it satisfies the operator equation

$$\mathcal{T}\tau^* = \tau^*. \tag{2}$$

The set of all fixed point of $\mathcal{T}$ is denoted by $\mathscr{F}(\mathcal{T})$.

A Banach space $\mathbb{E}$ is said to be uniformly convex (as introduced by Clarkson [1] in 1936) if for any $\epsilon$, $0 < \epsilon \leq 2$, the inequalities $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \epsilon$ imply that there exists a $\delta = \delta(\epsilon) > 0$ such that $\|\frac{(x+y)}{2}\| \leq 1 - \delta$.

Quite a number of physical problems are modeled as partial differential equations and ordinary differential equations in the form of initial value problems (IVPs) or boundary value problems (BVPs). Most times, these equations are in the form of nonlinear problems that are usually difficult (if not impossible) to solve through the use of analytical methods. In a way to circumvent this difficulty of obtaining the solutions to nonlinear problems through the use of the analytical method, the fixed point theory approach becomes useful by way of proving the existence and uniqueness of solutions to the problems of concern. Due to this advantage, since inception, fixed point theory has made a remarkable impact in mathematics and other areas of applied science, including its application to BVPs. For instance, the papers [2–4] and the references therein have been dedicated specifically to the use of fixed point iterative processes in approximating the solutions to BVPs via the use of Green's functions. The idea involved in fixed point theory is to transform any problem of focus into a fixed point equation as in (2) which thereafter solved for $\tau^*$ being the fixed point. In practice, the fixed point represents the approximate solution which is obtained more suitably through the use of a robust fixed point iterative scheme. The fixed point theory method has been applied in solving diverse problems in science and engineering (see, e.g., [5]) and the references therein.

While an ordinary differential equation is an equation involving differential coefficients to the integer order, a fractional differential equation exists as an equation involving differential coefficients to the fractional order.

The remarkable thing about a fractional differential equation is its wide range of applications. For example, it can be applied in modeling some physical phenomena in science and engineering, such as in optics [6]; in electrochemistry; in viscoelasticity [7]; in control theory; in biology; in fluid flow; and in other fields (see, for example, [8,9] and other references therein).

In the sequel, $0 < \alpha_n, \beta_n, \gamma_n < 1$ are parametric sequences of real numbers.

The aim of this paper is to answer the following question.

**Question** Is there a fixed point iterative scheme that can converge faster than other existing schemes in the literature and solve some problems in its application?

To answer the question above, we construct the following fixed point iterative scheme and call it the UO iterative scheme:

$$\begin{cases} v_o \in \mathcal{B} \\ r_n = \mathcal{T} v_n \\ s_n = (1-\alpha_n) r_n + \alpha_n \mathcal{T} r_n \\ t_n = \mathcal{T} s_n \\ u_n = (1-\beta_n) t_n + \beta_n \mathcal{T} t_n \\ v_{n+1} = (1-\gamma_n) u_n + \gamma_n \mathcal{T} u_n, \; n \in \mathbb{N}, \end{cases} \tag{3}$$

which, as will be shown in the subsequent sections, converges faster than some existing iterative schemes in the literature as outlined in the next section.

Our iterative scheme generalizes and extends other existing iterative schemes in the literature.

**Remark 1.** *Observe that the UO iterative scheme (3) is a five-step iterative iterative scheme, which is not as simple as one-step or two-step iterative schemes such as the Mann and Ishikawa iterative schemes.*

The remaining part of this paper is arranged as follows: Section 2 is dedicated to preliminary definitions and lemmas. Section 3 is for the main results which comprise the convergence results, rate of convergence, stability and data dependence. In Section 4, the application to an oxygen diffusion model is covered. Meanwhile, Section 5 is assigned to the application of our new scheme to BVPs, where the construction of the Green's function, the UO–Green iterative scheme, convergence analysis of the UO–Green iterative and numerical example are considered. Section 6 contains the conclusion.

## 2. Preliminary

Research in the area of fixed point theory via use of iterative scheme has experienced a surge due to construction of varying forms of iterative scheme that have been useful in application.

In 2012, Chugh et al. [10] introduced the CR iterative as follows:

$$\begin{cases} p_0 \in \mathcal{B} \\ r_n = (1-\gamma_n)p_n + \gamma_n \mathcal{T} p_n \\ q_n = (1-\beta_n)\mathcal{T} p_n + \beta_n \mathcal{T} r_n \\ p_{n+1} = (1-\alpha_n)q_n + \alpha_n \mathcal{T} q_n, \quad n \in \mathbb{N}. \end{cases} \tag{4}$$

Another iterative scheme is Picard-S, which was introduced by Gürsoy et al. [11] in 2014 and defined thus:

$$\begin{cases} z_n = (1-\beta_n)x_n + \beta_n \mathcal{T} x_n \\ y_n = (1-\alpha_n)\mathcal{T} x_n + \alpha_n \mathcal{T} z_n \\ x_{n+1} = \mathcal{T} y_n, \ n \in \mathbb{N}. \end{cases} \tag{5}$$

Abbas et al. (2022) [12] constructed the following AA iteration scheme:

$$\begin{cases} z_{n+1} = \mathcal{T} y_n \\ y_n = \mathcal{T}[(1-\gamma_n)\mathcal{T} d_n + \gamma_n \mathcal{T} w_n] \\ w_n = \mathcal{T}[(1-\beta_n)d_n + \beta_n \mathcal{T} d_n] \\ d_n = (1-\alpha_n)z_n + \alpha_n \mathcal{T} z_n, \ n \in \mathbb{N}, \end{cases} \tag{6}$$

which was used to approximate the solution to a delay fractional differential equation.

Uddin et al. [13], in 2022, introduced the following iterative scheme:

$$\begin{cases} g_0 \in \mathcal{B} \\ e_n = \mathcal{T}[(1-\alpha_n)g_n + \alpha_n \mathcal{T} g_n] \\ f_n = \mathcal{T} e_n \\ g_{n+1} = \mathcal{T} f_n, \ \ n \in \mathbb{N}, \end{cases} \tag{7}$$

and they were able to show that it converges faster than Thakur New, Vatan, M and M* iterations.

The following are, respectively, the $F^*$, Modified-SP [14] and Picard–Ishikawa [15] iterative schemes:

$$\begin{cases} p_0 \in \mathcal{B} \\ p_{n+1} = \mathcal{T} q_n \\ q_n = \mathcal{T}[(1-\alpha_n)p_n + \alpha_n \mathcal{T} p_n], \ \ n \in \mathbb{N}, \end{cases} \tag{8}$$

$$\begin{cases} x_0 \in \mathcal{B} \\ x_{n+1} = \mathcal{T} y_n \\ y_n = (1-\alpha_n)z_n + \alpha_n \mathcal{T} z_n \\ z_n = (1-\beta_n)x_n + \beta_n \mathcal{T} x_n, \ n \in \mathbb{N}, \end{cases} \tag{9}$$

$$\begin{cases} w_n = (1-\xi_n)u_n + \xi_n \mathcal{T} u_n, \\ v_n = (1-\varpi_n)u_n + \varpi_n \mathcal{T} w_n, \\ u_{n+1} = \mathcal{T} v_n, \ n \in \mathbb{N}. \end{cases} \tag{10}$$

Recently in 2023, Okeke et al. [16] introduced the AG iterative scheme, defined thus:

$$\begin{cases} u_0 = u \in \mathcal{B} \\ u_{n+1} = \mathcal{T} v_n \\ v_n = \mathcal{T}[(1-\alpha_n)w_n + \alpha_n \mathcal{T} w_n] \\ w_n = (1-\beta_n)\mathcal{T} u_n + \beta_n \mathcal{T} x_n \\ x_n = (1-\gamma_n)u_n + \gamma_n \mathcal{T} u_n, \ n \in \mathbb{N}, \end{cases} \tag{11}$$

which was used to approximate the fixed point of contraction mapping in a uniformly convex Banach space with applications.

**Definition 1.** *Two sequences, $\{u_n\}$ and $\{v_n\}$, are said to guarantee equivalence if* $\lim_{n\to\infty} \|u_n - v_n\| = 0$.

**Definition 2** ([17])**.** *Let $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ be an operator on a real Banach space E. Assume that $v_n \in \mathcal{B}$ and $v_{n+1} = f(\mathcal{T}, v_n)$ defines an iterative scheme which generates a sequence $\{v_n\}_{n=0}^{\infty}$ in $\mathcal{B}$. Assume, furthermore, that $\{v_n\}_{n=0}^{\infty}$ converges strongly to $\tau^* \in \mathcal{F}(\mathcal{T}) \neq \emptyset$, where $\mathcal{F}(\mathcal{T})$ is the set of all fixed points of $\mathcal{T}$. Assume that $\{p_n\}_{n=0}^{\infty}$ is an arbitrary bounded sequence in E and set $\epsilon_n = \|p_{n+1} - f(\mathcal{T}, p_n)\|$. Then,*

1. *The iterative scheme $\{v_n\}_{n=0}^{\infty}$ in a real Banach space E defined by $v_{n+1} = f(\mathcal{T}, v_n)$ is said to be $\mathcal{T}$-stable if $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\to\infty} \|p_n - \tau^*\| = 0$.*
2. *The iterative scheme $\{v_n\}_{n=0}^{\infty}$ defined by $v_{n+1} = f(\mathcal{T}, v_n)$ is said to be almost $\mathcal{T}$-stable if $\sum_{n=0}^{\infty} \epsilon_n < \infty$ implies that $\lim_{n\to\infty} \|p_n - \tau^*\| = 0$.*

**Definition 3** ([18])**.** *Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences of real numbers converging to a and b, respectively. If*

$$\lim_{n\to\infty} \frac{\|a_n - a\|}{\|b_n - b\|} = 0, \tag{12}$$

*then $\{a_n\}$ is said to converge to a faster than $\{b_n\}$ to b.*

**Definition 4** ([18])**.** *Suppose that for two fixed-point iterative processes $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$, both converging to the same fixed point p, the error estimates*

$$\|u_n - p\| \leq a_n, \text{ for all } n \in \mathbb{N},$$

$$\|v_n - p\| \leq b_n, \text{ for all } n \in \mathbb{N},$$

*exist, where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are two sequences of positive numbers converging to zero. If $\{a_n\}_{n=0}^{\infty}$ converges faster than $\{b_n\}_{n=0}^{\infty}$ then $\{u_n\}_{n=0}^{\infty}$ converges faster than $\{v_n\}_{n=0}^{\infty}$ to p.*

**Lemma 1** ([19])**.** *If $\rho \in [0, 1)$ is a real number and $\{\epsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \epsilon_n = 0$, then for any sequence of positive numbers, $\{p_n\}_{n=0}^{\infty}$ satisfying $p_{n+1} \leq \rho p_n + \epsilon_n$, $(n = 0, 1, 2, \ldots)$ such that $\lim_{n\to\infty} p_n = 0$.*

**Lemma 2** ([20])**.** *Let $\{p_n\}_{n=0}^{\infty}$ and $\{\epsilon_n\}_{n=0}^{\infty}$ be sequences of nonnegative numbers and $\delta \in [0, 1)$ such that*

$$\nu_{n+1} = \delta \nu_n + \epsilon_n \quad n \geq 0.$$

*If $\sum_{n=0}^{\infty} \epsilon_n < \infty$, then $\sum_{n=0}^{\infty} \nu_n < \infty$.*

**Lemma 3** ([21])**.** *Let $\{\xi_n\}$ be a nonnegative sequence for which one assumes there exists $n_0 \in \mathbb{N}$ such that all $n \geq n_0$, and suppose the following inequality is satisfied:*

$$\xi_{n+1} \leq (1 - \varphi_n)\xi_n + \varphi_n \varrho_n$$

*where $\varphi_n \in (0, 1)$, $\forall n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \varphi_n = \infty$ and $\varrho_n \geq 0$ $\forall n \in \mathbb{N}$. Then,*

$$0 \leq \limsup_{n\to\infty} \xi_n \leq \limsup_{n\to\infty} \varrho_n.$$

**Lemma 4** ([22])**.** *Let $\sigma_n$ be a nonnegative sequence satisfying the inequality*

$$\sigma_{n+1} \leq (1 - \eta_n)\sigma_n + \lambda_n$$

*with* $\eta_n \in [0,1]$, $\sum_{j=0}^{\infty} \eta_j = \infty$ *and* $\lambda_n = o(\eta_n)$. *Then,* $\lim_{n\to\infty} \sigma_n = 0$.

## 3. Main Results

We begin this section by establishing some useful convergence results for our newly developed iteration process in Banach spaces.

### 3.1. Convergence Analysis of the UO Iterative Process

**Theorem 1.** *Assume $\mathcal{B}$ is a nonempty closed convex subset of a Banach space $\mathbb{E}$ and $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ is a contraction mapping satisfying condition (1). Assume $\{v_n\}_{n=0}^{\infty}$ is an iterative sequence generated by the UO iterative scheme (3) with real sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in (0,1)$, satisfying $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{v_n\}_{n=0}^{\infty}$ converges to a unique fixed point $\tau^* \in \mathscr{F}(\mathcal{T})$.*

**Proof.** It can easily be verified that the Banach contraction principle guarantees the existence and uniqueness of $\tau^* \in \mathscr{F}(\mathcal{T})$. What is left is to show is that $\lim_{n\to\infty} \|v_n - \tau^*\| = 0$.

Using the contraction condition (1) and the iterative scheme (3), we have the following estimates:

$$\begin{aligned} \|r_n - \tau^*\| &= \|\mathcal{T} v_n - \tau^*\| \\ &= \|\mathcal{T} v_n - \mathcal{T} \tau^*\| \\ &\leq \delta \|v_n - \tau^*\|. \end{aligned} \tag{13}$$

Using (3) and (13),

$$\begin{aligned} \|s_n - \tau^*\| &= \|(1-\alpha_n) r_n + \alpha_n \mathcal{T} r_n - \tau^*\| \\ &\leq (1-\alpha_n)\|r_n - \tau^*\| + \alpha_n \|\mathcal{T} r_n - \tau^*\| \\ &\leq (1-\alpha_n)\|r_n - \tau^*\| + \alpha_n \delta \|r_n - \tau^*\| \\ &= [(1-\alpha_n) + \alpha_n \delta]\|r_n - \tau^*\| \\ &\leq [1 - (1-\delta)\alpha_n]\|r_n - \tau^*\| \\ &\leq \delta[1 - (1-\delta)\alpha_n]\|v_n - \tau^*\|, \end{aligned} \tag{14}$$

again, using (3) and (14),

$$\begin{aligned} \|t_n - \tau^*\| &= \|\mathcal{T} s_n - \tau^*\| \\ &\leq \delta \|s_n - \tau^*\| \\ &\leq \delta[1 - (1-\delta)\alpha_n]\|r_n - \tau^*\| \\ &\leq \delta^2[1 - (1-\delta)\alpha_n]\|v_n - \tau^*\|, \end{aligned} \tag{15}$$

furthermore, using (3) and (15),

$$\begin{aligned} \|u_n - \tau^*\| &= \|(1-\beta_n) u_n + \beta_n \mathcal{T} u_n - \tau^*\| \\ &\leq (1-\beta_n)\|t_n - \tau^*\| + \beta_n \|\mathcal{T} t_n - \tau^*\| \\ &\leq (1-\beta_n)\|t_n - \tau^*\| + \beta_n \delta \|t_n - \tau^*\| \\ &= [(1-\beta_n) + \beta_n \delta]\|t_n - \tau^*\| \\ &\leq [1 - (1-\delta)\beta_n]\|t_n - \tau^*\| \\ &\leq \delta^2[1 - (1-\delta)\alpha_n][1 - (1-\delta)\beta_n]\|v_n - \tau^*\|. \end{aligned} \tag{16}$$

Finally,

$$\begin{aligned} \|v_{n+1}-\tau^*\| &= \|(1-\gamma_n)u_n+\gamma_n\mathcal{T}u_n-\tau^*\| \\ &\leq (1-\gamma_n)\|u_n-\tau^*\|+\gamma_n\|\mathcal{T}u_n-\tau^*\| \\ &\leq (1-\gamma_n)\|u_n-\tau^*\|+\gamma_n\delta\|u_n-\tau^*\| \\ &= [(1-\gamma_n)+\gamma_n\delta]\|u_n-\tau^*\| \\ &\leq [1-(1-\delta)\gamma_n]\|u_n-\tau^*\| \\ &\leq \delta^2[1-(1-\delta)\gamma_n][1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\|v_n-\tau^*\|. \end{aligned} \tag{17}$$

Since $\delta \in [0,1)$, $[1-(1-\delta)\beta_n] < 1$ and $[1-(1-\delta)\gamma_n] < 1$, we have that

$$\|v_{n+1}-\tau^*\| \leq \delta^2[1-(1-\delta)\alpha_n]\|v_n-\tau^*\|.$$

Via induction, we have the following inequalities:

$$\begin{aligned} \|v_n-\tau^*\| &\leq [1-\alpha_{n-1}(1-\delta)]\|v_{n-1}-\tau^*\| \\ \|v_{n-1}-\tau^*\| &\leq \delta^2[1-\alpha_{n-2}(1-\delta)]\|v_{n-2}-\tau^*\| \\ &\vdots \\ \|v_1-\tau^*\| &\leq \delta^2[1-\alpha_0(1-\delta)]\|v_0-\tau^*\|. \end{aligned}$$

$$\begin{aligned} \|v_{n+1}-\tau^*\| &\leq \delta^{2(n+1)}\prod_{k=0}^{n}[1-\alpha_k(1-\delta)]\|v_0-\tau^*\| \\ &= \delta^{2(n+1)}\|v_0-\tau^*\|[1-(1-\delta)\alpha]^{n+1}. \end{aligned} \tag{18}$$

From elementary analysis, it is clear that $1-q \leq e^{-q}$ for $q \in (0,1)$. Consequent upon that fact and inequality (18), we have

$$\begin{aligned} \|v_{n+1}-\tau^*\| &\leq \prod_{k=0}^{n} e^{-\alpha_k(1-\delta)}\|v_0-\tau^*\| \\ &= \|v_0-\tau^*\|e^{-(1-\delta)\sum_{k=0}^{n}\alpha_k}. \end{aligned} \tag{19}$$

Taking the limit as $n \to \infty$ of both sides of (19), then, $\lim_{n\to\infty}\|v_n-\tau^*\| = 0$. □

**Theorem 2.** *Suppose $\mathcal{B}$ is a nonempty closed convex subset of a Banach space $\mathbb{E}$ and $\mathcal{T}:\mathcal{B}\to\mathcal{B}$ is a mapping satisfying condition (1) with a unique fixed point $\tau^* \in \mathscr{F}(\mathcal{T})$. Suppose that $\{v_n\}$ and $\{z_n\}$ are two iterative sequences generated by the UO iterative scheme and the AA iterative scheme, respectively, with real sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in (0,1)$ satisfying $\sum_{k=0}^{\infty}\alpha_k = \infty$. Then, the following are equivalent:*

1. $\lim_{n\to\infty}\|v_n-\tau^*\| = 0$
2. $\lim_{n\to\infty}\|z_n-\tau^*\| = 0.$

**Proof.** We start by showing that $(1) \implies (2)$; that is, if the UO iterative scheme converges to the fixed point $\tau^*$, then the $AA$ iterative also converge to the same fixed point $\tau^*$.

$$\begin{aligned} \|v_{n+1}-z_{n+1}\| &= \|(1-\gamma_n)u_n+\gamma_n\mathcal{T}u_n-\mathcal{T}y_n\| \\ &\leq (1-\gamma_n)\|u_n-\mathcal{T}y_n\|+\gamma_n\|\mathcal{T}u_n-\mathcal{T}y_n\| \\ &\leq (1-\gamma_n)\|u_n-\mathcal{T}y_n\|+\gamma_n\delta\|u_n-y_n\| \\ &\leq (1-\gamma_n)\|u_n-\mathcal{T}u_n\|+\delta\|u_n-y_n\|. \end{aligned} \tag{20}$$

$$\begin{aligned}
\|u_n - y_n\| &= \|(1-\beta_n)t_n + \beta_n \mathcal{T} t_n - \mathcal{T}[(1-\gamma_n)\mathcal{T} d_n + \gamma_n \mathcal{T} w_n]\| \\
&\leq (1-\beta_n)\|t_n - \mathcal{T}[(1-\gamma_n)\mathcal{T} d_n + \gamma_n \mathcal{T} w_n]\| \\
&\quad + \beta_n \|\mathcal{T} t_n - \mathcal{T}[(1-\gamma_n)\mathcal{T} d_n + \gamma_n \mathcal{T} w_n]\| \\
&\leq (1-\beta_n)\|t_n - \mathcal{T} t_n + \mathcal{T} t_n - \mathcal{T}[(1-\gamma_n)\mathcal{T} d_n + \gamma_n \mathcal{T} w_n]\| \\
&\quad + \beta_n \|\mathcal{T} t_n - \mathcal{T}[(1-\gamma_n)\mathcal{T} d_n + \gamma_n \mathcal{T} w_n]\| \\
&\leq (1-\beta_n)\|t_n - \mathcal{T} t_n\| + (1-\beta_n)\|\mathcal{T} t_n - \mathcal{T}[(1-\gamma_n)\mathcal{T} d_n + \gamma_n \mathcal{T} w_n]\| \\
&\quad + \beta_n \|\mathcal{T} t_n - \mathcal{T}[(1-\gamma_n)\mathcal{T} d_n + \gamma_n \mathcal{T} w_n]\| \\
&\leq (1-\beta_n)\|t_n - \mathcal{T} t_n\| + \|\mathcal{T} t_n - \mathcal{T}[(1-\gamma_n)\mathcal{T} d_n + \gamma_n \mathcal{T} w_n]\| \\
&\leq (1-\beta_n)\|t_n - \mathcal{T} t_n\| + \delta\|t_n - [(1-\gamma_n)\mathcal{T} d_n + \gamma_n \mathcal{T} w_n]\| \\
&\leq (1-\beta_n)\|t_n - \mathcal{T} t_n\| + \delta(1-\gamma_n)\|t_n - \mathcal{T} t_n\| + \delta\gamma_n\|t_n - \mathcal{T} w_n\| \\
&\leq (1-\beta_n)\|t_n - \mathcal{T} t_n\| + \delta(1-\gamma_n)\|t_n - \mathcal{T} t_n + \mathcal{T} t_n - \mathcal{T} d_n\| \\
&\quad + \delta\gamma_n\|t_n - \mathcal{T} t_n + \mathcal{T} t_n - \mathcal{T} w_n\| \\
&\leq (1-\beta_n)\|t_n - \mathcal{T} t_n\| + \delta(1-\gamma_n)\|t_n - \mathcal{T} t_n\| + \delta(1-\gamma_n)\|\mathcal{T} t_n - \mathcal{T} d_n\| \\
&\quad + \delta\gamma_n\|t_n - \mathcal{T} t_n\| + \delta\gamma_n\|\mathcal{T} t_n - \mathcal{T} w_n\| \\
&\leq \{(1-\beta_n) + \delta(1-\gamma_n) + \delta\gamma_n\}\|t_n - \mathcal{T} t_n\| + \delta^2(1-\gamma_n)\|t_n - d_n\| \\
&\quad + \delta^2\gamma_n\|t_n - w_n\| \\
&\leq [1-\beta_n + \delta]\|t_n - \mathcal{T} t_n\| + \delta^2(1-\gamma_n)\|t_n - d_n\| + \delta^2\gamma_n\|t_n - w_n\|.
\end{aligned} \tag{21}$$

$$\begin{aligned}
\|t_n - w_n\| &= \|\mathcal{T} s_n - \mathcal{T}[(1-\beta_n)d_n + \beta_n \mathcal{T} d_n]\| \\
&\leq \delta\|s_n - [(1-\beta_n)d_n + \beta_n \mathcal{T} d_n]\| \\
&\leq \delta(1-\beta_n)\|s_n - d_n\| + \delta\beta_n\|s_n - \mathcal{T} d_n\| \\
&\leq \delta(1-\beta_n)\|s_n - d_n\| + \delta\beta_n\|s_n - \mathcal{T} s_n + \mathcal{T} s_n - \mathcal{T} d_n\| \\
&\leq \delta(1-\beta_n)\|s_n - d_n\| + \delta\beta_n\|s_n - \mathcal{T} s_n\| + \delta^2\beta_n\|s_n - d_n\| \\
&\leq \delta\beta_n\|s_n - \mathcal{T} s_n\| + [\delta(1-\beta_n) + \delta^2\beta_n]\|s_n - d_n\|.
\end{aligned} \tag{22}$$

$$\begin{aligned}
\|s_n - d_n\| &= \|(1-\alpha_n)r_n + \alpha_n \mathcal{T} r_n - (1-\alpha_n)z_n - \alpha_n \mathcal{T} z_n\| \\
&\leq (1-\alpha_n)\|r_n - z_n\| + \alpha_n\|\mathcal{T} r_n - \mathcal{T} z_n\| \\
&\leq (1-\alpha_n)\|r_n - z_n\| + \delta\alpha_n\|r_n - z_n\| \\
&= [(1-\alpha_n) + \delta\alpha_n]\|r_n - z_n\|.
\end{aligned} \tag{23}$$

$$\begin{aligned}
\|r_n - z_n\| &= \|\mathcal{T} v_n - z_n\| \\
&\leq \|\mathcal{T} v_n - \mathcal{T} z_n + \mathcal{T} z_n - z_n\| \\
&\leq \delta\|v_n - z_n\| + \|z_n - \mathcal{T} z_n\|.
\end{aligned} \tag{24}$$

Put (24) in (23):

$$\|s_n - d_n\| \leq \delta[(1-\alpha_n) + \delta\alpha_n]\|v_n - z_n\| + [(1-\alpha_n) + \delta\alpha_n]\|z_n - \mathcal{T} z_n\|. \tag{25}$$

Put (25) in (22):

$$\begin{aligned}
\|t_n - w_n\| \leq \delta\beta_n\|s_n - \mathcal{T} s_n\| &+ [\delta(1-\beta_n) + \delta\beta_n]\times \\
&\{\delta[(1-\alpha_n) + \delta\alpha_n]\|v_n - z_n\| + [(1-\alpha_n) + \delta\alpha_n]\|z_n - \mathcal{T} z_n\|\}
\end{aligned} \tag{26}$$

$$\begin{aligned}
\|t_n - d_n\| &= \|\mathcal{T} s_n - (1-\alpha_n)z_n - \alpha_n \mathcal{T} z_n\| \\
&\leq (1-\alpha_n)\|\mathcal{T} s_n - z_n\| + \alpha_n\|\mathcal{T} s_n - \mathcal{T} z_n\| \\
&\leq (1-\alpha_n)\|\mathcal{T} s_n - \mathcal{T} z_n + \mathcal{T} z_n - z_n\| + \alpha_n\delta\|s_n - z_n\| \\
&\leq (1-\alpha_n)\delta\|s_n - z_n\| + (1-\alpha_n)\|\mathcal{T} z_n - z_n\| + \alpha_n\delta\|s_n - z_n\| \\
&\leq \delta\|s_n - z_n\| + (1-\alpha_n)\|\mathcal{T} z_n - z_n\|
\end{aligned} \tag{27}$$

$$
\begin{aligned}
\|s_n - z_n\| &= \|(1-\alpha_n)r_n + \alpha_n \mathcal{T} r_n - z_n\| \\
&\leq (1-\alpha_n)\|r_n - z_n\| + \alpha_n \|\mathcal{T} r_n - z_n\| \\
&\leq (1-\alpha_n)\|r_n - z_n\| + \alpha_n \|\mathcal{T} r_n - \mathcal{T} z_n + \mathcal{T} z_n - z_n\| \\
&\leq (1-\alpha_n)\|r_n - z_n\| + \alpha_n \delta \|r_n - z_n\| + \alpha_n \|\mathcal{T} z_n - z_n\| \\
&\leq [(1-\alpha_n) + \alpha_n \delta]\|r_n - z_n\| + \alpha_n \|z_n - \mathcal{T} z_n\| \\
&\leq [(1-\alpha_n) + \alpha_n \delta]\{\delta\|v_n - z_n\| + \|z_n - \mathcal{T} z_n\|\} + \alpha_n \|z_n - \mathcal{T} z_n\| \\
&\leq \delta[(1-\alpha_n) + \alpha_n \delta]\|v_n - z_n\| + [1 + \alpha_n \delta]\|z_n - \mathcal{T} z_n\|.
\end{aligned} \tag{28}
$$

Putting (28) in (27),

$$
\begin{aligned}
\|t_n - d_n\| \leq{} & \delta^2[1-(1-\delta)\alpha_n]\|v_n - z_n\| + [\delta + \delta^2\alpha_n]\|z_n - \mathcal{T} z_n\| \\
& + (1-\alpha_n)\|\mathcal{T} z_n - z_n\|,
\end{aligned} \tag{29}
$$

and, putting (26) and (29) in (21),

$$
\begin{aligned}
\|u_n - y_n\| \leq{} & [1-\beta_n+\delta]\|t_n - \mathcal{T} t_n\| + \delta^4(1-\gamma_n)[1-(1-\delta)\alpha_n]\|v_n - z_n\| \\
& + \delta^2(1-\gamma_n)[\delta + \delta^2\gamma_n]\|z_n - \mathcal{T} z_n\| + \delta^2(1-\gamma_n)(1-\alpha_n)\|z_n - \mathcal{T} z_n\| \\
& + \delta^3\beta_n\gamma_n\|s_n - \mathcal{T} s_n\| + \delta^4\gamma_n[1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\|v_n - z_n\| \\
& + \delta^3\gamma_n[1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\|z_n - \mathcal{T} z_n\|.
\end{aligned} \tag{30}
$$

Putting (30) in (20),

$$
\begin{aligned}
\|v_{n+1} - z_{n+1}\| \leq{} & (1-\gamma_n)\|u_n - \mathcal{T} u_n\| + \delta[1-\beta_n+\delta]\|t_n - \mathcal{T} t_n\| \\
& + \delta^5(1-\gamma_n)[1-(1-\delta)\alpha_n]\|v_n - z_n\| + \delta^3(1-\gamma_n)[\delta + \delta^2\alpha_n]\|z_n - \mathcal{T} z_n\| \\
& + \delta^3(1-\gamma_n)(1-\alpha_n)\|z_n - \mathcal{T} z_n\| + \delta^4\beta_n\gamma_n\|s_n - \mathcal{T} s_n\| \\
& + \delta^5\gamma_n[1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\|v_n - z_n\| \\
& + \delta^4\gamma_n[1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\|z_n - \mathcal{T} z_n\| \\
\leq{} & (1-\gamma_n)\|u_n - \mathcal{T} u_n\| + \delta[1-\beta_n+\delta]\|t_n - \mathcal{T} t_n\| + \delta^4\beta_n\gamma_n\|s_n - \mathcal{T} s_n\| \\
& + \Big\{\delta^5(1-\gamma_n)[1-(1-\delta)\alpha_n] \\
& \quad + \delta^5\gamma_n[1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\Big\}\|v_n - z_n\| \\
& + \Big\{\delta^3(1-\gamma_n)[\delta + \delta^2\alpha_n] \\
& \quad + \delta^4\gamma_n[1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\Big\}\|z_n - \mathcal{T} z_n\|.
\end{aligned} \tag{31}
$$

Since $\delta \in [0,1)$ and $\delta^5(1-\gamma_n) + \delta^5\gamma_n[1-(1-\delta)\beta_n] < 1$, we have

$$
\begin{aligned}
\|v_{n+1} - z_{n+1}\| \leq{} & (1-\gamma_n)\|u_n - \mathcal{T} u_n\| + \delta[1-\beta_n+\delta]\|t_n - \mathcal{T} t_n\| + \delta^4\beta_n\gamma_n\|s_n - \mathcal{T} s_n\| \\
& + \{\delta^3(1-\gamma_n)[\delta + \delta^2\alpha_n] + \delta^4\gamma_n[1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\}\|z_n - \mathcal{T} z_n\| \\
& + [1-(1-\delta)\alpha_n]\|v_n - z_n\|.
\end{aligned} \tag{32}
$$

Let

$$
\begin{aligned}
\eta_n &= (1-\delta)\alpha_n \in (0,1) \\
\pi_n &= \|v_n - z_n\| \\
c_n &= (1-\gamma_n)\|u_n - \mathcal{T} u_n\| + \delta[1-\beta_n+\delta]\|t_n - \mathcal{T} t_n\| + \delta^4\beta_n\gamma_n\|s_n - \mathcal{T} s_n\| \\
&\quad + \{\delta^3(1-\gamma_n)[\delta + \delta^2\alpha_n] + \delta^4\gamma_n[1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\}\|z_n - \mathcal{T} z_n\|.
\end{aligned}
$$

Moreover, using $\mathcal{T}\tau^* = \tau^*$ and $\|z_n - \tau^*\| \to 0$,

$$\lim_{n\to\infty} \|z_n - \mathcal{T}z_n\| = \lim_{n\to\infty} \|s_n - \mathcal{T}s_n\| = \lim_{n\to\infty} \|t_n - \mathcal{T}t_n\| = \lim_{n\to\infty} \|u_n - \mathcal{T}u_n\| = 0,$$

and it follows that $\frac{c_n}{\eta_n} \to 0$ as $n \to \infty$.

Clearly, (32) satisfies the conditions of Lemma 4 and, hence, $\lim_{n\to\infty} \|v_n - z_n\| = 0$. Since

$$\|z_n - \tau^*\| = \|v_n - z_n\| + \|z_n - \tau^*\|$$

we have

$$\lim_{n\to\infty} \|z_n - \tau^*\| = 0.$$

Next, we show that (2) $\implies$ (1):

$$\begin{aligned}
\|z_{n+1} - v_{n+1}\| &= \|\mathcal{T}y_n - (1-\gamma_n)u_n + \gamma_n \mathcal{T}u_n\| \\
&\leq (1-\gamma_n)\|\mathcal{T}y_n - u_n\| + \gamma_n\|\mathcal{T}y_n - \mathcal{T}u_n\| \\
&\leq (1-\gamma_n)\|\mathcal{T}y_n - u_n\| + \delta\gamma_n\|y_n - u_n\| \\
&\leq (1-\gamma_n)\|\mathcal{T}y_n - \mathcal{T}u_n + \mathcal{T}u_n - u_n\| + \delta\gamma_n\|y_n - u_n\| \\
&\leq (1-\gamma_n)\delta\|y_n - u_n\| + (1-\gamma_n)\|u_n - \mathcal{T}u_n\| + \delta\gamma_n\|y_n - u_n\| \\
&= \delta\|y_n - u_n\| + (1-\gamma_n)\|u_n - \mathcal{T}u_n\|
\end{aligned} \tag{33}$$

$$\begin{aligned}
\|y_n - u_n\| &= \|\mathcal{T}[(1-\gamma_n)\mathcal{T}d_n + \gamma_n\mathcal{T}w_n] - (1-\beta_n)t_n - \beta_n\mathcal{T}t_n\| \\
&\leq (1-\beta_n)\|\mathcal{T}[(1-\gamma_n)\mathcal{T}d_n + \gamma\mathcal{T}w_n] - t_n\| \\
&\quad + \beta_n\|\mathcal{T}[(1-\gamma_n)\mathcal{T}d_n + \gamma_n\mathcal{T}w_n] - \mathcal{T}t_n\| \\
&\leq (1-\beta_n)\|\mathcal{T}[(1-\gamma_n)\mathcal{T}d_n + \gamma_n\mathcal{T}w_n] - \mathcal{T}t_n + \mathcal{T}t_n - t_n\| \\
&\quad + \beta_n\delta\|(1-\gamma_n)\mathcal{T}d_n + \gamma_n\mathcal{T}w_n - t_n\| \\
&\leq (1-\beta_n)\delta\|(1-\gamma_n)\mathcal{T}d_n + \gamma_n\mathcal{T}w_n - t_n\| + (1-\beta_n)\|\mathcal{T}t_n - t_n\| \\
&\quad + \beta_n\delta(1-\gamma_n)\|\mathcal{T}d_n - t_n\| + \beta_n\gamma_n\delta\|\mathcal{T}w_n - t_n\| \\
&\leq (1-\beta_n)(1-\gamma_n)\delta\|\mathcal{T}d_n - t_n\| + (1-\beta_n)\gamma_n\delta\|\mathcal{T}w_n - t_n\| \\
&\quad + (1-\beta_n)\|\mathcal{T}t_n - t_n\| + \beta_n\delta^2(1-\gamma_n)\|d_n - t_n\| + \beta_n\delta(1-\gamma_n)\|\mathcal{T}t_n - t_n\| \\
&\quad + \beta_n\gamma_n\delta^2\|w_n - t_n\| + \beta_n\gamma_n\delta\|\mathcal{T}t_n - t_n\| \\
&\leq (1-\beta_n)(1-\gamma_n)\delta^2\|d_n - t_n\| + (1-\beta_n)(1-\gamma_n)\delta\|\mathcal{T}t_n - t_n\| \\
&\quad + (1-\beta_n)\gamma_n\delta^2\|w_n - t_n\| + (1-\beta_n)\gamma_n\delta\|\mathcal{T}t_n - t_n\| \\
&\quad + (1-\beta_n)\|\mathcal{T}t_n - t_n\| + \beta_n\delta^2(1-\gamma_n)\|d_n - t_n\| + \beta_n\delta(1-\gamma_n)\|\mathcal{T}t_n - t_n\| \\
&\quad + \beta_n\gamma_n\delta^2\|w_n - t_n\| + \beta_n\gamma_n\delta\|\mathcal{T}t_n - t_n\| \\
&= (1-\gamma_n)\delta^2\|d_n - t_n\| + \gamma_n\delta^2\|w_n - t_n\| + [\delta + (1-\beta_n)]\|\mathcal{T}t_n - t_n\|
\end{aligned} \tag{34}$$

$$\begin{aligned}
\|w_n - t_n\| &= \|\mathcal{T}[(1-\beta_n)d_n + \beta_n\mathcal{T}d_n] - \mathcal{T}s_n\| \\
&\leq \delta\|(1-\beta_n)d_n + \beta_n\mathcal{T}d_n - s_n\| \\
&\leq \delta(1-\beta_n)\|d_n - s_n\| + \beta_n\delta\|\mathcal{T}d_n - s_n\| \\
&\leq \delta(1-\beta_n)\|d_n - s_n\| + \beta_n\delta^2\|d_n - s_n\| + \beta_n\delta\|\mathcal{T}s_n - s_n\| \\
&= \delta[(1-\beta_n) + \beta_n\delta]\|d_n - s_n\| + \beta_n\delta\|\mathcal{T}s_n - s_n\|
\end{aligned} \tag{35}$$

$$
\begin{aligned}
\|d_n - s_n\| &= \|(1-\alpha_n)z_n + \alpha_n \mathcal{T} z_n - (1-\alpha_n)r_n - \alpha_n \mathcal{T} r_n\| \\
&\le (1-\alpha_n)\|z_n - r_n\| + \alpha_n \|\mathcal{T} z_n - \mathcal{T} r_n\| \\
&\le (1-\alpha_n)\|z_n - r_n\| + \alpha_n \delta \|z_n - r_n\| \\
&= [1-(1-\delta)\alpha_n]\|z_n - r_n\| \\
&\le [1-(1-\delta)\alpha_n]\|z_n - \mathcal{T} v_n\| \\
&\le [1-(1-\delta)\alpha_n]\|z_n - \mathcal{T} z_n + \mathcal{T} z_n - \mathcal{T} v_n\| \\
&\le [1-(1-\delta)\alpha_n]\|z_n - \mathcal{T} z_n\| + \delta[1-(1-\delta)\alpha_n]\|z_n - v_n\|
\end{aligned} \tag{36}
$$

Putting (36) in (35),

$$
\begin{aligned}
\|w_n - t_n\| &\le \delta[1-(1-\delta)\beta_n]\Big\{[1-(1-\delta)\alpha_n]\|z_n - \mathcal{T} z_n\| + \delta[1-(1-\delta)\alpha_n]\|z_n - v_n\|\Big\} \\
&\quad + \beta_n \delta \|\mathcal{T} s_n - s_n\| \\
&\le \delta[1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\|z_n - \mathcal{T} z_n\| \\
&\quad + \delta^2[1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\|z_n - v_n\| + \beta_n \delta \|\mathcal{T} s_n - s_n\|
\end{aligned} \tag{37}
$$

Next,

$$
\begin{aligned}
\|d_n - t_n\| &= \|(1-\alpha_n)z_n + \alpha_n \mathcal{T} z_n - \mathcal{T} s_n\| \\
&\le (1-\alpha_n)\|z_n - \mathcal{T} s_n\| + \alpha_n \|\mathcal{T} z_n - \mathcal{T} s_n\| \\
&\le (1-\alpha_n)\|z_n \mathcal{T} z_n + \mathcal{T} z_n - \mathcal{T} s_n\| + \alpha_n \delta \|z_n - s_n\| \\
&\le (1-\alpha_n)\|z_n - \mathcal{T} z_n\| + (1-\alpha_n)\delta\|z_n - s_n\| + \alpha_n \delta \|z_n - s_n\| \\
&= \delta\|z_n - s_n\| + (1-\alpha_n)\|z_n - \mathcal{T} z_n\| \\
&\le \delta\|z_n - (1-\alpha_n)r_n - \alpha_n \mathcal{T} r_n\| + (1-\alpha_n)\|z_n - \mathcal{T} z_n\| \\
&\le (1-\alpha_n)\delta\|z_n - r_n\| + \alpha_n \delta\|z_n - \mathcal{T} r_n\| + (1-\alpha_n)\|z_n - \mathcal{T} z_n\| \\
&\le (1-\alpha_n)\delta\|z_n - \mathcal{T} v_n\| + \alpha_n \delta\|z_n - \mathcal{T} z_n + \mathcal{T} z_n - \mathcal{T} r_n\| \\
&\quad + (1-\alpha_n)\|z_n - \mathcal{T} z_n\| \\
&\le (1-\alpha_n)\delta\|z_n - \mathcal{T} z_n + \mathcal{T} z_n - \mathcal{T} v_n\| + \alpha_n \delta\|z_n - \mathcal{T} z_n\| + \alpha_n \delta^2\|z_n - r_n\| \\
&\quad + (1-\alpha_n)\|z_n - \mathcal{T} z_n\| \\
&\le (1-\alpha_n)\delta\|z_n - \mathcal{T} z_n\| + (1-\alpha_n)\delta^2\|z_n - v_n\| + \alpha_n \delta\|z_n - \mathcal{T} z_n\| \\
&\quad + \alpha_n \delta^2\|z_n - \mathcal{T} v_n\| + (1-\alpha_n)\|z_n - \mathcal{T} z_n\| \\
&\le (1-\alpha_n)\delta\|z_n - \mathcal{T} z_n\| + (1-\alpha_n)\delta^2\|z_n - v_n\| + \alpha_n \delta\|z_n - \mathcal{T} z_n\| \\
&\quad + \alpha_n \delta^2\|z_n - \mathcal{T} z_n + \mathcal{T} z_n - \mathcal{T} v_n\| + (1-\alpha_n)\|z_n - \mathcal{T} z_n\| \\
&\le (1-\alpha_n)\delta\|z_n - \mathcal{T} z_n\| + (1-\alpha_n)\delta^2\|z_n - v_n\| + \alpha_n \delta\|z_n - \mathcal{T} z_n\| \\
&\quad + \alpha_n \delta^2\|z_n - \mathcal{T} z_n\| + \alpha_n \delta^3\|z_n - v_n\| + (1-\alpha_n)\|z_n - \mathcal{T} z_n\| \\
&= [(1-\alpha_n)\delta + \alpha_n \delta + \alpha_n \delta^2 + (1-\alpha_n)]\|z_n - \mathcal{T} z_n\| \\
&\quad + [(1-\alpha_n)\delta^2 + \alpha_n \delta^3]\|z_n - v_n\| \\
&\le [\delta + \alpha_n \delta^2 + (1-\alpha_n)]\|z_n - \mathcal{T} z_n\| + \delta^2[1-(1-\delta)\alpha_n]\|z_n - v_n\|
\end{aligned} \tag{38}
$$

Put (37) and (38) in (34):

$$
\begin{aligned}
\|y_n - u_n\| &\le (1-\gamma_n)\delta^2\Big\{[\delta + \alpha_n \delta^2 + (1-\alpha_n)]\|z_n - \mathcal{T} z_n\| + \delta^2[1-(1-\delta)\alpha_n]\|z_n - v_n\|\Big\} \\
&\quad + \gamma_n \delta^2\Big\{\delta[1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\|z_n - \mathcal{T} z_n\| \\
&\quad + \delta^2[1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\|z_n - v_n\| + \beta_n \delta\|s_n - \mathcal{T} s_n\|\Big\} \\
&\quad + [\delta + (1-\beta_n)]\|t_n - \mathcal{T} t_n\| \\
&\le (1-\gamma)\delta^2[\delta + \alpha_n \delta^2 + (1-\alpha_n)]\|z_n - \mathcal{T} z_n\| + (1-\gamma_n)\delta^4[1-(1-\delta)\alpha_n]\|z_n - v_n\|
\end{aligned}
$$

$$
\begin{aligned}
&+\gamma_n\delta^3[1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\|z_n-\mathcal{T}z_n\| \\
&+\gamma_n\delta^4[1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\|z_n-v_n\|+\beta_n\gamma_n\delta^3\|s_n-\mathcal{T}s_n\| \\
&+[\delta+(1-\beta_n)]\|t_n-\mathcal{T}t_n\| \\
=&\{(1-\gamma_n)\delta^2[\delta+\alpha_n\delta^2+(1-\alpha_n)]+\gamma_n\delta^3[1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\}\|z_n-\mathcal{T}z_n\| \\
&+\{(1-\gamma_n)\delta^4[1-(1-\delta)\alpha_n]+\gamma_n\delta^4[1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\}\|z_n-v_n\| \\
&+\beta_n\gamma_n\delta^3\|s_n-\mathcal{T}s_n\|+[\delta+(1-\beta_n)]\|t_n-\mathcal{T}t_n\|
\end{aligned} \tag{39}
$$

and, putting (39) in (33), we have

$$
\begin{aligned}
\|z_{n+1}-v_{n+1}\| \le& \Big\{(1-\gamma_n)\delta^3[\delta+\alpha_n\delta^2+(1-\alpha_n)] \\
&+\gamma_n\delta^4[1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\Big\}\|z_n-\mathcal{T}z_n\| \\
&+\Big\{(1-\gamma_n)\delta^5[1-(1-\delta)\alpha_n]+\gamma_n\delta^5[1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\Big\}\|z_n-v_n\| \\
&+\beta_n\gamma_n\delta^4\|s_n-\mathcal{T}s_n\|+\delta[\delta+(1-\beta_n)]\|t_n-\mathcal{T}t_n\|+(1-\gamma_n)\|u_n-\mathcal{T}u_n\| \\
\le& \Big\{(1-\gamma_n)\delta^3[\delta+\alpha_n\delta^2+(1-\alpha_n)] \\
&+\gamma_n\delta^4[1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\Big\}\|z_n-\mathcal{T}z_n\| \\
&+\beta_n\gamma_n\delta^4\|s_n-\mathcal{T}s_n\|+\delta[\delta+(1-\beta_n)]\|t_n-\mathcal{T}t_n\|+(1-\gamma_n)\|u_n-\mathcal{T}u_n\| \\
&+\{(1-\gamma_n)\delta^6+\gamma_n\delta^5[1-(1-\delta)\beta_n]\}[1-(1-\delta)\alpha_n]\|z_n-v_n\|
\end{aligned} \tag{40}
$$

Let

$$
\begin{aligned}
\eta_n &= (1-\delta)\alpha_n \in (0,1) \\
\pi_n &= \|z_n-v_n\| \\
c_n &= \Big\{(1-\gamma_n)\delta^3[\delta+\alpha_n\delta^2+(1-\alpha_n)]+\gamma_n\delta^4[1-(1-\delta)\beta_n]\times \\
&\quad [1-(1-\delta)\alpha_n]\Big\}\|z_n-\mathcal{T}z_n\|+\beta_n\gamma_n\delta^4\|s_n-\mathcal{T}s_n\| \\
&\quad +\delta[\delta+(1-\beta_n)]\|t_n-\mathcal{T}t_n\|+(1-\gamma_n)\|u_n-\mathcal{T}u_n\| \\
&\quad +\{(1-\gamma_n)\delta^5+\gamma_n\delta^5[1-(1-\delta)\beta_n]\}[1-(1-\delta)\alpha_n]\|z_n-v_n\|
\end{aligned}
$$

Using $\mathcal{T}\tau^*=\tau^*$ and $\|v_n-\tau^*\|\to 0$,

$$
\lim_{n\to\infty}\|z_n-\mathcal{T}z_n\|=\lim_{n\to\infty}\|s_n-\mathcal{T}s_n\|=\lim_{n\to\infty}\|t_n-\mathcal{T}t_n\|=\lim_{n\to\infty}\|u_n-\mathcal{T}u_n\|=0
$$

and it follows that $\frac{c_n}{\eta_n}\to 0$ as $n\to\infty$.

Hence, (40) satisfies the assumption of Lemma 4 and, as such, we have $\lim_{n\to\infty}\|z_n-v_n\|=0$ since $\|v_n-\tau^*\|=\|z_n-v_n\|+\|z_n-\tau^*\|\to 0$ as $n\to\infty$.

Therefore, $\lim_{n\to\infty}\|v_n-\tau^*\|=0$, thereby completing the proof. □

### 3.2. Rate of Convergence of Some Iteration Processes

**Theorem 3.** *Let $\mathcal{B}$ be a nonempty closed convex subset of a Banach space $\mathbb{E}$ and $\mathcal{T}:\mathcal{B}\to\mathcal{B}$ be a contraction mapping satisfying condition (1) and having the fixed point $\tau^*\in\mathscr{F}(\mathcal{T})\neq\emptyset$. Assume that $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}\in(0,1)$ are real sequences for $n\in\mathbb{N}$. Given that $v_0=z_0\in\mathcal{B}$, consider the iterative sequences $\{v_n\}_{n=0}^{\infty}$, $\{p_n\}_{n=0}^{\infty}$ and $\{x_n\}_{n=0}^{\infty}$ defined by the UO (3), CR (4) and Picard-S (5) iterative schemes, respectively. Then, the iterative sequence $\{v_n\}$ converges faster to the fixed point $\tau^*$ than $\{p_n\}$.*

**Proof.** From Theorem 1, we have that

$$\begin{aligned}\|v_{n+1}-\tau^*\| &\leq \delta^{2(n+1)}\prod_{k=0}^{n}[1-(1-\delta)\alpha_k]\|v_0-\tau^*\| \\ &= \delta^{2(n+1)}\|v_0-\tau^*\|[1-(1-\delta)\alpha]^{n+1}\end{aligned}$$

From the CR iterative scheme (4),

$$\begin{aligned}\|r_n-\tau^*\| &= \|(1-\gamma_n)p_n+\gamma_n\mathcal{T}p_n-\tau^*\| \\ &\leq (1-\gamma_n)\|p_n-\tau^*\|+\gamma_n\|\mathcal{T}p_n-\tau^*\| \\ &\leq (1-\gamma_n)\|p_n-\tau^*\|+\gamma_n\delta\|p_n-\tau^*\| \\ &= [1-(1-\delta)\gamma_n]\|p_n-\tau^*\|\end{aligned} \tag{41}$$

$$\begin{aligned}\|q_n-\tau^*\| &= \|(1-\beta_n)\mathcal{T}p_n+\beta_n\mathcal{T}r_n-\tau^*\| \\ &\leq (1-\beta_n)\|\mathcal{T}p_n-\tau^*\|+\beta_n\|\mathcal{T}r_n-\tau^*\| \\ &\leq (1-\beta_n\delta)\|p_n-\tau^*\|+\beta_n\delta\|r_n-\tau^*\| \\ &\leq (1-\beta_n)\delta\|p_n-\tau^*\|+\beta_n\delta[1-(1-\delta)\gamma_n]\|p_n-\tau^*\| \\ &= \{(1-\beta_n)\delta+\beta_n\delta[1-(1-\delta)\gamma_n]\}\|p_n-\tau^*\| \\ &\leq \delta[1-(1-\delta)\beta_n\gamma_n]\|p_n-\tau^*\|\end{aligned} \tag{42}$$

$$\begin{aligned}\|p_{n+1}-\tau^*\| &= \|(1-\alpha_n)q_n+\alpha_n\mathcal{T}q_n-\tau^*\| \\ &\leq (1-\alpha_n)\|q_n-\tau^*\|+\alpha_n\|\mathcal{T}q_n-\tau^*\| \\ &\leq (1-\alpha)\|q_n-\tau^*\|+\alpha_n\delta\|q_n-\tau^*\| \\ &= [1-(1-\delta)\alpha_n]\|q_n-\tau^*\|\end{aligned} \tag{43}$$

Putting (42) in (43), we have

$$\|p_{n+1}-\tau^*\| \leq \delta[1-(1-\delta)\alpha_n][1-(1-\delta)\beta_n\gamma_n]\|p_n-\tau^*\|$$

Since $\delta \in (0,1)$ and $\alpha_n, \beta_n, \gamma_n \in (0,1)$,

$$\|p_{n+1}-\tau^*\| \leq \delta[1-(1-\delta)\alpha_n]\|p_n-\tau^*\|.$$

Via induction, we have

$$\begin{aligned}\|p_{n+1}-\tau^*\| &\leq \delta^{(n+1)}\prod_{k=0}^{n}[1-\alpha_k(1-\delta)]\|p_0-\tau^*\| \\ &= \delta^{(n+1)}\|p_0-\tau^*\|[1-\alpha(1-\delta)]^{(n+1)}\end{aligned}$$

From the Picard-S iteration method (5) and the contraction condition (1), we have

$$\begin{aligned}\|z_n-\tau^*\| &= \|(1-\beta_n)x_n+\beta_n\mathcal{T}x_n-\tau^*\| \\ &\leq (1-\beta_n)\|x_n-\tau^*\|+\beta_n\|\mathcal{T}x_n-\tau^*\| \\ &\leq (1-\beta_n)\|x_n-\tau^*\|+\beta_n\delta\|x_n-\tau^*\| \\ &= [1-(1-\delta)\beta_n]\|x_n-\tau^*\|\end{aligned} \tag{44}$$

$$\begin{aligned}\|y_n-\tau^*\| &= \|(1-\alpha_n)\mathcal{T}x_n+\alpha_n\mathcal{T}z_n-\tau^*\| \\ &\leq (1-\alpha_n)\|\mathcal{T}x_n-\tau^*\|+\alpha_n\|\mathcal{T}z_n-\tau^*\| \\ &\leq (1-\alpha)\delta\|x_n-\tau^*\|+\alpha_n\delta\|z_n-\tau^*\| \\ &\leq (1-\alpha_n)\delta\|x_n-\tau^*\|+\alpha_n\delta\{[1-(1-\delta)\beta_n]\|x_n-\tau^*\|\} \\ &= [(1-\alpha)\delta+\alpha_n\delta[1-(1-\delta)\beta_n]]\|x_n-\tau^*\|\end{aligned} \tag{45}$$

$$\begin{aligned}\|x_{n+1}-\tau^*\| &= \|\mathcal{T}y_n-\tau^*\| \\ &\leq \delta\|y_n-\tau^*\|\end{aligned} \tag{46}$$

Putting (45) in (46), we have

$$\|x_{n+1} - \tau^*\| \leq \delta[(1-\alpha_n)\delta + \alpha_n\delta[1-(1-\delta)\beta_n]]\|x_n - \tau^*\|.$$

Since $\delta \in [0,1)$ and $\alpha_n, \beta_n \in (0,1)$, we have

$$[(1-\alpha_n)\delta + \alpha_n\delta[1-(1-\delta)\beta_n]] < 1$$

and it follows that

$$\begin{aligned} \|x_{n+1} - \tau^*\| &\leq \delta\|x_n - \tau^*\| \\ &\vdots \\ &\leq \delta^{n+1}\|x_n - \tau^*\| \\ &\leq \delta^{n+1}\|x_o - \tau^*\|. \end{aligned}$$

Let

$$\begin{aligned} a_n &= \delta^{2(n+1)}[1-(1-\delta)\alpha]^{n+1}\|v_0 - \tau^*\| \\ b_n &= \delta^{(n+1)}\|p_0 - \tau^*\|[1-\alpha(1-\delta)]^{(n+1)} \end{aligned}$$

and

$$c_n = \delta^{n+1}\|x_o - \tau^*\|$$

Set

$$\frac{a_n}{b_n} = \frac{\delta^{2(n+1)}[1-(1-\delta)\alpha]^{n+1}\|v_0 - \tau^*\|}{\delta^{(n+1)}\|p_0 - \tau^*\|[1-\alpha(1-\delta)]^{(n+1)}} \to 0 \text{ as } n \to \infty$$

and

$$\frac{a_n}{c_n} = \frac{\delta^{2(n+1)}[1-(1-\delta)\alpha]^{n+1}\|v_0 - \tau^*\|}{\delta^{n+1}\|x_o - \tau^*\|} \to 0 \text{ as } n \to \infty.$$

Hence, the UO iterative scheme (3) converges to $\tau^*$ faster than the CR and Picard-S iterative schemes. Therefore, the proof is complete. □

**Example 1.** *Let $\mathcal{B} = \mathbb{R}$. We define a mapping $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ by $\mathcal{T}x = \frac{3x+2}{4}$ which is a contraction mapping with the contraction constant $\delta = \frac{3}{4}$ and $\mathscr{F}(\mathcal{T}) = \{2\}$. If we choose $\alpha_n = \beta_n = \gamma_n = \frac{3}{4}$, then it is clear from Tables 1 and 2 and Figures 1 and 2 that our iterative scheme converges to the fixed point, 2, faster than all of the CR [10] as in (4), F* [14] as in (8), Picard-S [11] as in (5), Modified-SP [14] as in (9), Uddin et al. [13] as in (7) and Picard–Ishikawa [15] as in (10) methods.*

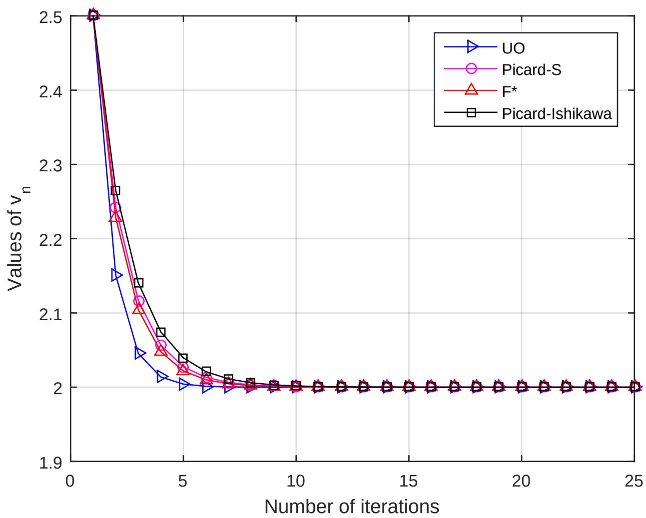


**Figure 1.** Graph corresponding to Table 1.

**Table 1.** Comparison of speed of convergence of some iterative schemes for Example 1.

| Step | UO | Picard-S | F* | Picard–Ishikawa |
|---|---|---|---|---|
| 1 | 2.5000000000 | 2.5000000000 | 2.5000000000 | 2.5000000000 |
| 2 | 2.1508560181 | 2.2416992188 | 2.2285156250 | 2.2651367188 |
| 3 | 2.0455150764 | 2.1168370247 | 2.1044387817 | 2.1405949593 |
| 4 | 2.0137324464 | 2.0564788352 | 2.0477317870 | 2.0745537723 |
| 5 | 2.0041432444 | 2.0273017807 | 2.0218149183 | 2.0395338851 |
| 6 | 2.0012500667 | 2.0131976381 | 2.0099700994 | 2.0209637692 |
| 7 | 2.0003771602 | 2.0063797176 | 2.0045566470 | 2.0111165299 |
| 8 | 2.0001137938 | 2.0030839455 | 2.0020825301 | 2.0058948005 |
| 9 | 2.0000343329 | 2.0014907745 | 2.0009517813 | 2.0031258561 |
| 10 | 2.0000103587 | 2.0007206380 | 2.0004349938 | 2.0016575585 |
| 11 | 2.0000031253 | 2.0003483553 | 2.0001988058 | 2.0008789592 |
| 12 | 2.0000009430 | 2.0001683944 | 2.0000908604 | 2.0004660887 |
| 13 | 2.0000002845 | 2.0000814016 | 2.0000415261 | 2.0002471545 |
| 14 | 2.0000000858 | 2.0000393494 | 2.0000189787 | 2.0001310595 |
| 15 | 2.0000000259 | 2.0000190214 | 2.0000086739 | 2.0000694973 |
| 16 | 2.0000000078 | 2.0000091949 | 2.0000039642 | 2.0000368526 |
| 17 | 2.0000000024 | 2.0000044448 | 2.0000018118 | 2.0000195420 |
| 18 | 2.0000000007 | 2.0000021486 | 2.0000008280 | 2.0000103626 |
| 19 | 2.0000000002 | 2.0000010386 | 2.0000003784 | 2.0000054950 |
| 20 | 2.0000000001 | 2.0000005021 | 2.0000001730 | 2.0000029139 |
| 21 | 2.0000000000 | 2.0000002427 | 2.0000000790 | 2.0000015451 |
| 22 | 2.0000000000 | 2.0000001173 | 2.0000000361 | 2.0000008193 |
| 23 | 2.0000000000 | 2.0000000567 | 2.0000000165 | 2.0000004345 |
| 24 | 2.0000000000 | 2.0000000274 | 2.0000000075 | 2.0000002304 |
| 25 | 2.0000000000 | 2.0000000133 | 2.0000000034 | 2.0000001222 |

**Table 2.** Comparison of speed of convergence of some iterative schemes for Example 1.

| Step | UO | CR | Uddin et al. | Modified SP |
|---|---|---|---|---|
| 1 | 2.5000000000 | 2.5000000000 | 2.5000000000 | 2.5000000000 |
| 2 | 2.1508560181 | 2.2618408203 | 2.1713867188 | 2.2475585938 |
| 3 | 2.0455150764 | 2.1371212304 | 2.0587468147 | 2.1225705147 |
| 4 | 2.0137324464 | 2.0718078709 | 2.0201368476 | 2.0606867685 |
| 5 | 2.0041432444 | 2.0376044636 | 2.0069023765 | 2.0300470621 |
| 6 | 2.0012500667 | 2.0196927672 | 2.0023659513 | 2.0148768169 |
| 7 | 2.0003771602 | 2.0103127406 | 2.0008109853 | 2.0073657677 |
| 8 | 2.0001137938 | 2.0054005929 | 2.0002779842 | 2.0036469182 |
| 9 | 2.0000343329 | 2.0028281914 | 2.0000952856 | 2.0018056519 |
| 10 | 2.0000103587 | 2.0014810719 | 2.0000326614 | 2.0008940093 |
| 11 | 2.0000031253 | 2.0007756102 | 2.0000111955 | 2.0004426394 |
| 12 | 2.0000009430 | 2.0004061728 | 2.0000038375 | 2.0002191584 |
| 13 | 2.0000002845 | 2.0002127052 | 2.0000013154 | 2.0001085091 |
| 14 | 2.0000000858 | 2.0001113898 | 2.0000004509 | 2.0000537247 |
| 15 | 2.0000000259 | 2.0000583328 | 2.0000001546 | 2.0000266000 |
| 16 | 2.0000000078 | 2.0000305478 | 2.0000000530 | 2.0000131701 |
| 17 | 2.0000000024 | 2.0000159973 | 2.0000000182 | 2.0000065208 |
| 18 | 2.0000000007 | 2.0000083775 | 2.0000000062 | 2.0000032285 |
| 19 | 2.0000000002 | 2.0000043871 | 2.0000000021 | 2.0000015985 |
| 20 | 2.0000000001 | 2.0000022975 | 2.0000000007 | 2.0000007914 |
| 21 | 2.0000000000 | 2.0000012031 | 2.0000000003 | 2.0000003919 |
| 22 | 2.0000000000 | 2.0000006301 | 2.0000000001 | 2.0000001940 |
| 23 | 2.0000000000 | 2.0000003300 | 2.0000000000 | 2.0000000961 |
| 24 | 2.0000000000 | 2.0000001728 | 2.0000000000 | 2.0000000476 |
| 25 | 2.0000000000 | 2.0000000905 | 2.0000000000 | 2.0000000235 |

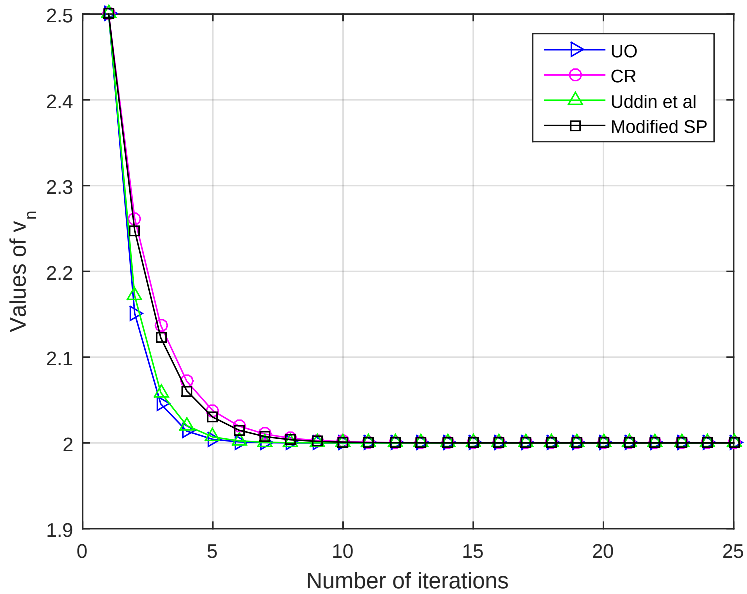


**Figure 2.** Graph corresponding to Table 2.

*3.3. Stability and Data Dependence Results*

**Theorem 4.** *Let $\mathbb{E}$ be a Banach space. Assume that $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ is a contraction mapping with $\delta \in [0,1)$ with a fixed point $\tau^* \in \mathscr{F}(\mathcal{T}) \neq \emptyset$. Assume further that $\{v_n\}_{n=0}^{\infty}$ is a sequence generated by the UO iterative scheme (3) and that it converges to $\tau^*$. Then, (3) is $\mathcal{T}$-stable.*

**Proof.** Assume that $\{p_n\}_{n=0}^{\infty}$ is an arbitrary sequence in $\mathcal{B}$ and let the sequence generated by the UO iterative scheme be $v_{n+1} = f(\mathcal{T}, v_n)$, which converges to a unique fixed point $\tau^*$.

Let $\epsilon_n = \|p_{n+1} - f(\mathcal{T}, p_n)\|$. Our aim is to show that $\lim_{n\to\infty} \epsilon_n = 0$ if and only if $\lim_{n\to\infty} \|p_n - \tau^*\| = 0$. Set $r_n = \mathcal{T} p_n$.

Suppose $\lim_{n\to\infty} \epsilon_n = 0$:

$$\begin{aligned} \|p_{n+1} - \tau^*\| &= \|p_{n+1} - f(\mathcal{T}, p_n) + f(\mathcal{T}, p_n) - \tau^*\| \\ &\leq \|p_{n+1} - f(\mathcal{T}, p_n)\| + \|f(\mathcal{T}, p_n) - \tau^*\| \\ &\leq \epsilon_n + \|f(\mathcal{T}, p_n) - \tau^*\| \\ &\leq \epsilon_n + \|(1-\gamma_n)u_n + \gamma_n \mathcal{T} u_n - \tau^*\| \\ &\leq \epsilon_n + (1-\gamma_n)\|u_n - \tau^*\| + \gamma_n \delta \|u_n - \tau^*\| \\ &= \epsilon_n + [1 - (1-\delta)\gamma_n]\|u_n - \tau^*\| \end{aligned} \tag{47}$$

Next,

$$\begin{aligned} \|u_n - \tau^*\| &= \|(1-\beta_n)y_n + \beta_n \mathcal{T} t_n - \tau^*\| \\ &\leq (1-\beta_n)\|t_n - \tau^*\| + \beta_n \|\mathcal{T} t_n - \tau^*\| \\ &\leq (1-\beta_n)\|t_n - \tau^*\| + \beta_n \delta \|t_n - \tau^*\| \\ &= [1 - (1-\delta)\beta_n]\|t_n = \tau^*\| \end{aligned} \tag{48}$$

$$\begin{aligned} \|t_n - \tau^*\| &= \|\mathcal{T} s_n - \tau^*\| \\ &\leq \delta \|s_n - \tau^*\| \end{aligned} \tag{49}$$

$$\begin{aligned} \|s_n - \tau^*\| &= \|(1-\alpha_n)r_n + \alpha_n \mathcal{T} r_n - \tau^*\| \\ &\leq (1-\alpha_n)\|r_n - \tau^*\| + \alpha_n \delta \|r_n - \tau^*\| \\ &= [1 - (1-\delta)\alpha_n]\|r_n - \tau^*\| \end{aligned} \tag{50}$$

and, combining (49) and (50), we have

$$\begin{aligned}\|t_n-\tau^*\| &\le \delta[1-(1-\delta)\alpha_n]\|r_n-\tau^*\| \\ &\le \delta[1-(1-\delta)\alpha_n]\|\mathcal{T}p_n-\tau^*\| \\ &\le \delta^2[1-(1-\delta)\alpha_n]\|p_n-\tau^*\|\end{aligned} \tag{51}$$

Putting (51) in (48), we have

$$\|u_n-\tau^*\| \le \delta^2[1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\|p_n-\tau^*\| \tag{52}$$

Again, putting (52) in (47), we have

$$\|p_{n+1}-\tau^*\| \le \epsilon_n+\delta^2[1-(1-\delta)\alpha_n][1-(1-\delta)\beta_n][1-(1-\delta)\gamma_n]\|p_n-\tau^*\|$$

Since $\delta \in [0,1)$, $\{\alpha_n\},\{\beta_n\},\{\gamma_n\} \in (0,1)$, from Lemma 1, we have that $\lim_{n\to\infty}\|p_n-\tau^*\|=0$.

Conversely, suppose $\lim_{n\to\infty}\|p_n-\tau^*\|=0$; then,

$$\begin{aligned}\epsilon_n &= \|p_{n+1}-f(\mathcal{T},p_n)\| \\ &\le \|p_{n+1}-\tau^*+\tau^*-f(\mathcal{T},p_n)\| \\ &\le \|p_{n+1}-\tau^*\|+\|\tau^*-f(\mathcal{T},p_n)\| \\ &\le \|p_{n+1}-\tau^*\|+\|(1-\gamma_n)u_n+\gamma_n\mathcal{T}u_n-\tau^*\| \\ &\le \|p_{n+1}-\tau^*\|+(1-\gamma_n)\|u_n-\tau^*\|+\gamma_n\delta\|u_n-\tau^*\| \\ &= \|p_{n+1}-\tau^*\|+[1-(1-\delta)\gamma_n]\|u_n-\tau^*\| \\ &\le \|p_{n+1}-\tau^*\|+[1-(1-\delta)\gamma_n][1-(1-\delta)\beta_n]\|t_n-\tau^*\| \\ &\le \|p_{n+1}-\tau^*\|+\delta[1-(1-\delta)\gamma_n][1-(1-\delta)\beta_n]\|s_n-\tau^*\| \\ &\le \|p_{n+1}-\tau^*\|+\delta[1-(1-\delta)\gamma_n][1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\|r_n-\tau^*\| \\ &\le \|p_{n+1}-\tau^*\|+\delta^2[1-(1-\delta)\gamma_n][1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\|p_n-\tau^*\|.\end{aligned}$$

Take the limit as $n\to\infty$ on both sides and note that $\lim_{n\to\infty}\epsilon_n=0$.

Hence, our new fixed point iterative scheme is $\mathcal{T}$-stable. □

Next, we show the near-$\mathcal{T}$-stability of our new iterative scheme.

**Theorem 5.** *Let $\mathbb{E}$, $\mathcal{B}$ and $\mathcal{T}$ remain the same as in Theorem 4, with $\mathcal{T}$ satisfying (1) for $\mathscr{F}(\mathcal{T})\neq\emptyset$. The iterative scheme is almost $\mathcal{T}$-stable.*

**Proof.** Let $\{p_n\}$ be an approximate sequence of $\{v_n\}$ in $\mathcal{B}$. Suppose that our new iterative scheme (3) is represented as $v_{n+1}=f(\mathcal{T},v_n)$, which converges to a fixed point $\tau^*$, and $\epsilon_n=\|p_{n+1}-f(\mathcal{T},p_n)\|$, $n\in\mathbb{N}$. It is our aim to show that $\sum_{n=0}^{\infty}\epsilon_n<\infty$ implies that $\lim_{n\to\infty}\|p_n-\tau^*\|=0$.

Let $\sum_{n=0}^{\infty}\epsilon_n<\infty$; then, via (3), we have

$$\begin{aligned}\|p_{n+1}-\tau^*\| &= \|p_{n+1}-f(\mathcal{T},p_n)+f(\mathcal{T},p_n)-\tau^*\| \\ &\|p_{n+1}-f(\mathcal{T},p_n)\|+\|f(\mathcal{T},p_n)-\tau^*\| \\ &\le \epsilon_n+\|f(\mathcal{T},p_n)-\tau^*\| \\ &\le \epsilon_n+\|(1-\gamma_n)u_n+\gamma_n\mathcal{T}u_n-\tau^*\| \\ &\le \epsilon_n+(1-\gamma_n)\|u_n-\tau^*\|+\gamma_n\delta\|u_n-\tau^*\| \\ &= \epsilon_n+[1-(1-\delta)\gamma_n]\|u_n-\tau^*\| \\ &\le \epsilon_n+[1-(1-\delta)\gamma_n][1-(1-\delta)\beta_n]\|t_n-\tau^*\| \\ &\le \epsilon_n+\delta[1-(1-\delta)\gamma_n][1-(1-\delta)\beta_n]\|s_n-\tau^*\|\end{aligned}$$

$$\leq \epsilon_n + \delta[1-(1-\delta)\gamma_n][1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\|r_n - \tau^*\|$$
$$\leq \epsilon_n + \delta^2[1-(1-\delta)\gamma_n][1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\|p_n - \tau^*\|$$

Set $\nu_n = \|p_n - \tau^*\|$ and, since $\delta \in [0,1)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in (0,1)$, $[1-(1-\delta)\alpha_n]$ $[1-(1-\delta)\beta_n][1-(1-\delta)\gamma_n] < 1$,

$$\nu_{n+1} \leq \delta^2 \nu_n + \epsilon_n$$

Again, since $\sum_{n=0}^{\infty} \epsilon_n < \infty$, then, via Lemma 2, we have $\sum_{n=0}^{\infty} \nu_n < \infty$. It implies that $\lim_{n\to\infty} \nu_n = 0$ (that is, $\lim_{n\to\infty} \|p_n - \tau^*\| = 0$). □

**Remark 2** ([23]). *An iterative scheme $\{v_n\}$ which is $\mathcal{T}$-stable is also almost $\mathcal{T}$. However, the converse is not true.*

**Theorem 6.** *Suppose $\mathcal{G}$ is an approximate operator of a contraction operator $\mathcal{T}$. Let $\{v_n\}_{n=0}^{\infty}$ be a sequence generated by the UO iterative scheme (3) for $\mathcal{T}$. Let the approximate scheme of the sequence $\{\bar{v}_n\}_{n=0}^{\infty}$ be*

$$\begin{cases} \bar{v}_0 = v \in \mathcal{B} \\ \bar{r}_n = \mathcal{G}\bar{v}_n \\ \bar{s}_n = (1-\alpha_n)\bar{r}_n + \alpha_n \mathcal{G}\bar{r}_n \\ \bar{t}_n = \mathcal{G}\bar{s}_n \\ \bar{u}_n = (1-\beta_n)\bar{t}_n + \beta_n \mathcal{G}\bar{t}_n \\ \bar{v}_{n+1} = (1-\gamma_n)\bar{u}_n + \gamma_n \mathcal{G}\bar{u}_n, \quad n \in \mathbb{N} \end{cases} \tag{53}$$

*where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are real parametric sequences in $(0,1)$ satisfying the condition; $\frac{1}{2} \leq \alpha_n$, $\forall n \in \mathbb{N}$. If $\mathcal{T}\tau^* = \tau^*$ and $\mathcal{G}s^* = s^*$ such that $\lim_{n\to\infty} \|\bar{v}_n - s^*\| = 0$, then, for $0 \leq \delta < 1$, $\|\tau^* - s^*\| \leq \frac{9\epsilon}{1-\delta}$, where $\epsilon > 0$ is a fixed constant.*

**Proof.** Using (3) and (53),

$$\begin{aligned} \|r_n - \bar{r}_n\| &= \|\mathcal{T}v_n - \mathcal{G}\bar{v}_n\| \\ &\leq \|\mathcal{T}v_n - \mathcal{T}\bar{v}_n + \mathcal{T}\bar{v}_n - \mathcal{G}\bar{v}_n\| \\ &\leq \|\mathcal{T}v_n - \mathcal{T}\bar{v}_n\| + \|\mathcal{T}\bar{v}_n - \mathcal{G}\bar{v}_n\| \\ &\leq \|\mathcal{T}v_n - \mathcal{T}\bar{v}_n\| + \epsilon \\ &\leq \delta\|v_n - \bar{v}_n\| + \epsilon \end{aligned} \tag{54}$$

$$\begin{aligned} \|s_n - \bar{s}_n\| &= \|(1-\alpha_n)r_n + \alpha_n\mathcal{T}r_n - (1-\alpha_n)\bar{r}_n - \alpha_n\mathcal{G}\bar{r}_n\| \\ &\leq (1-\alpha_n)\|r_n - \bar{r}_n\| + \alpha_n\|\mathcal{T}r_n - \mathcal{G}\bar{r}_n\| \\ &\leq (1-\alpha_n)\|r_n - \bar{r}_n\| + \alpha_n\|\mathcal{T}r_n - \mathcal{T}\bar{r}_n + \mathcal{T}\bar{r}_n - \mathcal{G}\bar{r}_n\| \\ &\leq (1-\alpha_n)\|r_n - \bar{r}_n\| + \alpha_n\|\mathcal{T}r_n - \mathcal{T}\bar{r}_n\| + \alpha_n\|\mathcal{T}\bar{r}_n - \mathcal{G}\bar{r}_n\| \\ &\leq (1-\alpha_n)\|r_n - \bar{r}_n\| + \alpha_n\delta\|r_n - \bar{r}_n\| + \alpha_n\epsilon \\ &= [1-(1-\delta)\alpha_n]\|r_n - \bar{r}_n\| + \alpha_n\epsilon \end{aligned} \tag{55}$$

Putting (54) in (55),

$$\|s_n - \bar{s}_n\| \leq \delta[1-(1-\delta)\alpha_n]\|v_n - \bar{v}_n\| + [1-(1-\delta)\alpha_n]\epsilon + \alpha_n\epsilon$$

Again, from (3) and (53),

$$\begin{aligned}\|t_n - \bar{t}_n\| &= \|\mathcal{T}s_n - \mathcal{G}\bar{s}_n\| \\ &\leq \|\mathcal{T}s_n - \mathcal{T}\bar{s}_n + \mathcal{T}\bar{s}_n - \mathcal{G}\bar{s}_n\| \\ &\leq \|\mathcal{T}s_n - \mathcal{T}\bar{s}_n\| + \|\mathcal{T}\bar{s}_n - \mathcal{G}\bar{s}_n\| \\ &\leq \delta\|s_n - \bar{s}_n\| + \epsilon \\ &\leq \delta\{\delta[1-(1-\delta)\alpha_n]\|v_n - \bar{v}_n\| + [1-(1-\delta)\alpha_n]\epsilon + \alpha_n\epsilon\} + \epsilon \\ &\leq \delta^2[1-(1-\delta)\alpha_n]\|v_n - \bar{v}_n\| + \delta[1-(1-\delta)\alpha_n]\epsilon + \delta\alpha_n\epsilon + \epsilon\end{aligned} \tag{56}$$

$$\begin{aligned}\|u_n - \bar{u}_n\| &= \|(1-\beta_n)t_n + \beta_n\mathcal{T}t_n - (1-\beta_n)\bar{t}_n - \beta_n\mathcal{G}\bar{t}_n\| \\ &\leq (1-\beta_n)\|t_n - \bar{t}_n\| + \beta_n\|\mathcal{T}t_n - \mathcal{G}\bar{t}_n\| \\ &\leq (1-\beta_n)\|t_n - \bar{t}_n\| + \beta_n\|\mathcal{T}t_n - \mathcal{T}\bar{t}_n + \mathcal{T}\bar{t}_n - \mathcal{G}\bar{t}_n\| \\ &\leq (1-\beta_n)\|t_n - \bar{t}_n\| + \beta_n\|\mathcal{T}t_n - \mathcal{T}\bar{t}_n\| + \beta_n\|\mathcal{T}\bar{t}_n - \mathcal{G}\bar{t}_n\| \\ &\leq (1-\beta_n)\|t_n - \bar{t}_n\| + \beta_n\delta\|t_n - \bar{t}_n\| + \beta\epsilon \\ &= [1-(1-\delta)\beta_n]\|t_n - \bar{t}_n\| + \beta_n\epsilon\end{aligned} \tag{57}$$

Putting (56) in (57),

$$\begin{aligned}\|u_n - \bar{u}_n\| \leq{}& [1-(1-\delta)\beta_n]\{\delta^2[1-(1-\delta)\alpha_n]\|v_n - \bar{v}_n\| + \delta[1-(1-\delta)\alpha_n]\epsilon \\ &+ \delta\alpha_n\epsilon + \epsilon\} + \beta_n\epsilon \\ \leq{}& \delta^2[1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\|v_n - \bar{v}_n\| + \delta[1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\epsilon \\ &+ \delta[1-(1-\delta)\beta_n]\alpha_n\epsilon + [1-(1-\delta)\beta_n]\epsilon + \beta_n\epsilon\end{aligned} \tag{58}$$

Next,

$$\begin{aligned}\|v_{n+1} - \bar{v}_{n+1}\| &= \|(1-\gamma_n)u_n + \gamma_n\mathcal{T}u_n - (1-\gamma_n)\bar{u}_n - \gamma_n\mathcal{G}\bar{u}_n\| \\ &\leq (1-\gamma_n)\|u_n - \bar{u}_n\| + \gamma_n\|\mathcal{T}u_n - \mathcal{G}\bar{u}_n\| \\ &\leq (1-\gamma_n)\|u_n - \bar{u}_n\| + \gamma_n\|\mathcal{T}u_n - \mathcal{T}\bar{u}_n + \mathcal{T}\bar{u}_n - \mathcal{G}\bar{u}_n\| \\ &\leq (1-\gamma_n)\|u_n - \bar{u}_n\| + \gamma_n\|\mathcal{T}u_n - \mathcal{T}\bar{u}_n\| + \gamma_n\|\mathcal{T}\bar{u}_n - \mathcal{G}\bar{u}_n\| \\ &\leq (1-\gamma_n)\|u_n - \bar{u}_n\| + \gamma_n\delta\|u_n - \bar{u}_n\| + \gamma_n\epsilon \\ &\leq [1-(1-\delta)\gamma_n]\|u_n - \bar{u}_n\| + \gamma_n\epsilon\end{aligned} \tag{59}$$

Putting (58) in (59),

$$\begin{aligned}\|v_{n+1} - \bar{v}_{n+1}\| \leq{}& [1-(1-\delta)\gamma_n]\Big\{\delta^2[1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\|v_n - \bar{v}_n\| \\ &+ \delta[1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\epsilon + \delta[1-(1-\delta)\beta_n]\alpha_n\epsilon \\ &+ [1-(1-\delta)\beta_n]\epsilon + \beta_n\epsilon\Big\} + \gamma_n\epsilon \\ \leq{}& \delta^2[1-(1-\delta)\gamma_n][1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\|v_n - \bar{v}_n\| \\ &+ \delta[1-(1-\delta)\gamma_n][1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n]\epsilon \\ &+ \delta[1-(1-\delta)\gamma_n][1-(1-\delta)\beta_n]\alpha_n\epsilon + [1-(1-\delta)\gamma_n][1-(1-\delta)\beta_n]\epsilon \\ &+ [1-(1-\delta)\gamma_n]\beta_n\epsilon + \gamma_n\epsilon\end{aligned}$$

Since $\delta \in [0,1)$ and $\{\alpha_n\},\{\beta_n\},\{\gamma_n\} \in (0,1)$, $n \in \mathbb{N}$, $\delta^2[1-(1-\delta)\gamma_n][1-(1-\delta)\beta_n] < 1$, $\delta[1-(1-\delta)\gamma_n][1-(1-\delta)\beta_n][1-(1-\delta)\alpha_n] < 1$, $\delta[1-(1-\delta)\gamma_n][1-(1-\delta)\beta_n] < 1$, $[1-(1-\delta)\gamma_n][1-(1-\delta)\beta_n] < 1$, $[1-(1-\delta)\gamma_n]\beta_n < 1$, $\gamma_n < 1$ and $1-\alpha_n \leq \alpha_n$, so that

$$\begin{aligned}\|v_{n+1}-\bar{v}_{n+1}\| &\leq [1-(1-\delta)\alpha_n]\|v_n-\bar{v}_n\|+\alpha_n\epsilon+4\epsilon \\ &\leq [1-(1-\delta)\alpha_n]\|v_n-\bar{v}_n\|+\alpha_n\epsilon+4(1-\alpha_n+\alpha_n)\epsilon \\ &\leq [1-(1-\delta)\alpha_n]\|v_n-\bar{v}_n\|+\alpha_n(1-\delta)\frac{9\epsilon}{(1-\delta)}.\end{aligned}$$

Let $\xi_n := \|v_{-}\bar{v}_n\|$, $\varphi_n := \alpha_n(1-\delta) \in (0,1)$ and $\varrho_n := \frac{9\epsilon}{(1-\delta)}$. From Lemma 3, it is clear that $0 \leq \limsup_{n\to\infty} \|v_n - \bar{v}_n\| \leq \limsup_{n\to\infty} \frac{9\epsilon}{1-\delta}$. Again, from Theorem 1, it is easy to confirm that $\lim_{n\to\infty} \|v_n - \tau^*\| = 0$. Consequently, given that $\lim_{n\to\infty} \|\bar{v}_n - s^*\| = 0$, we have $\|\tau^* - s^*\| \leq \frac{9\epsilon}{1-\delta}$.

Therefore, the proof is complete. □

## 4. Application to Oxygen Diffusion Model

Oxygen diffusion transport is a critical chain reaction happening seamlessly within the human body to make oxygen available to every cell by basically moving oxygen down a concentration gradient across tissue barriers, including the alveolar–capillary membrane, and across the extracellular matrix between the tissue capillaries and diffusion distance, which is related to the tissue capillary density. Oxygen diffuses from the air into the blood in the lungs and it does not have the same rate of consumption (see, e.g., [24]) as in a real situation, which tends to be influenced by the thermal energy of particles induced by its kinetic energy. Oxygen most particularly binds to hemoglobin (in a large volume) and dissolves in the blood plasma (in a minute volume), and it is altogether transported through the arteries to capillaries.

Our aim here is to study the model as developed by Srivastava and Rai [25]. The model shown below

$$\frac{\partial^{\xi}\mathcal{C}}{\partial t^{\xi}} - \lambda\frac{\partial^{\mu}\mathcal{C}}{\partial t^{\mu}} = \nabla(d\cdot\nabla\mathcal{C}) - K, \ \ \xi,\mu \in (0,1] \tag{60}$$

is based on a fractional diffusion equation where $\frac{\partial^{\xi}\mathcal{C}}{\partial t^{\xi}}$ is a fractional order derivative for $0 < \xi < 1$ representing the subdiffusion process, $\mathcal{C}(r,z,t)$ is the concentration of oxygen, $k(r,z,t)$ is the rate of consumption per volume of tissue and $d$ is the diffusion coefficient of oxygen. The net diffusion of oxygen to tissue is $\frac{\partial^{\xi}\mathcal{C}}{\partial t^{\xi}} - \lambda\frac{\partial^{\mu}\mathcal{C}}{\partial t^{\mu}}$, with $\lambda$ being the time lag in the concentration of oxygen $\mathcal{C}$ along the $z$-axis.

The equation (60) can be reduced to an integral equivalent:

$$\mathcal{C}(r,z,t) = \mathcal{C}(r,z,0)\Big(1-\lambda\frac{t^{\xi-\mu}}{\Gamma(\xi-\mu+1)}\Big) + \lambda\,\mathcal{D}_t^{-(\xi-\mu)}\,\mathcal{C} + \mathcal{D}_t^{-\xi}(\nabla(d\cdot\nabla\mathcal{C})) - \mathcal{D}_t^{-\xi}\,K \tag{61}$$

Equation (61) can alternatively be written as

$$\Psi(r,z,t) = \Psi(r,z,0)\Big(1-\lambda\frac{t^{\xi-\mu}}{\Gamma(\xi-\mu+1)}\Big) + \lambda\,\mathcal{D}_t^{-(\xi-\mu)}\,\Psi + \mathcal{D}_t^{-\xi}(\nabla(d\cdot\nabla\Psi) - K) \tag{62}$$

or

$$\Psi(r,z,t) = \mathcal{K}(\Psi_0) + \frac{1}{\Gamma(\xi)}\int_0^t \mathcal{H}(s,\Psi(s),K)ds \tag{63}$$

where $\mathcal{K}(\Psi_0) = \Psi(r,z,0)\Big[1-\lambda\frac{t^{\xi-\mu}}{\Gamma(\xi-\mu+1)}\Big]$ and $\mathcal{H}(s,\Psi,K) = \lambda\frac{\partial^{\mu}\Psi}{\partial t^{\mu}} + [\nabla(d\cdot\nabla\Psi) - K]$.

Let

$$\mathcal{T}\Psi(r,z,t) = \mathcal{K}(\Psi_0) + \frac{1}{\Gamma(\xi)}\int_0^t \mathcal{H}(s,\Psi(s),K)ds \tag{64}$$

be an integral operator.

We define, for $t \in [0,T]$ and the space $\mathcal{S} = ([0,T],\mathbb{R})$, the supremum norm:

$$\|\Psi\| = \sup_{t\in[0,T]}\{|\Psi(t)| : \Psi \in \mathcal{S}\}.$$

The following result will be useful in proving our main result in this section.

**Theorem 7.** *Suppose that the following conditions are satisfied:*

$(E_1)$ *There exists a constant $L_{\mathcal{H}} > 0$ such that*

$$|\mathcal{H}(t, \Psi_1(t), K) - \mathcal{H}(t, \Psi_2(t), K)| \leq L_{\mathcal{H}}|\Psi_1 - \Psi_2|$$

*for each $\Psi \in \mathcal{S}$ and $t \in [0, T]$.*

$(E_2)$ $\frac{L_{\mathcal{H}} T}{\Gamma(\xi)} < 1.$

*Then, (60) has a unique solution.*

Here, we are set to present our main result of this section.

**Theorem 8.** *Assume that the condition of theorem 7 holds. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in (0, 1)$ be real sequences of the iterative scheme (3) such that $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then, the diffusion model Equation (60) has a solution $c$ and the iterative scheme (3) converges to $c$.*

**Proof.** Consider the space $\mathcal{S} = ([0, T], \mathbb{R})$ with a supremum norm defined as

$$\|\Psi\| = \sup_{t \in [0,T]} \{|\Psi(t)| : \Psi \in \mathcal{S}\}.$$

Let $\{v_n\}$ be a sequence generated by the iterative scheme (3) for the operator $\mathcal{T} : \mathcal{S} \to \mathcal{S}$ defined by

$$\mathcal{T}\Psi(t) = \mathcal{K}(\Psi_0) + \frac{1}{\Gamma(\xi)} \int_0^t \mathcal{H}(s, \Psi(s), K) ds.$$

We want to show that $\{v_n\}$ converges to $c$ as $n$ converges to $\infty$.

From (3), (64) and the conditions of Theorem 7, we have

$$\begin{aligned}
\|r_n - c\| &= \|\mathcal{T} v_n - c\| \\
&\leq \|\mathcal{T} v_n - \mathcal{T} c\| \\
&\leq \max_{t \in [0,T]} |\mathcal{T} v_n(t) - \mathcal{T} c(t)| \\
&= \max_{t \in [0,T]} |\mathcal{K}(\Psi_0) + \frac{1}{\Gamma(\xi)} \int_0^t \mathcal{H}(s, v_n(s), K) ds - \mathcal{K}(\Psi_0) - \frac{1}{\Gamma(\xi)} \int_0^t \mathcal{H}(s, c(s), K) ds| \\
&= \frac{1}{\Gamma(\xi)} \max_{t \in [0,T]} \int_0^t |\mathcal{H}(s, v_n(s), K) - \mathcal{H}(s, c(s), K)| ds \\
&\leq \frac{L_{\mathcal{H}}}{\Gamma(\xi)} \max_{t \in [0,T]} \int_0^t |v_n(s) - c(s)| ds \\
&\leq \frac{L_{\mathcal{H}}}{\Gamma(\xi)} t \|v_n - c\|
\end{aligned} \tag{65}$$

$$\begin{aligned}
\|s_n - c\| &= \|(1 - \alpha_n) r_n + \alpha_n \mathcal{T} r_n - c\| \\
&\leq (1 - \alpha_n)\|r_n - c\| + \alpha_n \|\mathcal{T} r_n - \mathcal{T} c\| \\
&\leq (1 - \alpha_n)\|r_n - c\| + \alpha_n \max_{t \in [0,T]} |\mathcal{T} r_n - \mathcal{T} c| \\
&\leq (1 - \alpha_n)\|r_n - c\| + \alpha_n \max_{t \in [0,T]} \Big|\mathcal{K}(\Psi_0) + \frac{1}{\Gamma(\xi)} \int_0^t \mathcal{H}(s, r_n, K) ds \\
&\quad - \mathcal{K}(\Psi_0) - \frac{1}{\Gamma(\xi)} \int_0^t \mathcal{H}(s, c(s), K) ds\Big| \\
&\leq (1 - \alpha_n)\|r_n - c\| + \alpha_n \max_{t \in [0,T]} \Big|\frac{1}{\Gamma(\xi)} \int_0^t \mathcal{H}(s, r_n(s), K) ds
\end{aligned} \tag{66}$$

$$
\begin{aligned}
&- \frac{1}{\Gamma(\xi)}\int_0^t \mathcal{H}(s,c(s),K)ds\Big| \\
&= (1-\alpha_n)\|r_n - c\| + \frac{\alpha_n}{\Gamma(\xi)}\max_{t\in[0,T]}\Big|\int_0^t \mathcal{H}(s,r_n(s),K) - \mathcal{H}(s,c(s),K)\Big|ds \\
&\leq (1-\alpha_n)\|r_n - c\| + \frac{\alpha_n L_{\mathcal{H}}}{\Gamma(\xi)}\max_{t\in[0,T]}\int_0^t |r_n(s) - c(s)|ds \\
&\leq (1=\alpha_n)\|r_n - c\| + \frac{\alpha_n L_{\mathcal{H}}}{\Gamma(\xi)}T\|r_n - c\| \\
&= [1-(1-\frac{L_{\mathcal{H}}}{\Gamma(\xi)}T)\alpha_n]\|r_n - c\|
\end{aligned}
$$

$$
\begin{aligned}
\|t_n - c\| &= \|\mathcal{T}s_n - c\| \leq \|\mathcal{T}s_n - \mathcal{T}c\| \\
&\leq \max_{t\in[0,T]}|\mathcal{T}s_n(t) - \mathcal{T}c(t)| \\
&\leq \max_{t\in[0,T]}\Big|\mathcal{K}(\Psi_0) + \frac{1}{\Gamma(\xi)}\int_0^t \mathcal{H}(s,s_n(s),K)ds \\
&\quad - \mathcal{K}(\Psi_0) - \frac{1}{\Gamma(\xi)}\int_0^t \mathcal{H}(s,c(s),K)ds\Big| \\
&= \frac{1}{\Gamma(\xi)}\max_{t\in[0,T]}\Big|\int_0^t \mathcal{H}(s,s_n(s),K)ds - \int_0^t \mathcal{H}(s,c(s),K)ds\Big| \\
&\leq \frac{1}{\Gamma(\xi)}\max_{t\in[0,T]}\int_0^t |\mathcal{H}(s,s_n(s),K) - \mathcal{H}(s,c(s),K)|ds \\
&\leq \frac{L_{\mathcal{H}}}{\Gamma(\xi)}\max_{t\in[0,T]}\int_0^t |s_n(s) - c(s)|ds \\
&\leq \frac{L_{\mathcal{H}}T}{\Gamma(\xi)}\|s_n - c\|
\end{aligned}
\tag{67}
$$

Putting (66) in (67), we have

$$
\|t_n - c\| \leq \frac{l_{\mathcal{H}}}{\Gamma(\xi)}T[1-(1-\frac{L_{\mathcal{H}}}{\Gamma(\xi)}T)\alpha_n]\|r_n - c\|
$$

$$
\begin{aligned}
\|u_n - c\| &= \|(1-\beta_n)t_n + \beta_n\mathcal{T}t_n - c\| \\
&\leq (1-\beta_n)\|t_n - c\| + \beta_n\|\mathcal{T}t_n - c\| \\
&\leq (1-\beta_n)\|t_n - c\| + \beta_n\|\mathcal{T}t_n - \mathcal{T}c\| \\
&\leq (1-\beta_n)\|t_n - c\| + \beta_n\max_{t\in[0,T]}|\mathcal{T}t_n(s) - \mathcal{T}c(s)| \\
&\leq (1-\beta_n)\|t_n - c\| + \beta_n\max_{t\in[0,T]}\Big|\mathcal{K}(\Psi_0) + \frac{1}{\Gamma(\xi)}\int_0^t \mathcal{H}(s,t_n(s),K)ds \\
&\quad - \mathcal{K}(\Psi_0) - \frac{1}{\Gamma(\xi)}\int_0^t \mathcal{H}(s,c(s),K)ds\Big| \\
&= (1-\beta_n)\|t_n - c\| + \frac{\beta_n}{\Gamma(\xi)}\max_{t\in[0,T]}\Big|\int_0^t \mathcal{H}(s,t_n(s),K)ds - \int_0^t \mathcal{H}(s,c(s),K)ds\Big| \\
&\leq (1-\beta_n)\|t_n - c\| + \frac{\beta_n}{\Gamma(\xi)}\max_{t\in[0,T]}\int_0^t |\mathcal{H}(s,t_n(s),K) - \mathcal{H}(s,c(s),K)|ds \\
&\leq (1-\beta_n)\|t_n - c\| + \frac{\beta_n}{\Gamma(\xi)}L_{\mathcal{H}}\max_{t\in[0,T]}\int_0^t |t_n(s) - c(s)|ds \\
&\leq (1-\beta_n)\|t_n - c\| + \frac{\beta_n L_{\mathcal{H}}}{\Gamma(\xi)}T\|t_n - c\| \\
&= [1-(1-\frac{L_{\mathcal{H}}}{\Gamma(\xi)})\beta_n]\|t_n - c\|
\end{aligned}
\tag{68}
$$

$$\leq \frac{L_{\mathcal{H}}T}{\Gamma(\xi)}[1-(1-\frac{L_{\mathcal{H}}T}{\Gamma(\xi)})\alpha_n][1-(1-\frac{L_{\mathcal{H}}T}{\Gamma(\xi)})\beta_n]\|r_n-c\|$$

$$\begin{aligned}
\|v_{n+1}-c\| &= \|(1-\gamma_n)u_n+\gamma\mathcal{T}u_n-c\| \\
&\leq (1-\gamma_n)\|u_n-c\|+\gamma_n\|\mathcal{T}u_n-c\| \\
&\leq (1-\gamma_n)\|u_n-c\|+\gamma_n\|\mathcal{T}u_n-\mathcal{T}c\| \\
&\leq (1-\gamma_n)\|u_n-c\|+\gamma_n \max_{t\in[0,T]}|\mathcal{T}u_n-\mathcal{T}c| \\
&\leq (1-\gamma_n)\|u_n-c\|+\gamma_n \max_{t\in[0,T]}\Big|\mathcal{K}(\Psi_0)+\frac{1}{\Gamma(\xi)}\int_0^t \mathcal{H}(s,u_n(s),K)ds \\
&\quad -\mathcal{K}(\Psi_0)-\frac{1}{\Gamma(\xi)}\int_0^t \mathcal{H}(s,c(s),K)ds\Big| \\
&\leq (1-\gamma_n)\|u_n-c\|+\frac{\gamma_n}{\Gamma(\xi)}\max_{t\in[0,T]}\Big|\int_0^t \mathcal{H}(s,u_n(s),K)ds-\int_0^t \mathcal{H}(s,c(s),K)ds\Big| \\
&\leq (1-\gamma_n)\|u_n-c\|+\frac{\gamma_n}{\Gamma(\xi)}\max_{t\in[0,T]}\int_0^t |\mathcal{H}(s,u_n(s),K)-\mathcal{H}(s,c(s),K)|ds \\
&\leq (1-\gamma_n)\|u_n-c\|+\frac{\gamma_n Ł_{\mathcal{H}}}{\Gamma(\xi)}\max_{t\in[0,T]}\int_0^t |u_n(s)-c(s)|ds \\
&\leq (1-\gamma_n)\|u_n-c\|+\frac{\gamma_n L_{\mathcal{H}}}{\Gamma(\xi)}T\max_{t\in[0,T]}|u_n-c| \\
&\leq (1-\gamma_n)\|u_n-c\|+\frac{\gamma_n L_{\mathcal{H}}}{\Gamma(\xi)}T\|u_n-c\| \\
&= [1-(1-\frac{L_{\mathcal{H}}T}{\Gamma(\xi)})\gamma_n]\|u_n-c\|
\end{aligned} \tag{69}$$

Combining (65), (68) and (69), we have

$$\|v_{n+1}-c\| \leq \frac{L_{\mathcal{H}}^2T^2}{\Gamma^2(\xi)}\Big[1-\Big(1-\frac{L_{\mathcal{H}}}{\Gamma(\xi)}T\Big)\alpha_n\Big]\Big[1-\Big(1-\frac{L_{\mathcal{H}}}{\Gamma(\xi)}T\Big)\beta_n\Big]\Big[1-\Big(1-\frac{L_{\mathcal{H}}}{\Gamma(\xi)}T\Big)\gamma_n\Big]\|v_n-c\| \tag{70}$$

From assumption $(E_2)$ and the fact that $[1-(1-\frac{L_{\mathcal{H}}}{\Gamma(\xi)}T)\beta_n][1-(1-\frac{L_{\mathcal{H}}}{\Gamma(\xi)}T)\gamma_n] < 1$, (70) reduces to

$$\|v_{n+1}-c\| \leq [1-(1-\frac{L_{\mathcal{H}}}{\Gamma(\xi)}T)\alpha_n]\|v_n-c\|.$$

Via induction,

$$\|v_{n+1}-c\| \leq \|v_0-c\|\prod_{m=0}^{n}[1-(1-\frac{L_{\mathcal{H}}}{\Gamma(\xi)}T)\alpha_m]. \tag{71}$$

From classical analysis, $1-x \leq e^{-x}$ for $x \in [0,1]$.

$$\|v_{n+1}-c\| \leq \|v_0-c\|e^{-(1-\frac{L_{\mathcal{H}}T}{\Gamma(\xi)})\sum_{m=0}^{n}\alpha_m}$$

Taking the limit as $n \to \infty$, we have $\lim_{n\to\infty}\|v_n-c\| = 0$. Hence, the proof is complete. □

## 5. Application to Boundary Value Problem via Green's Function

### *5.1. Construction of Green's Function*

To construct the Green's function, we consider a third-order boundary value problem (BVP),

$$L[g] \equiv p_1(t)g'''(t)+p_2(t)g''(t)+p_3(t)g'(t)+p_4(t)g(t) = M(t) \tag{72}$$

where $t \in [a,b]$, with the corresponding boundary conditions (BCs)

$$
\begin{aligned}
B_{k_1}[g] &= \varphi_1 g(k_1) + \varphi_2 g'(k_1) + \varphi_3 g''(k_1) = \varphi \\
B_{k_2}[g] &= \varpi_1 g(k_2) + \varpi_2 g'(k_2) + \varpi_3 g''(k_2) = \varpi \\
B_{k_3}[g] &= \vartheta_1 g(k_3) + \vartheta_2 g'(k_3) + \vartheta_3 g''(k_3) = \vartheta
\end{aligned} \tag{73}
$$

for $k_3 = k_2$ or $k_3 = k_1$. From (72), $L[g]$ is linear and the righthand side can be written as $M(t, g(t), g'(t), g''(t))$. The righthand side could be linear or nonlinear; $\varphi$, $\varpi$, $\vartheta$ are constants.

The homogeneous part $L[g] = 0$ of (72) can be solved to obtain three linearly independent complementary solutions, $g_1, g_2$ and $g_3$, and will be used to obtain the Green's function, which is a piecewise function expressed as a linear combination of the linearly independent complementary solutions $g_1, g_2$ and $g_3$; thus,

$$
G(t,s) = \begin{cases} d_1 g_1 + d_2 g_2 + d_3 g_3, & a < t < s \\ e_1 g_1 + e_2 g_2 + e_3 g_3, & s < t < b, \end{cases} \tag{74}
$$

where $d_1, d_2, d_3, e_1, e_2, e_3$ are constants that can be determined accordingly through the hypotheses of the following axioms;

$(A_1)$ $G$ satisfies the associated boundary conditions:

$$
B_{k_1}[G(t,s)] = B_{k_2}[G(t,s)] = B_{k_3}[G(t,s)] = 0
$$

$(A_2)$ $G$ is continuous at $t = s$:

$$
d_1 g_1(s) + d_2 g_2(s) + d_3 g_3(s) = e_1 g_1(s) + e_2 g_2(s) + e_3 g_3(s)
$$

$(A_3)$ $G'$ is continuous at $t = s$:

$$
d_1 g_1'(s) + d_2 g_2'(s) + d_3 g_3'(s) = e_1 g_1'(s) + e_2 g_2'(s) + e_3 g_3'(s)
$$

$(A_4)$ $G''$ has jump discontinuity at $t = s$:

$$
d_1 g_1''(s) + d_2 g_2''(s) + d_3 g_3''(s) + \frac{1}{h(s)} = e_1 g_1''(s) + e_2 g_2''(s) + e_3 g_3''(s)
$$

If the Green's function $G(t,s)$ is the solution to the BVP (72), then it will satisfy the equation

$$
-L[G(t,s)] = \delta(t-s) \tag{75}
$$

where $\delta$ is the Kronecker Delta that is subject to the homogeneous boundary conditions

$$
B_{k_1}[G(t,s)] = B_{k_2}[G(t,s)] = B_{k_3}[G(t,s)] = 0.
$$

As a matter of fact, the righthand side of (75) will be $-\delta(t-s)$ for self-adjoint operators. The Green's function in (73) will satisfy the homogeneous equation $L[G(t,s)] = 0$ for $t \neq s$.

### *5.2. UO–Green Iterative Scheme*

Our aim here is to embed the Green's function obtained from the preceding section in the iterative scheme (3). This aim can be achieved by considering the nonlinear boundary value problem

$$
L[g] + N[g] = M(t,g), \tag{76}
$$

where $L[g]$ is linear in $g$, $N[g]$ is nonlinear in $g$ and $M(t,g)$ is a function in $g$ that could be linear or nonlinear. The general solution to (76) is given as $g = g_c + g_p$ where $g_c$ is the complementary solution obtained from the homogeneous part $L[g] = 0$ subject to the boundary conditions as expressed in axiom $(A_1)$.

Assume $g_p$ is the particular solution to the nonhomogeneous part of (76). We define an integral operator in terms of the Green's function, $G(t,s)$, and the particular solution, $g_p$.

$$\Omega[g_p] = \int_a^b G(t,s)L[g_p]ds. \tag{77}$$

Setting $g_p$ to $g$ for convenience so that (77) becomes

$$\Omega[g] = \int_a^b G(t,s)L[g]ds. \tag{78}$$

Obviously, $g$ is a fixed point if and only if $g$ is the solution to (76). Suppose $g_p - g = \int_a^b G(t,s)[M(t,g) - N[g]]ds$,

$$\begin{aligned}\Omega[g] &= \int_a^b G(t,s)[L[g] + N[g] - M(t,g) - N[g] + M(t,g)]ds \\ &\leq \int_a^b G(t,s)[L[g] + N[g] - M(t,g)]ds \\ &\quad + \int_a^b G(t,s)[M(t,g) - N[g]]ds \\ &= g + \int_a^b G(t,s)[L[g] + N[g] - M(t,g)]ds.\end{aligned}$$

Applying the UO iterative scheme (3), we have

$$\begin{cases} r_n = \Omega[v_n] \\ s_n = (1-\alpha_n)r_n + \alpha_n\Omega[r_n] \\ t_n = \Omega[s_n] \\ u_n = (1-\beta_n)t_n + \beta_n\Omega[t_n] \\ v_{n+1} = (1-\gamma_n)u_n + \gamma_n\Omega[u_n], \end{cases} \tag{79}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$ for all $n \in \mathbb{N}$. In an expanded form, (79) can be expressed as

$$\begin{aligned} r_n &= v_n + \int_a^b G(t,s)[L[v_n] + N[v_n] - M(t,v_n)]ds \\ s_n &= (1-\alpha_n)r_n + \alpha_n[r_n + \int_a^b G(t,s)[L[r_n] + N[r_n] - M(t,r_n)]ds] \\ t_n &= s_n + \int_a^b G(t,s)[L[s_n] + N[s_n] - M(t,s_n)]ds \\ u_n &= (1-\beta_n)t_n + \beta_n\Big\{t_n + \int_a^b G(t,s)[L[t_n] + N[t_n] - M(t,t_n)]ds\Big\} \\ v_{n+1} &= (1-\gamma_n)u_n + \gamma_n\Big\{u_n + \int_a^b G(t,s)[L[u_n] + N[u_n] - M(t,u_n)]ds\Big\} \end{aligned}$$

which reduces to

$$\begin{aligned} r_n &= v_n + \int_a^b G(t,s)[L[v_n] + N[v_n] - M(t,v_n)]ds \\ s_n &= r_n + \alpha_n \int_a^b G(t,s)[L[r_n] + N[r_n] - M(t,r_n)]ds \\ t_n &= s_n + \int_a^b G(t,s)[L[s_n] + N[s_n] - M(t,s_n)]ds \\ u_n &= t_n + \beta_n \int_a^b G(t,s)[L[t_n] + N[t_n] - M(t,t_n)]ds \end{aligned} \tag{80}$$

$$v_{n+1} = u_n + \gamma_n \int_a^b G(t,s)[L[u_n] + N[u_n] - M(t,u_n)]ds.$$

### 5.3. Convergence Analysis

It is our aim to show the convergence of our iterative scheme (3) to a solution to the BVP via Green's function. To achieve our aim, we consider the following BVP:

$$-g'''(t) = M(t, g(t), g'(t), g''(t))$$

with boundary conditions

$$g(1) = C_1,\ g''(1) = C_2,\ g(2) = C_3$$

When the homogeneous equation $g'''(t) = 0$ is solved, the following Green's function is obtained

$$G(t,s) = \begin{cases} d_1 t^2 + d_2 t + d_3, & 1 \le t \le s \le 2 \\ e_t^2 + e_2 t + e_3, & 1 \le s \le t \le 2 \end{cases} \tag{81}$$

If axioms $(A_1)$–$(A_4)$ are invoked, then real values for the constants $d_i, e_i$ $(i = 1,2,3)$ are obtained. Hence, (81) becomes

$$G(t,s) = \begin{cases} -\frac{s^2}{2} + 2s - 2 + (\frac{s^2}{2} - 2s + 2)t, & 1 \le t \le s \le 2 \\ -s^2 + 2s - 2 + (\frac{s^2}{2} - 2s + 2)t, & 1 \le s \le t \le 2. \end{cases} \tag{82}$$

Furthermore, the UO–Green iterative scheme (80) is given as

$$\begin{cases} r_n = \mathcal{T}_G v_n \\ s_n = (1-\alpha_n) r_n + \alpha_n \mathcal{T}_G r_n \\ t_n = \mathcal{T}_G s_n \\ u_n = (1-\beta_n) t_n + \beta_n \mathcal{T}_G t_n \\ v_{n+1} = (1-\gamma_n) u_n + \mathcal{T}_G u_n \end{cases} \tag{83}$$

where the operator $\mathcal{T}_G : C^2([1,2]) \to C^2([1,2])$ is defined as

$$\mathcal{T}_G(v) = v + \int_1^2 G(t,s)(v''' - M(s, v, v', v''))ds \tag{84}$$

The initial iterate $v_0$ to (83) satisfies the homogeneous equation $v_0''' = 0$ and the BCs: $v_0(1) = C_1,\ v_0''(1) = C_2$ and $v_0(2) = C_3$.

Suppose we use integration by parts for $\int_1^2 G(t,s)v'''ds$ in (84) and, noting that $\int_1^2 \frac{\partial^3 G(t,s)}{\partial s^3} g(s)ds = \int_1^2 \delta(t-s)g(s)ds$, we then have

$$\mathcal{T}_G(v) = (2-t)C_1 + \frac{1}{2}(t^2 - 3t + 2)C_2 + (t-1)C_3 - \int_1^2 G(t,s)M(s,v,v',v'')ds.$$

Our next aim is to prove that the operator $\mathcal{T}_G$ is a contraction on the Banach space $C^2([1,2])$ for the norm

$$\|v\|_{C^2} = \sum_{i=0}^{2} \sup_{s\in[1,2]} |v^{(i)}(s)|$$

under certain conditions on $M$. Moreover, we shall show that $\mathcal{T}_G$ is a Zamfirescu operator under certain hypotheses on $M$.

**Theorem 9.** *Let $M$, which appears in $\mathcal{T}_G$, satisfy the following Lipschitz condition:*

$$|M(s,v,v',v'') - M(s,\ell,\ell',\ell'')| \le \mu_1 |v(s) - \ell(s)| + \mu_2 |v'(s) - \ell'(s)| + \mu_3 |v''(s) - \ell''(s)| \tag{85}$$

*where $\mu_1, \mu_2$ and $\mu_3$ are positive constants such that*

$$\frac{1}{8}\max\{\mu_1, \mu_2, \mu_3\} \leq 1.$$

*The operator $\mathcal{T}_G$ is a contraction on the Banach space $C^2([1,2], \|\cdot\|_{C^2})$, and the sequence $\{v_n\}$ defined by the UO iterative scheme (3) converges to the fixed point of $\mathcal{T}_G$.*

**Proof.** Assume $v_1, v_2 \in C^2([1,2])$, so that by (85), we have

$$\begin{aligned}
&|\mathcal{T}_G(v_1) - \mathcal{T}_G(v_2)| \\
&\quad = |\int_1^2 G(t,s)M(s, v_1, v_1', v_1'')ds - \int_1^2 G(t,s)M(s, v_2, v_2', v_2'')ds| \\
&\quad = |\int_1^2 G(t,s)(M(s, v_1, v_1', v_1'') - M(s, v_2, v_2', v_2''))ds| \\
&\quad \leq \int_1^2 |G(t,s)||M(s, v_1, v_1', v_1'') - M(s, v_2, v_2', v_2'')|ds \\
&\quad \leq \Big(\sup_{[1,2]\times[1,2]} |G(t,s)|\Big) \int_1^2 |M(s, v_1, v_1', v_1'') - M(s, v_2, v_2', v_2'')|ds \\
&\quad = G(\frac{3}{4}, 1) \int_1^2 |M(s, v_1, v_1', v_1'') - M(s, v_2, v_2', v_2'')|ds \\
&\quad = \frac{1}{8} \int_1^2 |M(s, v_1, v_1', v_1'') - M(s, v_2, v_2', v_2'')|ds \\
&\quad \leq \frac{1}{8} \int_1^2 [\mu_1|v_1(s) - v_2(s)| + \mu_2|v_1'(s) - v_2'(s)| + \mu_3|v_1''(s) - v_2''(s)|]ds \\
&\quad \leq \frac{1}{8}\max\{\mu_1, \mu_2, \mu_3\} \int_1^2 \Big(\sum_{i=1}^{2} |v_1^{(i)}(s) - v_2^{(i)}(s)|\Big)ds \\
&\quad \leq \frac{1}{8}\max\{\mu_1, \mu_2, \mu_3\}\|v_1 - v_2\|_{C^2} \\
&\quad < \|v_1 - v_2\|_{C^2}.
\end{aligned}$$

which shows that $\mathcal{T}_G$ is a contraction.

Next, we want to show the strong convergence of the sequence $\{v_n\}$ defined by the UO iterative scheme (3) to the fixed point of the operator $\mathcal{T}_G$.

Since $\mathcal{T}_G$ is a contraction, it is clear from the known Banach contraction principle that the existence of a unique fixed point, $\tau^*$, of $\mathcal{T}_G$ in the Banach space $C^2([1,2], \|\cdot\|_{C^2})$ is certain. That is, we shall prove that $\lim_{n\to\infty} \|v_n - \tau^*\| = 0$.

From (83), we have

$$\begin{aligned}
\|r_n - \tau^*\| &= \|\mathcal{T}_G v_n - \tau^*\| \\
&\leq \|\mathcal{T}_G v_n - \mathcal{T}_G \tau^*\| \\
&\leq \delta\|v_n - \tau^*\|
\end{aligned} \tag{86}$$

$$\begin{aligned}
\|s_n - \tau^*\| &= (1-\alpha_n)r_n + \alpha_n \mathcal{T}_G r_n - \tau^*\| \\
&\leq (1-\alpha_n)\|r_n - \tau^*\| + \alpha_n\|\mathcal{T}_G r_n - \tau^*\| \\
&\leq (1-\alpha_n)\|r_n - \tau^*\| + \alpha_n\delta\|r_n - \tau^*\| \\
&= [1-(1-\delta)\alpha_n]\|r_n - tau^*\|,
\end{aligned} \tag{87}$$

$$\begin{aligned}
\|t_n - \tau^*\| &= \|\mathcal{T}_G s_n - \tau^*\| \\
&\leq \|\mathcal{T}_G s_n - \mathcal{T}_G \tau^*\| \\
&\leq \delta\|s_n - \tau^*\|,
\end{aligned} \tag{88}$$

$$\begin{aligned}\|u_n-\tau^*\| &= \|(1-\beta_n)t_n+\beta_n\mathcal{T}_G t_n-\tau^*\| \\ &\le (1-\beta_n)\|t_n-\tau^*\|+\beta_n\|\mathcal{T}_G t_n-\tau^*\| \\ &\le (1-\beta_n)\|t_n-\tau^*\|+\beta_n\|\mathcal{T}_G t_n-\mathcal{T}_G\tau^*\| \\ &= [(1-\beta_n)+\beta_n\delta]\|t_n-\tau^*\| \\ &\le [1-(1-\delta)\beta_n]\|t_n-\tau^*\|\end{aligned} \tag{89}$$

and

$$\begin{aligned}\|v_{n+1}-\tau^*\| &= \|(1-\gamma_n)u_n+\gamma_n\mathcal{T}_G u_n-\tau^*\| \\ &\le (1-\gamma_n)\|u_n-\tau^*\|+\gamma_n\|\mathcal{T}_G u_n-\tau^*\| \\ &\le (1-\alpha_n)\|u_n-\tau^8\|+\gamma_n\delta\|u_n-\tau^*\| \\ &= [1-(1-\delta)\gamma_n]\|u_n-\tau^*\|\end{aligned} \tag{90}$$

and, combining (86)–(90), we have

$$\|v_{n+1}-\tau^*\| \le \delta^2[1-(1-\delta)\alpha_n][1-(1-\delta)\beta_n][1-(1-\delta)\beta_n]\|v_n-\tau^*\|$$

Since $\delta\in[0,1)$ and $\alpha_n,\beta_n,\gamma_n\in(0,1)$, we can say that $[1-(1-\delta)\beta_n][1-(1-\delta)\gamma_n]<1$. It follows that

$$\|v_{n+1}-\tau^*\| \le \delta^2[1-(1-\delta)\alpha_n]\|v_n-\tau^*\|.$$

Inductively,

$$\|v_{n+1}-\tau^*\| \le \delta^{2(n+1)}[1-(1-\delta)\alpha_n]\|v_0-\tau^*\|$$

$$v_{n+1}-\tau^*\| \le \delta^{2(n+1)}\|v_0-\tau^*\|\prod_{k=0}^{n}[1-(1-\delta)\alpha_k]$$

From elementary analysis, it is clear that $1-x\le e^{-x}$ for $0<x<1$, so that

$$\begin{aligned}\|v_{n+1}-\tau^*\| &\le \delta^{2(n+1)}\|v_0-\tau^*\|^{n+1}\prod_{k=0}^{n}e^{-(1-\delta)\alpha_k} \\ &le\delta^{2(n+1)}\|v_0-\tau^*\|^{n+1}e^{-(1-\delta)\sum_{k=0}^{\infty}\alpha_k}\end{aligned}$$

Clearly, if $\sum_{k=0}^{\infty}\alpha_k=\infty$, such that $e^{-(1-\delta)\sum_{k=0}^{n}\alpha_k}\to 0$ as $n\to\infty$, then $\lim\limits_{n\to\infty}\|v_n-\tau^*\|=0$, thereby completing the proof. □

**Example 2.** *Consider the BVP*

$$g'''(t) = -tg''(t)-2t^2+t-2 \tag{91}$$

*with BCs*

$$g(0)=g'(0)=g'(1)=0 \tag{92}$$

*The exact solution is* $g(t)=\frac{t^2}{2}-\frac{t^3}{3}$.
*The corresponding Green's function is given as follows:*

$$G(t,s)=\begin{cases}\frac{(s-1)}{2}t^2, & 0<t<s\\ \frac{s(1-t)^2}{2}+\frac{s^2-s}{2}, & s<t<1\end{cases}$$

*Embedding the Green's function in the UO–Green fixed point iterative scheme (80), we have*

$$\begin{aligned}r_n &= v_n+\int_0^t\left[\frac{(s-1)}{2}t^2\right][g'''(t)+tg''(t)+2t^2-t+2]ds \\ &\quad+\int_t^1\left[\frac{s(1-t)^2}{2}+\frac{s^2-s}{2}\right][g'''(t)+tg''(t)+2t^2-t+2]ds \\ s_n &= r_n+\alpha_n\int_0^t\left[\frac{(s-1)}{2}t^2\right][g'''(t)+tg''(t)+2t^2-t+2]ds\end{aligned}$$

$$
\begin{aligned}
&+\alpha_n \int_t^1 \left[\frac{s(1-t)^2}{2} + \frac{s^2-s}{2}\right][g'''(t) + tg''(t) + 2t^2 - t + 2]ds \\
t_n &= s_n + \int_0^t \left[\frac{(s-1)}{2}t^2\right][g'''(t) + tg''(t) + 2t^2 - t + 2]ds \\
&+ \int_t^1 \left[\frac{s(1-t)^2}{2} + \frac{s^2-s}{2}\right][g'''(t) + tg''(t) + 2t^2 - t + 2]ds \\
u_n &= t_n + \beta_n \int_0^t \left[\frac{(s-1)}{2}t^2\right][g'''(t) + tg''(t) + 2t^2 - t + 2]ds \\
&+ \beta_n \int_t^1 \left[\frac{s(1-t)^2}{2} + \frac{s^2-s}{2}\right][g'''(t) + tg''(t) + 2t^2 - t + 2]ds \\
u_{n+1} &= u_n + \gamma_n \int_0^t \left[\frac{(s-1)}{2}t^2\right][g'''(t) + tg''(t) + 2t^2 - t + 2]ds \\
&+ \gamma_n \int_t^1 \left[\frac{s(1-t)^2}{2} + \frac{s^2-s}{2}\right][g'''(t) + tg''(t) + 2t^2 - t + 2]ds.
\end{aligned}
$$

*With a better choice of $\alpha_n, \beta_n$, and $\gamma_n \in (0,1)$, it is guaranteed that the UO–Green iterative scheme converges faster than the Picard–Green [26], Mann–Green [27], Khan–Green [28], Ishikawa–Green [29] and GA–Green [2].*

*Furthermore, the minimization of the $L^2$-norm of the residual error guarantees a perfect computation.*

## 6. Conclusions

The UO iterative scheme generalizes and extends other existing iterative schemes in the literature as shown in Example 1, where our scheme converges to the fixed point 2 faster than all of the CR, $F^*$ Picard-S, Modified-SP Uddin et al. and Picard–Ishikawa iterative schemes with visualization in tables and graphs. Our newly developed UO iteration process is applied in solving a multi-term fractional diffusion equation for oxygen delivery via a capillary of tissues, as found in [25]. Embedding the Green's function in the UO scheme (3) gives rise to the UO–Green iterative scheme, which is used to approximate the solution of a BVP.

**Author Contributions:** Conceptualization, G.A.O. and A.V.U.; Methodology, A.V.U. and N.H.A.; Software, N.H.A.; Validation, R.T.A.; Formal analysis, G.A.O. and N.H.A.; Investigation, G.A.O. and R.T.A.; Resources, N.H.A. and R.T.A.; Data curation, A.V.U.; Writing—original draft, A.V.U.; Writing—review & editing, G.A.O.; Supervision, R.T.A.; Project administration, N.H.A.; Funding acquisition, N.H.A. and R.T.A. All authors have read and agreed to the published version of the manuscript.

**Funding:** This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-RPP2023126).

**Data Availability Statement:** No new data were created or analyzed in this study. Data sharing is not applicable to this article.

**Acknowledgments:** This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-RPP2023126). The authors wish to thank the editor and the reviewers for their useful comments and suggestions.

**Conflicts of Interest:** The authors declare no conflicts of interest.

## References

1. Clarkson, J.A. Uniformly convex spaces. *Trans. Am. Math. Soc.* **1936**, *40*, 396–414. [CrossRef]
2. Okeke, G.A.; Ofem, A.E.; Işik, H. A faster iterative method for solving nonlinear third-order BVPs based on Green's function. *Bound. Value Probl.* **2022**, *2022*, 103. [CrossRef]
3. Okeke, G.A.; Udo, A.V.; Rasulov, Z. A novel Picard-Ishikawa-Green's iterative scheme for solving third order boundary value problems. *Math. Methods Appl. Sci.* **2024**, 1–15. [CrossRef]
4. Okeke, G.A.; Udo, A.V.; Alqahtani, R.T.; Kaplan, M.; Ahmed, W.E. A novel iterative scheme for solving delay differential equations and third order boundary value problems via Green's functions. *AIMS Math.* **2024**, *9*, 6468–6498. [CrossRef]

5. Younis, M.; Abdou, A.A.N. Novel fuzzy contractions and applications to engineering science. *Fractal Fract.* **2024**, *8*, 28. [CrossRef]
6. Longhi, S. Fractional Schr′odinger equation in optics. *Opt. Lett.* **2015**, *40*, 1117–1120. [CrossRef] [PubMed]
7. Srivastava, T.; Singh, A.P.; Agarwal, H. Modeling the under-actuated mechanical system with fractional order derivative. *Progr. Fract. Differ. Appl.* **2015**, *1*, 57–64.
8. Xuan, C.F.; Jin, C.; Wei, H. *Anomalous diffusion and fractional advection-diffusion equation*. *Acta Phys. Sin.* **2005**, *53*, 1113–1117. [CrossRef]
9. Zheng, L.; Zhang, X. Variational iteration method and homotopy perturbation method. In *Modeling and Analysis of Modern Fluid Problems*; Academic Press: Cambridge, MA, USA, 2017. [CrossRef]
10. Chugh, R.; Kumar, V.; Kumar, S. Strong convergence of a new three step iterative scheme in Banach spaces. *Am. J. Comput. Math.* **2012**, *2*, 345–357. [CrossRef]
11. Gürsoy, P.; Karakaya, V. A Picard-S hybrid iteration method for solving a differential equation with retarded argument. *arXiv* **2014**, arXiv:1403.2546v2.
12. Abbas, M.; Asghar, M.W.; De la Sen, M. Approximation of the solution of delay fractional differential equation using AA-iterative scheme. *Mathematics* **2022**, *10*, 273. [CrossRef]
13. Uddin, I.; Garodia, C.; Abdeljawad, T.; Mlaiki, N. Convergence analysis of a novel iteration process with application to a fractional differential equation. *Adv. Contin. Discret. Model.* **2022**, *2022*, 16. [CrossRef]
14. Çeliker, F.Ö. Convergence analysis for a Modified SP iterative method. *Sci. World J.* **2014**, *2014*, 840504. [CrossRef] [PubMed]
15. Okeke, G. Convergence of the Picard-Ishikawa hybrid iterative process with applications. *Afr. Mat.* **2019**, *30*, 817–835. . [CrossRef]
16. Okeke, G.A.; Udo, A.V.; Alqahtani, R.T.; Hussain, N. A faster iterative scheme for solving nonlinear fractional differential equation of the Caputo type. *AIMS Math.* **2023**, *8*, 28488–28516. [CrossRef]
17. Liu, Z.; Kang, S.M.; Cho, Y.J. Convergence and almost stability of Ishikawa iterative scheme with errors for m-accretive operators. *Comput. Math. Appl.* **2004**, *47*, 767–778. [CrossRef]
18. Berinde, V. Picard iteration converges faster than Mann iteration for a class of quasi-contractive operator. *Fixed Point Theory Appl.* **2004**, *2004*, 97–105. [CrossRef]
19. Berinde, V. On the stability of some fixed procedure. *Bul. Ştiinţ. Univ. Baia Mare Ser. B Mat.-Inform.* **2002**, *XVIII*, 7–14.
20. Berinde, V. Summable almost stability of fixed point iteration procedures. *Carpathian J. Math.* **2003** *19*, 81–88.
21. Şultuz, Ş.M.; Grosan, T. Data dependence for Ishikawa iteration when dealing with contractive-like operators. *Fixed Point Theory Appl.* **2008**, *2008*, 242916. [CrossRef]
22. Weng, X. Fixed point iteration for local strictly pseudo-contractive mapping. *Proc. Am. Math. Soc.* **1991**, *113*, 727–731. [CrossRef]
23. Osilike, M.O. Stability of the Mann and Ishikawa iteration procedure for $\phi$-strong pseudocontractions and nonlinear equations of the $\phi$-strongly accreetive type. *J. Math. Anal. Appl.* **1998**, *227*, 319–334. [CrossRef]
24. Leach, R.M.; Treacher, D.F. ABC of oxygen. Oxygen transport-2. Tissue hypoxia. *BMJ* **1998**, *317*, 1370–1373. [CrossRef]
25. Srivastava, V.; Rai, K.N. A multi-term fractional diffusion equation for oxygen delivery through a capillary to tissues. *Math. Comput. Model.* **2010**, *51*, 616–624. [CrossRef]
26. Khuri, S.A.; Sayfy, A.; Zaveri, A. A new iteration method based on Green's functions for the solution of PDEs. *Int. J. Comput. Math.* **2016**, *3*, 3091–3103. [CrossRef]
27. Khuri, S.A.; Sayfy, A. Numerical solution of functional differential equations: A Green's function-based iterative approach. *Int. J. Comput. Math.* **2018**, *95*, 1937–1949. [CrossRef]
28. Ali, F.; Ali, J.; Uddin, I. A novel approach for the solution of BVPs via Green's function and fixed point iterative method. *J. Appl. Math. Comput.* **2020**, *66*, 167–181. [CrossRef]
29. Khuri, S.A.; Louhichi, I. A new fixed point iteration method for nonlinear third-order BVPs. *Int. J. Comput. Math.* **2021**, *98*, 2220–2232. [CrossRef]