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On the Integrability of Persistent Quadratic Three-Dimensional Systems

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Abstract: We consider a nine-parameter familiy of 3D quadratic systems, $\dot{x} = x + P_2(x,y,z), \dot{y} = -y + Q_2(x,y,z), \dot{z} = -z + R_2(x,y,z)$, where P_2, Q_2, R_2 are quadratic polynomials, in terms of integrability. We find necessary and sufficient conditions for the existence of two independent first integrals of corresponding semi-persistent, weakly persistent, and persistent systems. Unlike some of the earlier works, which primarily focus on planar systems, our research covers three-dimensional spaces, offering new insights into the complex dynamics that are not typically apparent in lower dimensions.

Keywords: ordinary differential equations; three-dimensional systems; integrability problem; persistent systems

MSC: 34A34; 34C60

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1. Introduction

The center-focus problem is a basic and important problem in the qualitative theory of planar systems for ordinary differential equations (ODEs), and it asks one to distinguish between a center and a focus for a monodromic equilibrium with imaginary eigenvalues. Closely related to the center-focus problem is the problem of local integrability, which is the main topic of this paper. Poincaré [1] proved that a center (whose linear part is a linear center of a planar analytic differential system) is locally integrable and analytic. An equilibrium of an analytical differential system is called a *center* if it is surrounded by a family of periodic orbits.

For the center-focus problem, Lyapunov provided a method by calculating Lyapunov constants and judging their vanishment or lack thereof; see [2,3]. Generally, we are not able to calculate a large number of Lyapunov constants due to the limitations of computers. We need to know in which step the calculation of Lyapunov constants can be stopped (in the sense that we have obtained enough quantities/polynomials) not only using computer technology but also using some other analytic and algebraic tools; see [4,5].

For a center configuration, the structure is usually unstable, and it may disappear after a small perturbation. There is a natural and interesting question: For which class of systems can the center be possibly persistent and can parameter variations not destroy the center configuration? The phenomenon of persistency in the sense considered below was originally defined for planar systems by Cima et al. in [6]. They considered a differential system with a linear center at the origin in the form

$$\begin{aligned} \dot{x} &= -y + P(x,y) = -y + \textstyle\sum_{k=2}^{\infty} f_k(x,y), \\ \dot{y} &= x + Q(x,y) = x + \textstyle\sum_{k=2}^{\infty} g_k(x,y), \end{aligned} \tag{1}$$

or its equivalent complex form

$$\dot{z} = iz + P(z,\bar{z}) = iz + \sum_{k=2}^{\infty} P_k(z,\bar{z}), \tag{2}$$

where $z = x + iy$, $f_k(x,y)$ and $g_k(x,y)$ are real, homogeneous polynomials of degree k, and $P_k = f_k + ig_k$ are complex, homogeneous polynomials of degree k. The authors of [6] introduced the following definitions of persistent centers and weakly persistent centers.

Definition 1. *The origin of system* (3) *is called a weakly persistent center if it is, for any* $\mu \in \mathbb{R}$, *a center of the following system:*

$$\dot{z} = iz + P(z,\bar{z}) = iz + \mu \sum_{k=2}^{\infty} P_k(z,\bar{z}). \tag{3}$$

If the origin of system (3) *is a center for any* $\mu \in \mathbb{C}$, *it is called a persistent center.*

Clearly, in the space of parameters, the set of all persistent centers of system (3) is a subset of all weakly persistent centers. Five general subfamilies of persistent centers inside the family (3) have been found, and all persistent centers of the cubic family have been classified in [6].

Chen et al. [7] presented the existence of *persistent centers* (resp. *weakly persistent centers*) for the parametrized analytic complex system

$$\dot{x} = ix + \lambda F(x,y),\ \dot{y} = -iy + \mu G(x,y), \quad x,y \in \mathbb{C} \tag{4}$$

as $\lambda, \mu \in \mathbb{C}$ (resp. $\lambda = \mu \in \mathbb{C}$), where the lowest degree of the Taylor expansions of functions $F(x,y)$ and $G(x,y)$ is two. They found all conditions for persistent centers of cubic systems and all conditions for weakly persistent centers of complex cubic Lotka–Volterra systems.

Following the Poincaré–Lyapunov theorem [1,8], the origin is a center of (4) if it admits the local analytic first integral of the form

$$\psi_{1:-1}(x,y) = xy + O\Big(|x,y|^3\Big).$$

The authors of [9] further generalized the concept of persistent centers to persistent $p{:}-q$ resonant centers. The origin $(0,0)$ is a persistent (resp. weakly persistent) $p{:}-q$ resonant center of the system

$$\dot{x} = px + P(x,y), \quad \dot{y} = -qy + Q(x,y), \tag{5}$$

if it is a $p:-q$ resonant center of the system

$$\dot{x} = px + \lambda P(x,y), \quad \dot{y} = -qy + \mu Q(x,y) \tag{6}$$

for all $\lambda, \mu \in \mathbb{C}$ (resp. $\lambda = \mu \in \mathbb{C}$). They obtained necessary and sufficient conditions for the existence of a persistent $p{:}-q$ resonant center of a $p{:}-q$ resonant system with quadratic nonlinearities for $(p,q) = (1,2)$, $(p,q) = (1,3)$, $(p,q) = (1,4)$, $(p,q) = (1,5)$, and $(p,q) = (2,3)$ by applying the Darboux theory of integrability, the method of formal first integrals, and a recursive procedure based on blow-up transformation. It is well known that the computations for obtaining conditions for the existence of a $p{:}-q$ resonant center and its first integral become more demanding by increasing the values of p and q [10], and the consideration of (weakly) persistent $p{:}-q$ resonant centers is of great benefit [9].

In the next section, we elaborate on the theory of integrability for n-dimensional systems of ODEs whose matrix of linear approximation lacks eigenvalues that are equal to zero. In the third section, we address the integrability of three-dimensional quadratic (semi-) persistent systems.

2. Preliminaries

The notion of integrability holds significant importance in the qualitative theory of ordinary differential equations. Consider the n-dimensional system of ODEs given by the equation

$$\dot{x} = Ax + f(x), \tag{7}$$

where $x = (x_1, \ldots, x_n)^T$, A is an $n \times n$ diagonal matrix with eigenvalues $\mu = (\mu_1, \ldots, \mu_n)$ (where $\mu_k \neq 0$ for $k = 1, \ldots, n$), and $f(x) = (f_1(x), \ldots, f_n(x))^T$ consists of series starting with at least quadratic terms.

The first integral of system (7) is a non-constant function ψ satisfying

$$\mathcal{X}(\psi) := \sum_{i=1}^{n} \frac{\partial \psi}{\partial x_i} \dot{x}_i \equiv 0, \tag{8}$$

where $\mathcal{X}$ denotes the derivative with respect to the vector field associated with system (7). Let

$$\mathcal{R} = \{\alpha \in \mathbb{N}_0^n, |\langle \mu, \alpha \rangle = 0, |\alpha| > 0\}, \tag{9}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\langle \mu, \alpha \rangle$ is a dot product of the vectors μ and α, and let r_μ be the rank of the $\mathbb{Z}$-module spanned by the elements of $\mathcal{R}$. The system (7) can have $n-1$ locally analytic functionally independent first integrals only if $r_\mu = n - 1$. In such a case, integrals can be found in the form

$$\psi_s(x) = x^{\alpha^{(s)}} + \sum_{|\alpha| > |\alpha^{(s)}|} \psi_s^\alpha x^\alpha, \tag{10}$$

where $\alpha^{(s)}$, $s = 1, \ldots, n-1$, are linearly independent vectors from $\mathcal{R}$ (see [11]) and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.

Definition 2. *A system (7) is considered locally analytically (or formally) integrable (also referred to as complete local integrability) in a neighborhood of the origin if it admits $n-1$ functionally independent analytic (or formal) first integrals in a neighborhood of the origin.*

To identify integrable systems in a polynomial family (7), one can search for functions $\psi_s(x)$, $s = 1, \ldots, n-1$, of the form (10) satisfying

$$\mathcal{X}(\psi_s(x)) = \sum_{\substack{\alpha \in \mathcal{R} \\ j \geq 1}} p_s^\alpha (x^\alpha)^j, \tag{11}$$

where p_s^α are polynomials in the parameters of (7). Polynomials p_s^α are referred to as the integrability quantities of system (7).

The phenomenon of persistency can also be generalized to any analytic system (7). For $\lambda = (\lambda_1, \ldots, \lambda_n)$, define $\lambda \circ f(x) := (\lambda_1 f_1(x), \ldots, \lambda_n f_n(x))$. Assuming that the system (7) is integrable for some parameters at the origin, then the system (7) is persistently integrable at the origin if the system

$$\dot{x} = Ax + \lambda \circ f(x) \tag{12}$$

is integrable for every $\lambda = (\lambda_1, \ldots, \lambda_n)$ at the origin.

To determine the necessary conditions for the integrability of a polynomial system (12), one can choose $n-1$ linearly independent vectors $\alpha^{(1)}, \ldots, \alpha^{(n-1)}$ from $\mathcal{R}$ and search for functions of the form (10). Since $\mathcal{X}(\psi_s(x)) \equiv 0$ leads to $p_s^\alpha = 0$ for all $\alpha^{(s)} \in \mathcal{R}$ and any $\lambda = (\lambda_1, \ldots, \lambda_n)$, we obtain

$$p_s^\alpha = \sum_{|\alpha|>|\alpha^{(s)}|} p_s^{\alpha,\beta} \lambda^\beta = 0, \tag{13}$$

where $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ and $\lambda^\beta = \lambda_1^{\beta_1} \lambda_2^{\beta_2} \cdots \lambda_n^{\beta_n}$ and $|\beta|$ is equal to the degree of polynomial p_s^α. When considering quantities p_s^α as polynomials in λ, coefficients $p_s^{\alpha,\beta}$ play a crucial role in the analysis of persistent integrability. These coefficients are called the *persistent integrability quantities*.

For each $s = 1, \ldots, n-1$ the polynomials $p_s^{\alpha,\beta}$ generate an ideal, which we denote as $\mathcal{B}^{(s)}$, $s = 1, \ldots, n-1$. The variety $V\left(\mathcal{B}^{(s)}\right)$ yields conditions for the existence of the particular first integral $\psi_s(x)$. Next, we define $\mathcal{B} = \mathcal{B}^{(1)} + \cdots + \mathcal{B}^{(n-1)}$ and the variety $V(\mathcal{B})$ as the set of all points in the parameter space where the polynomials of $\mathcal{B}$ simultaneously vanish. This condition indicates the simultaneous existence of $n-1$ independent first integrals $\psi_s(x)$ in the form (10) in system (12), suggesting that the corresponding systems are integrable. Thus, $V(\mathcal{B})$ is referred to as the integrability variety of system (12). According to the Hilbert Basis Theorem [12], there exists $m \in \mathbb{N}$ such that $V(\mathcal{B}) = V(\mathcal{B}_m)$, where $\mathcal{B}_m$ is an ideal generated via the polynomials of the set $\left\{p_s^{\alpha,\beta} | s = 1, \ldots, n-1, |\alpha| \leq m\right\}$. One approach to finding such m is to compare the varieties $V(\mathcal{B}_i)$ for the first few consecutive $i \in \mathbb{N}$. Upon discovering a specific i for which $V(\mathcal{B}_i) = V(\mathcal{B}_{i+1})$, one can determine the irreducible decomposition of the variety $V(\mathcal{B}_i)$, thereby obtaining the necessary conditions for the integrability of system (12). Subsequently, it is essential to demonstrate that all identified systems are locally analytically integrable.

3. Integrability of 3D Persistent and Semi-Persistent Systems

For an analytic system (1), a classical problem in integrability theory is the existence or nonexistence of an analytic first integral defined in an open neighborhood of an isolated singular point. We call this problem the problem of local integrability. A linear center is a singular point of system (1) whose eigenvalues are imaginary. It is known that a center whose linear part is a linear center of an analytic differential system (1) is locally integrable and analytic [1].

Many works (see, for instance, [5,13,14]) have contributed to the solution of the problem of the integrability of planar polynomial differential systems since Dulac classified integrable quadratic systems with a (1:−1) resonant equilibrium [15].

However, the integrability of three-dimensional polynomial differential systems has not been studied to this extent. The most researched family of three-dimensional systems are the Lotka–Volterra systems; see [16–18]. Llibre et al. in [19] studied the Darboux integrability of a three-dimensional, forced-damped differential system. The integrability of three-dimensional quadratic systems in a neighborhood of a (0:−1:1) resonant equilibrium was investigated in [20,21].

Nonlinear differential systems in dimensions greater than two may (and usually do) admit chaotic dynamics (in the sense that the difference between the initial conditions grows exponentially with time). Therefore, in higher dimensions, it is very important to find conditions to significantly reduce the absence of chaotic dynamics.

In this paper, we deal with quadratic three-dimensional systems with (1:−1:−1)—resonant singularity at the origin:

$$\begin{aligned}
\dot{x} &= x + a_{12}xy + a_{13}xz + a_{23}yz = x + P_2(x,y,z) \\
\dot{y} &= -y + b_{12}xy + b_{13}xz + b_{23}yz = -y + Q_2(x,y,z) \\
\dot{z} &= -z + c_{12}xy + c_{13}xz + c_{23}yz = -z + R_2(x,y,z),
\end{aligned} \tag{14}$$

where $a_{ij}, b_{ij}, c_{ij} \in \mathbb{C}$ and P_2, Q_2, and R_2 are homogeneous polynomials of degree two.

For polynomial differential systems (14) (at least theoretically), we can look for parameter values for which the systems can be partially or completely integrable (when one or two independent first integrals exist, respectively) using the Darboux theory (see, e.g., [22,23]). For system (14), the monomials xy and xz are clearly the first integrals of its linear approximation. Therefore (cf. [24]), we look for two first integrals of the form

$$\psi_1(x,y,z) = xy + O\left(|x,y,z|^3\right) \quad \text{and} \quad \psi_2(x,y,z) = xz + O\left(|x,y,z|^3\right) \tag{15}$$

to prove the (complete) integrability of the system (14).

In [25], Dukarić et al. studied the local integrability and linearizability of (14), which does not belong to the Lotka–Volterra family. Note that the decomposition of the variety of the ideal generated via the integrability quantities (in [25], the authors used the name focus quantities) that corresponds to the existence of one first integral of the form ψ_1 from (15) for system (14) with $a_{23} = 0$ was performed using modular computations, i.e., in the finite field. This indicates the difficulty of computations. Therefore, in this paper, we focus the investigation on the persistent case (16), which simplifies the computations. (The integrability quantities of the non-persistent system are split into smaller persistent integrability quantities of the corresponding persistent system. The variety of the ideal generated via these smaller quantities is easier to decompose.)

In [26], Edneral et al. considered the center-focus problem on the center manifolds for a family of polynomial vector fields $\dot{x} = F(x)$ in $\mathbb{R}^3$, where $F(0) = 0$ and $dF(0)$ has one nonzero and two purely imaginary eigenvalues. They proved that, for each fixed value of the non-zero real eigenvalue, the set of such systems having a center on the local center manifold at the origin corresponds to a variety in the space of admissible coefficients.

We characterized the persistent integrable systems within family (14). In particular, system (14) is partially or completely persistently integrable if the system

$$\begin{aligned}
\dot{x} &= x + \lambda(a_{12}xy + a_{13}xz + a_{23}yz) = x + \lambda P_2(x,y,z) \\
\dot{y} &= -y + \mu(b_{12}xy + b_{13}xz + b_{23}yz) = -y + \mu Q_2(x,y,z) \\
\dot{z} &= -z + \nu(c_{12}xy + c_{13}xz + c_{23}yz) = -z + \nu R_2(x,y,z),
\end{aligned} \tag{16}$$

with $(\lambda, \mu, \nu) \in \mathbb{C}^3$ is partially or completely integrable.

In the case of $\lambda = \mu = \nu$, system (14) is weekly persistently integrable, and in the case of either $\lambda = \mu$ or $\lambda = \nu$ or $\mu = \nu$, it is semi-persistently integrable.

3.1. Integrability of Weakly Persistent Systems

If a system has homogeneous nonlinearities, the following result, which is the generalization of the result from [7], can be applied.

Proposition 1. *Suppose that*

$$\begin{aligned}
\dot{x} &= x + P_n(x,y,z) \\
\dot{y} &= -y + Q_n(x,y,z) \\
\dot{z} &= -z + R_n(x,y,z)
\end{aligned} \tag{17}$$

is a quasi-homogeneous system of degree n (i.e., P_n, Q_n, R_n are homogeneous polynomials of degree n). Then, system (17) is weakly persistently integrable if and only if it is integrable.

Proof. The corresponding weakly persistent system of (17) is

$$\begin{aligned} \dot{x} &= x + \lambda P_n(x,y,z) \\ \dot{y} &= -y + \lambda Q_n(x,y,z) \\ \dot{z} &= -z + \lambda R_n(x,y,z), \end{aligned} \tag{18}$$

where $n \geq 2$, and P_n, Q_n and R_n are homogeneous polynomials of degree n in variables x, y and z. We shall prove that system (17) is equivalent to system (18) up to a linear change in variables, so both systems must have the same integrability varieties. In fact, for any $\gamma \neq 0$, consider the linear change of variables

$$X = \gamma x, \quad Y = \gamma y, \quad Z = \gamma z.$$

Applying this change to system (17) and using the homogeneity of P_n, Q_n and R_n, we obtain the system

$$\begin{aligned} \dot{X} &= X + \gamma^{n-1} P_n(X,Y,Z) \\ \dot{Y} &= -Y + \gamma^{n-1} Q_n(X,Y,Z) \\ \dot{Z} &= -Z + \gamma^{n-1} R_n(X,Y,Z), \end{aligned}$$

Setting $\lambda = \gamma^{n-1}$, we arrive at (18). Since (14) is a quasi-homogeneous system of degree two, the conditions for weakly persistent integrability coincide with the conditions for the integrability of system (14). □

3.2. Persistent System (16)

Theorem 1. *System* (14) *is completely persistently integrable at the origin if and only if one of the following conditions holds:*

1. $c_{23} = c_{13} = c_{12} = b_{23} = b_{12} = a_{13} = a_{12} = 0$;
2. $c_{23} = c_{13} = c_{12} = b_{13} = a_{23} = a_{13} = a_{12} = 0$;
3. $c_{23} = c_{13} = b_{23} = b_{13} = b_{12} = a_{13} = a_{12} = 0$;
4. $c_{23} = b_{23} = a_{23} = a_{13} = a_{12} = 0$;
5. $c_{13} = c_{12} = b_{13} = b_{12} = 0$;
6. $c_{12} = b_{23} = b_{13} = b_{12} = a_{23} = a_{13} = a_{12} = 0$.

Proof. First, we calculate the required conditions for integrability. Subsequently, for each condition identified, we conduct a verification of its sufficiency. This entails demonstrating the presence of two first integrals either by proving their existence or identifying them in the existing literature. Note that, for system (16), this involves seeking two independent first integrals of the forms (15). Since the systems (14) (the non-persistent case of system (16)) have already been treated by other authors, we were able to verify the existence of two independent first integrals for each case in the existing literature.

To obtain the necessary conditions for the existence of two independent first integrals, we looked for functions ψ_1 and ψ_2 in the form (15) with undetermined coefficients such that

$$\frac{\partial \psi_i}{\partial x}\dot{x} + \frac{\partial \psi_i}{\partial y}\dot{y} + \frac{\partial \psi_i}{\partial z}\dot{z} \equiv 0, \quad i = 1,2. \tag{19}$$

Obstacles to the fulfillment of (19) give us integrability quantities for the existence of first integrals of the form (15) for system (16). The first three integrability quantities of system (16) obtained for the existence of the first integral $\psi_1(x,y,z)$ are as follows:

$$\begin{aligned} p_1^{(2,2,0)} &= a_{13}c_{12}\lambda\nu + b_{23}c_{12}\mu\nu, \\ p_1^{(2,1,1)} &= -a_{13}b_{12}\lambda\mu + a_{13}c_{13}\lambda\nu + b_{23}c_{13}\mu\nu - b_{13}c_{23}\mu\nu, \\ p_1^{(2,0,2)} &= -a_{13}b_{13}\lambda\mu + b_{13}b_{23}\mu^2. \end{aligned} \tag{20}$$

Since all integrability quantities need to be zero, we obtained from (20) the following 7 persistent integrability quantities:

$$
\begin{aligned}
p_1^{(2,2,0)(1,0,1)} &= a_{13}c_{12}, \\
p_1^{(2,2,0)(0,1,1)} &= b_{23}c_{12}, \\
p_1^{(2,1,1)(1,1,0)} &= -a_{13}b_{12}, \\
p_1^{(2,1,1)(1,0,1)} &= a_{13}c_{13}, \\
p_1^{(2,1,1)(0,1,1)} &= b_{23}c_{13} - b_{13}c_{23}, \\
p_1^{(2,0,2)(1,1,0)} &= -a_{13}b_{13}, \\
p_1^{(2,0,2)(0,2,0)} &= b_{13}b_{23}.
\end{aligned}
$$

Similarly, we obtained from the next four integrability quantities 31 persistent integrability quantities. When dealing with the first integral, $\psi_2(x, y, z)$, the process is similar, and we obtained 38 persistent integrability quantities from the first 7 integrability quantities. The simultaneous vanishing of all persistent integrability quantities provides conditions for a system to be integrable. Therefore, we had to find the variety of the ideal $\mathcal{B}$ consisting of 76 persistent integrability quantities.

With the help of the routine `minAssGTZ` [27] of the computer algebra system SINGULAR [28], which finds the minimal associate primes of a radical of a polynomial ideal with the algorithm of [29], we compute the decomposition of the variety of the ideal $\mathcal{B}$ and obtain 6 components listed in Theorem 1.

To establish the sufficiency of the obtained conditions, we draw upon the findings in [25], where the authors comprehensively established both necessary and sufficient conditions for the integrability of quadratic three-dimensional systems as delineated in (14). A closer examination reveals a correspondence between conditions 4 and 5 in Theorem 1 and conditions (5) and (1), respectively, in [[25], Theorem 3.1]. Furthermore, conditions 1, 2, 3, and 6 in Theorem 1 align with sub-conditions from (4), (3), (6), and (3), respectively, in [[25], Theorem 3.1].

Given that the authors of [25] successfully demonstrated the integrability of such systems, the completion of the proof for this theorem naturally follows. □

In [30], the authors proved the following relation between the algebraic partial integral and the corresponding cofactor of a two-dimensional system and the algebraic partial integral and its corresponding cofactor of the corresponding persistent system.

Lemma 1 ([30])**.** *Assume that system (6) admits algebraic partial integral $G(\boldsymbol{a}, \boldsymbol{b}, x, y)$ with the corresponding cofactor $L(\boldsymbol{a}, \boldsymbol{b}, x, y)$. Then,*

$$G(\boldsymbol{a}, \boldsymbol{b}, x, y) \equiv F(\lambda\boldsymbol{a}, \mu\boldsymbol{b}, x, y) \quad \text{and} \quad L(\boldsymbol{a}, \boldsymbol{b}, x, y) \equiv K(\lambda\boldsymbol{a}, \mu\boldsymbol{b}, x, y),$$

where $F(\boldsymbol{a}, \boldsymbol{b}, x, y)$ is the algebraic partial integral of system (5), and $K(\boldsymbol{a}, \boldsymbol{b}, x, y)$ is the corresponding cofactor.

Lemma 1 can be generalized to the three-dimensional systems as follows in the next theorem. Its proof is technically similar to the proof of Lemma 1 (see [30] for details).

Theorem 2. *Assume that system (16) admits the algebraic partial integral $F(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, x, y, z)$ with the corresponding cofactor $K(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, x, y, z)$. Then,*

$$F(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, x, y, z) \equiv f(\lambda\boldsymbol{a}, \mu\boldsymbol{b}, \nu\boldsymbol{c}, x, y, z) \quad \text{and} \quad K(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, x, y, z) \equiv k(\lambda\boldsymbol{a}, \mu\boldsymbol{b}, \nu\boldsymbol{c}, x, y, z),$$

where $f(\lambda\boldsymbol{a}, \mu\boldsymbol{b}, \nu\boldsymbol{c}, x, y, z)$ is the algebraic partial integral of system (14), and $k(\lambda\boldsymbol{a}, \mu\boldsymbol{b}, \nu\boldsymbol{c}, x, y, z)$ is the corresponding cofactor.

We demonstrate Theorem 2 using the following example. Under the conditions of case 4, Theorem 1, the corresponding system (16) is as follows:

$$\begin{aligned}
\dot{x} &= x \\
\dot{y} &= -y + \mu(b_{12}xy + b_{13}xz) \\
\dot{z} &= -z + \nu(c_{12}xy + c_{13}xz).
\end{aligned}$$

We can find three algebraic partial integrals:

$$\begin{aligned}
F_1 &= x, \\
F_2 &= y - \frac{z\left(b_{12}\mu - c_{13}\nu - \sqrt{(c_{13}\nu - b_{12}\mu)^2 + 4b_{13}c_{12}\mu\nu}\right)}{2c_{12}\nu}, \\
F_3 &= y - \frac{z\left(b_{12}\mu - c_{13}\nu + \sqrt{(c_{13}\nu - b_{12}\mu)^2 + 4b_{13}c_{12}\mu\nu}\right)}{2c_{12}\nu}
\end{aligned}$$

with the corresponding cofactors

$$\begin{aligned}
K_1 &= 1, \\
K_2 &= \frac{1}{2}\left(-2 + b_{12}x\mu + c_{13}x\nu - x\sqrt{(b_{12}\mu - c_{13}\nu)^2 + 4b_{13}c_{12}\mu\nu}\right), \\
K_3 &= \frac{1}{2}\left(-2 + b_{12}x\mu + c_{13}x\nu + x\sqrt{(b_{12}\mu - c_{13}\nu)^2 + 4b_{13}c_{12}\mu\nu}\right).
\end{aligned}$$

Now, we compare these results with the results obtained in [25], where the algebraic partial integrals $f_{2,3}$ are

$$f_{2,3} = y - \frac{z\left(b_{12} - c_{13} \pm \sqrt{(c_{13} - b_{12})^2 + 4b_{13}c_{12}}\right)}{2c_{12}}$$

and we also computed the corresponding cofactors, $k_{2,3}$

$$k_{2,3} = \frac{1}{2}\left(-2 + b_{12}x + c_{13}x \pm x\sqrt{(b_{12} - c_{13})^2 + 4b_{13}c_{12}}\right).$$

3.3. Semi-Persistent System (16) with $\lambda = \nu$

Theorem 3. *System (14) is completely semi-persistently integrable at the origin (with $\lambda = \nu$) if and only if one of the following conditions holds:*

1. $c_{23} = c_{13} = c_{12} = b_{23} = b_{12} = a_{13} = a_{12} = 0$;
2. $c_{13} = c_{12} = b_{13} = b_{12} = 0$;
3. $c_{13} = c_{12} = b_{13} = a_{23} = a_{13} = a_{12} + c_{23} = 0$;
4. $c_{13} = b_{23} = b_{13} = b_{12} = a_{13} = a_{12} - c_{23} = 0$;
5. $c_{12} = b_{23} = b_{13} = b_{12} = a_{23} = a_{13} = a_{12} = 0$;
6. $c_{23} = b_{23} = a_{23} = a_{13} = a_{12} = 0$.

Proof. In this scenario, we seek two independent first integrals, following the expressions in (15), specifically for the system (16) where $\lambda = \nu$. In both cases, the computation of the series in (15) up to degree 6 suffices. The resulting integrability quantities are split into persistent integrability quantities, yielding 21 when utilizing the first integral, $\psi_1(x, y, z)$, and 16 when utilizing $\psi_2(x, y, z)$.

To determine the necessary conditions for the integrability of system (16) when $\lambda = \nu$, our investigation focused on identifying the structure of the ideal $\mathcal{B}$, which encompasses the 37 persistent integrability quantities. Utilizing the routine `minAssGTZ` within the SINGULAR computer algebra system, we computed the decomposition of the variety of ideal $\mathcal{B}$, revealing 6 distinct components, as specified in Theorem 3.

The adequacy of these conditions is efficiently proven by drawing on findings from [25]. Conditions 2 and 6 are consistent with those stated in Theorem 1. Condition 4 of Theorem 3 is equivalent to condition (6) in [[25], Theorem 3.1]. Furthermore, conditions 1, 3, and 5 in Theorem 1 are considered sub-conditions stemming from conditions (4), (3), and (3) in [[25], Theorem 3.1], respectively. Thus, the outcomes presented in [25] support the proof of the conditions' sufficiency. □

3.4. Semi-Persistent System (16) with $\mu = \nu$

Theorem 4. *System (14) is completely semi-persistently integrable at the origin (with $\mu = \nu$) if and only if one of the following conditions holds:*

1. $c_{23} = c_{13} = c_{12} = b_{23} = b_{12} = a_{13} = a_{12} = 0$;
2. $c_{23} = c_{13} = c_{12} = b_{13} = a_{23} = a_{13} = a_{12} = 0$;
3. $c_{23} = c_{13} = b_{23} = b_{13} = b_{12} = a_{13} = a_{12} = 0$;
4. $c_{23} = c_{12} = b_{23} = b_{13} = b_{12} - c_{13} = a_{23} = 0$;
5. $c_{23} = c_{12} = b_{23} = b_{13} = b_{12} + c_{13} = a_{13} = a_{12} = 0$;
6. $c_{23} = b_{23} = a_{23} = a_{13} = a_{12} = 0$;
7. $c_{13} = c_{12} = b_{13} = b_{12} = 0$;
8. $c_{12} = b_{23} = b_{13} = b_{12} = a_{23} = a_{13} = a_{12} = 0$.

Proof. The method of formulating necessary conditions, as delineated in Theorem 4, is consistent with the approach taken in the two previous scenarios. Consequently, our exploration began with the quest for two independent first integrals, specifically within the framework of system (16) when $\mu = \nu$, guided by the formulations presented in (15). This process identified 18 persistent integrability quantities associated with both first integrals, $\psi_1(x, y, z)$ and $\psi_2(x, y, z)$.

Following this, we calculated the decomposition of the variety of ideal $\mathcal{B}$, which encompasses the 36 identified persistent integrability quantities, revealing 8 distinct components, as elaborated upon in Theorem 4.

In verifying the adequacy of these extracted conditions, we note that conditions 5, 6, and 7 in Theorem 4 correspond to conditions (2), (5), and (1) respectively, as found in [[25], Theorem 3.1]. Moreover, conditions 1, 2, 3, 4, and 8 in Theorem 4 are interpreted as sub-conditions emanating from conditions (4), (3), (6), (3), and (3) in [[25], Theorem 3.1], respectively. Thus, the demonstration of condition sufficiency is effectively achieved. □

3.5. Semi-Persistent System (16) with $\lambda = \mu$

Theorem 5. *System (14) is completely semi-persistently integrable at the origin (with $\lambda = \mu$) if and only if one of the following conditions holds:*

1. $c_{23} = c_{13} = c_{12} = b_{13} = a_{23} = a_{13} = a_{12} = 0$;
2. $c_{23} = c_{13} = c_{12} = b_{12} = a_{13} - b_{23} = a_{12} = 0$;
3. $c_{23} = c_{13} = b_{23} = b_{13} = b_{12} = a_{13} = a_{12} = 0$;
4. $c_{23} = b_{23} = a_{23} = a_{13} = a_{12} = 0$;
5. $c_{13} = c_{12} = b_{13} = b_{12}$;
6. $c_{12} = b_{13} = b_{12} = a_{23} = a_{13} + b_{23} = a_{12} = 0$.

Proof. In this scenario, we aimed to identify two independent first integrals in the formats specified in (15) for system (16) under the condition $\lambda = \mu$. By employing the first integral, $\psi_1(x, y, z)$, we uncovered 16 persistent integrability quantities, while the use of the first integral, $\psi_2(x, y, z)$, reveals 21 persistent integrability quantities. Subsequently, we proceeded to calculate the decomposition of the variety of the ideal $\mathcal{B}$, which consists of 37 obtained persistent integrability quantities, resulting in the identification of 6 distinct components, as outlined in Theorem 5.

To prove the sufficiency of the obtained conditions, we note that conditions 2, 3, 4, and 5 are conditions (4), (6), (5), and (1), respectively in [[25], Theorem 3.1], and conditions 1 and 6, in Theorem 5 are sub-conditions from condition (3) in [[25], Theorem 3.1]; thus, the proof is completed. □

4. Conclusions

The paper has successfully extended integrability theory to three-dimensional systems, specifically addressing persistent, semi-persistent, and weakly persistent quadratic systems. Through rigorous mathematical analysis, it has established necessary and sufficient conditions under which these systems exhibit complete integrability. This represents a significant step forward in understanding the complex dynamics of such systems. Notably, to the authors' knowledge, semi-persistent 3D systems had not been considered yet in the literature.

Identifying conditions under which three-dimensional systems can be integrable enriches the existing body of knowledge on differential systems and resonant centers. This work not only provides a foundation for further theoretical exploration but also contributes to the broader field of the qualitative theory of differential equations by highlighting the intricate relationships between system parameters and their integrability properties.

While this study is primarily theoretical, the insights gained have potential applications in various fields, including physics, engineering, and biological systems modeling. Understanding the conditions for integrability can help in the simplification of complex models, leading to more accurate predictions and analyses.

While thorough, this investigation primarily focused on a specific class of three-dimensional systems. An intriguing direction for future research involves the exploration of integrability conditions for a broader spectrum of systems, including those of higher dimensionality and a non-polynomial nature.

Of particular interest is the generalization of these findings to $p{:}-q{:}-r$ resonant persistent systems. Following [15], the origin of the complex system (6) is a $p{:}-q$ resonant center if there exists a local analytic first integral of the form

$$\psi_{p:-q}(x,y) = x^q y^p + O\Big(|x,y|^{p+q+1}\Big). \tag{21}$$

The notion of a resonant center can be generalized at least for family (14). Namely, system

$$\begin{aligned} \dot{x} &= px + P_2(x,y,z) \\ \dot{y} &= -qy + Q_2(x,y,z) \\ \dot{z} &= -rz + R_2(x,y,z), \end{aligned} \tag{22}$$

may be referred to as a $(p{:}-q{:}-r)$ resonant system. In this way, we considered the integrability of a $(1{:}-1{:}-1)$ resonant persistent quadratic system. However, one may consider the integrability of a $(p{:}-q{:}-r)$ system. By analogy to (21), system (22) may admit two independent first integrals of the form

$$\begin{aligned} \psi_1(x,y,z) &= x^q y^p + O\Big(|x,y,z|^{p+q+1}\Big), \\ \psi_2(x,y,z) &= x^r z^p + O\Big(|x,y,z|^{p+r+1}\Big). \end{aligned}$$

The investigation of this kind of system will be performed in a separate paper.

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