mathematics

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f-Biharmonic Submanifolds in Space Forms and f-Biharmonic Riemannian Submersions from 3-Manifolds

Ze-Ping Wang * and Li-Hua Qin

School of Mathematical Sciences, Guizhou Normal University, Guiyang 550025, China; 222100060209@gznu.edu.cn
* Correspondence: 201507006@gznu.edu.cn

Abstract: f-biharmonic maps are generalizations of harmonic maps and biharmonic maps. In this paper, we give some descriptions of f-biharmonic curves in a space form. We also obtain a complete classification of proper f-biharmonic isometric immersions of a developable surface in $\mathbb{R}^3$ by proving that a proper f-biharmonic developable surface exists only in the case where the surface is a cylinder. Based on this, we show that a proper biharmonic conformal immersion of a developable surface into $\mathbb{R}^3$ exists only in the case when the surface is a cylinder. Riemannian submersions can be viewed as a dual notion of isometric immersions (i.e., submanifolds). We also study f-biharmonicity of Riemannian submersions from 3-manifolds by using the integrability data. Examples are given of proper f-biharmonic Riemannian submersions and f-biharmonic surfaces and curves.

Keywords: biharmonic maps; f-biharmonic maps; Riemannian submersions; f-biharmonic curves; f-biharmonic submanifolds

MSC: 58E20; 53C12; 53C42

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Citation: Wang, Z.-P.; Qin, L.-H. f-Biharmonic Submanifolds in Space Forms and f-Biharmonic Riemannian Submersions from 3-Manifolds. *Mathematics* **2024**, *12*, 1184. https://doi.org/10.3390/math12081184

Academic Editor: Stéphane Puechmorel

Received: 9 March 2024
Revised: 6 April 2024
Accepted: 9 April 2024
Published: 15 April 2024

1. Introduction and Preliminaries

In this paper, one assumes that all manifolds, maps, and tensor fields studied are smooth unless there is a statement otherwise.

Recall that a *biharmonic map* $\phi : (M, g) \to (N, h)$ between Riemannian manifolds is a critical point of the bienergy functional

$$E^2(\phi, \Omega) = \frac{1}{2}\int_{\Omega} |\tau(\phi)|^2 v_g,$$

where Ω is a compact domain of M and $\tau(\phi) = \mathrm{Trace}_g \nabla d\phi$ denotes the tension field of ϕ. By calculating the first variation of the functional (see [1]), the biharmonic equation can be written as

$$\tau_2(\phi) := \mathrm{Trace}_g(\nabla^\phi \nabla^\phi - \nabla^\phi_{\nabla^M})\tau(\phi) - \mathrm{Trace}_g R^N(d\phi, \tau(\phi))d\phi = 0, \tag{1}$$

where R^N is the curvature operator of (N, h) defined by

$$R^N(X, Y)Z = [\nabla^N_X, \nabla^N_Y]Z - \nabla^N_{[X,Y]}Z.$$

An *f-biharmonic map* is a critical point of the f-bienergy functional for a map ϕ: $(M, g) \to (N, h)$ between Riemannian manifolds:

$$E_{2,f}(\phi) = \int_{\Omega} f|\tau(\phi)|^2 v_g,$$

where Ω is a compact subset of M and $f : M \to (0, +\infty)$. One finds that the f-biharmonic map equation is the Euler–Lagrange equation of the f-bienergy functional, which can be written as (see e.g., [2,3])

$$\tau_{2,f}(\phi) = -J^{\phi}(f\tau(\phi)) = f\tau_2(\phi) + (\Delta f)\tau(\phi) + 2\nabla^{\phi}_{\mathrm{grad} f}\tau(\phi) = 0. \tag{2}$$

Here $\tau(\phi)$ and $\tau_2(\phi)$ denote the tension and bitension fields of ϕ, respectively, and J^{ϕ} is the Jacobi operator of the map ϕ, defined by $J^{\phi}(X) = -\{\mathrm{Trace}_g(\nabla^{\phi}\nabla^{\phi} - \nabla^{\phi}_{\nabla^M})X - \mathrm{Trace}_g R^N(\mathrm{d}\phi, X)\mathrm{d}\phi\}$. Clearly, both harmonic maps and biharmonic maps are f-biharmonic. We also have the following obvious relationships:

$\{Harmonic\ \ maps\} \subset \{Biharmonic\ \ maps\} \subset \{f\text{-}Biharmonic\ \ maps\}.$

A submanifold is an f-biharmonic submanifold if the isometric immersion defining the submanifold is an f-biharmonic map. A Riemannian submersion between Riemannian manifolds is an f-biharmonic Riemannian submersion if the Riemannian submersion is f-biharmonic. We call f-biharmonic maps (respectively, submanifolds, Riemannian submersions) that are not biharmonic proper f-biharmonic maps (respectively, submanifolds, Riemannian submersions).

The f-biharmonic map was first introduced in [2]. Later, f-biharmonic submanifolds were studied in [3], where the author derived the f-biharmonic curve equation into a space form and also obtained a complete classification of proper f-biharmonic curves into $\mathbb{R}^3$. The paper [4] shows that neither circular cone and part of the standard sphere S^2 in $\mathbb{R}^3$ is f-biharmonic for any f, and a constant mean curvature surface in $\mathbb{R}^3$ is f-biharmonic if and only if it is a part of a circular cylinder or a plane. For some recent progress on f-biharmonic submanifolds and some results on f-biharmonic maps, we refer the readers to [2,3,5–12] and the references therein.

As the dual notion of biharmonic isometric immersions (i.e., biharmonic submanifolds), biharmonic Riemannian submersions were first studied by using the integrability data of an adapted frame of the Riemannian submersion in [4], where the authors showed that any biharmonic Riemannian submersion from a 3-space form onto a surface has to be harmonic and also constructed a family of proper biharmonic Riemannian submersions from a 3-dimensional warped product space. Following the idea from [4], we study f-biharmonic Riemannian submersions from 3-manifolds by using the integrability data.

In this paper, we first derive an f-biharmonic curve equation in a general Riemannian manifold. We then obtain some characterizations of f-biharmonic curves in a space form by giving the explicit functions f and the explicit curvatures of the curves. For f-biharmonic surfaces in $\mathbb{R}^3$, we give a complete classification of proper f-biharmonic isometric immersions of a developable surface into $\mathbb{R}^3$ by proving that a proper f-biharmonic developable surface into $\mathbb{R}^3$ exists only in the case when the surface is a cylinder. Based on this, we also show that a proper biharmonic conformal immersion of a developable surface into $\mathbb{R}^3$ exists only in the case where the surface is a cylinder in $\mathbb{R}^3$. Riemannian submersions can be viewed as the dual notion of isometric immersions (i.e., submanifolds). We also study f-biharmonicity of a Riemannian submersion from 3-manifolds by using the integrability data. Examples are given of proper f-biharmonic Riemannian submersions and f-biharmonic surfaces and curves.

2. f-Biharmonic Submanifolds in a Space Form

In this section, we characterize f-biharmonic curves in a space form by using the explicit functions f and the explicit curvatures of the curves. Many examples of proper f-biharmonic curves in $\mathbb{R}^n$ are obtained. We also give complete classifications of f-biharmonic isometric immersions and biharmonic conformal immersions of a developable surface into $\mathbb{R}^3$. An interesting result is that a cylinder whose directrix takes a proper $\bar{f}$-biharmonic curve in a 2-space form $N^2(K)$ of constant Gauss curvature K has to be a proper f-biharmonic cylinder for $f = \frac{\psi(v,K)}{c_1}\bar{f}$, where $\psi(v,K)$ given by (30) and a constant $c_1 > 0$. Based on this, one constructs infinitely many examples of proper f-biharmonic cylinders and biharmonic conformal immersions of cylinders into $\mathbb{R}^3$.

2.1. f-Biharmonic Curves in Space Forms

A Frenet frame $\{T_i\}_{i=1,2,\ldots,n}$ associated to an arc length parametrized curve $\gamma : (a,b) \to (N^n, h)$ (see, e.g., [13]) is an orthonormal frame which can be described by

$$\begin{cases} T_1 = d\gamma(\frac{\partial}{\partial s}) = \gamma', \\ \nabla_{T_1} T_1 = \kappa_1 T_2, \\ \nabla_{T_1} T_i = -\kappa_{i-1} T_{i-1} + k_i T_{i+1}, \forall\, i = 2,\ 3,\ \ldots,\ n-1, \\ \nabla_{T_1} T_n = -\kappa_{n-1} T_{n-1}, \end{cases}$$

where the functions $\kappa_i,\ i = 1, 2, \ldots, n-1$, are called the curvatures of the curve γ. It is well known (see [3]) that the curve is f-biharmonic with a function $f : (a,b) \to \mathbb{R}^+$ iff

$$f(\nabla^N_{\gamma'} \nabla^N_{\gamma'} \nabla^N_{\gamma'} \gamma' - R^N(\gamma', \nabla^N_{\gamma'} \gamma')\gamma') + 2f' \nabla^N_{\gamma'} \nabla^N_{\gamma'} \gamma' + f'' \nabla^N_{\gamma'} \gamma' = 0. \tag{3}$$

With respect to the Frenet frame, Equation (3) can be written as follows:

Lemma 1. *Let $\gamma : (a,b) \to (N^n, h)(n \geq 2)$ be a curve parametrized by arc length into an n-dimensional Riemannian manifold. Then, γ is an f-biharmonic curve iff:*

$$\begin{cases} \begin{aligned} -3\kappa_1\kappa_1' - 2\kappa_1^2 f'/f &= 0, \\ \kappa_1'' - \kappa_1\kappa_2^2 - \kappa_1^3 + \kappa_1 R^N(T_1, T_2, T_1, T_2) + \kappa_1' f''/f + 2\kappa_1' f'/f &= 0, \\ 2\kappa_1\kappa_2 + \kappa_1\kappa_2' + \kappa_1 R^N(T_1, T_2, T_1, T_3) + 2\kappa_1\kappa_2 f'/f &= 0, \\ \kappa_1\kappa_2\kappa_3 + \kappa_1 R^N(T_1, T_2, T_1, T_4) &= 0, \\ k_1 R(T_1, T_2, T_1, T_j) = 0, \qquad j = 5, \ldots, n. & \end{aligned} \end{cases} \tag{4}$$

Proof. A straightforward computation gives

$$\begin{aligned} &\tau(\gamma) = \nabla^N_{\gamma'} \gamma' = \kappa_1 F_2, \\ &\nabla^N_{\gamma'} \nabla^N_{\gamma'} \gamma' = -\kappa_1^2 T_1 + \kappa_1' T_2 + \kappa_1\kappa_2 T_3, \\ &\tau_2(\gamma) = \nabla^N_{\gamma'} \nabla^N_{\gamma'} \nabla^N_{\gamma'} \gamma' - R^N(\gamma', \nabla^N_{\gamma'} \gamma')\gamma' \\ &= -3\kappa_1\kappa_1' F_1 + (\kappa_1'' - \kappa_1\kappa_2^2 - \kappa_1^3 + \kappa_1 R^N(T_1, T_2, T_1, T_2))T_2 \\ &+(2\kappa_1'\kappa_2 + \kappa_1\kappa_2' + \kappa_1 R^N(T_1, T_2, T_1, T_3))T_3 \\ &+(\kappa_1\kappa_2\kappa_3 + \kappa_1 R^N(T_1, T_2, T_1, T_4))T_4 + \sum_{j=5}^{n} \kappa_1 R^N(T_1, T_2, T_1, T_j)F_j. \end{aligned} \tag{5}$$

We substitute (5) into (3) and compare the coefficients of both sides to obtain (4), from which the lemma follows. □

Applying Lemma 1, we have

Proposition 1 (see [3])**.** *Let $\gamma : (a,b) \to N^n(C)$ be a curve parametrized by arc length into an n-dimensional space form. Then, γ is f-biharmonic iff one of the following cases happens:*
(i) $\kappa_2 = 0, f = c_1\kappa_1^{-3/2}$ and the curvature κ_1 solves the following ODE

$$3\kappa_1'^2 - 2\kappa_1\kappa_1'' = 4\kappa_1^2(\kappa_1^2 - C); \tag{6}$$

(ii) $\kappa_2 \neq 0,\ \kappa_3 = 0,\ \kappa_2/\kappa_1 = c_3,\ f = c_1\kappa_1^{-3/2}$ and the curvature κ_1 solves the following ODE

$$3\kappa_1'^2 - 2\kappa_1\kappa_1'' = 4\kappa_1^2[(1 + c_3^2)\kappa_1^2 - C]. \tag{7}$$

It is critical to solve the *ODEs* (6) and (7) to describe f-biharmonic curves in space forms. So we need the following proposition.

Proposition 2. *For constants A and C, solving the following ODE*

$$3y'^2 - 2yy'' = 4y^2(Ay^2 - C), \tag{8}$$

we obtain all nonconstant solutions as

$$y = \begin{cases} \frac{1}{C_1 e^{2\sqrt{-C}s} + C_2 e^{-2\sqrt{-C}s} \pm \sqrt{4C_1C_2 + \frac{A}{C}}}, & \text{for } C < 0, \\ \frac{4C_1}{16A + C_1^2 (s + C_2)^2}, & \text{for } C = 0, \\ \frac{1}{C_1 \cos(2\sqrt{C}s) + C_2 \sin(2\sqrt{C}s) \pm \sqrt{C_1^2 + C_2^2 + \frac{A}{C}}}, & \text{for } C > 0, \end{cases}$$

where C_1, C_2 are constants and $C_1^2 + C_2^2 \neq 0$.

Proof. First of all, one sees that $y = 0$ is a solution of (8). For $AC > 0$, it is easy to check that (8) has constant solutions $y = \pm\sqrt{\frac{C}{A}}$.

From now on, we only need to consider that $y(s)$ is a nonconstant solution of (8). Putting $y = u^{-1}$ and substituting this into (8), we obtain

$$u'^2 - 2uu'' = 4(Cu^2 - A). \tag{9}$$

Putting $p = u' = \frac{du}{ds}$, then we obtain $p\frac{dp}{du} = u''(s)$, and hence, (9) turns into

$$\frac{d(p^2)}{du} - \frac{1}{u}p^2 + 4Cu - 4A/u = 0, \tag{10}$$

which is solved by $p^2 = -4Cu^2 + Bu - 4A$. This, together with $u' = p$, implies that

$$u'^2 = -4Cu^2 + Bu - 4A, \tag{11}$$

where B is a constant.

We take the derivative of both sides of (11) with respect to s and simplify the resulting equation to obtain

$$u'' = -4Cu + \frac{1}{2}B. \tag{12}$$

For the case $C > 0$, we solve (12) to obtain the general solution as

$$u = C_1 \cos(2\sqrt{C}s) + C_2 \sin(2\sqrt{C}s) + \tfrac{B}{8C}, \tag{13}$$

where C_1, C_2 are constants. Substituting (13) into (11) and simplifying the resulting equation, we have

$$B^2 = 64AC + 64C^2(C_1^2 + C_2^2), \tag{14}$$

which implies that

$$\begin{aligned} u &= C_1 \cos(2\sqrt{C}s) + C_2 \sin(2\sqrt{C}s) \pm \sqrt{\tfrac{A}{C} + (C_1^2 + C_2^2)}, \\ \text{(and hence) } y &= \frac{1}{C_1 \cos(2\sqrt{C}s) + C_2 \sin(2\sqrt{C}s) \pm \sqrt{\frac{A}{C} + C_1^2 + C_2^2}}, \end{aligned}$$

where constant $8C\sqrt{\frac{A}{C} + C_1^2 + C_2^2} = \pm B$.

In a similar way, we solve (8) for $C = 0$ and $C < 0$, respectively, to obtain $y = \frac{4C_1}{16A + C_1^2(s + C_2)^2}$ and $y = \frac{1}{C_1 e^{2\sqrt{-C}s} + C_2 e^{-2\sqrt{-C}s} \pm \sqrt{4C_1C_2 + \frac{A}{C}}}$, respectively, where C_1 and C_2 are constants.

Summarizing all results above we obtain the proposition. □

Remark 1. *Hereafter, $c_1 > 0$, C, c_3, $C_3 > 0$, C_4, C_5, C_6, $C_1 > 0$, $C_2 > 0$, $C_5^2 + C_6^2 \neq 0$, and $4C_1C_2 + \frac{1 + c_3^2}{C} > 0$ are assumed to be constant unless it is otherwise stated. It is convenient to introduce the following new function:*

$$\chi(s,c_3,C)=\begin{cases}\frac{1}{C_1e^{2\sqrt{-C}s}+C_2e^{-2\sqrt{-C}s}+\sqrt{4C_1C_2+\frac{1+c_3^2}{C}}}, & \text{for } C<0,\\ \frac{4C_3}{16(1+c_3^2)+C_3^2(s+C_4)^2}, & \text{for } C=0,\\ \frac{1}{C_5\cos(2\sqrt{C}s)+C_6\sin(2\sqrt{C}s)+\sqrt{\frac{1+c_3^2}{C}+(C_5^2+C_6^2)}}, & \text{for } C>0.\end{cases} \tag{15}$$

As an application of Proposition 2, we now give a characterization of a proper f-biharmonic curve in $N^n(C)$.

Theorem 1. *Let $\gamma:(a,b)\to N^n(C)$ be a curve parametrized by arc length into an n-dimensional space form. Then, γ is proper f-biharmonic iff one of the following cases happens:*
(i): γ is a curve with $\kappa_2=0$, $\kappa_1=\chi(s,0,C)$, and $f=c_1\kappa_1^{-3/2}$, or
(ii): γ is a curve with $\kappa_3=0$, $\kappa_2/\kappa_1=c_3\neq 0$, $\kappa_1=\chi(s,c_3,C)$, and $f=c_1\kappa_1^{-3/2}$.

Proof. Applying Proposition 1 and Proposition 2 with $A=1$ or $A=1+c_3^2$ to (8), we immediately obtain the theorem. □

Remark 2. *From Theorem 1, we can give the explicit function f for any proper f-biharmonic curve in a space form; in a sense, our result recovers Theorems 4.2 and 4.4 in [3].*

By Theorem 1, we see that a proper f-biharmonic curve in $\mathbb{R}^3$ is either a planar curve or a general helix.

Corollary 1 ([3]). *Let $\gamma:(a,b)\to\mathbb{R}^3$ be a curve parametrized by arc length. Then γ is proper f-biharmonic iff one of the following cases happens:*
(1) *γ is a planar curve with $\kappa_2=0$, $\kappa_1(s)=\frac{4C_3}{16+C_3^2(s+C_4)^2}$, and $f=c_1\kappa_1^{-3/2}$, or*
(2) *γ is a general helix with $\kappa_1(s)=\frac{4C_3}{16(1+c_3^2)+C_3^2(s+2C_4)^2}$, $\kappa_2/\kappa_1(s)=c_3\neq 0$, and $f=c_1\kappa_1^{-3/2}$.*

Remark 3. *A proper f-biharmonic map $\varphi:(M,g)\to(N,h)$ followed by a totally geodesic embedding $\psi:(N,h)\to(Q,k)$ is a proper f-biharmonic map $\psi\circ\varphi:(M,g)\to(Q,k)$.*

In fact, using the fact in [14] (page 371), one can easily check that $\tau(\psi\circ\varphi)=d\psi(\tau(\varphi))$, $\tau^2(\psi\circ\varphi)=d\psi(\tau^2(\varphi))$, and $\nabla^{\psi\circ\varphi}_{\mathrm{grad}f}\tau(\psi\circ\varphi)=d\psi\Big(\nabla^{\varphi}_{\mathrm{grad}f}\tau(\varphi)\Big)$. It follows that $\tau_{2,f}(\psi\circ\varphi)=d\psi\Big(\tau_{2,f}(\varphi)\Big)$ and, hence, $\tau_{2,f}(\psi\circ\varphi)=0$ iff $\tau_{2,f}(\varphi)=0$.

We know that there is no proper biharmonic curve in $\mathbb{R}^n$ (see, e.g., [11]). For more results on biharmonic curves, see [11,15–20] and the references therein. At the end of this section, we try to construct some explicit examples of proper f-biharmonic curves in a space form.

Proposition 3. *(1) If $\rho'(s)\neq\pm 1$ solves the following ODE*

$$\rho''+(1-\rho'^2)\tan\rho+\frac{\sqrt{1-\rho'^2}}{C_5\cos(2s)+C_6\sin(2s)+\sqrt{1+C_5^2+C_6^2}}=0, \tag{16}$$

then the curve $\gamma:(a,b)\to(S^2,d\rho^2+\cos^2\rho d\phi^2)$ with $\gamma(s)=(\rho(s),\int\frac{\sqrt{1-\rho'^2}}{\cos\rho}ds)$ followed by a totally geodesic embedding $\psi:S^2\to S^n$ is a proper f-biharmonic curve $\tilde{\gamma}=\psi\circ\gamma:(a,b)\to S^n$ for $f=c_1\Big(C_5\cos(2s)+C_6\sin(2s)+\sqrt{1+C_5^2+C_6^2}\Big)^{3/2}$.

(2) If $u'(s)\neq\pm 1$ solves the following ODE

$$u''(s)-(1-u'^2)+\frac{\sqrt{1-u'^2}}{C_1e^{2s}+C_2e^{-2s}+\sqrt{4C_1C_2-1}}=0, \tag{17}$$

then the curve $\gamma : (a, b) \to (H^2, du^2 + e^{2u}dv^2)$ *with* $\gamma(s) = (u(s), \int \frac{\sqrt{1-u'^2}}{e^u}ds)$ *followed by a totally geodesic embedding* $\psi : H^2 \to H^n$ *is a proper* f*-biharmonic curve* $\tilde{\gamma} = \psi \circ \gamma : (a, b) \to H^n$ *for* $f = c_1\big(C_1e^{2s} + C_2e^{-2s} + \sqrt{4C_1C_2 - 1}\big)^{3/2}$.

(3) If $x'(s) \neq \pm 1$ *solves the following ODE*

$$x''(s) + \frac{4C_3\sqrt{1-x'^2}}{16+C_3^2(s+C_4)^2} = 0, \tag{18}$$

then the planar curve $\gamma : (a, b) \to (\mathbb{R}^2, dx^2 + dy^2)$ *with* $\gamma(s) = (x(s), \int \sqrt{1 - x'^2}ds)$ *followed by a totally geodesic embedding* $\psi : \mathbb{R}^2 \to \mathbb{R}^n$ *is a proper* f*-biharmonic curve* $\tilde{\gamma} = \psi \circ \gamma : (a, b) \to \mathbb{R}^n$ *for* $f = c_1\left(\frac{4C_3}{16+C_3^2(s+C_4)^2}\right)^{-3/2}$.

Proof. Let $\gamma : (a, b) \to (S^2, d\rho^2 + \cos^2\rho d\phi^2)$, $\gamma(s) = (\rho(s),\ \phi(s))$ be a proper f-biharmonic curve parametrized by arc length. It follows from Statement (i) of Theorem 1 that the curve γ has the geodesic curvature as

$$\kappa_1(s) = \frac{1}{C_5\cos(2s)+C_6\sin(2s)+\sqrt{1+C_5^2+C_6^2}}.$$

Suppose that θ is the angle between the curve γ and $\mathbb{R}ho$ curves. Therefore, applying Liouville's formula for the geodesic curvature of curves on the surface S^2 in $\mathbb{R}^3$ yields

$$\theta'(s) - \sin\theta\tan\rho(s) = \kappa_1(s), \tag{19}$$

which, together with $\mathbb{R}ho' = \cos\theta$ and $\phi'\cos\rho = \sin\theta$, implies

$$\rho'' + (1 - \rho'^2)\tan\rho + \kappa_1\sqrt{1 - \rho'^2} = 0,$$

and hence, $\phi(s) = \int \frac{\sqrt{1-\rho'^2}}{\cos\rho}ds$. Since $\kappa_1(s) \neq 0$, we can check that $\sin\theta \neq 0$, and hence, $\cos\theta \neq \pm 1$ and $\rho' \neq \pm 1$. Hence, using Statement (i) of Theorem 1 and Remark 3, we obtain Statement (1).

Similar to Statement (1), we can obtain Statements (2) and (3). Summarizing all the above results, we obtain the proposition. □

We can apply Statement (3) of Proposition 3 to construct infinitely many examples of proper f-biharmonic curves in $\mathbb{R}^n$.

Example 1. *A family of planar curves* $\tilde{\gamma}(s) = \big(\frac{4\ln|\sqrt{16+C_3^2s^2}+C_3s|}{C_3},\ \frac{\sqrt{16+C_3^2s^2}}{C_3}, 0, \ldots, 0\big)$ *in* $\mathbb{R}^n$ *are proper* f*-biharmonic curves for* $f = c_1\big(\frac{4C_3}{16+C_3^2s^2}\big)^{-3/2}$, *where* c_1 *and* C_3 *are positive constants. Moreover, the curvature* $\kappa_1 = \frac{4C_3}{16+C_3^2s^2}$ *and the torsion* $\kappa_2 = 0$. *The example recovers the family of proper* f*-biharmonic curves found in [3].*

In fact, in the case of $C_3 > 0$ *and* $C_4 = 0$, *one finds* $x(s) = \frac{4\ln|\sqrt{16+C_3^2s^2}+C_3s|}{C_3}$ *to be a solution of (18), and hence,* $y(s) = \int \sqrt{1 - x'^2}ds = \frac{\sqrt{16+C_3^2s^2}}{C_3}$. *Note that the curvature* $\kappa_1 = \frac{4C_3}{16+C_3^2s^2}$ *and the torsion* $\kappa_2 = 0$. *Therefore, we apply Statement (3) of Proposition 3 to obtain the example.*

We give a family of proper f-biharmonic general helixes in $\mathbb{R}^3$.

Proposition 4. *If* $x'(s) \neq \pm\sin\omega$ *solves the following ODE*

$$x''(s) + \frac{\sqrt{\sin^2\omega - x'^2}}{\sin\omega}\frac{4C_3}{16(1+c_3^2)+C_3^2(s+C_4)^2} = 0, \tag{20}$$

where $\sin\omega\cos\omega \neq 0$ *and* $c_3 = \frac{\cos\omega}{\sin\omega}$ *are constants, then a general helix* $\gamma : (a,b) \to \mathbb{R}^3$ *with* $\gamma(s) = (x(s), \int \sqrt{\sin^2\omega - x'^2}ds, s\cos\omega)$ *followed by a totally geodesic embedding* $\psi : \mathbb{R}^3 \to \mathbb{R}^n$ *is a proper* f*-biharmonic curve* $\tilde{\gamma} = \psi \circ \gamma : (a,b) \to \mathbb{R}^n$ *for* $f = c_1(\frac{4C_3}{16(1+c_3^2)+C_3^2(s+C_4)^2})^{-3/2}$.

Proof. Let a general helix $\gamma : (a,b) \to \mathbb{R}^3$ with $\gamma(s) = (x(s), y(s), s\cos\omega)$ be a proper f-biharmonic curve parametrized by arc length, where constant $\cos\omega \neq 0$. By Corollary 1, the helix has the curvature $\kappa_1(s) = \frac{4C_3}{16(1+c_3^2)+C_3^2(s+2C_4)^2} \neq 0$ and the torsion $\kappa_2 = c_3\kappa_1(s) \neq 0$. Since $\gamma'(s) = (x'(s), y'(s), \cos\omega)$ and $|\gamma'(s)|^2 = x'^2(s) + y'^2 + \cos^2\omega = 1$, we may assume that $x'(s) = \sin\omega\cos\theta(s),\ y' = \sin\omega\sin\theta$. Therefore, we have

$$\begin{aligned}&\gamma' = (\cos\theta\sin\omega, \sin\theta\sin\omega, \cos\omega),\ \gamma'' = \theta'\sin\omega(-\sin\theta, \cos\theta, 0),\\ &\gamma''' = \theta''\sin\omega(-\sin\theta, \cos\theta, 0) - \theta'^2\sin\omega(\cos\theta, \sin\theta, 0).\end{aligned} \tag{21}$$

It follows that

$$\kappa_1(s) = \sin\omega\theta',\ \kappa_2 = \frac{(\gamma', \gamma'', \gamma''')}{\kappa_1^2} = \cos\omega\theta',\ \text{and hence } c_3 = \frac{\cos\omega}{\sin\omega}. \tag{22}$$

Combining these, together with $\kappa_1(s) \neq 0$ and $\kappa_2 \neq 0$, we have $\sin\omega\cos\omega \neq 0$ and $x' \neq \pm\sin\omega$. A direct computation gives $x''(s) = -\sin\omega\theta'\sin\theta = -\theta' y' = -\theta'\sqrt{\sin^2\omega - x'^2}$. Substituting this into the first equation of (22), we obtain (20). Clearly, $y = \int \sqrt{\sin^2\omega - x'^2}ds$. From these and using Remark 3, the proposition follows. □

We will look for some special solutions of (20).

Example 2. *A family of helixes* $\gamma(s) = (\frac{4\ln|\sqrt{32+C_3^2s^2}+C_3s|}{C_3}, \frac{\sqrt{64+2C_3^2s^2}}{2C_3}, \frac{\sqrt{2}}{2}s)$ *in* $\mathbb{R}^3$ *with the curvature* $\kappa_1 = \frac{4C_3}{32+C_3^2s^2}$ *and the torsion* $\kappa_2 = \frac{4C_3}{32+C_3^2s^2}$ *are proper* f*-biharmonic curves for* $f = c_1(\frac{4C_3}{32+C_3^2s^2})^{-3/2}$*, where* c_1 *and* C_3 *are positive constants. Moreover, a family of curves* $\bar{\gamma}(s) = (\frac{4\ln|\sqrt{32+C_3^2s^2}+C_3s|}{C_3}, \frac{\sqrt{64+2C_3^2s^2}}{2C_3}, \frac{\sqrt{2}s}{2}, 0, \ldots, 0)$ *in* $\mathbb{R}^n$ *are also proper* f*-biharmonic curves for* $f = c_1(\frac{4C_3}{32+C_3^2s^2})^{-3/2}$.

In fact, by taking $\sin\omega = \cos\omega = \frac{\sqrt{2}}{2}$, $c_3 = 1$ *and* $C_4 = 0$, *(20) becomes*

$$x''(s) + \frac{\sqrt{\frac{1}{2} - x'^2}}{\frac{\sqrt{2}}{2}}\frac{4C_3}{32+C_3^2s^2} = 0. \tag{23}$$

If $C_3 > 0$, *one finds that* $x(s) = \frac{4\ln|\sqrt{32+C_3^2s^2}+C_3s|}{C_3}$ *is a solution of (23), and hence,* $y(s) = \int\sqrt{1/2 - x'^2}ds = \frac{\sqrt{64+2C_3^2s^2}}{2C_3}$. *Clearly, the curvature* $\kappa_1 = \frac{4C_3}{32+C_3^2s^2}$ *and the torsion* $\kappa_2 = \frac{4C_3}{32+C_3^2s^2}$ *in this case. Thus, we apply Proposition 4 to obtain the example.*

2.2. *f-Biharmonic Isometric Immersions and Biharmonic Conformal Immersions of a Developable Surface into* $\mathbb{R}^3$

The equation for f-biharmonic hypersurfaces in a space form can be stated as follows:

Lemma 2 (see, e.g., [3,10])**.** *A hypersurface* $\phi : M^m \to N^{m+1}(C)$ *in a space form of constant sectional curvature* C *with mean curvature vector field* $\eta = H\xi$ *is* f*-biharmonic if and only if the function* H *satisfies the following equation:*

$$\begin{cases}\Delta(fH) - (fH)[|A|^2 - mC] = 0,\\ A\,(\mathrm{grad}\,(fH)) + \frac{m}{2}(fH)\mathrm{grad}\,H = 0,\end{cases} \tag{24}$$

where $\mathrm{Ric}^N : T_qN \to T_qN$ *is the Ricci operator of* $N^{m+1}(C)$ *defined by* $\langle \mathrm{Ric}^N(Z), W\rangle = \mathrm{Ric}^N(Z,W)$ *and* A *is the shape operator of the hypersurface* M^m *with respect to the unit normal vector field* ξ.

We are ready to give a characterization of proper f-biharmonic cylinders in $\mathbb{R}^3$.

Theorem 2. *Let* $X : (M^2, g) \to (\mathbb{R}^3, h = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$, $X(s,v) = a(s) + vb$ *be a cylinder with the mean curvature function* H*, where* b *is a unit constant vector in* $\mathbb{R}^3$ *and* $a(s)$ *is an immersed regular curve in* $\mathbb{R}^3$ *parametrized by arc length with the geodesic curvature* $\kappa_1(s)$ *satisfying* $h(a', b) = 0$*. Then,* X *is a proper* f*-biharmonic cylinder iff one of the following cases happens:*
(1) κ_1 *is a nonzero constant, and the directrix* $a(s)$ *of the cylinder is (a part of) a circle with radius* $\frac{1}{|\kappa_1|}$*. Moreover, the mean curvature* $H = \frac{\kappa_1}{2}$ *and the function* $f = d_1 e^{\kappa_1 v} + d_2 e^{-\kappa_1 v}$*, where* d_1 *and* d_2 *are some constants, or*
(2) $\kappa_1(s)$ *is nonconstant, and the directrix* $a(s)$ *of the cylinder has the geodesic curvature* $\kappa_1(s) = \kappa_N(s, K)$ *given by (31) and the geodesic torsion* $\kappa_2 = 0$*, where* K *is a constant. Furthermore, the function* $H = \frac{\kappa_N(s,K)}{2}$ *and the function* $f = \psi(v, K)\kappa_N^{-3/2}(s, K)$*, where* $\psi(v, K)$*, given by (30).*

Proof. A straightforward computation gives

$$\begin{aligned} &X_s = a'(s),\ X_v = b,\ X_{ss} = a''(s),\ X_{sv} = 0,\ X_{vv} = 0,\\ &N = \frac{X_s \times X_v}{|X_s \times X_v|} = a' \times b,\ \ |a''(s)| = \kappa(s), \end{aligned}$$

where κ is the curvature of the curve $a(s)$. By a further computation, the first fundamental form I and the second fundamental form II of the cylinder are given by

$$I = g = \mathrm{d}s^2 + \mathrm{d}v^2,\ \ II = \kappa_N(s)\mathrm{d}s^2,$$

where κ_N is the normal curvature of $a(s)$. It follows that $\{e_1 = \frac{\partial}{\partial s},\ e_2 = \frac{\partial}{\partial v},\ \xi = N\}$ forms an orthonormal frame adapted to the cylinder with the normal vector field ξ, $A(e_1) = II(e_1,e_1)e_1 + II(e_1,e_2)e_2 = \kappa_N e_1$, $A(e_2) = II(e_2,e_1)e_1 + II(e_2,e_2)e_2 = 0$, and $H = \frac{\kappa_N}{2}$. Substituting these into the f-biharmonic Equation (24) with $m = 2$ and $C = 0$, we have

$$\begin{cases} \kappa_N^2 \frac{\partial \ln f}{\partial s} = -\frac{3}{2}\kappa_N'(s)\kappa_N,\\ \kappa_N''(s) - \kappa_N^3 + \kappa_N[\frac{\partial^2 \ln f}{\partial s^2} + \frac{\partial^2 \ln f}{\partial v^2} + (\frac{\partial \ln f}{\partial s})^2 + (\frac{\partial \ln f}{\partial v})^2] + 2\frac{\partial \ln f}{\partial s}\kappa_N'(s) = 0. \end{cases} \tag{25}$$

If $\kappa_N = 0$, i.e., $H = 0$, then the surface is harmonic and hence biharmonic, not proper f-biharmonic. From now on, we assume that $\kappa_N \neq 0$. If κ_N is a nonzero constant, then (25) turns into

$$\frac{\partial \ln f}{\partial s} = 0,\ \ -\kappa_N^2 + \frac{\partial^2 \ln f}{\partial v^2} + (\frac{\partial \ln f}{\partial v})^2 = 0, \tag{26}$$

which is solved by $f = d_1 e^{v\kappa_N} + d_2 e^{-v\kappa_N}$, where d_1 and d_2 are constants. Note that the curve $a(s)$ can be viewed as (a part of) a circle of radius $\frac{1}{|\kappa_N|}$.

We now assume that κ is nonconstant and apply the first equation of (25) to have

$$\ln f = -\frac{3}{2}\ln|\kappa_N(s)| + \ln \psi(v),\ \text{and hence}\ f = \psi(v)\kappa_N^{-3/2}(s), \tag{27}$$

where $\psi(v)$ is a positive function. Substituting (27) into the second equation of (25) and simplifying the resulting equation, we obtain

$$-2\kappa_N\kappa_N''(s) + 3\kappa_N'^2(s) - 4\kappa_N^4 + 4\kappa_N^2(s)\frac{\psi''(v)}{\psi(v)} = 0. \tag{28}$$

Equation (28) implies that for any s, v, we have

$$3\kappa_N'^2(s) - 2\kappa_N\kappa_N''(s) = 4\kappa_N^2(\kappa_N^2 - K)\ \text{ and }\ \frac{\psi''(v)}{\psi(v)} = K, \tag{29}$$

where K is a constant. The second equation of (29) is equivalent to $\psi''(v) = K\psi(v)$, solved by

$$\psi(v) = \psi(v,K) = \begin{cases} d_5 \sin(\sqrt{-K}\, v) + d_6 \cos(\sqrt{-K}\, v), & \text{for } K < 0, \\ d_3 v + d_4, & \text{for } K = 0, \\ d_1 e^{\sqrt{K}\, v} + d_2 e^{-\sqrt{K}\, v}, & \text{for } K > 0, \end{cases} \tag{30}$$

where d_i, $i = 1, 2, \ldots, 6$, are some constants.

An interesting thing is that the first equation of (29) happens to be the type of *ODE* (8) in Proposition 2 when $A = 1$ and $C = K$, which is solved by

$$\kappa_N(s) = \kappa_N(s,K) = \begin{cases} \frac{1}{C_1 e^{2\sqrt{-K}\, s} + C_2 e^{-2\sqrt{-K}\, s} + \sqrt{4C_1C_2 + \frac{1}{K}}}, & \text{for } K < 0, \\ \frac{4C_3}{16 + C_3^2 (s + C_4)^2}, & \text{for } K = 0, \\ \frac{1}{C_5 \cos(2\sqrt{K}\, s) + C_6 \sin(2\sqrt{K}\, s) + \sqrt{\frac{1}{K} + (C_5^2 + C_6^2)}}, & \text{for } K > 0, \end{cases} \tag{31}$$

where C_i, $i = 1, 2, \ldots, 6$, are some constants.

It is not difficult to check that $\nabla_{e_1} e_1 = \kappa_N e_3$, $\nabla_{e_1} e_3 = -\kappa_N e_1$, and $\nabla_{e_1}(-e_2) = 0$, and hence $\{e_1,\ e_3 = N,\ -e_2\}$ forms the Frenet frame along the curve $a(s)$. This implies that the curve $a(s)$ has the geodesic curvature $\kappa_1 = \kappa_N$ and the geodesic torsion $\kappa_2 = 0$.

Summarizing all the above results, we complete the proof of the theorem. □

Remark 4. *Theorem 2 recovers the classification result of proper f-biharmonic cylinders with constant mean curvature in [3].*

Corollary 2. *Let $X : (M^2, g) \to \mathbb{R}^3$ be a cylinder with $X(s,v) = a(s) + vb$ and nonconstant mean curvature function H, where b is a unit constant vector in $\mathbb{R}^3$ and $a(s)$ is an immersed regular curve in $\mathbb{R}^3$ parametrized by arc length with the geodesic curvature $\kappa_1(s)$ satisfying $h(a', b) = 0$. Then, X is proper f-biharmonic if and only if the directrix $a(s)$ is a proper $\bar{f}$-biharmonic curve in a 2-space form $N^2(K) \subset \mathbb{R}^3$ with constant Gauss curvature K for $\bar{f} = c_1 \kappa_1^{-3/2}$, where $c_1 > 0$ is a constant and $\kappa_1(s) = \kappa_N(s,K)$ given by (31). Moreover, the function $f = \frac{\psi(v,K)}{c_1} \bar{f} = \psi(v,K) \kappa_N^{-3/2}(s,K)$, where $\psi(v,K)$, given by (30).*

Proof. From the proof of Theorem 2 and Statement (i) of Proposition 1, together with the assumption that H is nonconstant, one sees that the directrix $a(s)$ on the surface has the geodesic torsion $\kappa_2 = 0$ and nonconstant geodesic curvature $\kappa_1 = \kappa_N = 2H$ solving the 1st equation of (29) which is a proper $\bar{f}$-biharmonic curve equation in a space form of constant sectional curvature K for $\bar{f} = c_1 \kappa_1^{-3/2}$, where constant $c_1 > 0$. This implies that the directrix $a(s)$ can be viewed as a proper $\bar{f}$-biharmonic curve in a 2-space form $N^2(K)$ of constant Gauss curvature K. Note that $f = \frac{\psi(v,K)}{c_1} \bar{f}$, where $\psi(v,K)$ given by (30). □

Proposition 5. *There is neither a proper f-biharmonic cone nor a tangent surface in $\mathbb{R}^3$.*

Proof. Let $b(s)$ be a curve on a unit sphere parametrized by arc length (i.e., a spherical curve). Consider a cone $X : (M^2, g) \to (\mathbb{R}^3, h = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$ into $\mathbb{R}^3$ with $X(s,v) = a + vb(s)$, where a is a constant vector. A simple computation, we obtain the first and the second fundamental forms I and II of the cone as

$$I = g = v^2 \mathrm{d}s^2 + \mathrm{d}v^2, \quad II = vw(s)\mathrm{d}s^2,$$

where $w(s) = b'' \cdot (b' \times b)$. One can check that $\{e_1 = \frac{1}{v}\frac{\partial}{\partial s}, e_2 = \frac{\partial}{\partial v},\ \xi = b' \times b\}$ constitutes an orthonormal frame adapted to the surface with ξ being normal to the surface. By a straightforward computation, we have $A(e_1) = II(e_1, e_1)e_1 + II(e_1, e_2)e_2 = \frac{w}{v} e_1$,

$A(e_2) = II(e_2, e_1)e_1 + II(e_2, e_2)e_2 = 0$, and $H = \frac{w(s)}{2v}$. Substituting these into the second equation of (24) and simplifying the resulting equation yields

$$H[3e_1(H) + 2He_1(\ln f)] = 0, \; He_2(H) = H\frac{\partial}{\partial v}\left(\frac{w(s)}{2v}\right) = 0. \tag{32}$$

The second equation of (32) implies that $-\frac{H^2}{v^2} = 0$ and hence $H = 0$, i.e., the cone has to be minimal. It follows that any f-biharmonic cone in $\mathbb{R}^3$ has to be harmonic and hence biharmonic but not proper f-biharmonic.

For a tangent surface in $\mathbb{R}^3$, let $a(s)$ be a curve parametrized by arc length and let $X : (M^2, g) \to \mathbb{R}^3$ be a tangent surface with $X(s, v) = a(s) + va'(s)$. By simple computation, the first and the second fundamental forms I and II of the tangent surface are as follows:

$$I = g = (1 + v^2\kappa^2)\mathrm{d}s^2 + 2\mathrm{d}s\mathrm{d}v + \mathrm{d}v^2, \quad II = -v\kappa\tau\mathrm{d}s^2. \tag{33}$$

Here, $\kappa = \kappa(s)$ and $\tau = \tau(s)$ denote the curvature and torsion of $a(s)$, respectively.

One can easily check that $\{\varepsilon_1 = \frac{1}{\sqrt{1+v^2\kappa^2}}\frac{\partial}{\partial s}, \; \varepsilon_2 = \frac{1}{v\kappa\sqrt{(1+v^2\kappa^2}}\frac{\partial}{\partial s} - \frac{\sqrt{1+v^2\kappa^2}}{v\kappa}\frac{\partial}{\partial v},$ $\xi = \frac{a''\times a'}{\kappa}\}$ forms an orthonormal frame adapted to the surface with the normal vector field ξ. Suppose that $A(\varepsilon_1) = k_1e_1 + k_{12}\varepsilon_2$ and $A(\varepsilon_2) = k_{21}\varepsilon_1 + k_2\varepsilon_2$. A direct computation gives $k_1 = II(\varepsilon_1, \varepsilon_1) = -\frac{v\kappa\tau}{1+v^2\kappa^2}$, $k_2 = II(\varepsilon_2, \varepsilon_2) = -\frac{\tau}{v\kappa(1+v^2\kappa^2)}$ and $k_{12} = k_{21} = II(e_1, e_2) = -\frac{\tau}{1+v^2\kappa^2}$. It follows that the mean curvature $H = \frac{k_1+k_2}{2} = -\frac{\tau}{2v\kappa}$ and $k_1k_2 = \frac{\tau^2}{(1+v^2\kappa^2)^2}$. Note that $H = 0$ is equivalent to $\tau = 0$; then, the surface is not proper f-biharmonic in this case.

Hereafter, assume that $H \neq 0$. Let $\{e_1 = \cos\theta\varepsilon_1 + \sin\theta\varepsilon_2, \; e_2 = -\sin\theta\varepsilon_1 + \cos\theta\varepsilon_2, \; \xi\}$ be another orthonormal frame adapted to the surface with the normal vector field ξ such that $A(e_1) = \lambda_1 e_1$ and $A(e_2) = \lambda_2 e_2$, where θ is the angle between e_1 and ε_1. Since the tangent surface is flat, it follows from the Gauss equation of the tangent surface that $\lambda_1\lambda_2 = 0$. Without a loss of generality, we may suppose that $\lambda_2 = 0$ and hence $\lambda_1 = 2H \neq 0$. We can easily conclude that $k_1 = 2H\cos^2\theta, \; k_2 = 2H\sin^2\theta, \; k_{12} = 2H\sin\theta\cos\theta, \; \cos^2\theta = \frac{v^2\kappa^2}{1+v^2\kappa^2}$, and $\sin^2\theta = \frac{1}{1+v^2\kappa^2}$. If the tangent surface is f-biharmonic, by (24) with $m = 2$ and $C = 0$, we have

$$\Delta(fH) - 4fH^3 = 0, \; 2He_1(fH) + fHe_1(H) = 0, \; fHe_2(H) = 0, \tag{34}$$

which implies that $e_2(H) = 0$. Therefore, we obtain $[-\sin\theta\varepsilon_1 + \cos\theta\varepsilon_2](H) = 0$, i.e., $\sin^2\theta(\varepsilon_1(H))^2 - \cos^2\theta(\varepsilon_2(H))^2 = 0$. A further computation gives

$$\begin{aligned} 0 &= \sin^2\theta[\varepsilon_1(-\tfrac{\tau}{2v\kappa})]^2 - \cos^2\theta[\varepsilon_2(-\tfrac{\tau}{2v\kappa})]^2 \\ &= \tfrac{1}{1+v^2\kappa^2}\left(\tfrac{1}{\sqrt{1+v^2\kappa^2(s)}}\tfrac{\partial}{\partial s}(\tfrac{\tau}{2v\kappa})\right)^2 - \tfrac{v^2\kappa^2}{1+v^2\kappa^2}\left(\tfrac{1}{v\kappa\sqrt{(1+v^2\kappa^2}}\tfrac{\partial}{\partial s}(\tfrac{\tau}{2v\kappa}) - \tfrac{\sqrt{1+v^2\kappa^2}}{v\kappa}\tfrac{\partial}{\partial v}(\tfrac{\tau}{2v\kappa})\right)^2 \\ &= -\tfrac{(\frac{\tau}{\kappa})^2}{4v^4} - \tfrac{[(\frac{\tau}{\kappa})^2]'(s)}{4v^3(1+v^2\kappa^2)}. \end{aligned} \tag{35}$$

It is easy to check that (35) implies that

$$\tau^2(s)\; v^2 + [(\tfrac{\tau}{\kappa})^2]'(s)\; v + (\tfrac{\tau}{\kappa})^2(s) = 0, \tag{36}$$

which, together with any $s, \; v$, implies that $\tau = 0$. This contradicts the assumption that $\tau \neq 0$. Combining these, any f-biharmonic tangent surface in $\mathbb{R}^3$ is harmonic but not proper f-biharmonic.

Summarizing all the above results, the proposition follows. □

Applying Theorem 2 and Proposition 5, we have

Corollary 3. *A proper f-biharmonic developable surface in $\mathbb{R}^3$ exists only in the case when the developable surface is either a circle cylinder or a cylinder containing a directrix with nonconstant geodesic curvature $\kappa_1(s) = \kappa_N(s, K)$ given by (31) and the geodesic torsion $\kappa_2(s) = 0$.*

We will construct a family of proper f-biharmonic cylinders whose directrices are proper $\bar{f}$-biharmonic curves in $\mathbb{R}^2$, where $f = \frac{d_3 v + d_4}{c_1}\bar{f}$, and d_3, d_4 and c_1 are constants.

Example 3. *A family of cylinders $X : (M^2, g = ds^2 + dv^2) \to \mathbb{R}^3$, $X(s, v) = (\frac{4\ln|\sqrt{16+C_3^2 s^2}+C_3 s|}{C_3}, \frac{\sqrt{16+C_3^2 s^2}}{C_3}, v)$ are proper f-biharmonic for $f = \frac{c_1(d_3 v+d_4)(16+C_3^2 s^2)^{3/2}}{(4C_3)^{3/2}}$, where d_3, d_4, c_1, and C_3 are positive constants. This family of cylinders followed by a totally geodesic embedding $\phi : \mathbb{R}^3 \to \mathbb{R}^n$ are proper f-biharmonic submanifolds $\phi \circ X : (M^2, g) \to \mathbb{R}^n$.*

In fact, it is easy to check that $X(s, v) = a(s) + vb$, where $a(s) = (\frac{4\ln|\sqrt{16+C_3^2 s^2}+C_3 s|}{C_3}, \frac{\sqrt{16+C_3^2 s^2}}{C_3}, 0)$ are a family of curves in $\mathbb{R}^2 \subset \mathbb{R}^3$ and $b = (0, 0, 1)$ is a unit constant vector in $\mathbb{R}^3$. Clearly, $|a'|^2 = 1$ and $h(a', b) = 0$. Therefore, by Example 1, any curve $a(s)$ is a proper $\bar{f}$-biharmonic curve in $\mathbb{R}^2 \subset \mathbb{R}^3$ for $\bar{f} = c_1(\frac{4C_3}{16+C_3^2 s^2})^{-3/2}$, where $c_1 > 0$ is a constant. Then, applying Corollary 2, we obtain the example.

We know from [3] that a map $\phi : (M^2, g) \to (N^n, h)$ is f-biharmonic iff the conformal immersion $\phi : (M^2, f^{-1}g) \to (N^n, h)$ is a biharmonic map. Therefore, applying Theorem 2 and Proposition 5, we have the following corollaries:

Corollary 4. *Under the same assumptions as in Theorem 2, we have the following:*
(1) If κ_1 is a nonzero constant and $f = d_1 e^{\kappa_1 v} + d_2 e^{-\kappa_1 v} > 0$, where d_1 and d_2 are constants, then the conformal immersion $X : (M^2, f^{-1}(ds^2 + dv^2)) \to \mathbb{R}^3$ with $X(s, v) = a(s) + vb$ is proper biharmonic, or
(2) If $\kappa_1(s) = \kappa_N(s, K)$ and $\psi(v, K)$ given by (31) and (30), respectively, and $f = \psi(v, K)\kappa_N^{-3/2}(s, K)$, where K is a constant, then the conformal immersion $X : (M^2, f^{-1}(ds^2 + dv^2)) \to \mathbb{R}^3$ with $X(s, v) = a(s) + vb$ is proper biharmonic.

Corollary 5. *We have the following:*
(1) There cannot exist a proper biharmonic conformal immersion of a cone or a tangent surface into $\mathbb{R}^3$.
(2) A conformal immersion of a developable surface into $\mathbb{R}^3$ is proper biharmonic iff the surface is either a circle cylinder or a cylinder containing a directrix with nonconstant geodesic curvature $\kappa_1(s) = \kappa_N(s, K)$ given by (31) and the torsion $\kappa_2(s) = 0$.

Remark 5. *Corollary 5 recovers the result in [9] which states that there is not a proper biharmonic conformal immersion of a circular cone into $\mathbb{R}^3$.*

Adopting the same notations as in Example 3 and applying Example 3 and Corollary 4, we have the following:

Example 4. *The conformal immersions of a family of cylinders $X : (M^2, f^{-1}(ds^2 + dv^2)) \to \mathbb{R}^3$, $X(s, v) = (\frac{4\ln|\sqrt{16+C_3^2 s^2}+C_3 s|}{C_3}, \frac{\sqrt{16+C_3^2 s^2}}{C_3}, v)$ are proper biharmonic with $f = \frac{c_1(d_3 v+d_4)(16+C_3^2 s^2)^{3/2}}{(4C_3)^{3/2}}$.*

3. f-Biharmonic Riemannian Submersions from 3-Dimensional Riemannian Manifolds

In this section, we study f-biharmonicity of a Riemannian submersion from a 3-dimensional Riemannian manifold by using the integrability data of an adapted frame of the Riemannian submersion. We also construct a family of proper f-biharmonic Riemannian submersions from Sol space.

Recall that if an orthonormal frame $\{e_1, e_2, e_3\}$ is adapted to a Riemannian submersion $\pi : (M^3, g) \to (N^2, h)$ with vertical vector field e_3, then we immediately have (see [4])

$$\begin{aligned}
&[e_1, e_3] = \kappa_1 e_3,\ [e_2, e_3] = \kappa_2 e_3,\ [e_1, e_2] = f_1 e_1 + f_2 e_2 - 2\sigma e_3,\\
&\nabla_{e_1} e_1 = -f_1 e_2,\ \nabla_{e_1} e_2 = f_1 e_1 - \sigma e_3,\ \nabla_{e_1} e_3 = \sigma e_2,\\
&\nabla_{e_2} e_1 = -f_2 e_2 + \sigma e_3,\ \nabla_{e_2} e_2 = f_2 e_1,\ \nabla_{e_2} e_3 = -\sigma e_1,\\
&\nabla_{e_3} e_1 = -\kappa_1 e_3 + \sigma e_2, \nabla_{e_3} e_2 = -\sigma e_1 - \kappa_2 e_3, \nabla_{e_3} e_3 = \kappa_1 e_1 + \kappa_2 e_2,
\end{aligned} \tag{37}$$

where $f_1,\ f_2,\ \kappa_1,\ \kappa_2$, and $\sigma \in C^\infty(M)$ are called the integrability data of the adapted frame of π. The bitension field of the map π is given by

$$\begin{aligned}
&\tau_2(\pi) = [-\Delta^M \kappa_1 - f_1 e_1(\kappa_2) - e_1(\kappa_2 f_1) - f_2 e_2(\kappa_2) - e_2(\kappa_2 f_2)\\
&+\kappa_1\kappa_2 f_1 + \kappa_2^2 f_2 + \kappa_1\{-K^N + f_1^2 + f_2^2\}]\varepsilon_1 + (-\Delta^M \kappa_2 + f_1 e_1(\kappa_1)\\
&+e_1(\kappa_1 f_1) + f_2 e_2(\kappa_1) + e_2(\kappa_1 f_2) - \kappa_1\kappa_2 f_2 - \kappa_1^2 f_1 + \kappa_2\{-K^N + f_1^2 + f_2^2\})\varepsilon_2,
\end{aligned} \tag{38}$$

where $d\pi(e_i) = \varepsilon_i,\ i = 1, 2$.

Proposition 6. *Let $\pi : (M^3, g) \to (N^2, h)$ be a Riemannian submersion with the adapted frame $\{e_1,\ e_2,\ e_3\}$ and the integrability data $\{f_1, f_2, \kappa_1,\ \kappa_2,\ \sigma\}$. Then, π is f-biharmonic iff*

$$\begin{cases}
-\Delta^M(f\kappa_1) - 2\sum\limits_{i=1}^{2} f_i e_i(f\kappa_2) - f\kappa_2 \sum\limits_{i=1}^{2}(e_i(f_i) - \kappa_i f_i) + f\kappa_1\left(-K^N + \sum\limits_{i=1}^{2} f_i^2\right) = 0,\\
-\Delta^M(f\kappa_2) + 2\sum\limits_{i=1}^{2} f_i e_i(f\kappa_1) + f\kappa_1 \sum\limits_{i=1}^{2}(e_i(f_i) - \kappa_i f_i) + f\kappa_2\left(-K^N + \sum\limits_{i=1}^{2} f_i^2\right) = 0,
\end{cases} \tag{39}$$

where $K^N = e_1(f_2) - e_2(f_1) - f_1^2 - f_2^2$ denotes the Gauss curvature of (N^2, h).

Proof. A straightforward computation using (37) gives

$$\begin{aligned}
&\tau(\pi) = \nabla^\pi_{e_i} d\pi(e_i) - d\pi(\nabla^M_{e_i} e_i) = -d\pi(\nabla^M_{e_3} e_3) = -\kappa_1\varepsilon_1 - \kappa_2\varepsilon_2,\\
&\Delta^M f = \sum_{i=1}^{3} e_i e_i(f) + f_1 e_2(f) - f_2 e_1(f) - \kappa_1 e_1(f) - \kappa_2 e_2(f),\\
&2\nabla^\pi_{\mathrm{grad} f}\tau(\pi) = -2\{\langle \mathrm{grad} f, \mathrm{grad}\kappa_1\rangle + \kappa_2 f_1 e_1(f) + \kappa_2 f_2 e_2(f)\}\varepsilon_1\\
&-2\{\langle \mathrm{grad} f, \mathrm{grad}\kappa_2\rangle - \kappa_1 f_1 e_1(f) - \kappa_1 f_2 e_2(f)\}\varepsilon_2.
\end{aligned} \tag{40}$$

Substituting (40) and (38) into (2) and simplifying the resulting equation yields

$$\begin{aligned}
&0 = \tau_{2,f}(\pi) = -J^\pi(f\tau(\pi))\\
&= f\tau^2(\pi) + (\Delta^M f)\tau(\pi) + 2\nabla^\pi_{\mathrm{grad} f}\tau(\pi)\\
&= [-\Delta^M(f\kappa_1) - f_1 e_1(f\kappa_2) - e_1(f\kappa_2 f_1) - f_2 e_2(f\kappa_2) - e_2(f\kappa_2 f_2)\\
&+f\kappa_2(\kappa_1 f_1 + \kappa_2 f_2) + f\kappa_1\{-K^N + f_1^2 + f_2^2\}]\varepsilon_1,\\
&+[-\Delta^M(f\kappa_2) + f_1 e_1(f\kappa_1) + e_1(f\kappa_1 f_1) + f_2 e_2(f\kappa_1) + e_2(f\kappa_1 f_2)\\
&-f\kappa_1(\kappa_1 f_1 + \kappa_2 f_2) + f\kappa_2\{-K^N + f_1^2 + f_2^2\}]\varepsilon_2.
\end{aligned}$$

from which the proposition follows. □

Remark 6. *By taking* $\mathrm{grad} f = \delta e_3$*, the authors in [5] derived an f-biharmonic equation for a Riemannian submersion $\pi : (M^3, g) \to (N^2, h)$ by using the integrability data $\{f_1, f_2, \kappa_1,\ \kappa_2,\ \sigma,\ \delta\}$ (i.e., Theorem 2 in [5]). Note that* $\mathrm{grad} f = e_1(f)e_1 + e_2(f)e_2 + e_3(f)e_3$ *is not always parallel to the vertical vector field e_3. So, our Proposition 6 recovers Theorem 2 in [5].*

When the integrability data $\kappa_2 = 0$, we have the following corollary.

Corollary 6. *Let $\pi : (M^3, g) \to (N^2, h)$ be a Riemannian submersion with the adapted frame $\{e_1,\ e_2,\ e_3\}$ and the integrability data $\{f_1, f_2, \kappa_1,\ \kappa_2,\ \sigma\}$ with $\kappa_2 = 0$. Then π is f-biharmonic iff*

$$\begin{cases} -\Delta^M(f\kappa_1) + f\kappa_1\{-K^N + f_1^2 + f_2^2\} = 0, \\ 2f_1e_1(f\kappa_1) + 2f_2e_2(f\kappa_1) + f\kappa_1\{e_1(f_1) + +e_2(f_2)\} - f\kappa_1^2 f_1 = 0. \end{cases} \tag{41}$$

Example 5. *For any positive constants A, B and the function $f = Ae^{\sqrt{2}z} + Be^{-\sqrt{2}z}$, the Riemannian submersion $\pi : (\mathbb{R}^3, g_{Sol} = e^{2z}\mathrm{d}x^2 + e^{-2z}\mathrm{d}y^2 + \mathrm{d}z^2) \to (\mathbb{R}^2, e^{-2z}\mathrm{d}y^2 + \mathrm{d}z^2)$ with $\pi(x, y, z) = (y, z)$ is proper f-biharmonic from Sol space.*

In fact, it is not difficult to find that the frame $\{e_1 = \frac{\partial}{\partial z},\ e_2 = e^z\frac{\partial}{\partial y},\ e_3 = e^{-z}\frac{\partial}{\partial x}\}$ on Sol space is an adapted frame of the Riemannian submersion with e_3 being vertical. A straightforward computation gives

$$\begin{aligned} &[e_1, e_3] = -e_3,\ [e_2, e_3] = 0,\ [e_1, e_2] = e_2,\ f_1 = \kappa_2 = \sigma = 0,\ -f_2 = \kappa_1 = -1, \\ &\text{(and hence) } K^N = e_1(f_2) - e_2(f_1) - f_1^2 - f_2^2 = -1. \end{aligned} \tag{42}$$

Substituting (42) into (41) yields

$$f_{zz} + e^{-2z}f_{xx} - 2f = 0,\ f_y = 0. \tag{43}$$

We find $f = Ae^{\sqrt{2}z} + Be^{-\sqrt{2}z}$ to be a special solution of (43), where A and B are positive constants. The bitension field $\tau_2(\pi) = -2\frac{\partial}{\partial z} \neq 0$ implies that π is not biharmonic. Thus, we obtain the example.

4. f-Biharmonic Riemannian Submersions from 3-Space Forms

A biharmonic Riemannian submersion from a 3-space form into a surface has to be harmonic (cf. [4]). We want to know if there exists a proper f-biharmonic Riemannian submersion from a 3-space form. Let $\pi : (M^3(c), g) \to (N^2(c), h)$ be a Riemannian submersion from a 3-space form with constant sectional curvature c onto a surface with constant Gauss curvature c. It follows from Lemma 3.2 in [4] that we can choose an orthonormal frame $\{e_1, e_2, e_3\}$ adapted to the Riemannian submersion with the integrability data $\{f_1, f_2, \kappa_1, \kappa_2, \sigma\}$ and $\kappa_2 = 0$ such that

$$\begin{aligned} &e_1(\sigma) - 2\kappa_1\sigma = 0,\ e_1(\kappa_1) + \sigma^2 - \kappa_1^2 = c,\ e_3(\sigma) = \kappa_1 f_1 = 0,\ K^N - 3\sigma^2 = c, \\ &e_3(\kappa_1) = e_2(\sigma) = e_2(\kappa_1) = 0,\ \sigma^2 - \kappa_1 f_2 = c,\ K^N = e_1(f_2) - e_2(f_1) - f_1^2 - f_2^2. \end{aligned} \tag{44}$$

Remark 7. *Note that for $\kappa_1 = \kappa_2 = 0$, π is harmonic and hence biharmonic. We now suppose that $\kappa_1 \neq 0$, $\kappa_2 = 0$ and the Gauss curvature of the base space $K^N = c$. A simple computation using (44) gives*

$$\begin{gathered} \kappa_1 \neq 0,\ \sigma = f_1 = e_2(\kappa_1) = e_3(\kappa_1) = 0, \\ \kappa_1 f_2 = -c,\ e_1(\kappa_1) = \kappa_1^2 + c,\ K^N = e_1(f_2) - f_2^2 = c. \end{gathered} \tag{45}$$

Here, a Riemannian submersion $\pi : (M^3(c), g) \to (N^2(c), h)$ may be the map as $\pi_U : (M^3(c) \supseteq U, g) \to (V \subseteq N^2(c), h)$ from a subset $M^3(c) \supseteq U$ to a subset $V \subseteq N^3(c)$. In spite of this, we denote the map $\pi_U : (M^3(c) \supseteq U, g) \to (V \subseteq N^2(c), h)$ as $\pi : (M^3(c), g) \to (N^2(c), h)$ later in the rest of this section by abuse of notations.

Theorem 3. *Let $\pi : (M^3(c), g) \to (N^2(c), h)$ be a Riemannian submersion with an adapted frame $\{e_1,\ e_2,\ e_3\}$ and the integrability data $f_1, f_2, \kappa_1,\ \sigma$ and $\kappa_2 = 0$. Then, π is proper f-biharmonic iff*

$$\begin{cases} -\Delta^M(f\kappa_1) + f\kappa_1(-c + c^2/\kappa_1^2) = 0, \\ ce_2(f) = 0, \\ -\Delta^M(\kappa_1) + \kappa_1(-c + c^2/\kappa_1^2) \neq 0. \end{cases} \tag{46}$$

Proof. Since the Gauss curvature of the base space $K^N = c$, $\kappa_2 = 0$ and the assumption that π is proper f-biharmonic, then we have $\kappa_1 \neq 0$. From these and Remark 7, we conclude that $f_2 = -c/\kappa_1$ and hence $e_2(f_2) = 0$. Substituting this and (45) into (41), we have $-\Delta^M(f\kappa_1) + f\kappa_1(-c + c^2/\kappa_1^2) = 0$ and $ce_2(f) = 0$. It is not difficult to check that the bitension field $\tau_2(\pi) = [-\Delta^M(\kappa_1) + \kappa_1(-c + c^2/\kappa_1^2)]\varepsilon_1$. Combining these, we conclude that π is proper f-biharmonic iff (46) holds, from which we obtain the theorem. □

Applying Theorem 3 with $c = 0$, we obtain the following:

Proposition 7. *For the positive function $f = \rho$, the Riemannian submersion $\pi : (\mathbb{R}^3\backslash\{0\}, \mathrm{d}\rho^2 + \mathrm{d}z^2 + \rho^2\mathrm{d}\theta^2) \to (\mathbb{R}^2\backslash\{0\}, \mathrm{d}\rho^2 + \mathrm{d}z^2)$, $\pi(\rho, z, \theta) = (\rho, z)$ is proper f-biharmonic.*

Proof. It is easy to find that the frame $\{e_1 = \frac{\partial}{\partial\rho},\ e_2 = \frac{\partial}{\partial z},\ e_3 = \frac{1}{\rho}\frac{\partial}{\partial\theta}\}$ on $(\mathbb{R}^3\backslash\{0\}, \mathrm{d}\rho^2 + \mathrm{d}z^2 + \rho^2\mathrm{d}\theta^2)$ is adapted to the Riemannian submersion π with $\mathrm{d}\pi(e_3) = 0$ and $\mathrm{d}\pi(e_i) = \varepsilon_i$, $i = 1, 2$. Here, $\{\varepsilon_1 = \frac{\partial}{\partial\rho},\ \varepsilon_2 = \frac{\partial}{\partial z}\}$ is an orthonormal frame on $(\mathbb{R}^2\backslash\{0\}, \mathrm{d}\rho^2 + \mathrm{d}z^2)$. A straightforward computation gives

$$[e_1, e_3] = -\tfrac{1}{\rho}e_3, [e_2, e_3] = 0, [e_1, e_2] = 0,\ f_1 = f_2 = \sigma = \kappa_2 = 0,\ \kappa_1 = -\tfrac{1}{\rho} \neq 0. \tag{47}$$

Using (37) and (47) with $c = 0$, (46) reduces to

$$\Delta\left(\frac{f}{\rho}\right) = 0. \tag{48}$$

Clearly, $f = \rho$ is a special solution of (48). Thus, we obtain the proposition. □

Applying Theorem 3 with $c = -1$, we have the following:

Proposition 8. *For any positive constants C_1, C_2 and the function $f = C_1e^{(1+\sqrt{3})\rho} + C_2e^{(1-\sqrt{3})\rho}$, the Riemannian submersion $\pi : (H^3, \mathrm{d}\rho^2 + e^{-2\rho}\mathrm{d}z^2 + e^{-2\rho}\mathrm{d}\theta^2) \to (H^2, \mathrm{d}\rho^2 + e^{-2\rho}\mathrm{d}z^2)$, $\pi(\rho, z, \theta) = (\rho, z)$ is proper f-biharmonic.*

Proof. One can check that the frame $\{e_1 = \frac{\partial}{\partial\rho},\ e_2 = e^\rho\frac{\partial}{\partial z},\ e_3 = e^\rho\frac{\partial}{\partial\theta}\}$ on $(H^3, \mathrm{d}\rho^2 + e^{-2\rho}\mathrm{d}z^2 + e^{-2\rho}\mathrm{d}\theta^2)$ forms an adapted frame of the Riemannian submersion π, with e_3 being vertical. A simple computation yields

$$[e_1, e_3] = e_3, [e_2, e_3] = 0, [e_1, e_2] = e_2,\ f_1 = \sigma = \kappa_2 = 0,\ \kappa_1 = f_2 = 1. \tag{49}$$

By (37) and (49), together with $c = -1$, (46) reduces to

$$-\Delta^M f + 2f = 0,\ e_2(f) = 0. \tag{50}$$

We can easily check that $f = C_1e^{(1+\sqrt{3})\rho} + C_2e^{(1-\sqrt{3})\rho}$ satisfies (50), where C_1 and C_2 are positive constants. Thus, we obtain the proposition. □

As an application of Theorem 3 with $c = 1$, we have the following:

Proposition 9. *For a positive function $f = f(\rho)$ defined on an open interval $I \subset (0, \frac{\pi}{2})$ that solves the following ODE*

$$f'' + \frac{2\cos 2\rho - 4}{\sin 2\rho}f' + \frac{1 + 2\sin^2\rho}{\sin^2\rho}f = 0, \tag{51}$$

then the Riemannian submersion $\pi : (S^3 \supset I \times S^1 \times S^1, \mathrm{d}\rho^2 + \cos^2\rho\mathrm{d}z^2 + \sin^2\rho\mathrm{d}\theta^2) \to (S^2 \supset I \times S^1, \mathrm{d}\rho^2 + \cos^2\rho\mathrm{d}z^2)$, $\pi(\rho, z, \theta) = (\rho, z)$ is proper f-biharmonic.

Proof. One can easily check that $\{e_1 = \frac{\partial}{\partial \rho},\ e_2 = \frac{1}{\cos\rho}\frac{\partial}{\partial z},\ e_3 = \frac{1}{\sin\rho}\frac{\partial}{\partial\theta}\}$ on $(I \times S^1 \times S^1, \mathrm{d}\rho^2 + \cos^2\rho \mathrm{d}z^2 + \sin^2\rho \mathrm{d}\theta^2)$ is an adapted frame of the Riemannian submersion π, with e_3 being vertical. A direct computation gives

$$\begin{aligned} &[e_1, e_3] = -\cot\rho e_3, [e_2, e_3] = 0, [e_1, e_2] = \tan\rho e_2, \\ &f_1 = \sigma = \kappa_2 = 0,\ \kappa_1 = -\cot\rho, f_2 = \tan\rho. \end{aligned} \tag{52}$$

It is easy to check that $-\Delta^M(\kappa_1) + \kappa_1(-1 + 1/\kappa_1^2) = \frac{1+\sin^2\rho\cos 2\rho}{\sin^3\rho\cos\rho} \neq 0$. Substituting (52) and $c = 1$ into (46), then we have

$$-\Delta^M(-f\cot\rho) - f\cot\rho(-1 + \tan^2\rho) = 0 \text{ and } e_2(f) = 0. \tag{53}$$

We now consider a special solution of (53) as the form $f = f(\rho)$. Substituting this, (37), and (52) into (53) and simplifying the resulting equation, we obtain (51). By the theory of ODE, we conclude that there exists a local solution to (51). Moreover, one finds that if y is a solution of (51) on $I \subset (0, \frac{\pi}{2})$, then $-y$ is also a solution of (51) on I. It follows that there is a positive function solution $f = f(\rho)$ to (51) on I. Thus, we obtain the proposition. $\square$

Author Contributions: Conceptualization, Z.-P.W.; methodology, Z.-P.W.; software, Z.-P.W. and L.-H.Q.; validation, Z.-P.W. and L.-H.Q.; formal analysis, Z.-P.W. and L.-H.Q.; supervision, Z.-P.W.; writing—original draft, Z.-P.W.; writing—review and editing, Z.-P.W. All authors have read and agreed to the published version of the manuscript.

Funding: Ze-Ping Wang was supported by the Scientific and Technological Project in Guizhou Province (Grant no. Qiankehe Platform Talents [2018]5769-04) and by the Natural Science Foundation of China (No. 11861022).

Data Availability Statement: No new data were created.

Acknowledgments: The first named author would like to thank Ye-Lin Ou for his guidance, discussions, suggestions, and stimulating questions during the preparation of this work. A part of the work was conducted when Ze-Ping Wang was a visiting scholar at Yunnan University in Fall 2023. He would like to express his gratitude to Han-Chun Yang for his invitation and to Yunnan University for the hospitality.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

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