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Multi-Pursuer and One-Evader Evasion Differential Game with Integral Constraints for an Infinite System of Binary Differential Equations

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Abstract: In the Hilbert space l_2, a differential evasion game involving multiple pursuers is considered. Integral constraints are imposed on player control functions. The pursuers are tasked with bringing the state of a system back to the origin of l_2, while the evader simultaneously tries to avoid it. It is assumed that the energy of the evader is greater than the total energy of the pursuers. In this paper, we contribute to the solution of the differential evasion game with multiple pursuers by building an exact strategy for the evader.

Keywords: differential game; control; evasion strategy; infinite system of differential equations; integral constraint

MSC: 91A23; 49N75

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1. Introduction

There are many works dedicated to differential games in finite-dimensional Euclidean spaces (see, for example, [1,2]). In recent years, multiplayer differential games have attracted increasing interest. In the multi-pursuer and multi-evader differential game studied in [3], the pursuers are faster than the evaders. It has been established that the optimal evading strategy in the game with multiple pursuers and one evader depends only on those pursuers catching the evader simultaneously. Using Apollonian circles, pursuers are classified as redundant and active. Based on this, a dynamic allocation algorithm for the pursuers is proposed to solve the problem.

A differential pursuit–evasion game of multiple pursuers and one evader is studied in [4] in the presence of dynamic environmental disturbances. Pursuers are classified as active pursuers, guards, and redundant pursuers. The conditions under which the game can be terminated have been obtained. To avoid interception with one or more pursuers, the evasion strategy was built based on Apollonius' circle in pursuit–evasion situations in [5].

In [6], the authors present differential pursuit–evasion games with an overview of recent advances in the area, highlighting important contributions and describing recent results using a classification based on the number of players. Additionally, the two cutters and the escape ship are studied, as well as the differential pursuit–evasion games of active target defense. Furthermore, the works [7–13] are dedicated to various differential pursuit and evasion games.

The evasion strategy was built for the evader to avoid many pursuers in [14,15], when the players' control functions are subject to integral constraints. The work [16] is dedicated to the game of the optimal approach of multiple pursuers to an evader subject to a fixed terminal time. Necessary and sufficient conditions for completing a differential game were obtained in [17] for a multi-pusuer and one-evader differential game on manifolds with the Euclidean metric.

An interesting differential game has been studied in [18], where the players of the game are a group of pursuers, an evader, and a group of defenders of the evader. All players have the same dynamic and inertial capabilities. The evader and the group of defenders play cooperatively against the pursuers. The necessary and sufficient conditions for the simultaneous multiple capture of the evader have been obtained. Furthermore, the article [19] is dedicated to a simultaneous multi-capture differential game. In [20], an n prisoner's dilemma game model for network interaction of multiple players is studied and a strategy involving punishment is proposed. The article [21] is dedicated to a detailed investigation of multi-player differential games.

Modeling control problems as partial differential equations (PDEs) is a common approach in the field of control theory, especially when dealing with distributed parameter systems. It is worth noting that the complexity of solving PDEs and designing controllers for distributed parameter systems can be challenging. A large literature has studied control problems for partial differential equations. A time-optimal control problem was studied for the first time in the work [22] for the parabolic-type equation. For a more in-depth understanding, we refer the readers to the book [23].

Differential games for PDEs involve dynamic interactions between multiple players in which the evolution of the system is described by the PDEs. The first works that studied differential game problems for PDEs are [24,25]. The decomposition method has been used by many researchers to study control or differential clearance problems for PDEs (see, for example, [26–33]) to obtain such problems for an infinite system of ordinary differential equations (ODEs). Despite the simplicity of the ODEs in the system, it is it difficult to investigate the control problems of such systems due to the infinite number of ODEs in the system.

The papers [26,27,29] motivated us to study differential games for the infinite system of differential equations independently of the PDEs. Several studies have been conducted on control and differential game problems described by infinite systems of differential equations (see, for example, [31,34]).

If, to model control processes, exhaustible resources such as fuel, energy, resources, etc., are bounded, then the control functions are bounded by integral constraints. Therefore, an integral constraint on the control functions is one of the most important constraints.

For an infinite system of binary differential equations, the optimal strategies for players in a differential game of a chaser and an evader were constructed in [34] when the control functions are subject to integral constraints. In the present paper, we study a differential evasion game of many pursuers and one evader with integral constraints for the same infinite system of binary differential equations in the Hilbert space l_2. We prove that, if the control resource of the evader is greater than or equal to the total control resource of the pursuers, then evasion is possible from any initial position of the infinite system. Furthermore, we build an evasion strategy for the evader.

2. Statement of the Problem

We recall that the vector space of all sequences of real numbers:

$$l_2 = \left\{ \xi = (\xi_1, \xi_2, \dots) \mid \sum_{n=1}^{\infty} \xi_n^2 < \infty \right\}$$

is a Hilbert space with the inner product and norm:

$$\langle \xi, \eta \rangle = \sum_{n=1}^{\infty} \xi_n \eta_n, \;\; ||\xi|| = \sqrt{\langle \xi, \xi \rangle}.$$

We consider a differential game described by the following infinite system of differential equations:

$$\begin{aligned} \dot{x}_{ij} &= -\alpha_i x_{ij} - \beta_i y_{ij} + u_{ij}^1 - v_i^1, \quad x_{ij}(0) = x_{ij}^0, \\ \dot{y}_{ij} &= \;\;\beta_i x_{ij} - \alpha_i y_{ij} + u_{ij}^2 - v_i^2, \quad y_{ij}(0) = y_{ij}^0, \end{aligned} \tag{1}$$

where x_{ij}, $y_{ij} \in \mathbb{R}$, $i = 1, 2, ..., m$, $j = 1, 2, ...$, are the state variables, α_i, β_i are given real numbers with $\alpha_i \geq 0$, u_{ij}^1, u_{ij}^2 are the control parameters of the i-th pursuer, $i = 1, 2, \ldots, m$, and v_i^1, v_i^2 are the control parameters of the evader:

$$x_i = (x_{i1}^0, x_{i2}^0, ...) \in l_2, \;\; y_i = (y_{i1}^0, y_{i2}^0, ...) \in l_2, \;\; i = 1, 2, \ldots, m.$$

Let $u_{ij} = (u_{ij}^1, u_{ij}^2)$, $v_j = (v_j^1, v_j^2)$, $\psi_{ij}^0 = (x_{ij}^0, y_{ij}^0)$, $\psi_{ij}(t) = (x_{ij}(t), y_{ij}(t))$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots$. Also, we let

$$v(t) = (v_1(t), v_2(t), ...), \;\; ||v(t)|| = \left(\sum_{j=1}^{\infty} \left(\left(v_j^1(t) \right)^2 + \left(v_j^2(t) \right)^2 \right) \right)^{1/2},$$

$$u_i(t) = (u_{i1}(t), u_{i2}(t), ...), \;\; ||u_i(t)|| = \left(\sum_{j=1}^{\infty} \left(\left(u_{ij}^1(t) \right)^2 + \left(u_{ij}^2(t) \right)^2 \right) \right)^{1/2},$$

$$\psi_i^0 = (\psi_{i1}^0, \psi_{i2}^0, ...), \;\; \|\psi_i^0\| = \left(\sum_{j=1}^{\infty} \left(\left(x_{ij}^0 \right)^2 + \left(y_{ij}^0 \right)^2 \right) \right)^{1/2},$$

$$\psi_i(t) = (\psi_{i1}(t), \psi_{i2}(t), ...), \;\; \|\psi_i(t)\| = \left(\sum_{j=1}^{\infty} \left(x_{ij}^2(t) + y_{ij}^2(t) \right) \right)^{1/2}.$$

We assume that $\psi_i^0 \neq 0$ for all $i = 1, 2, \ldots, m$.

Definition 1. *Vector functions $u_i(t) = (u_{i1}(t), u_{i2}(t), \ldots)$, $i \in \{1, 2, \ldots, m\}$ and $v(t) = (v_1(t), v_2(t), \ldots)$ with Borel measurable coordinates $u_{ij}(t)$ and $v_j(t)$, $j = 1, 2, \ldots$, such that*

$$\int_0^T ||u_i(t)||^2 dt \leq \rho_i^2, \quad \int_0^T ||v(t)||^2 dt \leq \sigma^2 \tag{2}$$

are called admissible controls of the i-th pursuer and evader, respectively, where ρ_i and σ are given positive numbers and T is a given positive number.

Definition 2. *We call a continuous function $V(t, u_1, \ldots, u_m)$, $V : \mathbb{R}_+ \times l_2 \times \cdots \times l_2 \to l_2$, a strategy of the evader if initial-value problems (1) have unique solutions $\psi_1(t), \psi_2(t), \ldots, \psi_m(t)$, $0 \leq t \leq T$, at $v = V(t, u_1, \ldots, u_m)$ and any admissible controls $u_1 = u_1(t), \ldots, u_m = u_m(t)$ of the pursuers, and the following integral constraint:*

$$\int_0^T ||V(t, u_1(t), \ldots, u_m(t))||^2 dt \leq \sigma^2$$

is satisfied.

Definition 3. *If there exists a strategy $V_0(t, u_1, \dots, u_m)$, $0 \le t \le T$, of the evader such that for arbitrary admissible controls $u_i(t)$, $0 \le t \le T$, of pursuers, the solutions of initial-value problems (1), $\psi_i(t) = (\psi_{i1}(t), \psi_{i2}(t), \dots)$, $i = 1, \dots, m$, are not equal to zero for any t, $0 \le t \le T$, i.e., $\psi_i(t) \ne 0$ for all $0 \le t \le T$ and $i = 1, 2, \dots, m$, then we say that evasion is possible in game (1).*

Note that, on the time interval $[0, T]$, the evader uses a strategy V_0 and the pursuers apply arbitrary admissible controls $u_i(t)$, $i = 1, 2, \dots, m$.

The problem is to construct a strategy V_0, $0 \le t \le T$, for the evader such that evasion is possible in game (1).

3. The Main Result

The following statement is the main result of the paper.

Theorem 1. *If*

$$\sum_{j=1}^{m} \rho_j^2 \le \sigma^2,$$

then, for any initial states ψ_i^0, $i = 1, 2, \dots, m$, evasion is possible in the game described by the infinite system of Equation (1).

Proof. We rewrite system (1) as follows:

$$\dot{\psi}_{ij} = B_j \psi_{ij} + u_{ij} - v_j, \quad \psi_{ij}(0) = \psi_{ij}^0, \quad i = 1, 2, ..., m, \quad j = 1, 2, ..., \tag{3}$$

where

$$B_j = \begin{bmatrix} -\alpha_j & -\beta_j \\ \beta_j & -\alpha_j \end{bmatrix}, \quad j = 1, 2, ...$$

Observe that

$$e^{B_j t} = e^{-\alpha_j t} \begin{bmatrix} \cos \beta_j t & -\sin \beta_j t \\ \sin \beta_j t & \cos \beta_j t \end{bmatrix}, \quad j = 1, 2, ...$$

Then, the solution of (1) (that is (3)) takes the form

$$\psi_{ij}(t) = e^{B_j t} \xi_{ij}(t), \quad \xi_{ij}(t) = \psi_{ij}^0 + \int_0^t e^{-B_j s} (u_{ij}(s) - v_j(s)) ds. \tag{4}$$

The solutions $\psi_i(t) = (\psi_{i1}(t), \psi_{i2}(t), \dots)$, $0 \le t \le T$, of the infinite system of differential equations (1) are considered in the space of continuous functions $h(t) = (h_1(t), h_2(t), \dots) \in l_2$ with absolutely continuous coordinates $h_i(t)$ defined on the interval $0 \le t \le T$. Applying techniques similar to [32] (see Ch. III, Sec. 1 and 2), one can establish that $\psi_i(t) = (\psi_{i1}(t), \psi_{i2}(t), \dots) \in l_2$, $0 \le t \le T$, for any fixed positive T. Therefore, the integration in (2) is over $[0, T]$.

From the fact that the matrices $e^{B_j t}$ are not singular, we conclude that $\psi_{ij}(t) = 0$ if and only if $\xi_{ij}(t) = 0$, $i = 1, 2, ..., m$, $j = 1, 2, \dots$. Hence, to prove the theorem, it is sufficient to show that, for some strategy of the evader, $\xi_i(t) = (\xi_{i1}(t), \xi_{i2}(t), \dots) \ne 0$ for all $0 \le t \le T$ and $i = 1, 2, \dots, m$.

The condition $\psi_1^0 = (\psi_{11}^0, \psi_{12}^0, \dots) \ne 0$ implies that at least one of the components $\psi_{1j}^0 \in \mathbb{R}^2$, $j = 1, 2, \dots$, of ψ_1^0 is not equal to 0, meaning that there exists a positive integer n_1 such that $\psi_{1n_1}^0 \ne 0$. Similarly, it follows from the condition $\psi_2^0 = (\psi_{21}^0, \psi_{22}^0, \dots) \ne 0$ that $\psi_{2n_2}^0 \ne 0$ for some positive integer n_2, and so on. Finally, $\psi_m^0 = (\psi_{m1}^0, \psi_{m2}^0, \dots) \ne 0$ implies that $\psi_{mn_m}^0 \ne 0$ for some positive integer n_m.

We denote $n = \max_{i=1,\ldots,m} n_i$. Then, obviously, $\Psi_i^0 = (\psi_{i1}^0, \psi_{i2}^0, \ldots, \psi_{in}^0) \neq 0$ for all $i = 1, 2, \ldots, m$. Note that the vector Ψ_i^0 consists of the first n components of ψ_i^0. We can assume, by increasing n if necessary, that $2n \geq m$. In this way, we obtain

$$\Psi_i^0 = (\psi_{i1}^0, \psi_{i2}^0, \ldots, \psi_{in}^0) \neq 0, \quad \Psi_i^0 \in \mathbb{R}^{2n}, \ i = 1, 2, \ldots, m. \tag{5}$$

Let

$$\Xi_i(t) = (\xi_{i1}(t), \xi_{i2}(t), ..., \xi_{in}(t)), \ \ i = 1, ..., m,$$

where

$$\xi_{ij}(t) = \psi_{ij}^0 + \int_0^t e^{-B_j s}(u_{ij}(s) - v_j(s))ds, \ \ i = 1, 2, ..., m, \ \ j = 1, 2, ..., n.$$

Clearly, $\Xi_i(t)$ consists of the first n components of $\xi_i(t)$.

To prove the theorem, it is sufficient to establish that $\Xi_i(t) \neq 0$, $i = 1, ..., m$, for a strategy of the evader since these inequalities imply that

$$\xi_i(t) = (\xi_{i1}(t), \xi_{i2}(t), \ldots) \neq 0, \ \ i = 1, 2, ..., m, \ \ 0 \leq t \leq T,$$

and so, $\psi_i(t) \neq 0$, $i = 1, 2, .., m$. In this way, we have reduced the game in the Hilbert space l_2 to a game in the finite-dimensional Euclidean space $\mathbb{R}^{2n}$.

Since the number m of points $\Psi_i^0 \in \mathbb{R}^{2n}$ does not exceed the dimension $2n$ of the space $\mathbb{R}^{2n}$, that is $m \leq 2n$, we conclude that there is a unit vector:

$$p = (p_1, p_2, ..., p_n) \in \mathbb{R}^{2n}, \ \ |p| = 1, \ \ p_j \in \mathbb{R}^2,$$

such that the inner product $\langle \Psi_i^0, p \rangle \geq 0$ for all $i = 1, ..., m$, that is

$$\sum_{j=1}^{n} \langle \psi_{ij}^0, p_j \rangle \geq 0, \quad i = 1, 2, ..., m, \tag{6}$$

where

$$|p| = \left(\sum_{j=1}^{n} |p_j|^2 \right)^{1/2}, \ \ |p_j| = \left(p_{j1}^2 + p_{j2}^2 \right)^{1/2}.$$

As the vector $p \in \mathbb{R}^{2n}$, we can choose an orthonormal vector to the hyperplane passing through the points Ψ_i^0, $i = 1, 2, \ldots, m$.

We let the evader apply the following strategy:

$$\begin{aligned} &v_j(t) = -\frac{e^{-B_j^* t} p_j}{\sqrt{\sum_{k=1}^{n} |e^{-B_k^* t} p_k|^2}} \sqrt{\sum_{k=1}^{m} ||u_k(t)||^2}, \quad j = 1, ..., n \\ &v_j(t) = 0, \quad j = n+1, n+2, \ldots \end{aligned}, \qquad t \geq 0, \tag{7}$$

where B_j^* denotes the transpose of the matrix B_j.

The admissibility of the strategy (7) follows from the following relations:

$$\begin{aligned} \int_0^T \|v(t)\|^2 dt &= \int_0^T \frac{\|e^{-B_j^* t} p\|^2}{\|e^{-B_j^* t} p\|^2} \sum_{k=1}^{m} \|u_k(t)\|^2 dt \\ &= \sum_{k=1}^{m} \int_0^T \|u_k(t)\|^2 dt \leq \sum_{k=1}^{m} \rho_k^2 \leq \sigma^2. \end{aligned}$$

We prove that, if the evader applies strategy (7), then $\Xi_i(t) \neq 0$ for all $t \geq 0$ and $i = 1, 2, \ldots, m$. Assume the contrary: Let

$$\Xi_q(\tau) = 0 \tag{8}$$

for some $\tau \geq 0$ and $q \in \{1, 2, ..., m\}$. Then, by (6), we have for $i = q$ that

$$\sum_{j=1}^{n} \langle \psi_{qj}^0, p_j \rangle \geq 0, \tag{9}$$

and hence,

$$\begin{aligned}
\langle \Xi_q(\tau), p \rangle &= \sum_{j=1}^{n} \langle \xi_{qj}(\tau), p_j \rangle \\
&= \sum_{j=1}^{n} \langle \psi_{qj}^0, p_j \rangle + \sum_{j=1}^{n} \int_0^\tau \langle e^{-B_j s}(u_{qj}(s) - v_j(s)), p_j \rangle ds \\
&\geq \int_0^\tau \sum_{j=1}^{n} \langle e^{-B_j s} u_{qj}(s), p_j \rangle ds - \int_0^\tau \sum_{j=1}^{n} \langle e^{-B_j s} v_j(s), p_j \rangle ds
\end{aligned} \tag{10}$$

By (7), the second integral in (10) can be transformed as follows:

$$\begin{aligned}
\int_0^\tau \sum_{j=1}^{n} \langle e^{-B_j s} v_j(s), p_j \rangle ds &= \int_0^\tau \sum_{j=1}^{n} \langle v_j(s), e^{-B_j^* s} p_j \rangle ds \\
&= \int_0^\tau \frac{\sum_{j=1}^{n} |e^{-B_j^* s} p_j|^2}{\sqrt{\sum_{k=1}^{n} |e^{-B_k^* s} p_k|^2}} \sqrt{\sum_{j=1}^{n} |u_{qj}(s)|^2} ds \\
&= \int_0^\tau \sqrt{\sum_{j=1}^{n} |e^{-B_j^* s} p_j|^2} \sqrt{\sum_{j=1}^{n} |u_{qj}(s)|^2} ds.
\end{aligned} \tag{11}$$

By using the Cauchy inequality $\langle a, b \rangle \leq |a||b|$, for the integrand of the first integral in (10), we obtain that

$$\begin{aligned}
\int_0^\tau \sum_{j=1}^{n} \langle e^{-B_j s} u_{qj}(s), p_j \rangle ds &= \int_0^\tau \sum_{j=1}^{n} \langle u_{qj}(s), e^{-B_j^* s} p_j \rangle ds \\
&\geq -\int_0^\tau \sqrt{\sum_{j=1}^{n} |u_{qj}(s)|^2} \sqrt{\sum_{j=1}^{n} |e^{-B_j^* s} p_j|^2} ds \\
&\geq -\int_0^\tau ||u_q(s)|| \sqrt{\sum_{j=1}^{n} |e^{-B_j^* s} p_j|^2} ds,
\end{aligned} \tag{12}$$

where the equality sign in the first inequality holds when

$$u_{qj}(s) = -\lambda \, |e^{-B_j^* s} p_j|, \;\; j = 1, 2, ..., n, \quad 0 \leq s \leq \tau,$$

for some real number $\lambda > 0$, and the equality sign in the second inequality holds when

$$||u_q(s)|| = \sqrt{\sum_{j=1}^{n} |u_{qj}(s)|^2}, \quad u_{qj}(s) = 0, \;\; j = n+1, n+2, ..., \quad 0 \le s \le \tau.$$

Thus, the equality signs in the inequalities in (12) hold when

$$\begin{array}{ll} u_{qj}(s) & = -\lambda\, |e^{-B_j^* s} p_j|, \;\; j = 1, 2, ..., n \\ u_{qj}(s) & = 0, \;\; j = n+1, n+2, ... \end{array}, \quad 0 \le s \le \tau, \tag{13}$$

for some real number $\lambda > 0$. We obtain from (13) that

$$\sum_{j=1}^{n} |u_{qj}(s)|^2 = \lambda^2 \sum_{j=1}^{n} |e^{-B_j^* s} p_j|^2, \quad 0 \le s \le \tau,$$

and so,

$$\lambda^2 = \frac{\sum\limits_{j=1}^{n} |u_{qj}(s)|^2}{\sum\limits_{j=1}^{n} |e^{-B_j^* s} p_j|^2} = \frac{||u_q(s)||^2}{\sum\limits_{j=1}^{n} |e^{-B_j^* s} p_j|^2}. \tag{14}$$

Hence, the equality sign in (12) holds if λ in (13) is defined as follows:

$$\lambda = \frac{||u_q(s)||}{\sqrt{\sum\limits_{j=1}^{n} |e^{-B_j^* s} p_j|^2}}. \tag{15}$$

Thus, the equality sign in (12) holds if

$$\begin{array}{ll} u_{qj}(s) & = -\dfrac{e^{-B_j^* s} p_j}{\sqrt{\sum\limits_{k=1}^{n} |e^{-B_k^* s} p_k|^2}}\, ||u_q(s)||, \quad j = 1, 2, ..., n \\ u_{qj}(s) & = 0, \quad j = n+1, n+2, ... \end{array}, \quad 0 \le s \le \tau. \tag{16}$$

Combining (7), (10), (11), and (12), we obtain

$$\begin{aligned} \langle \Xi_q(\tau), e \rangle \;\; & \ge \;\; -\int_0^{\tau} ||u_q(s)|| \sqrt{\sum_{j=1}^{n} |e^{-B_j^* s} p_j|^2} ds \\ & \quad + \int_0^{\tau} \sqrt{\sum_{k=1}^{m} ||u_k(s)||^2} \cdot \sqrt{\sum_{j=1}^{n} |e^{-B_j^* s} p_j|^2} ds \\ & = \int_0^{\tau} \left(\sqrt{\sum_{k=1}^{m} ||u_k(s)||^2} - ||u_q(s)|| \right) \sqrt{\sum_{j=1}^{n} |e^{-B_j^* s} p_j|^2} ds. \end{aligned} \tag{17}$$

Clearly,

$$\sqrt{\sum_{k=1}^{m} ||u_k(s)||^2} \ge ||u_q(s)||, \quad 0 \le s \le \tau, \quad (1 \le q \le m) \tag{18}$$

then (17) implies that

$$\langle \Xi_q(\tau), e \rangle \ge 0. \tag{19}$$

We now find the conditions, under which the equality sign occurs in (19). By (8), we obtain $\langle \Xi_p(\tau), p\rangle = 0$. Hence, the equality sign in (10) holds. This implies that $\sum_{j=1}^{n}(\psi_{qj}^0, p_j) = 0$ in (12), and that u_q is defined by (16), and that $\sqrt{\sum_{k=1}^{m} ||u_k(s)||^2} - ||u_q(s)|| = 0$ in (17), and hence, $u_k(s) = 0, 0 \leq s \leq \tau, k \in \{1, 2, \ldots, m\} \setminus \{q\}$. Then,

$$\sqrt{\sum_{k=1}^{m} ||u_k(s)||^2} = ||u_q(s)||,$$

and therefore, (7) takes the form

$$\left.\begin{aligned} v_j(t) &= -\frac{e^{-B_j^* t} p_j}{\sqrt{\sum_{k=1}^{n} |e^{-B_k^* t} p_k|^2}} \cdot ||u_q(t)||, \quad j = 1, \ldots, n \\ v_j(t) &= 0, \quad j = n+1, n+2, \ldots \end{aligned}\right., \quad 0 \leq t \leq \tau. \tag{20}$$

Comparing (16) and (20), we conclude that

$$v(t) = u_q(t), \quad 0 \leq t \leq \tau. \tag{21}$$

Substituting (21) into (4) for $i = q$, we obtain

$$\xi_{qj}(\tau) = \psi_{qj}^0 + \int_0^\tau e^{-B_j s}(v_j(s) - v_j(s))ds = \psi_{qj}^0, \quad j = 1, 2, \ldots, n.$$

By the assumption $\Xi_q(\tau) = 0$ (see Equation (8)), consequently, $\xi_{qj}(\tau) = \psi_{qj}^0 = 0$, $j = 1, 2, \ldots, n$, and so,

$$\Psi_q^0 = (\psi_{q1}^0, \psi_{q2}^0, \ldots, \psi_{qn}^0) = 0,$$

which contradicts condition (5).

Hence, $\Xi_i(t) \neq 0$, and so, $\xi_i(t) \neq 0$ for all $t \geq 0$ and $i = 1, 2, \ldots, m$. The proof of the theorem is complete. □

4. Conclusions

We have studied an evasion game problem for an infinite system of two-block differential equations in the Hilbert space l_2. In the work [34], the game was studied in the case of one pursuer with integral constraints on the players' controls. In the present paper, we studied a differential many pursuers evasion game with integral constraints on the players' controls.

Our contributions are as follows: (i) if the total control resource of the pursuers is less than or equal to the control resource of the evader, then evasion is possible; (ii) the reduction of the game in the Hilbert space l_2 to an equivalent differential game in the finite-dimensional space $\mathbb{R}^{2n}$; (iii) the construction of an explicit evasion strategy that guarantees evasion.

It should be noted that the main difficulty in solving differential game problems with integral constraints on player control functions arises from optimizing the expenditure of player control resources. Therefore, building an explicit escape strategy that guarantees escape is the most challenging part of solving evasion games with integral constraints.

For future work, we recommend studying a differential escape game with countably many pursuers.

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