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Global Weak Solution for Phase Transition Equations with Polarization

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Abstract: This paper deals with a phase transition model with polarization which describes the thermodynamic, electromagnetic, and polarization properties of ferromagnetic–ferroelectric materials. The existence of the global weak solution for the phase transition equations with polarization is rigorously established through the viscosity vanishing argument.

Keywords: phase transition equations; polarization; viscosity vanishing argument; global weak solution

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1. Introduction

It is well-known that the magnetism of matter can be divided into diamagnetism, paramagnetism, ferromagnetism, antiferromagnetism, and ferrimagnetism. Among them, ferromagnetism can be found in metals like iron, cobalt, and nickel, as well as in numerous alloys comprising these elements. There exists spontaneous magnetization in ferromagnetic materials [1]. The theory of spontaneous magnetization can be used to explain many ferromagnetic properties, such as the effect of temperature on ferromagnetism. When the temperature is below Curie temperature θ_c, the spin magnetic moments remain aligned, even after the external magnetic field is removed. When the temperature rises above Curie temperature θ_c, the spontaneous magnetization disappears, and the material reverts to the paramagnetic phase [2].

There has recently been a lot of progress in the study and development of the theory of the paramagnetic–ferromagnetic transition [3–8]. In [5,6], Berti proposes a model for the dynamics of a magnetization vector in a ferromagnetic body. The model provides a detailed depiction of the three-dimensional evolution of thermodynamic and electromagnetic properties inherent in ferromagnetic materials, which can account for temperature-induced transitions from the paramagnetic to the ferromagnetic regime. The model is applicable to a wide range of temperatures [9], while the well-known Landau–Lifschitz equation can only describe magnetization dynamics at low temperatures [10–13]. The evolution of the ferromagnetic material is described by the phase transition equations referenced in [6], which are formulated as follows:

$$\gamma\partial_t\mathbf{M} = \nu_1\Delta\mathbf{M} - \theta_c(|\mathbf{M}|^2 - 1)\mathbf{M} - \theta\mathbf{M} + \mathbf{H}, \tag{1}$$

$$c_1\partial_t(\ln\theta) + c_2\partial_t\theta = k_0\Delta(\ln\theta) + \mathbf{M}\cdot\partial_t\mathbf{M} + k\Delta\theta + \hat{r}, \tag{2}$$

where $\mathbf{M}$ denotes the magnetization vector, $\mathbf{H}$ denotes the magnetic field, θ is the absolute temperature, $\gamma, \nu_1, c_1, c_2, k_0, k$ are strictly positive constants, and θ_c is the Curie temperature. $\hat{r}$ is a known function of x, t. The well-posedness and long-time behavior results are proved first in [14] via energy methods for the above model with $c_1 = k_0 = 0$ without the magnetic field $\mathbf{H}$. Existence and uniqueness, continuous dependence on the data, and regularity results are proved for a more general model without the magnetic field $\mathbf{H}$ in [15,16]. The existence and uniqueness of the global weak solution for the above equations with Maxwell equations are proved in [6], some limit problems for this model are obtained in [17], and the fractional version of the model is obtained in [18].

In this paper, we study the phase transition model incorporating polarization which describes the thermodynamic, electromagnetic, and polarization properties of materials but which has not been considered in the above references. Due to the laws $k(\theta) = k_1\theta$ and $c(\theta) = \frac{c_2}{2}\theta^2$ in [18], both heat conductivity and specific heat depend on the absolute temperature; thus, we assume that $c_1 = k_0 = 0$. A great variety of assumptions about heat conductivity and specific heat can be found in many references, for instance, [19].

The phase transition equations with polarization considered in this paper can be written as follows:

$$\gamma\partial_t\mathbf{M} = \nu_1\Delta\mathbf{M} - \theta_c(|\mathbf{M}|^2 - 1)\mathbf{M} - \theta\mathbf{M} + \mathbf{H}, \tag{3}$$

$$c\partial_t\theta = \mathbf{M}\cdot\partial_t\mathbf{M} + k\Delta\theta + \hat{r}, \tag{4}$$

$$\nabla\times\mathbf{H} = \partial_t(\mathbf{E}+\mathbf{P}) + \sigma\mathbf{E}, \tag{5}$$

$$\nabla\times\mathbf{E} = -\partial_t\mathbf{H} - \beta\partial_t\mathbf{M}, \tag{6}$$

$$\partial_t^2\mathbf{P} + \lambda^2\mathrm{curl}^2\mathbf{P} + \mu\partial_t\mathbf{P} = \nu(\mathbf{E} - \hat{\mathbf{E}}(\mathbf{P})), \tag{7}$$

where $\lambda > 0$ represents the velocity of light within the internal field and $\sigma \geq 0$ denotes the constant conductivity, while the constant β can be interpreted as the magnetic permeability of free space. μ, ν are positive constants which can be found in [20]; $\mathbf{H}(x,t) = (H_1(x,t), H_2(x,t), H_3(x,t))$ denotes the magnetic field; $\mathbf{E} = (E_1(x,t), E_2(x,t), E_3(x,t))$ denotes the electric field; $\mathbf{P}(x,t) = (P_1(x,t), P_2(x,t), P_3(x,t))$ represents the electric polarization; and $\mathrm{curl}^2\mathbf{P} = \mathrm{curl}(\mathrm{curl}\mathbf{P}) = \nabla\times\nabla\times\mathbf{P}$, $\hat{\mathbf{E}}(\mathbf{P}) = 2\mathbf{P}\Phi'(|\mathbf{P}|^2)$ is the equilibrium electric field.

The periodic conditions are

$$\begin{gathered}\mathbf{M}(x+2De_i,t) = \mathbf{M}(x,t),\ \theta(x+2De_i,t) = \theta(x,t),\\ \mathbf{H}(x+2De_i,t) = \mathbf{H}(x,t),\ \mathbf{E}(x+2De_i,t) = \mathbf{E}(x,t),\\ \mathbf{P}(x+2De_i,t) = \mathbf{P}(x,t),\ \partial_t\mathbf{P}(x+2De_i,t) = \partial_t\mathbf{P}(x,t),\\ x\in\Omega\in\mathbb{R}^3,\ t\geq 0,\ i=1,2,3\end{gathered} \tag{8}$$

and the initial values are

$$\begin{gathered}\mathbf{M}(x,0) = \mathbf{M}_0(x),\ \theta(x,0) = \theta_0(x),\ \mathbf{H}(x,0) = \mathbf{H}_0(x),\\ \mathbf{E}(x,0) = \mathbf{E}_0(x),\ \mathbf{P}(x,0) = \mathbf{P}_0(x),\ \partial_t\mathbf{P}(x,0) = \partial_t\mathbf{P}_0(x),\ \ x\in\Omega\subset\mathbb{R}^3,\end{gathered} \tag{9}$$

where $D > 0$ is a constant, (e_1, e_2, e_3) denotes the unit orthogonal basis of $\mathbb{R}^3$, and

$$\Omega = \{x = (x_1, x_2, x_3) |\ |x_i| \leq D; (i = 1,2,3)\}.$$

Set $Q_T = \{(x,t)|x\in\Omega, 0 < t \leq T\}$.

When an electric field is applied to a medium composed of a large number of atoms or molecules, such as a dielectric medium, under the action of the external electric field, the electric charges appear on the dielectric's surface as well as in its internal nonuniform regions. This phenomenon is referred to as polarization. The polarization of the dielectric causes the positive and negative charges of the molecules in the dielectric to shift or change the orientation, and there are many electric dipoles arranged in the direction of the external electric field in the dielectric, which changes the original electric field distribution of the whole dielectric. Thus, the electric polarization $\mathbf{P}$ (the dipole polarized per unit volume) not only determines the volume charge density but also dictates the charge density on the surface of the polarized dielectric; one interested in polarization can refer to the literature [21–23].

All ferroelectric materials produce spontaneous polarization within a temperature range. When a ferroelectric material is combined with a ferromagnetic material, a new material called a multiferroic material is obtained. These multiferroic materials have ferroelectric and ferromagnetic phases simultaneously inside and show both ferroelectric and ferromagnetic properties outside. Moreover, the materials also show the magnetoelectric effect, which is the induction of a magnetic field by an electric field [24,25]. This magnetic field can also induce electric polarization. It is quite important to analyze thermodynamic, electromagnetic, and polarization properties since the magnetoelectric conversion function makes the materials widely used in microsensors [26], microelectromechanical systems MEMS [27], high-density information storage [28], transducers [29], and other fields [30].

In fact, the coupling of the phase transition equations with polarization $\mathbf{P}$ can be obtained from the full Maxwell system

$$\partial_t \mathbf{B} = -\text{curl } \mathbf{E}, \ \ \partial_t \mathbf{D} + \sigma \mathbf{E} = \text{curl } \mathbf{H}. \tag{10}$$

In the above system, the definitions of electric $\mathbf{D}$ and magnetic displacements $\mathbf{B}$ are given by

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \ \ \mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}), \tag{11}$$

where ε_0 represents the permittivity of free space, while μ_0 denotes the magnetic permeability of free space. From $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 \mathbf{E} + \chi_e \varepsilon_0 \mathbf{E} = \varepsilon_r \varepsilon_0 \mathbf{E} = \varepsilon \mathbf{E}$, we can also derive (4) by the similar method in [6]. If we regard polarization $\mathbf{P}$ as an internal field, then we can derive the electric polarization Equation (7), more details can be found in [20]. Substituting (11) into (10), we obtain a couple system of $\mathbf{M}, \theta, \mathbf{E}, \mathbf{H}$, and $\mathbf{P}$, that is, (3)–(7).

The earliest research of multiferroic materials can be traced back to the 1950s. In recent years, with the development of material synthesis and technology, these materials are applied to more areas, such as smart sensors, electron spin devices, and so on. The study of partial differential equations arising in these materials has very important practical significance. Greenberg considers the existence and asymptotic behavior of multiple solutions for the so-called nonlinear Maxwell systems (5)–(7) for a simple case in [20], which models the dynamics of electric field $\mathbf{E}$, magnetic field $\mathbf{H}$, and polarization $\mathbf{P}$ (without $\mathbf{M}, \theta$) in ferroelectric materials. For a general case, Habib and Kamel establish the global existence, uniqueness, and regularity of weak solutions in [31]. Jochmann considers the asymptotic behavior of the solution for Maxwell equations with nonlinear polarization and field-dependent currents in [32]. Ding and Guo obtain the global existence of a periodic weak solution for the Landau–Lifshitz–Maxwell system with electric polarization $\mathbf{P}$ in [33].

To the best of our knowledge, our work presented in this paper seems to be the first rigorous treatment of a mathematical model for the dynamics of magnetization $\mathbf{M}$, absolute temperature θ, magnetic field $\mathbf{H}$, electric field $\mathbf{E}$, and electric polarization $\mathbf{P}$ arising in ferromagnetic–ferroelectric materials. We establish the existence of the global weak solution for the phase transition equations with polarization in (3)–(7). The results of this paper can not only provide a theoretical basis for physicists but also develop the theoretical study of strongly coupled partial differential equations.

In the process of obtaining the weak solution of the system, the main difficulty comes from the fact that the system (3)–(7) is strongly coupled. It is difficult to obtain a weak solution through the semigroup method as attempted by Habib and Kamel in [31], so we employ the Galerkin method. Noticing that Equation (7) lacks compactness, we employ the viscosity vanishing method to obtain the weak solution for the viscosity problem to overcome this difficulty. Therefore, we replace (7) with the following viscosity approximation

$$\partial_t^2 \mathbf{P} + \lambda^2 \text{curl}^2 \mathbf{P} + \mu \partial_t \mathbf{P} - \varepsilon \Delta \mathbf{P} = \nu(\mathbf{E} - 2\mathbf{P}\Phi'(|\mathbf{P}|^2)). \tag{12}$$

Firstly, we consider the viscosity system (3)–(6), (12) with initial value conditions (8) and (9) by the Galerkin method. Secondly, we consider a more regular class of weak solutions because of the lack of compactness in the limit procedure and obtain the uniform estimation in ε of solutions to the viscosity problem. Finally, we let ε tend to 0, and then we can obtain the desired weak solution to the original problem (3)–(9).

The rest of this paper is organized as follows. The subsequent section introduces the preliminaries requisite for the forthcoming proof procedure, along with the definition of a weak solution for the system and the main result, Theorem 1, whose proof is deferred to Section 4. In Section 3, the Galerkin method is employed to establish the existence of the weak solution for the viscosity problem (3)–(6), (8), (9), and (12). In Section 4, we derive the uniform estimates in ε of the approximate solutions for the viscosity problem. Then, taking the limit $\varepsilon \to 0$, we establish the existence of the global weak solution for the problem (3)–(9). In Section 5, we provide a concise recapitulation of the foremost challenges encountered, the methodologies employed, novel contributions made, and the main result associated with the phase transition model with polarization.

2. Preliminaries and Main Result

In this section, we first introduce some necessary assumptions and the definition of the weak solution to phase transition equations with polarization. Then we present the main theorem of our model.

Suppose that $\Phi : \mathbb{R}^+ \to \mathbb{R}$ is a C^2 convex function ensuring that

$$|\Phi'(r)| \le C_0, \ |r\Phi''(r)| \le C_1, \forall r \ge 0, \tag{13}$$

and the function $\Phi(r^2)$ attains its unique minimum at some point r_0^2. These assumptions ensure that $\forall r \ge 0$, $|r\Phi''(r^2)| \le C_2$, where we denote $C_2 = C_0 + 2C_1$. Thus, we derive

$$|x\Phi'(|x|^2) - y\Phi'(|y|^2)| \le C_2|x-y|, \ \forall x, y \in \mathbb{R}^3. \tag{14}$$

Much more about the equilibrium relation may be found in [22].

For simplicity, we introduce notation as follows:

$$\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_p, \ \ p \ge 2, \ \ f_t = \partial_t f = \frac{\partial f}{\partial t}, \ f \text{ is a function of } (x,t).$$

Let us now define the weak solution to the problem (3)–(9).

Definition 1. *A 2D-periodic vector function* $(\mathbf{M}(x,t), \theta(x,t), \mathbf{E}(x,t), \mathbf{H}(x,t), \mathbf{P}(x,t)) \in (L^\infty(0,T;H^1(\Omega)),\ L^\infty(0,T;L^2(\Omega)), L^\infty(0,T;L^2(\Omega)), L^\infty(0,T;L^2(\Omega)), W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)))$ *is called a weak solution to (3)–(9) if for any test functions* $\phi(x,t), \chi(x,t) \in C^1(Q_T)$ *with* $\phi(x,t)|_{t=T} = \chi(x,t)|_{t=T} = 0$, *the following equations hold:*

$$\begin{aligned} &\gamma \iint_{Q_T} \mathbf{M} \cdot \phi_t dxdt - \nu_1 \iint_{Q_T} \nabla \mathbf{M} \cdot \nabla \phi dxdt - \iint_{Q_T} \theta_c(|\mathbf{M}|^2 - 1)\mathbf{M} \cdot \phi dxdt \\ &\quad - \iint_{Q_T} \theta \mathbf{M} \cdot \phi dxdt + \iint_{Q_T} \mathbf{H} \cdot \phi dxdt + \gamma \int_\Omega \mathbf{M}_0 \cdot \phi(x,0) dx = 0, \end{aligned} \tag{15}$$

$$\begin{aligned} &c \iint_{Q_T} \theta \cdot \chi_t dxdt + \iint_{Q_T} \mathbf{M} \cdot \mathbf{M}_t \cdot \chi dxdt - k \iint_{Q_T} \nabla \theta \cdot \nabla \chi dxdt \\ &\quad + \iint_{Q_T} \hat{r} \cdot \chi dxdt + c \int_\Omega \theta_0 \cdot \chi(x,0) dx = 0, \end{aligned} \tag{16}$$

$$\begin{aligned} &\iint_{Q_T} (\mathbf{E} + \mathbf{P}) \cdot \phi_t(x,t) e^{\sigma t} dxdt + \sigma \iint_{Q_T} e^{\sigma t} \mathbf{P} \cdot \phi dxdt \\ &\quad + \iint_{Q_T} e^{\sigma t} \nabla \times \phi \cdot \mathbf{H} dxdt + \int_\Omega (\mathbf{E}_0 + \mathbf{P}_0) \cdot \phi(x,0) dx = 0, \end{aligned} \tag{17}$$

$$\iint_{Q_T} (\mathbf{H} + \beta \mathbf{M}) \cdot \phi_t dxdt - \iint_{Q_T} (\nabla \times \phi) \cdot \mathbf{E} dxdt + \int_\Omega (\mathbf{H}_0 + \beta \mathbf{M}_0) \cdot \phi(x,0) dx = 0, \tag{18}$$

$$\begin{aligned} &\iint_{Q_T} \mathbf{P}_t \cdot \phi_t dxdt - \lambda^2 \iint_{Q_T} \operatorname{curl} \mathbf{P} \cdot \operatorname{curl} \phi dxdt - \mu \iint_{Q_T} \mathbf{P}_t \cdot \phi dxdt \\ &\quad + \nu \iint_{Q_T} \mathbf{E} \cdot \phi dxdt - 2\nu \iint_{Q_T} \Phi'(|\mathbf{P}|^2)\mathbf{P} \cdot \phi dxdt + \int_\Omega \partial_t \mathbf{P}_0 \cdot \phi(x,0) dx = 0. \end{aligned} \tag{19}$$

The main result is as follows:

Theorem 1. *Assume that initial value data* $(\mathbf{M}_0(x), \theta_0(x), \mathbf{E}_0(x), \mathbf{H}_0(x), \mathbf{P}_0(x), \partial_t \mathbf{P}_0(x)) \in (H^1(\Omega), L^2(\Omega), L^2(\Omega), L^2(\Omega), H^1(\Omega), L^2(\Omega)), \hat{r}(x,t) \in L^2(0,T; H^1(\Omega))$ *and* $(\mathrm{div}(\mathbf{H}_0), \mathrm{div}(\mathbf{E}_0), \mathrm{div}(\mathbf{P}_0), \nabla \mathrm{div}(\mathbf{P}_0), \mathrm{div}(\partial_t \mathbf{P}_0)) \in L^2(\Omega)$. *The constants* $\gamma, \nu_1, c, k, \sigma, \beta, \lambda, \mu, \nu$ *are positive. Then the problem* (3)–(9) *has at least one global weak solution* $\big(\mathbf{M}(x,t), \theta(x,t), \mathbf{E}(x,t), \mathbf{H}(x,t), \mathbf{P}(x,t)\big)$ *satisfying*

$$\begin{aligned}
&\mathbf{M}(x,t) \in L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega)) \cap C^{(0,\frac{1}{2})}(0,T; L^2(\Omega)), \\
&\theta(x,t) \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)) \cap C^{(0,\frac{1}{2})}(0,T; H^{-1}(\Omega)), \\
&\mathbf{E}(x,t) \in L^\infty(0,T; L^2(\Omega)) \cap C^{(0,\frac{1}{2})}(0,T; H^{-1}(\Omega)), \\
&\mathbf{H}(x,t) \in L^\infty(0,T; L^2(\Omega)) \cap C^{(0,\frac{1}{2})}(0,T; H^{-1}(\Omega)), \\
&\mathbf{P}(x,t) \in L^\infty(0,T; H^1(\Omega)) \cap C^{(0,\frac{1}{2})}(0,T; L^2(\Omega)), \\
&\partial_t \mathbf{P}(x,t) \in L^\infty(0,T; L^2(\Omega)) \cap C^{(0,\frac{1}{2})}(0,T; H^{-1}(\Omega)).
\end{aligned} \tag{20}$$

We give the proof of Theorem 1 in Section 4, after deriving the weak solution to the viscosity system (3)–(6), (12) and obtaining the uniform estimates in the following sections.

3. Solutions to the Viscosity Problem

In this section, we aim to demonstrate the existence of the global weak solution for the viscosity problem (3)–(6), (12) with (8) and (9). Firstly, we construct the Galerkin approximate solutions of the viscosity problem and establish a priori estimates of the approximate solutions; then, we give the proof of the existence of the weak solutions for the viscosity problem (3)–(6), (12) with (8) and (9).

Define $\omega_n(x), n = 1, 2, \cdots$ as the unit eigenfunctions that satisfy the equations

$$\Delta \omega_n + \lambda_n \omega_n = 0, \;\; \omega_n(x - De_i) = \omega_n(x + De_i), \; i = 1, 2, 3,$$

where λ_n, $n = 1, 2, \cdots$ are the distinct eigenvalues associated with each eigenfunction. Similarly, we consider $\vartheta_n(x), n = 1, 2, \cdots$ the eigenfunctions for the eigenvalue problem $\Delta \vartheta_n + \kappa_n \vartheta_n = 0, n = 1, 2, \cdots$ under periodic boundary conditions with κ_n, $n = 1, 2, \cdots$ being the corresponding eigenvalues.

Denote the approximate solution of the problem (3)–(6), (12) with (8) and (9) by $\mathbf{M}_N^\varepsilon(x,t)$, $\theta_N^\varepsilon(x,t)$, $\mathbf{E}_N^\varepsilon(x,t)$, $\mathbf{H}_N^\varepsilon(x,t)$, $\mathbf{P}_N^\varepsilon(x,t)$ in the following form:

$$\begin{aligned}
&\mathbf{M}_N^\varepsilon(x,t) = \sum_{s=1}^N \alpha_{sN}(t) \omega_s(x), \;\; \theta_N^\varepsilon(x,t) = \sum_{s=1}^N \beta_{sN}(t) \vartheta_s(x), \;\; \mathbf{E}_N^\varepsilon(x,t) = \sum_{s=1}^N \gamma_{sN}(t) \omega_s(x), \\
&\mathbf{H}_N^\varepsilon(x,t) = \sum_{s=1}^N \zeta_{sN}(t) \omega_s(x), \;\; \mathbf{P}_N^\varepsilon(x,t) = \sum_{s=1}^N \xi_{sN}(t) \omega_s(x),
\end{aligned}$$

where $\alpha_{sN}(t)$, $\beta_{sN}(t), \gamma_{sN}(t), \zeta_{sN}(t), \xi_{sN}(t)$ $(s = 1, 2, \cdots, N,\ N = 1, 2, \cdots)$ satisfy the following system of ordinary differential equations:

$$\begin{aligned}
\int_\Omega \gamma \mathbf{M}_{Nt}^\varepsilon \omega_s(x) dx = &- \int_\Omega \nu_1 \nabla \mathbf{M}_N^\varepsilon \nabla \omega_s(x) dx - \int_\Omega \theta_c (|\mathbf{M}_N^\varepsilon|^2 - 1) \mathbf{M}_N^\varepsilon \omega_s(x) dx \\
&- \int_\Omega \theta_N^\varepsilon \mathbf{M}_N^\varepsilon \omega_s(x) dx + \int_\Omega \mathbf{H}_N^\varepsilon \omega_s(x) dx,
\end{aligned} \tag{21}$$

$$\int_\Omega c \theta_{Nt}^\varepsilon \vartheta_s(x) dx = \int_\Omega \mathbf{M}_N^\varepsilon \cdot \mathbf{M}_{Nt}^\varepsilon \vartheta_s(x) dx + \int_\Omega k \nabla \theta_N^\varepsilon \nabla \vartheta_s(x) dx + \int_\Omega \hat{r} \vartheta_s(x) dx, \tag{22}$$

$$\begin{aligned}
&\int_\Omega (\mathbf{E}^\varepsilon_{Nt} + \mathbf{P}^\varepsilon_{Nt})\omega_s(x)dx + \sigma \int_\Omega (\mathbf{E}^\varepsilon_N + \mathbf{P}^\varepsilon_N)\omega_s(x)dx = \int_\Omega (\nabla \times \mathbf{H}^\varepsilon_N)\omega_s(x)dx \\
&\qquad\qquad + \sigma \int_\Omega \mathbf{P}^\varepsilon_N \omega_s(x)dx \qquad (23)\\
&\int_\Omega (\mathbf{H}^\varepsilon_{Nt} + \beta \mathbf{M}^\varepsilon_{Nt})\omega_s(x)dx = -\int_\Omega (\nabla \times \mathbf{E}^\varepsilon_N)\omega_s(x)dx, \qquad (24)\\
&\int_\Omega \mathbf{P}^\varepsilon_{Ntt}\omega_s(x)dx + \lambda^2 \int_\Omega \mathrm{curl}^2 \mathbf{P}^\varepsilon_N \omega_s(x)dx + \mu \int_\Omega \mathbf{P}^\varepsilon_{Nt}\omega_s(x)dx \\
&\quad - \varepsilon \int_\Omega \Delta \mathbf{P}^\varepsilon_N \omega_s(x)dx = \nu \int_\Omega \mathbf{E}^\varepsilon_N \omega_s(x)dx - 2\nu \int_\Omega \mathbf{P}^\varepsilon_N \Phi'(|\mathbf{P}^\varepsilon_N|^2)\omega_s(x)dx \qquad (25)
\end{aligned}$$

with the initial conditions

$$\begin{aligned}
\alpha_{sN}(0) &= \int_\Omega \mathbf{M}^\varepsilon_N(x,0)\omega_s(x)dx = \int_\Omega \mathbf{M}_0(x)\omega_s(x)dx = \alpha_{0s},\\
\beta_{sN}(0) &= \int_\Omega \theta^\varepsilon_N(x,0)\vartheta_s(x)dx = \int_\Omega \theta_0(x)\vartheta_s(x)dx = \beta_{0s},\\
\gamma_{sN}(0) &= \int_\Omega \mathbf{E}^\varepsilon_N(x,0)\omega_s(x)dx = \int_\Omega \mathbf{E}_0(x)\omega_s(x)dx = \gamma_{0s},\\
\zeta_{sN}(0) &= \int_\Omega \mathbf{H}^\varepsilon_N(x,0)\omega_s(x)dx = \int_\Omega \mathbf{H}_0(x)\omega_s(x)dx = \zeta_{0s},\\
\xi_{sN}(0) &= \int_\Omega \mathbf{P}^\varepsilon_N(x,0)\omega_s(x)dx = \int_\Omega \mathbf{P}_0(x)\omega_s(x)dx = \xi_{0s},\\
\xi'_{sN}(0) &= \int_\Omega \mathbf{P}^\varepsilon_{Nt}(x,0)\omega_s(x)dx = \int_\Omega \mathbf{P}_1(x)\omega_s(x)dx = \xi'_{0s}.
\end{aligned} \qquad (26)$$

Obviously, there holds

$$\begin{aligned}
&\int_\Omega \mathbf{M}^\varepsilon_{Nt}\omega_s(x)dx = \alpha'_{sN}(t),\ \int_\Omega \theta^\varepsilon_{Nt}\vartheta_s(x)dx = \beta'_{sN}(t),\ \int_\Omega \mathbf{E}^\varepsilon_{Nt}\omega_s(x)dx = \gamma'_{sN}(t)\\
&\int_\Omega \mathbf{H}^\varepsilon_{Nt}\omega_s(x)dx = \zeta'_{sN}(t),\ \int_\Omega \mathbf{P}^\varepsilon_{Nt}\omega_s(x)dx = \xi'_{sN}(t).
\end{aligned} \qquad (27)$$

According to the standard theory of nonlinear ordinary differential equations, it can be inferred that the problem (21)–(26) admits a unique local solution. To establish the existence of the solution of (21)–(26), we need the following estimates.

Lemma 1. *Assume that* $(\mathbf{M}_0(x), \theta_0(x), \mathbf{E}_0(x), \mathbf{H}_0(x), \mathbf{P}_0(x), \partial_t \mathbf{P}_0(x)) \in (H^1(\Omega), L^2(\Omega), L^2(\Omega), L^2(\Omega), H^1(\Omega), L^2(\Omega))$, $\hat{r}(x,t) \in L^2(0,T; H^1(\Omega))$, *then the following estimates can be derived for the solutions to problem* (21)–(26)*:*

$$\begin{aligned}
&\sup_{0\le t\le T}\Big\{ \|\mathbf{M}^\varepsilon_N(\cdot,t)\|^2_{H^1} + \|\theta^\varepsilon_N(\cdot,t)\|^2_2 + \|\mathbf{E}^\varepsilon_N(\cdot,t)\|^2_2 + \|\mathbf{H}^\varepsilon_N(\cdot,t)\|^2_2 \\
&\quad + \|\mathbf{P}^\varepsilon_N(\cdot,t)\|^2_2 + \|\mathbf{M}^\varepsilon_{Nt}(\cdot,t)\|^2_2 + \|\nabla\theta^\varepsilon_N(\cdot,t)\|^2_2 + \|\mathbf{E}^\varepsilon_N(\cdot,t)\|^2_2 \\
&\quad + \|\mathrm{curl}\ \mathbf{P}^\varepsilon_N(\cdot,t)\|^2_2 + \varepsilon\|\nabla \mathbf{P}^\varepsilon_N\|^2_2 + \|\mathbf{P}^\varepsilon_{Nt}(\cdot,t)\|^2_2 \Big\} + \int_0^t \|\mathbf{M}^\varepsilon_{Nt}(\cdot,t)\|^2_2 d\tau \le C, \qquad (28)
\end{aligned}$$

where C is a constant which is independent of N, D, and ε.

Proof. By multiplying Equation (21) by $\alpha'_{sN}(t)$ and then summing up the outcomes for all $s = 1, 2, ..., N$, we derive

$$\frac{1}{2}\frac{d}{dt}\left[\nu_1\|\nabla \mathbf{M}_N^\varepsilon\|_2^2 + \frac{\theta_c}{2}\|\mathbf{M}_N^\varepsilon\|_4^4\right] + \gamma\|\mathbf{M}_{Nt}^\varepsilon\|_2^2 + \int_\Omega \theta_N^\varepsilon \mathbf{M}_N^\varepsilon \mathbf{M}_{Nt}^\varepsilon dx$$
$$- \theta_c \int_\Omega \mathbf{M}_N^\varepsilon \mathbf{M}_{Nt}^\varepsilon dx - \int_\Omega \mathbf{H}_N^\varepsilon \mathbf{M}_{Nt}^\varepsilon dx = 0. \tag{29}$$

Multiplying (22) by $\beta_{sN}(t)$ and then summing up the outcomes for all $s = 1, 2, \cdots, N$, we obtain

$$\frac{c}{2}\frac{d}{dt}\|\theta_N^\varepsilon\|_2^2 + k\|\nabla\theta_N^\varepsilon\|_2^2 - \int_\Omega \theta_N^\varepsilon \mathbf{M}_N^\varepsilon \mathbf{M}_{Nt}^\varepsilon dx - \int_\Omega \hat{r}\theta_N^\varepsilon dx = 0. \tag{30}$$

By taking the scalar product of $\gamma_{sN}(t)$ with (23) and the scalar product of $\zeta_{sN}(t)$ with (24), respectively, adding the two resulting equalities together, and subsequently summing up the outcomes for all $s = 1, 2, ..., N$, we derive

$$\frac{1}{2}\frac{d}{dt}\left[\|\mathbf{H}_N^\varepsilon\|_2^2 + \|\mathbf{E}_N^\varepsilon\|_2^2\right] + \sigma\|\mathbf{E}_N^\varepsilon\|_2^2 + \int_\Omega \mathbf{P}_{Nt}^\varepsilon \cdot \mathbf{E}_N^\varepsilon dx + \beta\int_\Omega \mathbf{M}_{Nt}^\varepsilon \cdot \mathbf{H}_N^\varepsilon dx = 0. \tag{31}$$

Multiplying (23) by $\gamma_{sN}(t) + \xi_{sN}(t)$, summing up the results for all $s = 1, 2, \cdots, N$, and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{E}_N^\varepsilon + \mathbf{P}_N^\varepsilon\|_2^2 + \sigma\|\mathbf{E}_N^\varepsilon + \mathbf{P}_N^\varepsilon\|_2^2 = \int_\Omega (\nabla \times \mathbf{H}_N^\varepsilon)(\mathbf{E}_N^\varepsilon + \mathbf{P}_N^\varepsilon)dx + \sigma\int_\Omega \mathbf{P}_N^\varepsilon(\mathbf{E}_N^\varepsilon + \mathbf{P}_N^\varepsilon)dx. \tag{32}$$

It follows from (31) and (32) that

$$\frac{1}{2}\frac{d}{dt}\left[2\|\mathbf{H}_N^\varepsilon\|_2^2 + \|\mathbf{E}_N^\varepsilon\|_2^2 + \|\mathbf{E}_N^\varepsilon + \mathbf{P}_N^\varepsilon\|_2^2\right] + \sigma\|\mathbf{E}_N^\varepsilon\|_2^2 + \sigma\|\mathbf{E}_N^\varepsilon + \mathbf{P}_N^\varepsilon\|_2^2$$
$$+ \int_\Omega \mathbf{P}_{Nt}^\varepsilon \cdot \mathbf{E}_N^\varepsilon dx + 2\beta\int_\Omega \mathbf{M}_{Nt}^\varepsilon \cdot \mathbf{H}_N^\varepsilon dx$$
$$= \int_\Omega (\nabla \times \mathbf{H}_N^\varepsilon) \cdot \mathbf{P}_N^\varepsilon dx + \sigma\int_\Omega \mathbf{P}_N^\varepsilon(\mathbf{E}_N^\varepsilon + \mathbf{P}_N^\varepsilon)dx. \tag{33}$$

By the fact $\|\mathbf{P}_N^\varepsilon\|_2^2 - 2\|\mathbf{E}_N^\varepsilon\|_2^2 \le 2\|\mathbf{E}_N^\varepsilon + \mathbf{P}_N^\varepsilon\|_2^2 \le 3\|\mathbf{E}_N^\varepsilon + \mathbf{P}_N^\varepsilon\|_2^2$, we can rewrite (33) as follows:

$$\frac{1}{2}\frac{d}{dt}\left[6\|\mathbf{H}_N^\varepsilon\|_2^2 + \|\mathbf{E}_N^\varepsilon\|_2^2 + \|\mathbf{P}_N^\varepsilon\|_2^2\right] + 3\sigma\|\mathbf{E}_N^\varepsilon\|_2^2 + 3\sigma\|\mathbf{E}_N^\varepsilon + \mathbf{P}_N^\varepsilon\|_2^2$$
$$+ 3\int_\Omega \mathbf{P}_{Nt}^\varepsilon \cdot \mathbf{E}_N^\varepsilon dx + 6\beta\int_\Omega \mathbf{M}_{Nt}^\varepsilon \cdot \mathbf{H}_N^\varepsilon dx$$
$$= 3\int_\Omega (\nabla \times \mathbf{H}_N^\varepsilon) \cdot \mathbf{P}_N^\varepsilon dx + 3\sigma\int_\Omega \mathbf{P}_N^\varepsilon(\mathbf{E}_N^\varepsilon + \mathbf{P}_N^\varepsilon)dx. \tag{34}$$

From (29), (30), and (34), we obtain

$$\frac{1}{2}\frac{d}{dt}\left[\nu_1\|\nabla\mathbf{M}_N^\varepsilon\|_2^2 + \frac{\theta_c}{2}\|\mathbf{M}_N^\varepsilon\|_4^4 + c\|\theta_N^\varepsilon\|_2^2 + 6\|\mathbf{H}_N^\varepsilon\|_2^2 + \|\mathbf{E}_N^\varepsilon\|_2^2 + \|\mathbf{P}_N^\varepsilon\|_2^2\right] + 3\sigma\|\mathbf{E}_N^\varepsilon\|_2^2$$
$$+ 3\sigma\|\mathbf{E}_N^\varepsilon + \mathbf{P}_N^\varepsilon\|_2^2 + \gamma\|\mathbf{M}_{Nt}^\varepsilon\|_2^2 + 3\int_\Omega \mathbf{P}_{Nt}^\varepsilon \cdot \mathbf{E}_N^\varepsilon dx + (6\beta - 1)\int_\Omega \mathbf{M}_{Nt}^\varepsilon \cdot \mathbf{H}_N^\varepsilon dx$$
$$=\theta_c\int_\Omega \mathbf{M}_N^\varepsilon \mathbf{M}_{Nt}^\varepsilon dx + \int_\Omega \hat{r}\theta_N^\varepsilon dx + 3\int_\Omega (\nabla \times \mathbf{H}_N^\varepsilon) \cdot \mathbf{P}_N^\varepsilon dx + 3\sigma\int_\Omega \mathbf{P}_N^\varepsilon(\mathbf{E}_N^\varepsilon + \mathbf{P}_N^\varepsilon)dx. \tag{35}$$

To deal with the term $\int_\Omega \mathbf{M}^\varepsilon_{Nt} \cdot \mathbf{H}^\varepsilon_N dx$, we can multiply (24) by $(6\beta-1)\alpha_{sN}(t)$. Then, summing up the results for all $s = 1, 2, \cdots, N$, we obtain

$$(6\beta-1)\int_\Omega \mathbf{H}^\varepsilon_{Nt} \cdot \mathbf{M}^\varepsilon_N dx + \frac{\beta(6\beta-1)}{2}\frac{d}{dt}\|\mathbf{M}^\varepsilon_N\|_2^2 + (6\beta-1)\int_\Omega (\nabla \times \mathbf{E}^\varepsilon_N)\cdot \mathbf{M}^\varepsilon_N dx = 0. \tag{36}$$

Adding (35) and (36), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\nu_1\|\nabla \mathbf{M}^\varepsilon_N\|_2^2 + \frac{\theta_c}{2}\|\mathbf{M}^\varepsilon_N\|_4^4 + c\|\theta^\varepsilon_N\|_2^2 + 6\|\mathbf{H}^\varepsilon_N\|_2^2 + \|\mathbf{E}^\varepsilon_N\|_2^2 + \|\mathbf{P}^\varepsilon_N\|_2^2\right] \\
&+ 3\sigma\|\mathbf{E}^\varepsilon_N\|_2^2 + 3\sigma\|\mathbf{E}^\varepsilon_N + \mathbf{P}^\varepsilon_N\|_2^2 + \gamma\|\mathbf{M}^\varepsilon_{Nt}\|_2^2 + 3\int_\Omega \mathbf{P}^\varepsilon_{Nt}\cdot \mathbf{E}^\varepsilon_N dx \\
&\quad + (6\beta-1)\frac{d}{dt}\int_\Omega \mathbf{M}^\varepsilon_N \mathbf{H}^\varepsilon_N dx + \frac{\beta(6\beta-1)}{2}\frac{d}{dt}\|\mathbf{M}^\varepsilon_N\|_2^2 \\
&= \theta_c \int_\Omega \mathbf{M}^\varepsilon_N \mathbf{M}^\varepsilon_{Nt} dx + \int_\Omega \hat{r}\theta^\varepsilon_N dx + 3\sigma\int_\Omega (\nabla\times \mathbf{H}^\varepsilon_N)\cdot \mathbf{P}^\varepsilon_N dx + 3\sigma\int_\Omega \mathbf{P}^\varepsilon_N(\mathbf{E}^\varepsilon_N + \mathbf{P}^\varepsilon_N)dx \\
&\quad - (6\beta-1)\int_\Omega (\nabla\times \mathbf{E}^\varepsilon_N)\cdot \mathbf{M}^\varepsilon_N dx.
\end{aligned} \tag{37}$$

Multiplying (25) by $\xi'_{sN}(t)$, summing up the results for all $s = 1, 2, \cdots, N$, and integrating by parts, we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\|\mathbf{P}^\varepsilon_{Nt}\|_2^2 + \lambda^2\|\mathrm{curl}\ \mathbf{P}^\varepsilon_N\|_2^2 + \varepsilon\|\nabla \mathbf{P}^\varepsilon_N\|_2^2\right] + \mu\|\mathbf{P}^\varepsilon_{Nt}\|_2^2 \\
&= \nu\int_\Omega \mathbf{E}^\varepsilon_N \mathbf{P}^\varepsilon_{Nt} dx - 2\nu\int_\Omega \mathbf{P}^\varepsilon_N \Phi'(|\mathbf{P}^\varepsilon_N|^2)\mathbf{P}^\varepsilon_{Nt} dx.
\end{aligned} \tag{38}$$

Summing (37) with (38), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\nu_1\|\nabla \mathbf{M}^\varepsilon_N\|_2^2 + \frac{\theta_c}{2}\|\mathbf{M}^\varepsilon_N\|_4^4 + c\|\theta^\varepsilon_N\|_2^2 + 6\|\mathbf{H}^\varepsilon_N\|_2^2 + \|\mathbf{E}^\varepsilon_N\|_2^2 + \|\mathbf{P}^\varepsilon_N\|_2^2 + \|\mathbf{P}^\varepsilon_{Nt}\|_2^2\right. \\
&\quad \left.+ \lambda^2\|\mathrm{curl}\ \mathbf{P}^\varepsilon_N\|_2^2 + \varepsilon\|\nabla \mathbf{P}^\varepsilon_N\|_2^2\right] + 3\sigma\|\mathbf{E}^\varepsilon_N\|_2^2 + 3\sigma\|\mathbf{E}^\varepsilon_N + \mathbf{P}^\varepsilon_N\|_2^2 + \gamma\|\mathbf{M}^\varepsilon_{Nt}\|_2^2 + \mu\|\mathbf{P}^\varepsilon_{Nt}\|_2^2 \\
&\quad + (6\beta-1)\frac{d}{dt}\int_\Omega \mathbf{M}^\varepsilon_N \mathbf{H}^\varepsilon_N dx + \frac{\beta(6\beta-1)}{2}\frac{d}{dt}\|\mathbf{M}^\varepsilon_N\|_2^2 \\
&= \theta_c \int_\Omega \mathbf{M}^\varepsilon_N \mathbf{M}^\varepsilon_{Nt} dx + \int_\Omega \hat{r}\theta^\varepsilon_N dx + 3\sigma\int_\Omega (\nabla\times \mathbf{H}^\varepsilon_N)\cdot \mathbf{P}^\varepsilon_N dx + 3\sigma\int_\Omega \mathbf{P}^\varepsilon_N(\mathbf{E}^\varepsilon_N + \mathbf{P}^\varepsilon_N)dx \\
&\quad - (6\beta-1)\int_\Omega (\nabla\times \mathbf{E}^\varepsilon_N)\cdot \mathbf{M}^\varepsilon_N dx + (\nu-3)\int_\Omega \mathbf{E}^\varepsilon_N \mathbf{P}^\varepsilon_{Nt} dx - 2\nu\int_\Omega \mathbf{P}^\varepsilon_N \Phi'(|\mathbf{P}^\varepsilon_N|^2)\mathbf{P}^\varepsilon_{Nt} dx.
\end{aligned} \tag{39}$$

By Hölder inequality and Young inequality, we have

$$\begin{aligned}
&\theta_c \int_\Omega \mathbf{M}^\varepsilon_N \mathbf{M}^\varepsilon_{Nt} dx + \int_\Omega \hat{r}\theta^\varepsilon_N dx \le \frac{\gamma}{2}\|\mathbf{M}^\varepsilon_{Nt}\|_2^2 + C(\|\mathbf{M}_N\|_2^2 + \|\theta_N\|_2^2 + \|\hat{r}\|_2^2), \\
&3\sigma\int_\Omega (\nabla\times \mathbf{H}^\varepsilon_N)\cdot \mathbf{P}^\varepsilon_N dx \le C(\|\nabla\times \mathbf{P}^\varepsilon_N\|_2^2 + \|\mathbf{H}^\varepsilon_N\|_2^2), \\
&3\sigma\int_\Omega \mathbf{P}^\varepsilon_N(\mathbf{E}^\varepsilon_N + \mathbf{P}^\varepsilon_N)dx - 3\sigma\|\mathbf{E}^\varepsilon_N + \mathbf{P}^\varepsilon_N\|_2^2 \le C(\|\mathbf{E}^\varepsilon_N\|_2^2 + \|\mathbf{P}^\varepsilon_N\|_2^2), \\
&\quad - (6\beta-1)\int_\Omega (\nabla\times \mathbf{E}^\varepsilon_N)\cdot \mathbf{M}^\varepsilon_N dx \le C(\|\nabla \mathbf{M}^\varepsilon_N\|_2^2 + \|\mathbf{E}^\varepsilon_N\|_2^2), \\
&\quad (\nu-3)\int_\Omega \mathbf{E}^\varepsilon_N \mathbf{P}^\varepsilon_{Nt} dx - \mu\|\mathbf{P}^\varepsilon_{Nt}\|_2^2 \le C(\|\mathbf{P}^\varepsilon_{Nt}\|_2^2 + \|\mathbf{E}^\varepsilon_N\|_2^2)
\end{aligned} \tag{40}$$

and

$$-2\nu\int_\Omega \mathbf{P}^\varepsilon_N \Phi'(|\mathbf{P}^\varepsilon_N|^2)\mathbf{P}^\varepsilon_{Nt} dx \le C_0\nu(\|\mathbf{P}^\varepsilon_{Nt}\|_2^2 + \|\mathbf{P}^\varepsilon_N\|_2^2),$$

where C_0 is given by (13).

Thus, inserting the above estimates into inequality (39), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[\nu_1\|\nabla \mathbf{M}_N^\varepsilon\|_2^2 + \frac{\theta_c}{2}\|\mathbf{M}_N^\varepsilon\|_4^4 + c\|\theta_N^\varepsilon\|_2^2 + 6\|\mathbf{H}_N^\varepsilon\|_2^2 \\
&+ \|\mathbf{E}_N^\varepsilon\|_2^2 + \|\mathbf{P}_N^\varepsilon\|_2^2 + \|\mathbf{P}_{Nt}^\varepsilon\|_2^2 + \lambda^2\|\text{curl } \mathbf{P}_N^\varepsilon\|_2^2 + \varepsilon\|\nabla \mathbf{P}_N^\varepsilon\|_2^2\Big] \\
&+ (6\beta-1)\frac{d}{dt}\int_\Omega \mathbf{M}_N^\varepsilon \mathbf{H}_N^\varepsilon dx + \frac{\beta(6\beta-1)}{2}\frac{d}{dt}\|\mathbf{M}_N^\varepsilon\|_2^2 + \frac{\gamma}{2}\|\mathbf{M}_{Nt}^\varepsilon\|_2^2 \\
&\leq C(1 + \|\mathbf{M}_N\|_4^4 + \|\nabla \mathbf{M}_N^\varepsilon\|_2^2 + \|\theta_N\|_2^2 + \|\hat{r}\|_2^2 + \|\mathbf{E}_N^\varepsilon\|_2^2 \\
&\quad + \|\mathbf{H}_N^\varepsilon\|_2^2 + \|\mathbf{P}_N^\varepsilon\|_2^2 + \|\nabla \mathbf{M}_N^\varepsilon\|_2^2 + \|\mathbf{P}_{Nt}^\varepsilon\|_2^2).
\end{aligned} \tag{41}
$$

Integrating the inequality (41) with the variant t, we have

$$
\begin{aligned}
&\frac{1}{2}\Big[\nu_1\|\nabla \mathbf{M}_N^\varepsilon\|_2^2 + \theta_c\|\mathbf{M}_N^\varepsilon\|_4^4 + c\|\theta_N^\varepsilon\|_2^2 + 6\|\mathbf{H}_N^\varepsilon\|_2^2 + \|\mathbf{E}_N^\varepsilon\|_2^2 \\
&+ \|\mathbf{P}_N^\varepsilon\|_2^2 + \|\mathbf{P}_{Nt}^\varepsilon\|_2^2 + \lambda^2\|\text{curl } \mathbf{P}_N^\varepsilon\|_2^2 + \varepsilon\|\nabla \mathbf{P}_N^\varepsilon\|_2^2\Big] + \frac{\gamma}{2}\int_0^t \|\mathbf{M}_{Nt}^\varepsilon\|_2^2 d\tau \\
&\leq \bar{C} + C\int_0^t (1 + \|\mathbf{M}_N\|_4^4 + \|\nabla \mathbf{M}_N^\varepsilon\|_2^2 + \|\theta_N\|_2^2 + \|\mathbf{E}_N^\varepsilon\|_2^2 + \|\mathbf{H}_N^\varepsilon\|_2^2 \\
&\quad + \|\mathbf{P}_N^\varepsilon\|_2^2 + \|\nabla \mathbf{M}_N^\varepsilon\|_2^2 + \|\mathbf{P}_{Nt}^\varepsilon\|_2^2)d\tau + |6\beta-1|\int_\Omega |\mathbf{M}_N^\varepsilon||\mathbf{H}_N^\varepsilon|dx.
\end{aligned} \tag{42}
$$

Therefore, choosing δ_0 big enough to make the coefficient $6 - \frac{|6\beta-1|}{2\delta}$ positive, we derive

$$
\begin{aligned}
&\nu_1\|\nabla \mathbf{M}_N^\varepsilon\|_2^2 + \frac{\theta_c}{2}\|\mathbf{M}_N^\varepsilon\|_4^4 + c\|\theta_N^\varepsilon\|_2^2 + \left(6 - \frac{|6\beta-1|}{2\delta}\right)\|\mathbf{H}_N^\varepsilon\|_2^2 + \\
&\|\mathbf{E}_N^\varepsilon\|_2^2 + \|\mathbf{P}_N^\varepsilon\|_2^2 + \|\mathbf{P}_{Nt}^\varepsilon\|_2^2 + \lambda^2\|\text{curl } \mathbf{P}_N^\varepsilon\|_2^2 + \varepsilon\|\nabla \mathbf{P}_N^\varepsilon\|_2^2 + \gamma\int_0^t \|\mathbf{M}_{Nt}^\varepsilon\|_2^2 d\tau \\
&\leq C_\delta + C\int_0^t (1 + \|\mathbf{M}_N\|_4^4 + \|\nabla \mathbf{M}_N^\varepsilon\|_2^2 + \|\theta_N\|_2^2 + \|\mathbf{E}_N^\varepsilon\|_2^2 \\
&\quad + \|\mathbf{H}_N^\varepsilon\|_2^2 + \|\mathbf{P}_N^\varepsilon\|_2^2 + \|\nabla \mathbf{M}_N^\varepsilon\|_2^2 + \|\mathbf{P}_{Nt}^\varepsilon\|_2^2)d\tau.
\end{aligned}
$$

Then by Gronwall inequality, we can obtain the estimates (28). This completes the proof of Lemma 1. $\square$

Remark 1. *In fact, by Equation (21) and estimates (28), we can obtain* $\int_0^t \|\Delta \mathbf{M}_N\|_2^2 dx \leq C$ *easily.*

Lemma 2. *Subject to the conditions stated in Lemma 1, we can deduce the following estimates for solutions* $(\mathbf{M}_N^\varepsilon(x,t), \theta_N^\varepsilon(x,t), \mathbf{E}_N^\varepsilon(x,t), \mathbf{H}_N^\varepsilon(x,t), \mathbf{P}_N^\varepsilon(x,t))$ *of the initial value problem (21)–(26),*

$$
\|\mathbf{M}_{Nt}^\varepsilon\|_{H^{-1}(\Omega)} + \|\mathbf{H}_{Nt}^\varepsilon\|_{H^{-1}(\Omega)} + \|\mathbf{E}_{Nt}^\varepsilon\|_{H^{-1}(\Omega)} + \|\mathbf{P}_{Ntt}^\varepsilon\|_{H^{-1}(\Omega)} \leq C, \tag{43}
$$

$$
\|\theta_{Nt}^\varepsilon\|_{L^2(0,T;H^{-1}(\Omega))} \leq C, \tag{44}
$$

where C is independent of ε, N, *and* D, *and* $H^{-1}(\Omega)$ *denotes the dual space of* $H^1(\Omega)$.

Proof. $\forall \varphi \in H^2$, we have

$$
\varphi = \varphi_N + \overline{\varphi}_N, \quad \varphi_N = \sum_{s=1}^{N} \beta_s \omega_s(x), \quad \overline{\varphi}_N = \sum_{s=N+1}^{\infty} \beta_s \omega_s(x).
$$

For $s \geq N+1$, $\int_\Omega \mathbf{M}^\varepsilon_{Nt}\omega_s(x)dx = 0$. Then by Lemma 1 and the Hölder inequality, we obtain

$$\begin{aligned}
&\left|\int_\Omega \mathbf{M}^\varepsilon_{Nt}\varphi dx\right| = \left|\int_\Omega \mathbf{M}^\varepsilon_{Nt}\varphi_N(x)dx\right| \\
=&\left|\frac{\nu}{\gamma}\int_\Omega \nabla\mathbf{M}^\varepsilon_N\nabla\varphi_N dx + \frac{\theta_c}{\gamma}\int_\Omega (|\mathbf{M}^\varepsilon_N|^2-1)\mathbf{M}^\varepsilon_N\varphi_N dx\right. \\
&\left.+\frac{1}{\gamma}\int_\Omega \theta_N\mathbf{M}^\varepsilon_N\varphi_N dx + \frac{1}{\gamma}\int_\Omega \mathbf{H}^\varepsilon_N\varphi_N dx\right| \\
\leq& C\Big[\|\nabla\mathbf{M}^\varepsilon_N\|_2\|\nabla\varphi_N\|_2 + (\|\mathbf{M}^\varepsilon_N\|_6^6 + \|\mathbf{M}^\varepsilon_N\|_2^2)\|\varphi_N\|_2 \\
&+\|\mathbf{M}^\varepsilon_N\|_4\|\theta^\varepsilon_N\|_2\|\varphi_N\|_4 + \|\mathbf{H}^\varepsilon_N\|_2\|\varphi_N\|_2\Big] \\
\leq& C\|\varphi_N\|_{H^1(\Omega)} \leq C\|\varphi\|_{H^1(\Omega)}.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\left|\int_\Omega \mathbf{E}^\varepsilon_{Nt}\phi dx\right| = \left|\int_\Omega \mathbf{E}^\varepsilon_{Nt}\phi_N dx\right| \\
&= \left|\int_\Omega \nabla\times\mathbf{H}^\varepsilon_N\cdot\varphi_N dx + \int_\Omega \mathbf{P}^\varepsilon_{Nt}\varphi_N dx + \sigma\int_\Omega \mathbf{E}^\varepsilon_N\varphi_N dx\right| \\
&\leq C(\|\mathbf{H}^\varepsilon_N\|_2 + \|\nabla\varphi_N\|_2 + \|\mathbf{P}^\varepsilon_{Nt}\|_2 + \|\varphi_N\|_2 + \|\mathbf{E}^\varepsilon_N\|_2)\|\varphi_N\|_2 \\
&\leq C\|\varphi\|_{H^1(\Omega)},
\end{aligned}$$

$$\begin{aligned}
\left|\int_\Omega \mathbf{H}^\varepsilon_{Nt}\phi dx\right| = \left|\int_\Omega \mathbf{H}^\varepsilon_{Nt}\phi_N dx\right| &= \left|\int_\Omega (\nabla\times\mathbf{E}^\varepsilon_N)\varphi_N + \beta\mathbf{M}^\varepsilon_{Nt}\varphi_N dx\right| \\
&\leq C(\|\mathbf{E}_N\|_2\|\nabla\varphi_N\|_2 + \|\varphi\|_{H^1(\Omega)}) \\
&\leq C_1\|\varphi\|_{H^1(\Omega)}
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_\Omega \mathbf{P}^\varepsilon_{Ntt}\phi dx\right| = \left|\int_\Omega \mathbf{P}^\varepsilon_{Ntt}\phi_N dx\right| \\
&= \left|\lambda^2\int_\Omega \mathrm{curl}^2\,\mathbf{P}^\varepsilon_N\varphi_N dx + \varepsilon\int_\Omega \Delta\mathbf{P}^\varepsilon_N\varphi_N dx + \mu\int_\Omega \mathbf{P}^\varepsilon_{Nt}\varphi_N dx\right. \\
&\quad\left.+\int_\Omega \nu(\mathbf{E}^\varepsilon_N - 2\mathbf{P}^\varepsilon_N\Phi'(|\mathbf{P}^\varepsilon_N|^2))\varphi_N dx\right| \\
&\leq C(\|\mathrm{curl}\,\mathbf{P}^\varepsilon_N\|_2\|\nabla\varphi_N\|_2 + \|\nabla\mathbf{P}^\varepsilon_N\|_2\|\nabla\varphi_N\|_2 + \|\mathbf{P}^\varepsilon_{Nt}\|_2\|\varphi\|_2 \\
&+\|\mathbf{E}^\varepsilon_N\|_2\|\varphi\|_2 + \|\mathbf{P}^\varepsilon_N\|_2\|\varphi\|_2) \\
&\leq C_1\|\varphi\|_{H^1(\Omega)}.
\end{aligned}$$

Let $\Phi \in L^2(0,T;H^1(\Omega))$; by (22) and Lemma 1, we have

$$\begin{aligned}
&\left|\iint_{Q_T} \theta^\varepsilon_{Nt}\Phi dxdt\right| \\
&\leq \frac{1}{c}\left|\int_0^T \|\mathbf{M}^\varepsilon_N\|_4\|\mathbf{M}^\varepsilon_{Nt}\|_2\|\Phi\|_4 + k\|\nabla\theta^\varepsilon_N\|_2\|\nabla\Phi\|_2 + \|\hat{r}\|_2\|\Phi\|_2 dt\right| \\
&\leq C\int_0^T \|\Phi\|^2_{H^1(\Omega)}dt,
\end{aligned}$$

where the constant C is independent of ε, N, and D . The proof is completed. $\square$

Lemma 3. *Subject to the conditions stated in Lemma 1, we obtain the following estimates for solutions* $(\mathbf{M}_N^\varepsilon(x,t), \theta_N^\varepsilon(x,t), \mathbf{E}_N^\varepsilon(x,t), \mathbf{H}_N^\varepsilon(x,t), \mathbf{P}_N^\varepsilon(x,t))$ *of problem* (21)–(26),

$$\begin{aligned}
&\|\mathbf{M}_N^\varepsilon(\cdot,t_1)-\mathbf{M}_N^\varepsilon(\cdot,t_2)\|_{H^{-1}(\Omega)}+\|\theta_N^\varepsilon(\cdot,t_1)-\theta_N^\varepsilon(\cdot,t_2)\|_{H^{-1}(\Omega)}\le C|t_1-t_2|^{\frac12},\\
&\|\mathbf{H}_N^\varepsilon(\cdot,t_1)-\mathbf{H}_N^\varepsilon(\cdot,t_2)\|_{H^{-1}(\Omega)}+\|\mathbf{E}_N^\varepsilon(\cdot,t_1)-\mathbf{E}_N^\varepsilon(\cdot,t_2)\|_{H^{-1}(\Omega)}\le C|t_1-t_2|^{\frac12},\\
&\|\mathbf{P}_N^\varepsilon(\cdot,t_1)-\mathbf{P}_N^\varepsilon(\cdot,t_2)\|_2\le C|t_2-t_1|^{\frac12},\\
&\|\mathbf{P}_{Nt}^\varepsilon(\cdot,t_1)-\mathbf{P}_{Nt}^\varepsilon(\cdot,t_2)\|_{H^{-1}(\Omega)}\le C|t_1-t_2|^{\frac12},\quad \forall t_1,t_2\ge 0,
\end{aligned}$$

where the constant C *is independent of* $N, D,$ *and* ε.

Proof. It follows from Lemma 2 that

$$\begin{aligned}
\|\mathbf{M}_N^\varepsilon(\cdot,t_1)-\mathbf{M}_N^\varepsilon(\cdot,t_2)\|_2=&\left\|\int_{t_1}^{t_2}\mathbf{M}_{Nt}^\varepsilon dt\right\|_{H^{-1}(\Omega)}\le\int_{t_1}^{t_2}\|\mathbf{M}_{Nt}^\varepsilon\|_{H^{-1}(\Omega)}dt\\
\le|t_2-t_1|^{\frac12}&\left(\int_0^T\|\mathbf{M}_{Nt}^\varepsilon\|_{H^{-1}(\Omega)}^2dt\right)^{\frac12}\\
\le&C|t_2-t_1|^{\frac12}.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\|\theta_N^\varepsilon(\cdot,t_1)-\theta_N^\varepsilon(\cdot,t_2)\|_{H^{-1}}\le C|t_2-t_1|^{\frac12},\\
&\|\mathbf{E}_N^\varepsilon(\cdot,t_1)-\mathbf{E}_N^\varepsilon(\cdot,t_2)\|_{H^{-1}}\le C|t_2-t_1|^{\frac12},\\
&\|\mathbf{H}_N^\varepsilon(\cdot,t_1)-\mathbf{H}_N^\varepsilon(\cdot,t_2)\|_{H^{-1}}\le C|t_2-t_1|^{\frac12},\\
&\|\mathbf{P}_{Nt}^\varepsilon(\cdot,t_1)-\mathbf{P}_{Nt}^\varepsilon(\cdot,t_2)\|_{H^{-1}}\le C|t_2-t_1|^{\frac12}.
\end{aligned}$$

At the same time, we have

$$\begin{aligned}
\|\mathbf{P}_N^\varepsilon(\cdot,t_1)-\mathbf{P}_N^\varepsilon(\cdot,t_2)\|_2=&\left\|\int_{t_1}^{t_2}\mathbf{P}_{Nt}^\varepsilon dt\right\|_2\\
&\le|t_2-t_1|^{\frac12}\left(\int_0^T\|\mathbf{P}_{Nt}^\varepsilon\|_2^2dt\right)^{\frac12}\\
&\le C|t_2-t_1|^{\frac12}.
\end{aligned}$$

This lemma is proved. □

Based on ordinary differential equation theory, Lemmas 1–3 obtained above, the following lemma can be easily established.

Lemma 4. *Under the conditions stated in Lemma 1, there exists a unique global solution* $(\alpha_{sN}(t), \beta_{sN}(t), \gamma_{sN}(t), \zeta_{sN}(t), \xi_{sN}(t))$ $(s=1,2,\ldots,N,\ t\in[0,T],\ \forall T>0)$ *of the initial value problem* (21)–(26)*. Furthermore, this solution is continuously differentiable.*

To establish the existence of the weak solution for the viscosity problem (3)–(6), (8), (9), and (12), we require the lemmas in [12] stated below.

Lemma 5. *If* u_n *converges strongly to* u *in* $L^2(Q_T)$ *and* v_n *converges weakly to* v *in* $L^2(Q_T)$*, then* u_nv_n *converges to* uv *in* $L^1(Q_T)$ *and in the distribution sense.*

Lemma 6. *Suppose that $X \subset E \subset Y$ are Banach spaces and $X \hookrightarrow\hookrightarrow E$. If $1 \le q \le \infty$ or $1 \le r \le \infty$, then the following embeddings are compact:*

$$\text{(i)} \quad L^q(0,T,X) \cap \{\varphi : \frac{\partial \varphi}{\partial t} \in L^1(0,T;Y)\} \hookrightarrow\hookrightarrow L^q(0,T,E), \tag{45}$$

$$\text{(ii)} \quad L^\infty(0,T,X) \cap \{\varphi : \frac{\partial \varphi}{\partial t} \in L^r(0,T;Y)\} \hookrightarrow\hookrightarrow C(0,T,E). \tag{46}$$

Similar to Definition 1, we can define the weak solution for the viscosity problem (3)–(6), (8), (9), and (12); then, by the uniform estimates of the approximate solution, we prove the existence of the weak solution for the viscosity problem (3)–(6), (8), (9), and (12).

It follows from the uniform estimates of the approximate solution $\{\mathbf{M}_N^\varepsilon(x,t), \theta_N^\varepsilon(x,t), \mathbf{E}_N^\varepsilon(x,t), \mathbf{H}_N^\varepsilon(x,t), \mathbf{P}_N^\varepsilon(x,t)\}$ in Lemmas 1 and 2, the Sobolev embedding theorem, and the Lions–Aubin lemma that there exists a subsequence, which for simplicity we continue to denote as $\{\mathbf{M}_N^\varepsilon(x,t), \theta_N^\varepsilon(x,t), \mathbf{E}_N^\varepsilon(x,t), \mathbf{H}_N^\varepsilon(x,t), \mathbf{P}_N^\varepsilon(x,t)\}$, such that

$$\mathbf{M}_N^\varepsilon(x,t) \rightharpoonup \mathbf{M}^\varepsilon(x,t) \;\; \textit{weak} * \textit{ in } L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)), \tag{47}$$

$$\mathbf{M}_N^\varepsilon(x,t) \rightharpoonup \mathbf{M}^\varepsilon(x,t) \;\; \textit{weak} * \textit{ in } L^6(Q_T), \tag{48}$$

$$\mathbf{M}_N^\varepsilon(x,t) \to \mathbf{M}^\varepsilon(x,t) \;\; \textit{strongly in } L^{6-\varrho}(Q_T), \varrho > 0, \tag{49}$$

$$\mathbf{M}_{Nt}^\varepsilon(x,t) \rightharpoonup \mathbf{M}_t^\varepsilon(x,t) \;\; \textit{weak} * \textit{ in } L^\infty(0,T;H^{-1}(\Omega)), \tag{50}$$

$$\theta_N^\varepsilon(x,t) \rightharpoonup \theta^\varepsilon(x,t) \;\; \textit{weak} * \textit{ in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \tag{51}$$

$$\theta_{Nt}^\varepsilon(x,t) \rightharpoonup \theta_t^\varepsilon(x,t) \;\; \textit{weak} * \textit{ in } L^2(0,T;H^{-1}(\Omega)), \tag{52}$$

$$\theta_N^\varepsilon(x,t) \to \theta^\varepsilon(x,t) \;\; \textit{strongly in } L^2(0,T;L^2(\Omega)), \tag{53}$$

$$\mathbf{E}_N^\varepsilon(x,t) \rightharpoonup \mathbf{E}^\varepsilon(x,t) \;\; \textit{weak} * \textit{ in } L^\infty(0,T;L^2(\Omega)), \tag{54}$$

$$\mathbf{H}_N^\varepsilon(x,t) \rightharpoonup \mathbf{H}^\varepsilon(x,t) \;\; \textit{weak} * \textit{ in } L^\infty(0,T;L^2(\Omega)), \tag{55}$$

$$\mathbf{P}_N^\varepsilon(x,t) \rightharpoonup \mathbf{P}^\varepsilon(x,t) \;\; \textit{weak} * \textit{ in } L^\infty(0,T;H^1(\Omega)), \tag{56}$$

$$\mathbf{P}_{Nt}^\varepsilon(x,t) \rightharpoonup \mathbf{P}_t^\varepsilon(x,t) \;\; \textit{weak} * \textit{ in } L^\infty(0,T;L^2(\Omega)), \tag{57}$$

$$\text{curl } \mathbf{P}_N^\varepsilon(x,t) \rightharpoonup \text{curl } \mathbf{P}^\varepsilon(x,t) \;\; \textit{weak} * \textit{ in } L^\infty(0,T;L^2(\Omega)). \tag{58}$$

For any test function $\psi(x,t), \varpi(x,t) \in C^1(\overline{Q_T})$ with $\psi(x,t)|_{t=T} = \varpi(x,t)|_{t=T} = 0$, we introduce an approximate sequence as follows:

$$\psi_N(x,t) = \sum_{s=1}^N a_s(t)\omega_n(x), \;\; a_n(t) = \int_\Omega \psi(x,t)\omega_n(x)dx,$$

$$\varpi_N(x,t) = \sum_{s=1}^N b_s(t)\omega_n(x), \;\; b_n(t) = \int_\Omega \varpi(x,t)\omega_n(x)dx,$$

then

$$\psi_N \to \psi, \varpi_N \to \varpi \text{ in } C^1(Q_T) \text{ and in } L^p(Q_T), \forall p > 1. \tag{59}$$

By taking the scalar product of $a_s(t)$ with (21) and (23), respectively, the scalar product of $b_s(t)$ with (22), and the scalar product of $e^{\sigma t}a_s(t)$ with (24) and $a_s(t)$ with (25), then summing up the products with respect to $s = 1, 2, \cdots, N$ and integrating by parts, we obtain

$$\gamma \iint_{Q_T} \mathbf{M}_N^\varepsilon \cdot \psi_{Nt} dxdt - \nu_1 \iint_{Q_T} \nabla \mathbf{M}_N^\varepsilon \cdot \nabla \psi_N dxdt - \iint_{Q_T} \theta_c(|\mathbf{M}_N^\varepsilon|^2 - 1)\mathbf{M}_N^\varepsilon \cdot \psi_N dxdt$$
$$- \iint_{Q_T} \theta_N \mathbf{M}_N^\varepsilon \cdot \psi_N dxdt + \iint_{Q_T} \mathbf{H}_N^\varepsilon \cdot \psi_N dxdt$$
$$+ \gamma \int_\Omega \mathbf{M}_N^\varepsilon(x,0) \cdot \psi_N(x,0) dx = 0, \tag{60}$$

$$c \iint_{Q_T} \theta_N^\varepsilon \cdot \varpi_{Nt} dxdt + \iint_{Q_T} \mathbf{M}_N^\varepsilon \mathbf{M}_{Nt}^\varepsilon \cdot \varpi_N dxdt - k \iint_{Q_T} \nabla \theta_N^\varepsilon \cdot \nabla \varpi_N dxdt$$
$$+ \iint_{Q_T} \hat{r}(x,t) \cdot \varpi_N dxdt + c \int_\Omega \theta_N^\varepsilon(x,0) \cdot \varpi_N(x,0) dx = 0, \tag{61}$$

$$\iint_{Q_T} (\mathbf{E}_{Nt}^\varepsilon + \mathbf{P}_{Nt}^\varepsilon) e^{\sigma t} \psi_{Nt} dxdt + \iint_{Q_T} e^{\sigma t} \nabla \times \psi_N \cdot \mathbf{H}_N^\varepsilon dxdt$$
$$+ \sigma \iint_{Q_T} e^{\sigma t} \mathbf{P}_N^\varepsilon \cdot \psi_N dxdt + \int_\Omega (\mathbf{E}_N^\varepsilon(x,0) + \mathbf{P}_N^\varepsilon(x,0)) \cdot \psi_N(x,0) dx = 0, \tag{62}$$

$$\iint_{Q_T} (\mathbf{H}_N^\varepsilon + \beta \mathbf{M}_N^\varepsilon) \cdot \psi_{Nt} dxdt - \iint_{Q_T} (\nabla \times \psi_N) \cdot \mathbf{E}_N^\varepsilon dxdt$$
$$+ \int_\Omega (\mathbf{H}_N^\varepsilon(x,0) + \beta \mathbf{M}_N^\varepsilon(x,0)) \cdot \psi(x,0) dx = 0, \tag{63}$$

$$\iint_{Q_T} \mathbf{P}_{Nt}^\varepsilon \cdot \psi_{Nt} dxdt - \lambda^2 \iint_{Q_T} \mathrm{curl} \mathbf{P}_N^\varepsilon \cdot \mathrm{curl} \psi_N dxdt - \mu \iint_{Q_T} \mathbf{P}_{Nt}^\varepsilon \cdot \psi_N dxdt$$
$$+ \nu \iint_{Q_T} \mathbf{E}_N^\varepsilon \cdot \psi_N dxdt - \varepsilon \iint_{Q_T} \nabla \mathbf{P}_N^\varepsilon \cdot \nabla \psi_N dx$$
$$- 2\nu \iint_{Q_T} \Phi'(|\mathbf{P}_N^\varepsilon|^2) \mathbf{P}_N^\varepsilon \cdot \psi_N dxdt + \int_\Omega \mathbf{P}_{Nt}^\varepsilon(x,0) \cdot \phi(x,0) dx = 0. \tag{64}$$

In order to prove that $\{\mathbf{M}_N^\varepsilon(x,t), \theta_N^\varepsilon(x,t), \mathbf{E}_N^\varepsilon(x,t), \mathbf{H}_N^\varepsilon(x,t), \mathbf{P}_N^\varepsilon(x,t)\}$ is a weak solution of (3)–(6), (8), (9), and (12), we should set $N \to \infty$ in (60)–(64). From Lemmas 5 and 6 and (47)–(59), it suffices to deal with the nonlinear terms in (60)–(64).

Firstly, by Lemma 6, we obtain

$$L^\infty(0,T,H^1(\Omega)) \cap \{\varphi : \frac{\partial \varphi}{\partial t} \in L^\infty(0,T;L^2(\Omega))\} \hookrightarrow\hookrightarrow C(0,T;L^2(\Omega)) \subset L^2(0,T;L^2(\Omega)).$$

Thus by (56) and (57), we know that there exists a subsequence of $\mathbf{P}_N^\varepsilon$ which is still denoted by $\mathbf{P}_N^\varepsilon$, such that

$$\mathbf{P}_N^\varepsilon(x,t) \to \mathbf{P}^\varepsilon(x,t), \quad \text{strongly in } L^\infty(0,T;L^2(\Omega)). \tag{65}$$

Secondly, by (14) and (65), we have

$$\left| \iint_{Q_T} \Phi'(|\mathbf{P}_N^\varepsilon|^2) \mathbf{P}_N^\varepsilon \cdot \psi_N dxdt - \iint_{Q_T} \Phi'(|\mathbf{P}^\varepsilon|^2) \mathbf{P}^\varepsilon \cdot \psi dxdt \right|$$
$$= \left| \iint_{Q_T} (\Phi'(|\mathbf{P}_N^\varepsilon|^2) \mathbf{P}_N^\varepsilon - \Phi'(|\mathbf{P}^\varepsilon|^2) \mathbf{P}^\varepsilon) \cdot \psi_N dxdt + \iint_{Q_T} \Phi'(|\mathbf{P}^\varepsilon|^2) \mathbf{P}^\varepsilon \cdot (\psi_N - \psi) dxdt \right|$$
$$\le C_1 \iint_{Q_T} |\mathbf{P}_N^\varepsilon - \mathbf{P}^\varepsilon| |\psi_N| dxdt + C_2 \|\mathbf{P}^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \int_0^T \|\psi_N - \psi\|_2 dt$$
$$\le C_1 \|\mathbf{P}_N^\varepsilon - \mathbf{P}^\varepsilon\|_{L^2(0,T;L^2(\Omega))} \|\psi_N\|_{L^2(0,T;L^2(\Omega))} + C_2 \|\mathbf{P}^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \int_0^T \|\psi_N - \psi\|_2 dt$$
$$\to 0, \quad as \quad N \to \infty.$$

From (47) and (59), we have

$$\begin{aligned}
&\iint_{Q_T} (\nabla \mathbf{M}_N^\varepsilon \cdot \nabla \psi_N - \nabla \mathbf{M}^\varepsilon \cdot \nabla \psi) dxdt \\
=&\iint_{Q_T} (\nabla \mathbf{M}_N^\varepsilon - \nabla \mathbf{M}^\varepsilon) \cdot \nabla \psi_N + \nabla \mathbf{M}^\varepsilon (\nabla \psi_N - \nabla \psi) dxdt \\
\leq&\int_0^T \|\nabla \mathbf{M}_N^\varepsilon - \nabla \mathbf{M}^\varepsilon\|_2 \|\nabla \psi_N\|_2 dt + \int_0^T \|\nabla \mathbf{M}^\varepsilon\|_2 \|\nabla \psi_N - \nabla \psi\|_2 dt \to 0, \ \ as \ \ N \to \infty.
\end{aligned}$$

Equations (49) and (59) imply that

$$\iint_{Q_T} (|\mathbf{M}_N^\varepsilon|^2 - 1) \mathbf{M}_N^\varepsilon \cdot \psi_N dxdt \to \iint_{Q_T} (|\mathbf{M}^\varepsilon|^2 - 1) \mathbf{M}^\varepsilon \cdot \psi dxdt, \ \ as \ \ N \to \infty.$$

Note that

$$\begin{aligned}
&\iint_{Q_T} (\nabla \times \psi_N) \cdot \mathbf{E}_N^\varepsilon dxdt - \iint_{Q_T} (\nabla \times \psi) \cdot \mathbf{E}^\varepsilon \, dxdt \\
=&\iint_{Q_T} \nabla \times (\psi_N - \psi) \cdot \mathbf{E}_N^\varepsilon dxdt + \iint_{Q_T} \nabla \times \psi \cdot \mathbf{E}_N^\varepsilon dxdt - \iint_{Q_T} (\nabla \times \psi) \cdot \mathbf{E}^\varepsilon \, dxdt \\
=&\iint_{Q_T} \nabla \times (\psi_N - \psi) \cdot \mathbf{E}_N^\varepsilon dxdt + \iint_{Q_T} (\nabla \times \psi) \cdot (\mathbf{E}_N^\varepsilon - \mathbf{E}^\varepsilon) dxdt \\
\leq&\left(\iint_{Q_T} |\nabla (\psi_N - \psi)|^2 dxdt \right)^{\frac{1}{2}} \|\mathbf{E}_N^\varepsilon\|_{L^2(Q_T)} + \left| \iint_{Q_T} (\nabla \times \psi) \cdot (\mathbf{E}_N^\varepsilon - \mathbf{E}^\varepsilon) dxdt \right| \\
&\to 0, \ \ as \ \ N \to \infty.
\end{aligned}$$

Similarly, we can prove that as $N \to \infty$.

$$\begin{aligned}
&\iint_{Q_T} \mathbf{H}_N^\varepsilon \cdot \psi_{Nt} dxdt \to \iint_{Q_T} \mathbf{H}^\varepsilon \cdot \psi_t dxdt, \\
&\iint_{Q_T} \mathbf{E}_N^\varepsilon \cdot (\psi_{Nt} e^{\sigma t}) dxdt \to \iint_{Q_T} \mathbf{E}^\varepsilon \cdot (\psi_t e^{\sigma t}) dxdt, \\
&\iint_{Q_T} e^{\sigma t} (\nabla \times \psi_N) \cdot \mathbf{H}_N dxdt \to \iint_{Q_T} e^{\sigma t} (\nabla \times \psi) \cdot \mathbf{H} dxdt, \\
&\iint_{Q_T} \mathbf{M}_N^\varepsilon \cdot \psi_{Nt} dxdt \to \iint_{Q_T} \mathbf{M}^\varepsilon \cdot \psi_t dxdt, \\
&\iint_{Q_T} \mathbf{M}_N^\varepsilon \mathbf{M}_{Nt}^\varepsilon \cdot \varpi_N dxdt \to \iint_{Q_T} \mathbf{M}^\varepsilon \mathbf{M}^\varepsilon{}_t \cdot \varpi dxdt, \\
&\iint_{Q_T} \theta_N^\varepsilon \mathbf{M}_N^\varepsilon \cdot \psi_N dxdt \to \iint_{Q_T} \theta^\varepsilon \mathbf{M}^\varepsilon \cdot \psi dxdt, \\
&\iint_{Q_T} \theta_N^\varepsilon \cdot \varpi_{Nt} dxdt \to \iint_{Q_T} \theta^\varepsilon \cdot \varpi_t dxdt, \\
&\iint_{Q_T} \nabla \theta_N^\varepsilon \cdot \nabla \varpi_N dxdt \to \iint_{Q_T} \nabla \theta^\varepsilon \cdot \nabla \varpi dxdt.
\end{aligned}$$

Finally, taking $N \to \infty$ in (60)–(64), we obtain the limit function $(\mathbf{M}^\varepsilon(x,t), \theta^\varepsilon(x,t), \mathbf{E}^\varepsilon(x,t), \mathbf{H}^\varepsilon(x,t), \mathbf{P}^\varepsilon(x,t))$, which is a global weak solution of the viscosity problem (3)–(6), (8), (9), and (12). Therefore, we obtain the existence of the result of the global weak solution for the viscosity problem (3)–(6), (8), (9), and (12).

Lemma 7. *Assume that initial value data* $(\mathbf{M}_0(x), \theta_0(x), \mathbf{E}_0(x), \mathbf{H}_0(x), \mathbf{P}_0(x), \partial_t \mathbf{P}_0(x)) \in (H^1(\Omega), L^2(\Omega), L^2(\Omega), L^2(\Omega), H^1(\Omega), L^2(\Omega)), \hat{r}(x,t) \in L^2(0,T; H^1(\Omega))$. *The constants* γ,

$\nu_1, c, k, \sigma, \beta, \lambda, \mu, \nu$ are positive. Then the problem (3)–(6), (8), (9), and (12) has at least one global weak solution $\left(\mathbf{M}^\varepsilon(x,t), \theta^\varepsilon(x,t), \mathbf{E}^\varepsilon(x,t), \mathbf{H}^\varepsilon(x,t), \mathbf{P}^\varepsilon(x,t)\right)$ such that

$$\begin{aligned}
&\mathbf{M}^\varepsilon(x,t) \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap C^{(0,\frac{1}{2})}(0,T;L^2(\Omega)),\\
&\theta^\varepsilon(x,t) \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap C^{(0,\frac{1}{2})}(0,T;H^{-1}(\Omega)),\\
&\mathbf{E}^\varepsilon(x,t) \in L^\infty(0,T;L^2(\Omega)) \cap C^{(0,\frac{1}{2})}(0,T;H^{-1}(\Omega)),\\
&\mathbf{H}^\varepsilon(x,t) \in L^\infty(0,T;L^2(\Omega)) \cap C^{(0,\frac{1}{2})}(0,T;H^{-1}(\Omega)),\\
&\mathbf{P}^\varepsilon(x,t) \in L^\infty(0,T;H^1(\Omega)) \cap C^{(0,\frac{1}{2})}(0,T;L^2(\Omega)),\\
&\partial_t\mathbf{P}^\varepsilon(x,t) \in L^\infty(0,T;L^2(\Omega)) \cap C^{(0,\frac{1}{2})}(0,T;H^{-1}(\Omega)).
\end{aligned} \tag{66}$$

4. Existence of Global Weak Solutions

The global weak solution for the viscosity problem (3)–(6), (8), (9), and (12) was obtained for fixed $\varepsilon > 0$ in the above section. In this section, our aim is to establish uniform estimates in terms of ε for the solutions of the viscosity problem. Then letting $\varepsilon \to 0$, we will obtain the global weak solution to the problem (3)–(9).

We now introduce the Sobolev space and state the lemmas that will be applied in this section.

Definition 2 ([33]). *Define the space $H_p(\mathrm{curl}, \Omega)$ as follows:*

$$H_p(\mathrm{curl}, \Omega) = \{V \in L^2(\Omega); V \text{ is } 2D-\text{periodic and curl } V \in L^2(\Omega)\},$$

equipped with the norm

$$\|V\|_{H_p(\mathrm{curl},\Omega)} = (\|V\|^2_{L^2(\Omega)} + \|\mathrm{curl}\ V\|^2_{L^2(\Omega)})^{\frac{1}{2}}.$$

Define the space $H_p(\mathrm{div}, \Omega)$ by

$$H_p(\mathrm{div}, \Omega) = \{V \in L^2(\Omega); V \text{ is } 2D-\text{periodic and div } V \in L^2(\Omega)\},$$

equipped with the norm

$$\|V\|_{H_p(\mathrm{div},\Omega)} = (\|V\|^2_{L^2(\Omega)} + \|\mathrm{div}\ V\|^2_{L^2(\Omega)})^{\frac{1}{2}}.$$

We set

$$X_p(\Omega) = H_p(\mathrm{curl}, \Omega) \cap H_p(\mathrm{div}, \Omega),$$

with the norm

$$\|V\|_{X_p(\Omega)} = (\|V\|^2_{L^2(\Omega)} + \|\mathrm{curl}\ V\|^2_{L^2(\Omega)} + \|\mathrm{div}\ V\|^2_{L^2(\Omega)})^{\frac{1}{2}}.$$

Lemma 8 ([33]). *Suppose that $\Omega = \{x = (x_1, x_2, x_3); |x_i| \le D, i = 1,2,3\}$ and $Q \in X_p(\Omega)$, then one has $Q \in X_{H^1(\Omega)}$ and the following identity holds:*

$$\|Q\|^2_{H^1(\Omega)} = \|Q\|^2_{X_p(\Omega)}. \tag{67}$$

From the estimates and the convergence in Section 3, we easily obtain the following lemma.

Lemma 9. *Assume that* $(\mathbf{M}_0(x), \theta_0(x), \mathbf{E}_0(x), \mathbf{H}_0(x), \mathbf{P}_0(x), \partial_t \mathbf{P}_0(x)) \in (H^1(\Omega), L^2(\Omega), L^2(\Omega), L^2(\Omega), H^1(\Omega), L^2(\Omega))$, $\hat{r}(x,t) \in L^2(0,T;H^1(\Omega))$, *then for the solutions of the viscosity problem* (3)–(6), (8), (9), *and* (12), *we derive the estimates as follows:*

$$\sup_{0\le t\le T}\Big\{\|\mathbf{M}^\varepsilon(\cdot,t)\|_{H^1}^2+\|\theta^\varepsilon(\cdot,t)\|_2^2+\|\mathbf{E}^\varepsilon(\cdot,t)\|_2^2+\|\mathbf{H}^\varepsilon(\cdot,t)\|_2^2+\|\mathbf{P}^\varepsilon(\cdot,t)\|_2^2+\|\mathbf{M}_t^\varepsilon(\cdot,t)\|_2^2$$
$$+\|\nabla\theta^\varepsilon(\cdot,t)\|_2^2+\|\mathbf{E}^\varepsilon(\cdot,t)\|_2^2+\|\mathrm{curl}\ \mathbf{P}^\varepsilon(\cdot,t)\|_2^2+\|\mathbf{P}_{Nt}^\varepsilon(\cdot,t)\|_2^2\Big\}\le C, \tag{68}$$

$$\int_0^t \|\mathbf{M}_t^\varepsilon(\cdot,t)\|_2^2 d\tau \le C, \tag{69}$$

where C is a constant which is independent of ε and D.

Next, we need to prove that $\|\nabla\mathbf{P}^\varepsilon\|_{L^2(\Omega)}$ remains uniformly bounded within $L^\infty(0,T;L^2(\Omega))$. To achieve this, we study the compatibility conditions linked to the viscosity problem, which is defined by the equations below.

$$\partial_t(e^\varepsilon+p^\varepsilon)+\sigma e^\varepsilon=0, \tag{70}$$
$$\partial_t(h^\varepsilon+\beta\nabla\cdot\mathbf{M}^\varepsilon)=0, \tag{71}$$
$$\partial_t^2 p^\varepsilon+\mu\partial_t p^\varepsilon-\varepsilon\Delta p^\varepsilon-\nu e^\varepsilon+2\nu\Phi'(|\mathbf{P}^\varepsilon|^2)p^\varepsilon=-4\nu\Phi^{(2)}(|\mathbf{P}^\varepsilon|^2){\mathbf{P}^\varepsilon}_i{\mathbf{P}^\varepsilon}_j\frac{\partial\mathbf{P}_j^\varepsilon}{\partial x_i}, \tag{72}$$

where $e^\varepsilon=\mathrm{div}\ \mathbf{E}^\varepsilon$, $h^\varepsilon=\mathrm{div}\ \mathbf{P}^\varepsilon$, $p^\varepsilon=\mathrm{div}\ \mathbf{P}^\varepsilon$, ${\mathbf{P}^\varepsilon}_i$ is the i-th component of $\mathbf{P}^\varepsilon$ and the relation

$$\mathrm{div}(\Phi'(|\mathbf{P}^\varepsilon|^2)\mathbf{P}^\varepsilon)=\Phi'(|\mathbf{P}^\varepsilon|^2)p^\varepsilon+2\Phi^{(2)}(|\mathbf{P}^\varepsilon|^2){\mathbf{P}^\varepsilon}_i{\mathbf{P}^\varepsilon}_j\frac{\partial\mathbf{P}_j^\varepsilon}{\partial x_i}.$$

To obtain the $L^2(\Omega)$ estimate of $\nabla\mathbf{P}^\varepsilon(\cdot,t)$, we shall suppose that

$$\mathrm{div}(\mathbf{H}_0),\ \mathrm{div}(\mathbf{E}_0),\ \mathrm{div}(\mathbf{P}_0),\ \nabla\mathrm{div}(\mathbf{P}_0),\ \mathrm{div}(\partial_t\mathbf{P}_0)\in L^2(\Omega). \tag{73}$$

Define $\omega_n(x), n=1,2,\cdots$ as the unit eigenfunctions that satisfy the equations

$$\Delta\omega_n+\lambda_n\omega_n=0,\ \ \omega_n(x-De_i)=\omega_n(x+De_i),\ \ i=1,2,3,$$

where λ_n, $n=1,2,\cdots$ are the distinct eigenvalues associated with each eigenfunction. $\{\omega_n(x)\}$ consists of the orthogonal normal of $L^2(\Omega)$. Assume that the approximate solution of the problem (70)–(72) has the following form:

$$e_N^\varepsilon(x,t)=\sum_{s=1}^N\gamma_{sN}^\varepsilon(t)\omega_s(x),\ \ h_N^\varepsilon(x,t)=\sum_{s=1}^N\zeta_{sN}^\varepsilon(t)\omega_s(x),\ \ p_N^\varepsilon(x,t)=\sum_{s=1}^N\xi_{sN}^\varepsilon(t)\omega_s(x),$$

where $\gamma_{sN}^\varepsilon(t),\zeta_{sN}^\varepsilon(t),\xi_{sN}^\varepsilon(t)(s=1,2,\cdots,N,\quad N=1,2,\cdots)$ are determined by the following equations:

$$\int_\Omega(\partial_t(e_N^\varepsilon+p_N^\varepsilon)+\sigma e_N^\varepsilon)\omega_s(x)dx=0, \tag{74}$$

$$\int_\Omega(\partial_t(h_N^\varepsilon+\beta\nabla\cdot\mathbf{M}_N^\varepsilon))\omega_s(x)dx=0, \tag{75}$$

$$\int_\Omega(\partial_t^2 p_N^\varepsilon+\mu\partial_t p_N^\varepsilon-\varepsilon\Delta p_N^\varepsilon-\nu e_N^\varepsilon+2\nu\Phi'(|\mathbf{P}_N^\varepsilon|^2)p_N^\varepsilon)\omega_s(x)dx$$
$$=-4\nu\int_\Omega\Phi^{(2)}(|\mathbf{P}_N^\varepsilon|^2)\mathbf{P}_{Ni}^\varepsilon\mathbf{P}_{Nj}^\varepsilon\frac{\partial\mathbf{P}_{Nj}^\varepsilon}{\partial x_i}\omega_s(x)dx, \tag{76}$$

with initial conditions

$$\int_\Omega \mathbf{M}_N^\varepsilon(x,0)\omega_s(x)dx = \int_\Omega \mathbf{M}_0(x)\omega_s(x)dx, \int_\Omega e_N^\varepsilon(x,0)\omega_s(x)dx = \int_\Omega e_0(x)\omega_s(x)dx,$$
$$\int_\Omega h_N^\varepsilon(x,0)\omega_s(x)dx = \int_\Omega h_0(x)\omega_s(x)dx, \int_\Omega p_N^\varepsilon(x,0)\omega_s(x)dx = \int_\Omega p_0(x)\omega_s(x)dx, \quad (77)$$
$$\int_\Omega p_{Nt}^\varepsilon(x,0)\omega_s(x)dx = \int_\Omega \partial_t p_0(x)\omega_s(x)dx.$$

Multiplying (74) by $3\gamma_{sN}^\varepsilon(t)$ and $2(\gamma_{sN}^\varepsilon(t)+\xi_{sN}^\varepsilon(t))$, respectively, then summing up with respect to $s=1,2,\cdots,N$, we obtain

$$\frac{3}{2}\frac{d}{dt}\|e_N^\varepsilon\|^2 + 3\sigma\|e_N^\varepsilon\|_2^2 + 3\int_\Omega e_N^\varepsilon p_{Nt}^\varepsilon dx = 0, \quad (78)$$
$$\frac{d}{dt}\|e_N^\varepsilon + p_N^\varepsilon\|_2^2 + 2\sigma\|e_N^\varepsilon + p_N^\varepsilon\|_2^2 - 2\sigma\int_\Omega (|e_N^\varepsilon + p_N^\varepsilon|)p_N^\varepsilon dx = 0. \quad (79)$$

Multiplying (75) by $\xi_{sN}^\varepsilon(t)'$, then summing up for all $s=1,2,\cdots,N$, we derive

$$\frac{1}{2}\frac{d}{dt}\{\|\partial_t p_N^\varepsilon\|_2^2 + \varepsilon\|\nabla p_N^\varepsilon\|^2\} + \mu\|\partial_t p_N^\varepsilon\|_2^2 - \nu\int_\Omega e_N^\varepsilon\partial_t p_N^\varepsilon dx$$
$$+ 2\nu\int_\Omega \Phi'(|\mathbf{P}_N^\varepsilon|^2)p_N^\varepsilon\partial_t p_N^\varepsilon dx + 4\nu\int_\Omega \Phi^{(2)}(|\mathbf{P}_N^\varepsilon|^2)\mathbf{P}_{Ni}^\varepsilon\mathbf{P}_{Nj}^\varepsilon\frac{\partial \mathbf{P}_{Nj}^\varepsilon}{\partial x_i}\partial_t p_N^\varepsilon dx = 0. \quad (80)$$

From (78)–(80), we obtain

$$\frac{1}{2}\frac{d}{dt}\left\{3\|e_N^\varepsilon\|_2^2 + \|e_N^\varepsilon + p_N^\varepsilon\|_2^2 + \|\partial_t p_N^\varepsilon\|_2^2 + \varepsilon\|\nabla p_N^\varepsilon\|_2^2\right\}$$
$$+ \mu\|\partial_t p_N^\varepsilon\|_2^2 + 2\sigma\|e_N^\varepsilon + p_N^\varepsilon\|_2^2$$
$$=(\nu-3)\int_\Omega e_N^\varepsilon\partial_t p_N^\varepsilon dx - 2\nu\int_\Omega \Phi'(|\mathbf{P}_N^\varepsilon|^2)p_N^\varepsilon\partial_t p_N^\varepsilon dx -$$
$$- 4\nu\int_\Omega \Phi^{(2)}(|\mathbf{P}_N^\varepsilon|^2)\mathbf{P}_{Ni}^\varepsilon\mathbf{P}_{Nj}^\varepsilon\frac{\partial \mathbf{P}_{Nj}^\varepsilon}{\partial x_i}\partial_t p_N^\varepsilon dx + 2\sigma\int_\Omega (|e_N^\varepsilon + p_N^\varepsilon|)p_N^\varepsilon dx + 3\sigma\|e_N^\varepsilon\|_2^2$$
$$\le C(\|e_N^\varepsilon\|_2^2 + \|p_N^\varepsilon\|_2^2 + \|p_{Nt}^\varepsilon\|_2^2) + C_1\|\nabla\mathbf{P}^\varepsilon\|_2^2 + C_2. \quad (81)$$

Multiplying (75) by $\zeta_{sN}^\varepsilon(t)$, then summing up with respect to $s=1,2,\cdots,N$, and by Hölder inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\|h_N^\varepsilon\|_2^2 \le C\|h_N^\varepsilon\|_2^2. \quad (82)$$

By Lemma 1, we have $\|\mathbf{M}_N^\varepsilon\|_{H^1(\Omega)}^2 + \|\theta_N^\varepsilon\|_{L^2(\Omega)}^2 \le C$, with which we obtain

$$\sup_{0\le t\le T}\left\{\|\mathbf{M}_N^\varepsilon\|_{H^1(\Omega)}^2 + \|\theta_N^\varepsilon\|_{L^2(\Omega)}^2 + \|e_N^\varepsilon\|_2^2 + \|h_N^\varepsilon\|_2^2 + \|p_N^\varepsilon\|_2^2 + \|\partial_t p_N^\varepsilon\|_2^2 + \|\nabla p_N^\varepsilon\|_2^2\right\} \le C \quad (83)$$

from (81) and (82) and the Gronwall inequality.

Therefore, by the same method of the proof for Lemma 7, we can obtain the result that (74)–(77) has at least one global weak solution.

Integrating the inequality (81) with respect to t, we have

$$3\|e_N^\varepsilon(\cdot,t)\|_2^2 + 2\|e_N^\varepsilon(\cdot,t) + p_N^\varepsilon(\cdot,t)\|_2^2 + \|p_{Nt}^\varepsilon(\cdot,t)\|_2^2 + \varepsilon\|\nabla p_N^\varepsilon(\cdot,t)\|_2^2$$
$$\le 2C\int_0^t (\|e_N^\varepsilon\|_2^2 + \|p_N^\varepsilon\|_2^2 + \|p_{Nt}^\varepsilon\|_2^2)d\tau + 2C_1\int_0^t \|\nabla\mathbf{P}^\varepsilon\|_2^2 d\tau + C_2', \quad (84)$$

where $C_2' = 3\|\mathrm{div}(\mathbf{E}_0)\|_2^2 + 2\|\mathrm{div}(\mathbf{H}_0) + \mathrm{div}(\mathbf{P}_0)\|_2^2 + \|\mathrm{div}(\partial_t\mathbf{P}_0)\|_2^2 + \varepsilon\|\nabla\mathrm{div}(\mathbf{P}_0)\|_2^2 + 2C_2$ is a constant from hypothesis (73). Thus, by the inequality $\|p_N^\varepsilon(\cdot,t)\|_2^2 \le 2\|p_N^\varepsilon(\cdot,t) + e_N^\varepsilon(\cdot,t)\|_2^2 + 2\|p_N^\varepsilon(\cdot,t)\|_2^2$, we have

$$
\begin{aligned}
&\|e_N^\varepsilon(\cdot,t)\|_2^2+\|p_N^\varepsilon(\cdot,t)\|_2^2+\|p_{Nt}^\varepsilon(\cdot,t)\|_2^2+\varepsilon\|\nabla p_N^\varepsilon(\cdot,t)\|_2^2\\
\leq& 2C\int_0^t(\|e_N^\varepsilon\|_2^2+\|p_N^\varepsilon\|_2^2+\|p_{Nt}^\varepsilon\|_2^2)d\tau+2C_1\int_0^t\|\nabla \mathbf{P}^\varepsilon\|_2^2 d\tau+C_2',
\end{aligned}\tag{85}
$$

and by the Gronwall inequality, we find

$$
\begin{aligned}
&\|e_N^\varepsilon(\cdot,t)\|_2^2+\|p_N^\varepsilon(\cdot,t)\|_2^2+\|p_{Nt}^\varepsilon(\cdot,t)\|_2^2\\
\leq&(2C_1\int_0^t\|\nabla \mathbf{P}^\varepsilon\|_2^2 d\tau+C_2')(1+2Cte^{2Ct})\\
\leq& C_3+C_4\int_0^t\|\nabla \mathbf{P}^\varepsilon\|_2^2 d\tau.
\end{aligned}
$$

Therefore, we obtain the following estimate:

$$
\|p_N^\varepsilon(\cdot,t)\|_2^2\leq C_3+C_4\int_0^t\|\nabla \mathbf{P}^\varepsilon\|_2^2 d\tau. \tag{86}
$$

Using Lemma 8 for $\mathbf{P}^\varepsilon(\cdot,t)$, we have

$$
\begin{aligned}
\|\nabla \mathbf{P}^\varepsilon(\cdot,t)\|_2^2\leq& C\Big(\|\nabla\times \mathbf{P}^\varepsilon(\cdot,t)\|_2^2+\|\mathrm{div}\ \mathbf{P}^\varepsilon(\cdot,t)\|_2^2+\|\mathbf{P}^\varepsilon(\cdot,t)\|_2^2\Big)\\
\leq& C_5+C_6\int_0^t\|\nabla \mathbf{P}^\varepsilon\|_2^2 d\tau.
\end{aligned}
$$

By the Gronwall inequality, one obtains

$$
\|\nabla \mathbf{P}^\varepsilon(\cdot,t)\|_2^2\leq C, \tag{87}
$$

where C is independent of ε. Combing (83) and (87), we obtain that $\{\mathbf{P}^\varepsilon\}$ is uniformly bounded in $L^\infty(0,T;H^1(\Omega))$.

Proof of Theorem 1. By the above estimates, uniform in ε of $\mathbf{M}^\varepsilon,\theta^\varepsilon,\mathbf{E}^\varepsilon,\mathbf{H}^\varepsilon$, and $\mathbf{P}^\varepsilon$ for the viscosity problem, letting $\varepsilon\to 0$ in Equations (3)–(6), (8), (9), and (12), we can obtain the global weak solution of problem (3)–(9). The proof of Theorem 1 is completed. □

5. Conclusions

In this paper, we established the global existence of a weak solution to a phase transition model with polarization. To the best of our knowledge, our work seems to be the first rigorous treatment of a mathematical model for the dynamics of magnetization $\mathbf{M}$, absolute temperature θ, magnetic field $\mathbf{H}$, electric field $\mathbf{E}$, and electric polarization $\mathbf{P}$ arising in ferromagnetic–ferroelectric materials. From a mathematical perspective, the main challenge arises because Equation (7) lacks the compactness necessary to derive the H^1-norm for $\mathbf{P}$. In fact, from Equation (7), we can only obtain the $L^\infty(0,T;L^2(\Omega))$ estimates for curl$\mathbf{P}$ but cannot have the $L^\infty(0,T;L^2(\Omega))$ estimates for div$\mathbf{P}$. To overcome this difficulty, we apply the viscosity vanishing argument to obtain the weak solution for the following viscosity problem:

$$
\begin{aligned}
&\gamma\partial_t\mathbf{M}=\nu_1\Delta\mathbf{M}-\theta_c(|\mathbf{M}|^2-1)\mathbf{M}-\theta\mathbf{M}+\mathbf{H},\\
&c\partial_t\theta=\mathbf{M}\cdot\partial_t\mathbf{M}+k\Delta\theta+\hat{r},\\
&\nabla\times\mathbf{H}=\partial_t(\mathbf{E}+\mathbf{P})+\sigma\mathbf{E},\\
&\nabla\times\mathbf{E}=-\partial_t\mathbf{H}-\beta\partial_t\mathbf{M},\\
&\partial_t^2\mathbf{P}+\lambda^2\mathrm{curl}^2\mathbf{P}+\mu\partial_t\mathbf{P}-\varepsilon\Delta\mathbf{P}=\nu(\mathbf{E}-2\mathbf{P}\Phi'(|\mathbf{P}|^2)).
\end{aligned}
$$

Initially, we obtained a global weak solution for the viscosity problem with a fixed $\varepsilon > 0$. Subsequently, we derived additional a priori estimates for the div-component using a more regular class of weak solutions. This allowed us to establish uniform a priori estimates in ε for solutions to the viscosity problem, ensuring compactness in the limiting process. Finally, by passing to the limit as $\varepsilon \to 0$, we obtained the global weak solution to the original problem.

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