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On Higher-Order Generalized Fibonacci Hybrinomials: New Properties, Recurrence Relations and Matrix Representations

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Abstract: This paper presents a comprehensive survey of the generalization of hybrid numbers and hybrid polynomials, particularly in the fields of mathematics and physics. In this paper, by using higher-order generalized Fibonacci polynomials, we introduce higher-order generalized Fibonacci hybrid polynomials called higher-order generalized Fibonacci hybrinomials. We obtain some special cases and algebraic properties of the higher-order generalized Fibonacci hybrinomials, such as the recurrence relation, generating function, exponential generating function, Binet formula, Vajda's identity, Catalan's identity, Cassini's identity and d'Ocagne's identity. We also present three different matrices whose components are higher-order generalized Fibonacci hybrinomials, higher-order generalized Fibonacci polynomials and Lucas polynomials. By using these matrices, we obtain some identities related to these newly established hybrinomials.

Keywords: higher-order Fibonacci polynomials; generalized Fibonacci polynomials; hybrid numbers; hybrid polynomials; recurrence relation; matrix representation

MSC: 11B37; 11B39; 11R52; 05A15

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1. Introduction

The importance of certain integer sequences extends beyond the boundaries of pure and applied mathematics, permeating various scientific disciplines such as physics and engineering. One of the best known integer sequences is the Fibonacci sequence, named after the famous Italian mathematician Leonardo Pisano, more commonly known as Fibonacci. The Fibonacci sequence has many applications in various fields, including mathematics, physics and engineering. This has made it a subject of interest to many researchers [1,2]. The study of Fibonacci and Lucas numbers has been the subject of extensive research by mathematicians in the literature. The Fibonacci sequence $\{F_n\}$ and Lucas sequence $\{L_n\}$ are, respectively, defined by the following recurrence relations: for $n \in \mathbb{N} \cup \{0\}$ (here, $\mathbb{N}$ is the set of positive integers),

$$F_{n+2} = F_{n+1} + F_n$$

and

$$L_{n+2} = L_{n+1} + L_n,$$

where $F_0 = 0$, $F_1 = 1$, $L_0 = 2$ and $L_1 = 1$. The Binet formulas for the Fibonacci numbers and Lucas numbers are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and

$$L_n = \alpha^n + \beta^n,$$

where α and β are the roots of the characteristic equation $x^2 - x - 1 = 0$. One such generalization is the Fibonacci divisor, also known as higher-order Fibonacci numbers, which were studied by Pashaev and Özvatan (see, for details, [3,4]). Higher-order Fibonacci numbers (or Fibonacci divisor, conjugate to F_s) are defined for $s \geq 1$ integers as follows:

$$F_n^{(s)} = \frac{F_{ns}}{F_s} = \frac{(\alpha^s)^n - (\beta^s)^n}{\alpha^s - \beta^s}. \tag{1}$$

As F_{ns} is divisible by F_s, the ratio $\frac{F_{ns}}{F_s}$ is an integer. Therefore, all higher-order Fibonacci numbers, namely, $F_n^{(s)}$, are integers. For $s = 1$, the higher-order Fibonacci number $F_n^{(1)}$ becomes an ordinary Fibonacci number. For $s = 1, 2, 3, 4, 5$ and $n \in \mathbb{N}$, the first few numbers of the higher-order Fibonacci numbers $F_n^{(s)}$ are as follows:

- For $s = 1$, $F_n^{(1)} = F_n = 1, 1, 2, 3, \ldots$;
- For $s = 2$, $F_n^{(2)} = F_{2n} = 1, 3, 8, 21, \ldots$;
- For $s = 3$, $F_n^{(3)} = \frac{1}{2}F_{3n} = 1, 4, 17, 72, \ldots$;
- For $s = 4$, $F_n^{(4)} = \frac{1}{3}F_{4n} = 1, 7, 48, 329, \ldots$;
- For $s = 5$, $F_n^{(5)} = \frac{1}{5}F_{5n} = 1, 11, 122, 1353, \ldots$.

Pashaev [4] presented the critical characteristics of higher-order Fibonacci numbers and simultaneously demonstrated their application in various physical examples.

In [5], Horadam defined the general polynomial sequence of second order as

$$w_n(\gamma) = p(\gamma)w_{n-1}(\gamma) + q(\gamma)w_{n-2}(\gamma),$$

where

$$w_0(\gamma) = c_0,\ w_1(\gamma) = c_1\gamma^d,\ p(\gamma) = c_2\gamma^d,\ q(\gamma) = c_3\gamma^d$$

in which c_0, c_1, c_2, c_3 are constants and $d = 0$ or 1. Lee and Asci [6] defined and studied (p,q)-Fibonacci and (p,q)-Lucas polynomials extensively. These polynomials are also called generalized Fibonacci and Lucas polynomials. For $n \geq 2$, the genaralized Fibonacci and Lucas polynomials are defined by

$$U_{n+2}(\gamma) = p(\gamma)U_{n+1}(\gamma) - q(\gamma)U_n(\gamma), \tag{2}$$

$$V_{n+2}(\gamma) = p(\gamma)V_{n+1}(\gamma) - q(\gamma)V_n(\gamma), \tag{3}$$

where $U_0(\gamma) = 0$, $U_1(\gamma) = 1$, $V_0(\gamma) = 2$, and $V_1(\gamma) = p(\gamma)$. The generalized Fibonacci and Lucas polynomials can be expressed by

$$U_n(\gamma) = \frac{\eta^n(\gamma) - \theta^n(\gamma)}{\eta(\gamma) - \theta(\gamma)}, \tag{4}$$

$$V_n(\gamma) = \eta^n(\gamma) + \theta^n(\gamma), \tag{5}$$

where $\eta(\gamma)$ and $\theta(\gamma)$ are the roots of the characteristic equation $t^2 - p(\gamma)t + q(\gamma) = 0$ with $p^2(\gamma) - 4q(\gamma) \geqslant 0$.

Similar to Equation (1), one can present the higher-order generalized Fibonacci polynomial for $s \in \mathbb{N}$ as

$$U_n^{(s)}(\gamma) = \frac{U_{ns}(\gamma)}{U_s(\gamma)} = \frac{\eta^{sn}(\gamma) - \theta^{sn}(\gamma)}{\eta^s(\gamma) - \theta^s(\gamma)}, \tag{6}$$

where $U_0^{(s)}(\gamma) = 0$ and $U_1^{(s)}(\gamma) = 1$. For $\gamma = 1$, the higher-order generalized Fibonacci polynomial $U_n^{(s)}(\gamma)$ becomes the higher-order generalized Fibonacci numbers studied by Kızılateş and Kibar [7]. These polynomials have the following algebraic properties as follows:

- For $n \in \mathbb{N}$, the recurrence relation for higher-order generalized Fibonacci polynomials is as follows:
$$U_{n+1}^{(s)}(\gamma) = V_s(\gamma)U_n^{(s)}(\gamma) - q^s(\gamma)U_{n-1}^{(s)}(\gamma), \tag{7}$$
where $V_s(\gamma)$ are the generalized Lucas polynomials.
- The following identities hold for:
$$U_{-n}^{(s)}(\gamma) = -q^{sn}(\gamma)U_n^{(s)}(\gamma),$$
$$U_{-n}^{(-s)}(\gamma) = -q^s(\gamma)U_n^{(s)}(\gamma)$$
and
$$U_n^{(-s)}(\gamma) = q^{s(1-n)}(\gamma)U_n^{(s)}(\gamma).$$
- The generating function of higher-order generalized Fibonacci polynomials is
$$\sum_{n=0}^{\infty} U_n^{(s)}(\gamma)t^n = \frac{t}{1 - V_s(\gamma)t + q^s(\gamma)t^2}.$$

Complex, hyperbolic and dual numbers are well-known two-dimensional number systems that have been extensively studied in geometric, mathematical and physical applications by numerous researchers. Özdemir [8] was the first to define the set of hybrid numbers, denoted by $\mathbb{K}$, which includes both complex and dual numbers, as well as hyberbolic numbers. He also provided various theorems, properties and matrix forms related to hybrid numbers. A hybrid number is defined as

$$\mathbb{K} = \left\{ u + v\mathbf{i} + w\boldsymbol{\epsilon} + r\mathbf{h} : u, v, w, r \in \mathbb{R},\ \mathbf{i}^2 = -\mathbf{1},\ \boldsymbol{\epsilon}^2 = 0,\ \mathbf{h}^2 = \mathbf{1},\ \mathbf{ih} = -\mathbf{hi} = \boldsymbol{\epsilon} + \mathbf{i} \right\},$$

where $\mathbb{R}$ is the set of real numbers. With the hybrid numbers, we can perform some properties and operations. Namely, taking two hybrid numbers $\mathbf{Z}_1 = u_1 + v_1\mathbf{i} + w_1\boldsymbol{\epsilon} + r_1\mathbf{h}$ and $\mathbf{Z}_2 = u_2 + v_2\mathbf{i} + w_2\boldsymbol{\epsilon} + r_2\mathbf{h}$, we get

(a). $\mathbf{Z}_1 = \mathbf{Z}_2$ if and only if $u_1 = u_2$, $v_1 = v_2$, $w_1 = w_2$, $r_1 = r_2$ (equality);
(b). $\mathbf{Z}_1 + \mathbf{Z}_2 = (u_1 + u_2) + (v_1 + v_2)\mathbf{i} + (w_1 + w_2)\boldsymbol{\epsilon} + (r_1 + r_2)\mathbf{h}$ (addition);
(c). $\mathbf{Z}_1 - \mathbf{Z}_2 = (u_1 - u_2) + (v_1 - v_2)\mathbf{i} + (w_1 - w_2)\boldsymbol{\epsilon} + (r_1 - r_2)\mathbf{h}$ (subtraction);
(d). $k\mathbf{Z}_1 = ku_1 + kv_1\mathbf{i} + kw_1\boldsymbol{\epsilon} + kr_1\mathbf{h}$ (multiplication by scalar $k \in \mathbb{R}$).

The hybrid product is obtained by distributing the terms to the right while preserving the order of unit multiplication. Then, the values of the following are written, substituting each product of units with the corresponding equalities: $\mathbf{i}^2 = -\mathbf{1}$, $\boldsymbol{\epsilon}^2 = 0$, $\mathbf{h}^2 = \mathbf{1}$, $\mathbf{ih} = -\mathbf{hi} = \boldsymbol{\epsilon} + \mathbf{i}$. From these equalities, we can obtain the product of any two hybrid units. Table 1 shows the multiplication table for the basis of hybrid numbers.

Table 1. Multiplication table for $\mathbb{K}$.

	$\mathbf{1}$	$\mathbf{i}$	$\boldsymbol{\epsilon}$	$\mathbf{h}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{i}$	$\boldsymbol{\epsilon}$	$\mathbf{h}$
$\mathbf{i}$	$\mathbf{i}$	$-\mathbf{1}$	$\mathbf{1} - \mathbf{h}$	$\boldsymbol{\epsilon} + \mathbf{i}$
$\boldsymbol{\epsilon}$	$\boldsymbol{\epsilon}$	$\mathbf{h} + \mathbf{1}$	0	$-\boldsymbol{\epsilon}$
$\mathbf{h}$	$\mathbf{h}$	$-\boldsymbol{\epsilon} - \mathbf{i}$	$\boldsymbol{\epsilon}$	$\mathbf{1}$

Let $\mathbf{Z} = u + v\mathbf{i} + w\boldsymbol{\epsilon} + r\mathbf{h}$ be any hybrid number. The conjugate of $\mathbf{Z}$ is defined by

$$\bar{\mathbf{Z}} = u - v\mathbf{i} - w\boldsymbol{\epsilon} - r\mathbf{h}.$$

The real number

$$C(\mathbf{Z}) = \mathbf{Z}\,\bar{\mathbf{Z}} = \bar{\mathbf{Z}}\,\mathbf{Z} = u^2 + (v - w)^2 - w^2 - r^2 = u^2 + v^2 - 2vw - r^2,$$

is called the character of the hybrid number $\mathbf{Z}$. Hybrid numbers are classified as space-like, time-like, or light-like based on the character of $\mathbf{Z}$. Specifically, a hybrid number is space-like if $C(\mathbf{Z})$ is less than zero, time-like if $C(\mathbf{Z})$ is greater than zero and light-like if $C(\mathbf{Z})$ equals zero. It is important to note that hybrid number multiplication is not commutative, but it does have the property of associativity. The set of hybrid numbers forms a non-commutative ring under addition and multiplication (please see [8]).

After Özdemir's paper, hybrid numbers, whose components are defined by the homogeneous recurrence relation with constant coefficients, have been studied by a large number of researchers since 2018 (please see [9–15]). In [16], Kızılateş and Kone introduced Fibonacci divisor hybrid numbers that generalize the Fibonacci hybrid numbers defined by Szynal-Liana and Wloch [9]. The Fibonacci divisor hybrid numbers (or higher-order Fibonacci hybrid numbers) are defined by

$$F\mathbb{H}_n^{(s)} = F_n^{(s)} + F_{n+1}^{(s)}\mathbf{i} + F_{n+2}^{(s)}\boldsymbol{\epsilon} + F_{n+3}^{(s)}\mathbf{h}.$$

They also gave some properties of these numbers. Szynal-Liana and Wloch [17] defined and studied a family of special polynomials and special numbers related to the Fibonacci and Lucas hybrid polynomials, namely Fibonacci and Lucas hybrinomials. In [18], Szynal-Liana et al. defined the Pell hybrinomials and gave some properties of them. Then, Kızılateş [19] defined and studied Horadam hybrinomials which are a generalization of Fibonacci and Lucas hybrinomials. The Horadam hybrinomials are defined by

$$\mathbb{H}_n(x) = h_n(x) + h_{n+1}(x)\mathbf{i} + h_{n+2}(x)\boldsymbol{\epsilon} + h_{n+3}(x)\mathbf{h}. \tag{8}$$

where $h_n(x) = h_n(x; a, b; p, q)$ are the Horadam polynomials [5] defined by

$$h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x), \quad n \geq 3 \tag{9}$$

with the initial values $h_1(x) = a$ and $h_2(x) = bx$. For some other papers, please see [20–23].

Motivated by the above papers, especially articles [17–19], in this paper, we define higher-order generalized Fibonacci hybrinomials. Our definition includes not only Fibonacci and Fibonacci-type hybrinomials existing in the literature, but also new hybrid polynomials and numbers depending on the parameter s that generalize these numbers and polynomials. We give the recurrence relation, the generating functions, and the Binet-like formula of higher-order Fibonacci hybrid polynomials. We also give some identities such as Vajda's identity and its special cases. In the third part, we define some matrices whose elements are higher-order generalized Fibonacci hybrinomials and higher-order generalized Fibonacci and Lucas polynomials. We give the relations between these matrices. We also use these relations to obtain some identities for the newly defined family of hybrid polynomials and numbers.

2. Higher-Order Generalized Fibonacci Hybrinomials

In this paper, we define higher-order generalized Fibonacci hybrinomials as follows.

Definition 1. *For $n \in \mathbb{N} \cup \{0\}$, the n^{th} higher-order generalized Fibonacci hybrinomials are defined by*

$$\mathbb{KU}_n^{(s)}(\gamma) = U_n^{(s)}(\gamma) + U_{n+1}^{(s)}(\gamma)\mathbf{i} + U_{n+2}^{(s)}(\gamma)\boldsymbol{\epsilon} + U_{n+3}^{(s)}(\gamma)\mathbf{h}.$$

In fact, higher-order generalized Fibonacci hybrinomials contain several important hybrinomials and hybrid numbers as special cases. We give the following to illustrate the facts:

- For $p(\gamma) = \gamma$ and $q(\gamma) = -1$, the higher-order generalized Fibonacci hybrinomials $\mathbb{KU}_n^{(s)}(\gamma)$ become the higher-order Fibonacci hybrinomials $\mathbb{KF}_n^{(s)}(\gamma)$;

- For $p(\gamma) = 2\gamma$ and $q(\gamma) = -1$, the higher-order generalized Fibonacci hybrinomials $\mathbb{KU}_n^{(s)}(\gamma)$ become the higher-order Pell hybrinomials $\mathbb{KP}_n^{(s)}(\gamma)$;
- For $p(\gamma) = 1$ and $q(\gamma) = -1$, the higher-order generalized Fibonacci hybrinomials $\mathbb{KU}_n^{(s)}$ become the higher-order Fibonacci hybrid numbers $\mathbb{KF}_n^{(s)}$ [16];
- For $p(\gamma) = 2$ and $q(\gamma) = -1$, the higher-order generalized Fibonacci hybrinomials $\mathbb{KU}_n^{(s)}$ become the higher-order Pell hybrid numbers $\mathbb{KP}_n^{(s)}$;
- For $p(\gamma) = 1$ and $q(\gamma) = -2\gamma$, the higher-order generalized Fibonacci hybrinomials $\mathbb{KU}_n^{(s)}$ become the higher-order Jacobsthal hybrinomials $\mathbb{KJ}_n^{(s)}(\gamma)$;
- For $p(\gamma) = 1$ and $q(\gamma) = -2$, the higher-order generalized Fibonacci hybrinomials $\mathbb{KU}_n^{(s)}$ become the higher-order Jacobsthal hybrid numbers $\mathbb{KJ}_n^{(s)}$.

The following Lemma is crucial to our main results.

Lemma 1. *Let $\widetilde{\eta}(\gamma)$ and $\widetilde{\theta}(\gamma)$ be defined as follows:*

$$\widetilde{\eta}(\gamma) := 1 + \eta^s(\gamma)\mathbf{i} + \eta^{2s}(\gamma)\boldsymbol{\epsilon} + \eta^{3s}(\gamma)\mathbf{h}$$

and

$$\widetilde{\theta}(\gamma) := 1 + \theta^s(\gamma)\mathbf{i} + \theta^{2s}(\gamma)\boldsymbol{\epsilon} + \theta^{3s}(\gamma)\mathbf{h}.$$

Then, we have

$$\widetilde{\eta}(\gamma)\widetilde{\theta}(\gamma) = \Phi - \Omega q^s(\gamma)\boldsymbol{\omega} \tag{10}$$

and

$$\widetilde{\theta}(\gamma)\widetilde{\eta}(\gamma) = \Phi + \Omega q^s(\gamma)\boldsymbol{\omega}, \tag{11}$$

where

$$\Omega = (\eta^s(\gamma) - \theta^s(\gamma)),$$
$$\Phi = 1 - q^s(\gamma) + q^{3s}(\gamma) + q^s(\gamma)V_s(\gamma) + V_s(\gamma)\mathbf{i} + V_{2s}(\gamma)\boldsymbol{\epsilon} + V_{3s}(\gamma)\mathbf{h}$$

and

$$\boldsymbol{\omega} = U_2^{(s)}(\gamma)\mathbf{i} + U_2^{(s)}(\gamma)\boldsymbol{\epsilon} - \mathbf{h}.$$

Proof. We first prove (10). Due to Table 1, we can compute

$$\begin{aligned}
\widetilde{\eta}(\gamma)\widetilde{\theta}(\gamma) &= \Big(1 + \eta^s(\gamma)\mathbf{i} + \eta^{2s}(\gamma)\boldsymbol{\epsilon} + \eta^{3s}(\gamma)\mathbf{h}\Big)\Big(1 + \theta^s(\gamma)\mathbf{i} + \theta^{2s}(\gamma)\boldsymbol{\epsilon} + \theta^{3s}(\gamma)\mathbf{h}\Big) \\
&= 1 + \theta^s(\gamma)\mathbf{i} + \theta^{2s}(\gamma)\boldsymbol{\epsilon} + \theta^{3s}(\gamma)\mathbf{h} \\
&\quad + \eta^s(\gamma)\mathbf{i}\Big(1 + \theta^s(\gamma)\mathbf{i} + \theta^{2s}(\gamma)\boldsymbol{\epsilon} + \theta^{3s}(\gamma)\mathbf{h}\Big) \\
&\quad + \eta^{2s}(\gamma)\boldsymbol{\epsilon}\Big(1 + \theta^s(\gamma)\mathbf{i} + \theta^{2s}(\gamma)\boldsymbol{\epsilon} + \theta^{3s}(\gamma)\mathbf{h}\Big) \\
&\quad + \eta^{3s}(\gamma)\mathbf{h}\Big(1 + \theta^s(\gamma)\mathbf{i} + \theta^{2s}(\gamma)\boldsymbol{\epsilon} + \theta^{3s}(\gamma)\mathbf{h}\Big) \\
&= 1 - q^s(\gamma) + \eta^s(\gamma)\theta^s(\gamma)(\theta^s(\gamma) + \eta^s(\gamma)) + q^{3s}(\gamma) \\
&\quad + \Big(V_s(\gamma) + \eta^s(\gamma)\theta^s(\gamma)\Big(\theta^{2s}(\gamma) - \eta^{2s}(\gamma)\Big)\Big)\mathbf{i} \\
&\quad + \Big(V_{2s}(\gamma) + \eta^s(\gamma)\theta^s(\gamma)\Big(\theta^{2s}(\gamma) - \eta^{2s}(\gamma)\Big) - \eta^{2s}(\gamma)\theta^{2s}(\gamma)(\theta^s(\gamma) - \eta^s(\gamma))\Big)\boldsymbol{\epsilon} \\
&\quad + (V_{3s}(\gamma) - \eta^s(\gamma)\theta^s(\gamma)(\theta^s(\gamma) - \eta^s(\gamma)))\mathbf{h} \\
&= 1 - q^s(\gamma) + q^s(\gamma)V_s(\gamma) + q^{3s}(\gamma) \\
&\quad + \Big(V_s(\gamma) - q^s(\gamma)\Big(\eta^{2s}(\gamma) - \theta^{2s}(\gamma)\Big)\Big)\mathbf{i} \\
&\quad + \Big(V_{2s}(\gamma) - q^s(\gamma)\Big(\eta^{2s}(\gamma) - \theta^{2s}(\gamma)\Big) + q^{2s}(\gamma)(\eta^s(\gamma) - \theta^s(\gamma))\Big)\boldsymbol{\epsilon} \\
&\quad + (V_{3s}(\gamma) + q^s(\gamma)(\eta^s(\gamma) - \theta^s(\gamma)))\mathbf{h} \\
&= 1 - q^s(\gamma) + q^{3s}(\gamma) + q^s(\gamma)V_s(\gamma)
\end{aligned}$$

$$+\left(V_s(\gamma) - q^s(\gamma)\left(\eta^{2s}(\gamma) - \theta^{2s}(\gamma)\right)\frac{\eta^s(\gamma) - \theta^s(\gamma)}{\eta^s(\gamma) - \theta^s(\gamma)}\right)\mathbf{i}$$
$$+\left(V_{2s}(\gamma) - q^s(\gamma)\left(\eta^{2s}(\gamma) - \theta^{2s}(\gamma)\right)\frac{\eta^s(\gamma) - \theta^s(\gamma)}{\eta^s(\gamma) - \theta^s(\gamma)} + q^{2s}(\gamma)(\eta^s(\gamma) - \theta^s(\gamma))\right)\boldsymbol{\epsilon}$$
$$+(V_{3s}(\gamma) + q^s(\gamma)(\eta^s(\gamma) - \theta^s(\gamma)))\mathbf{h}.$$

Substituting

$$(\eta^s(\gamma) - \theta^s(\gamma)) \to \Omega,$$
$$1 - q^s(\gamma) + q^{3s}(\gamma) + q^s(\gamma)V_s(\gamma) + V_s(\gamma)\mathbf{i} + V_{2s}(\gamma)\boldsymbol{\epsilon} + V_{3s}(\gamma)\mathbf{h} \to \Phi$$

and

$$U_2^{(s)}(\gamma)\mathbf{i} + \left(U_2^{(s)}(\gamma) - q^s(\gamma)\right)\boldsymbol{\epsilon} - \mathbf{h} \to \boldsymbol{\omega},$$

in the last equation, we conclude

$$\widetilde{\eta}(\gamma)\widetilde{\theta}(\gamma) = \Phi - \Omega q^s(\gamma)\boldsymbol{\omega}.$$

Following a similar argument as the proof of (10), one can verify (11). The proof of Lemma 1 is complete. □

Next, we give the Binet formula for higher-order generalized Fibonacci hybrinomials of parameter s. Using this formula, we derive some properties of these hybrinomials.

Theorem 1. *Let $n \in \mathbb{N} \cup \{0\}$. The Binet formula for the higher-order generalized Fibonacci hybrinomials $\mathbb{KU}_n^{(s)}(\gamma)$ is*

$$\mathbb{KU}_n^{(s)}(\gamma) = \frac{\widetilde{\eta}(\gamma)(\eta^s(\gamma))^n - \widetilde{\theta}(\gamma)(\theta^s(\gamma))^n}{\eta^s(\gamma) - \theta^s(\gamma)}. \tag{12}$$

Proof. By using (6), we obtain

$$\begin{aligned}
\mathbb{KU}_n^{(s)} &= U_n^{(s)}(\gamma) + U_{n+1}^{(s)}(\gamma)\mathbf{i} + U_{n+2}^{(s)}(\gamma)\boldsymbol{\epsilon} + U_{n+3}^{(s)}(\gamma)\mathbf{h} \\
&= \left(\frac{\eta^{sn}(\gamma) - \theta^{sn}(\gamma)}{\eta^s(\gamma) - \theta^s(\gamma)}\right) + \frac{\eta^{s(n+1)}(\gamma) - \theta^{s(n+1)}(\gamma)}{\eta^s(\gamma) - \theta^s(\gamma)}\mathbf{i} \\
&\quad + \left(\frac{\eta^{s(n+2)}(\gamma) - \theta^{s(n+2)}(\gamma)}{\eta^s(\gamma) - \theta^s(\gamma)}\right)\boldsymbol{\epsilon} + \left(\frac{\eta^{s(n+3)}(\gamma) - \theta^{s(n+3)}(\gamma)}{\eta^s(\gamma) - \theta^s(\gamma)}\right)\mathbf{h} \\
&= \frac{1}{\eta^s(\gamma) - \theta^s(\gamma)}\left[\begin{array}{c} \eta^{sn}(\gamma)\left(1 + \eta^s(\gamma)\mathbf{i} + (\eta^s(\gamma))^2\boldsymbol{\epsilon} + (\eta^s(\gamma))^3\mathbf{h}\right) \\ -\theta^{sn}(\gamma)\left(1 + (\theta^s(\gamma))\mathbf{i} + (\theta^s(\gamma))^2\boldsymbol{\epsilon} + (\theta^s(\gamma))^3\mathbf{h}\right) \end{array}\right] \\
&= \frac{\widetilde{\eta}(\gamma)(\eta^s(\gamma))^n - \widetilde{\theta}(\gamma)(\theta^s(\gamma))^n}{\eta^s(\gamma) - \theta^s(\gamma)}.
\end{aligned}$$

The proof is complete. □

Theorem 2. *The generating function for the higher-order generalized Fibonacci hybrinomials $\mathbb{KU}_n^{(s)}(\gamma)$ is*

$$\sum_{n=0}^{\infty} \mathbb{KU}_n^{(s)}(\gamma)t^n = \frac{\widetilde{\eta}(\gamma) - \widetilde{\theta}(\gamma) - t(\widetilde{\eta}(\gamma)\theta^s(\gamma) - \widetilde{\theta}(\gamma)\eta^s(\gamma))}{(\eta^s(\gamma) - \theta^s(\gamma))(1 - V_s(\gamma)t + q^s(\gamma)t^2)}. \tag{13}$$

Proof. Using (12) and (5), we find that

$$\sum_{n=0}^{\infty} \mathbb{KU}_n^{(s)}(\gamma)t^n = \sum_{n=0}^{\infty} \frac{\widetilde{\eta}(\gamma)\eta^{sn}(\gamma) - \widetilde{\theta}(\gamma)\theta^{sn}(\gamma)}{\eta^s(\gamma) - \theta^s(\gamma)}t^n$$

$$
\begin{aligned}
&= \frac{\widetilde{\eta}(\gamma)}{\eta^s(\gamma)-\theta^s(\gamma)}\sum_{n=0}^{\infty}\eta^{sn}(\gamma)t^n - \frac{\widetilde{\theta}(\gamma)}{\eta^s(\gamma)-\theta^s(\gamma)}\sum_{n=0}^{\infty}\theta^{sn}(\gamma)t^n \\
&= \frac{1}{\eta^s(\gamma)-\theta^s(\gamma)}\left(\frac{\widetilde{\eta}(\gamma)}{1-\eta^s(\gamma)t} - \frac{\widetilde{\theta}(\gamma)}{1-\theta^s(\gamma)t}\right) \\
&= \frac{1}{\eta^s(\gamma)-\theta^s(\gamma)}\frac{\widetilde{\eta}(\gamma)-\widetilde{\eta}(\gamma)\theta^s(\gamma)t-\widetilde{\theta}(\gamma)+\widetilde{\theta}(\gamma)\eta^s(\gamma)t}{(1-\eta^s(\gamma)t)(1-\theta^s(\gamma)t)} \\
&= \frac{1}{\eta^s(\gamma)-\theta^s(\gamma)}\frac{\widetilde{\eta}(\gamma)-\widetilde{\theta}(\gamma)-t\left(\widetilde{\eta}(\gamma)\theta^s(\gamma)-\widetilde{\theta}(\gamma)\eta^s(\gamma)\right)}{1-(\theta^s(\gamma)+\eta^s(\gamma))t+\eta^s(\gamma)\theta^s(\gamma)t^2} \\
&= \frac{\widetilde{\eta}(\gamma)-\widetilde{\theta}(\gamma)-t(\widetilde{\eta}(\gamma)\theta^s(\gamma)-\widetilde{\theta}(\gamma)\eta^s(\gamma))}{(\eta^s(\gamma)-\theta^s(\gamma))(1-V_s(\gamma)t+q^s(\gamma)t^2)}.
\end{aligned}
$$

The proof is complete. □

Theorem 3. *For $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, the generating function of the higher-order generalized Fibonacci hybrinomials $\mathbb{KU}^{(s)}_{n+m}(\gamma)$ is*

$$
\sum_{n=0}^{\infty}\mathbb{KU}^{(s)}_{n+m}(\gamma)t^n = \frac{\mathbb{KU}^{(s)}_{m}(\gamma)-\mathbb{KU}^{(s)}_{m-1}(\gamma)q^s(\gamma)t}{1-V_s(\gamma)t+q^s(\gamma)t^2}.
$$

Proof. By virtue of (12) and after some calculations, we have

$$
\begin{aligned}
\sum_{n=0}^{\infty}\mathbb{KU}^{(s)}_{n+m}(\gamma)t^n &= \sum_{n=0}^{\infty}\left(\frac{\widetilde{\eta}(\gamma)\eta^{s(n+m)}(\gamma)-\widetilde{\theta}(\gamma)\theta^{s(n+m)}(\gamma)}{\eta^s(\gamma)-\theta^s(\gamma)}\right)t^n \\
&= \frac{1}{\eta^s(\gamma)-\theta^s(\gamma)}\left(\sum_{n=0}^{\infty}\widetilde{\eta}(\gamma)\eta^{s(n+m)}(\gamma)t^n-\sum_{n=0}^{\infty}\widetilde{\theta}(\gamma)\theta^{s(n+m)}(\gamma)t^n\right) \\
&= \frac{1}{\eta^s(\gamma)-\theta^s(\gamma)}\left(\widetilde{\eta}(\gamma)\eta^{sm}(\gamma)\sum_{n=0}^{\infty}\eta^{sn}(\gamma)t^n-\widetilde{\theta}(\gamma)\theta^{sm}(\gamma)\sum_{n=0}^{\infty}\theta^{sn}(\gamma)t^n\right) \\
&= \frac{1}{\eta^s(\gamma)-\theta^s(\gamma)}\left(\frac{\widetilde{\eta}(\gamma)\eta^{sm}(\gamma)(1-\theta^s(\gamma)t)-\widetilde{\theta}(\gamma)\theta^{sm}(\gamma)(1-\eta^s(\gamma)t)}{(1-\eta^s(\gamma)t)(1-\theta^s(\gamma)t)}\right) \\
&= \frac{\widetilde{\eta}(\gamma)\eta^{sm}(\gamma)-\widetilde{\theta}(\gamma)\theta^{sm}(\gamma)}{\eta^s(\gamma)-\theta^s(\gamma)}\frac{1}{1-V_s(\gamma)t+q^s(\gamma)t^2} \\
&\quad -\frac{\eta^s(\gamma)\theta^s(\gamma)t\left(\widetilde{\eta}(\gamma)\eta^{s(m-1)}(\gamma)-\widetilde{\theta}(\gamma)\theta^{s(m-1)}(\gamma)\right)}{(\eta^s(\gamma)-\theta^s(\gamma))(1-V_s(\gamma)t+q^s(\gamma)t^2)} \\
&= \frac{\mathbb{KU}^{(s)}_{m}(\gamma)}{1-V_s(\gamma)t+q^s(\gamma)t^2}-\frac{\left(\widetilde{\eta}(\gamma)\eta^{s(m-1)}(\gamma)-\widetilde{\theta}(\gamma)\theta^{s(m-1)}(\gamma)\right)}{(\eta^s(\gamma)-\theta^s(\gamma))} \\
&\quad \times\frac{q^s(\gamma)t}{(1-V_s(\gamma)t+q^s(\gamma)t^2)} \\
&= \frac{\mathbb{KU}^{(s)}_{m}(\gamma)-\mathbb{KU}^{(s)}_{m-1}(\gamma)q^s(\gamma)t}{1-V_s(\gamma)t+q^s(\gamma)t^2}.
\end{aligned}
$$

The proof is complete. □

Theorem 4. *The exponential generating function for the higher-order generalized Fibonacci hybrinomials $\mathbb{KU}_n^{(s)}(\gamma)$ is*

$$\sum_{n=0}^{\infty}\mathbb{KU}_n^{(s)}(\gamma)\frac{t^n}{n!}=\frac{\widetilde{\eta}(\gamma)e^{\eta^s(\gamma)t}-\widetilde{\theta}(\gamma)e^{\theta^s(\gamma)t}}{\eta^s(\gamma)-\theta^s(\gamma)}. \tag{14}$$

Proof. Using the Binet formula for the $\mathbb{KU}_n^{(s)}(\gamma)$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathbb{KU}_n^{(s)}(\gamma)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\left(\frac{\widetilde{\eta}(\gamma)\eta^{sn}(\gamma)-\widetilde{\theta}(\gamma)\theta^{sn}(\gamma)}{\eta^s(\gamma)-\theta^s(\gamma)}\right)\frac{t^n}{n!}\\
&= \frac{1}{\eta^s(\gamma)-\theta^s(\gamma)}\left(\widetilde{\eta}(\gamma)\sum_{n=0}^{\infty}\frac{\eta^{sn}(\gamma)}{n!}t^n-\widetilde{\theta}(\gamma)\sum_{n=0}^{\infty}\frac{\theta^{sn}(\gamma)}{n!}t^n\right)\\
&= \frac{1}{\eta^s(\gamma)-\theta^s(\gamma)}\left(\widetilde{\eta}(\gamma)e^{\eta^s(\gamma)t}-\widetilde{\theta}(\gamma)e^{\theta^s(\gamma)t}\right)\\
&= \frac{\widetilde{\eta}(\gamma)e^{\eta^s(\gamma)t}-\widetilde{\theta}(\gamma)e^{\theta^s(\gamma)t}}{\eta^s(\gamma)-\theta^s(\gamma)}.
\end{aligned}$$

The proof is complete. □

Theorem 5. *For $n\in\mathbb{N}$, the recurrence relation for the higher-order generalized Fibonacci hybrinomials $\mathbb{KU}_n^{(s)}(\gamma)$ is*

$$\mathbb{KU}_{n+1}^{(s)}(\gamma)=\mathbb{KU}_n^{(s)}(\gamma)V_s(\gamma)-q^s(\gamma)\mathbb{KU}_{n-1}^{(s)}(\gamma). \tag{15}$$

Proof. Applying (12) and (5) gives

$$\begin{aligned}
\mathbb{KU}_{n+1}^{(s)}(\gamma) &= \frac{\widetilde{\eta}(\gamma)\eta^{s(n+1)}(\gamma)-\widetilde{\theta}(\gamma)\theta^{s(n+1)}(\gamma)}{\eta^s(\gamma)-\theta^s(\gamma)}\\
&= \frac{1}{\eta^s(\gamma)-\theta^s(\gamma)}\left(\widetilde{\eta}(\gamma)\eta^{sn}(\gamma)\eta^s(\gamma)-\widetilde{\theta}(\gamma)\theta^{sn}(\gamma)\theta^s(\gamma)\right)\\
&= \frac{1}{\eta^s(\gamma)-\theta^s(\gamma)}\left[\begin{array}{c}\widetilde{\eta}(\gamma)\eta^{sn}(\gamma)\eta^s(\gamma)-\widetilde{\theta}(\gamma)\theta^{sn}(\gamma)\eta^s(\gamma)\\ +\widetilde{\theta}(\gamma)\theta^{sn}(\gamma)\eta^s(\gamma)-\widetilde{\theta}(\gamma)\theta^{sn}(\gamma)\theta^s(\gamma)\end{array}\right]\\
&= \frac{1}{\eta^s(\gamma)-\theta^s(\gamma)}\left(\widetilde{\eta}(\gamma)\eta^{sn}(\gamma)-\widetilde{\theta}(\gamma)\theta^{sn}(\gamma)\right)\eta^s(\gamma)\\
&\quad+\frac{1}{\eta^s(\gamma)-\theta^s(\gamma)}\left(\widetilde{\theta}(\gamma)\theta^{sn}(\gamma)\eta^s(\gamma)-\widetilde{\theta}(\gamma)\theta^{sn}(\gamma)\theta^s(\gamma)\right)\\
&= \mathbb{KU}_n^{(s)}(\gamma)\eta^s(\gamma)+\frac{1}{\eta^s(\gamma)-\theta^s(\gamma)}\left(\widetilde{\theta}(\gamma)\theta^{sn}(\gamma)\eta^s(\gamma)-\widetilde{\theta}(\gamma)\theta^{sn}(\gamma)\theta^s(\gamma)\right)\\
&= \mathbb{KU}_n^{(s)}(\gamma)(\eta^s(\gamma)+\theta^s(\gamma))-\mathbb{KU}_n^{(s)}(\gamma)\theta^s(\gamma)\\
&\quad+\frac{1}{\eta^s(\gamma)-\theta^s(\gamma)}\left[\widetilde{\theta}(\gamma)\theta^{sn}(\gamma)\eta^s(\gamma)-\widetilde{\theta}(\gamma)\theta^{sn}(\gamma)\theta^s(\gamma)\right]\\
&= \mathbb{KU}_n^{(s)}(\gamma)V_s(\gamma)+\frac{1}{\eta^s(\gamma)-\theta^s(\gamma)}\left(\begin{array}{c}\widetilde{\theta}(\gamma)\theta^{sn}(\gamma)\theta^s(\gamma)-\widetilde{\eta}(\gamma)\eta^{sn}(\gamma)\theta^s(\gamma)\\ +\widetilde{\theta}(\gamma)\theta^{sn}(\gamma)\eta^s(\gamma)-\widetilde{\theta}(\gamma)\theta^{sn}(\gamma)\theta^s(\gamma)\end{array}\right)\\
&= \mathbb{KU}_n^{(s)}(\gamma)V_s(\gamma)+\frac{1}{\eta^s(\gamma)-\theta^s(\gamma)}\left[-\widetilde{\eta}(\gamma)\eta^{sn}(\gamma)\theta^s(\gamma)+\widetilde{\theta}(\gamma)\theta^{sn}(\gamma)\eta^s(\gamma)\right]\\
&= \mathbb{KU}_n^{(s)}(\gamma)V_s(\gamma)+\frac{(\eta(\gamma)\theta(\gamma))^s}{\eta^s(\gamma)-\theta^s(\gamma)}\left[\widetilde{\theta}(\gamma)\theta^{s(n-1)}(\gamma)-\widetilde{\eta}(\gamma)\eta^{s(n-1)}(\gamma)\right]\\
&= \mathbb{KU}_n^{(s)}(\gamma)V_s(\gamma)-q^s(\gamma)\left[\frac{\widetilde{\eta}(\gamma)(\eta^s(\gamma))^{n-1}-\widetilde{\theta}(\gamma)(\theta^s(\gamma))^{n-1}}{\eta^s(\gamma)-\theta^s(\gamma)}\right]
\end{aligned}$$

$$= \mathbb{KU}_{n}^{(s)}(\gamma)V_s(\gamma) - q^s(\gamma)\mathbb{KU}_{n-1}^{(s)}(\gamma).$$

The required proof is complete. □

Table 1 shows us that the multiplication operation in the hybrid numbers is not commutative. Starting from this, we now give Vajda's identity, which generalizes Catalan's, Cassini's, and d'Ocagne's identities that are well known in the literature.

Theorem 6 (Vajda's identity)**.** *For any integers n, m and r, we have*

$$\mathbb{KU}_{n+m}^{(s)}(\gamma)\mathbb{KU}_{n+r}^{(s)}(\gamma) - \mathbb{KU}_{n}^{(s)}(\gamma)\mathbb{KU}_{n+m+r}^{(s)}(\gamma) = q^{sn}(\gamma)U_m^{(s)}(\gamma)\left[\Phi U_r^{(s)}(\gamma) + q^s(\gamma)\boldsymbol{\omega}V_{sr}(\gamma)\right]. \quad (16)$$

Proof. Using Binet formula for the higher-order generalized Fibonacci hybrinomials, we have

$$\begin{aligned}
&\mathbb{KU}_{n+m}^{(s)}(\gamma)\mathbb{KU}_{n+r}^{(s)}(\gamma) - \mathbb{KU}_{n}^{(s)}(\gamma)\mathbb{KU}_{n+m+r}^{(s)}(\gamma) \\
&= \frac{1}{(\eta^s(\gamma)-\theta^s(\gamma))^2}\begin{bmatrix} -\widetilde{\eta}(\gamma)\widetilde{\theta}(\gamma)(\eta^s(\gamma))^{n+m}(\theta^s(\gamma))^{n+r} - \widetilde{\theta}(\gamma)\widetilde{\eta}(\gamma)(\theta^s(\gamma))^{n+m}(\eta^s(\gamma))^{n+r} \\ +\widetilde{\eta}(\gamma)\widetilde{\theta}(\gamma)(\eta^s(\gamma))^{n}(\theta^s(\gamma))^{n+m+r} + \widetilde{\theta}(\gamma)\widetilde{\eta}(\gamma)(\theta^s(\gamma))^{n}(\eta^s(\gamma))^{n+m+r} \end{bmatrix} \\
&= \frac{(\eta^s(\gamma)\theta^s(\gamma))^n}{(\eta^s(\gamma)-\theta^s(\gamma))^2}\begin{bmatrix} -\widetilde{\eta}(\gamma)\widetilde{\theta}(\gamma)\eta^{sm}(\gamma)\theta^{sr}(\gamma) - \widetilde{\theta}(\gamma)\widetilde{\eta}(\gamma)\theta^{sm}(\gamma)\eta^{sr}(\gamma) \\ +\widetilde{\eta}(\gamma)\widetilde{\theta}(\gamma)\theta^{s(m+r)}(\gamma) + \widetilde{\theta}(\gamma)\widetilde{\eta}(\gamma)\eta^{s(m+r)}(\gamma) \end{bmatrix} \\
&= \frac{q^{sn}(\gamma)}{(\eta^s(\gamma)-\theta^s(\gamma))^2}\begin{bmatrix} -\widetilde{\eta}(\gamma)\widetilde{\theta}(\gamma)\theta^{sr}(\gamma)(\eta^{sm}(\gamma)-\theta^{sm}(\gamma)) \\ +\widetilde{\theta}(\gamma)\widetilde{\eta}(\gamma)\eta^{sr}(\gamma)(\eta^{sm}(\gamma)-\theta^{sm}(\gamma)) \end{bmatrix} \\
&= \frac{q^{sn}(\gamma)}{(\eta^s(\gamma)-\theta^s(\gamma))^2}(\eta^{sm}(\gamma)-\theta^{sm}(\gamma))\left[-\widetilde{\eta}(\gamma)\widetilde{\theta}(\gamma)\theta^{sr}(\gamma) + \widetilde{\theta}(\gamma)\widetilde{\eta}(\gamma)\eta^{sr}(\gamma)\right] \\
&= \frac{q^{sn}(\gamma)}{(\eta^s(\gamma)-\theta^s(\gamma))^2}(\eta^{sm}(\gamma)-\theta^{sm}(\gamma))[-(\Phi-\Omega q^s(\gamma)\boldsymbol{\omega})\theta^{sr}(\gamma) + (\Phi+\Omega q^s(\gamma)\boldsymbol{\omega})\eta^{sr}(\gamma)] \\
&= \frac{q^{sn}(\gamma)U_m^{(s)}(\gamma)}{\eta^s(\gamma)-\theta^s(\gamma)}[-\Phi\theta^{sr}(\gamma) + \Omega q^s(\gamma)\boldsymbol{\omega}\theta^{sr}(\gamma) + \Phi\eta^{sr}(\gamma) + \Omega q^s(\gamma)\boldsymbol{\omega}\eta^{sr}(\gamma)] \\
&= \frac{q^{sn}(\gamma)U_m^{(s)}(\gamma)}{\eta^s(\gamma)-\theta^s(\gamma)}[\Phi(\eta^{sr}(\gamma)-\theta^{sr}(\gamma)) + \Omega q^s(\gamma)\boldsymbol{\omega}(\theta^{sr}(\gamma)+\eta^{sr}(\gamma))] \\
&= \frac{q^{sn}(\gamma)U_m^{(s)}(\gamma)}{\eta^s(\gamma)-\theta^s(\gamma)}\left[\Phi\Omega U_r^{(s)}(\gamma) + \Omega q^s(\gamma)\boldsymbol{\omega}V_{sr}(\gamma)\right] \\
&= \frac{q^{sn}(\gamma)U_m^{(s)}(\gamma)}{\eta^s(\gamma)-\theta^s(\gamma)}\Omega\left[\Phi U_r^{(s)}(\gamma) + q^s(\gamma)\boldsymbol{\omega}V_{sr}(\gamma)\right] \\
&= q^{sn}(\gamma)U_m^{(s)}(\gamma)\left[\Phi U_r^{(s)}(\gamma) + q^s(\gamma)\boldsymbol{\omega}V_{sr}(\gamma)\right].
\end{aligned}$$

The required proof is complete. □

Remark 1. *As applications of Theorem 6, we obtain the following results.*

(i) *When taking $m \to -r$, we derive Catalan's identity for $\mathbb{KU}_{n}^{(s)}(\gamma)$ as follows:*

$$\mathbb{KU}_{n-r}^{(s)}(\gamma)\mathbb{KU}_{n+r}^{(s)}(\gamma) - \left(\mathbb{KU}_{n}^{(s)}(\gamma)\right)^2 = q^{sn}(\gamma)U_{-r}^{(s)}(\gamma)\left[\Phi U_r^{(s)}(\gamma) + q^s(\gamma)\boldsymbol{\omega}V_{sr}(\gamma)\right].$$

(ii) *When taking $r = -m = 1$, we obtain Cassini's identity for $\mathbb{KU}_{n}^{(s)}(\gamma)$ as follows:*

$$\mathbb{KU}_{n-1}^{(s)}(\gamma)\mathbb{KU}_{n+1}^{(s)}(\gamma) - \left(\mathbb{KU}_{n}^{(s)}(\gamma)\right)^2 = -q^{sn+s}(\gamma)[\Phi + q^s(\gamma)\boldsymbol{\omega}V_s(\gamma)].$$

(iii) *When taking $r \to j - n$, and $m = 1$, we have d'Ocagne's identity for $\mathbb{KU}_n^{(s)}(\gamma)$ as follows:*

$$\mathbb{KU}_{n+1}^{(s)}(\gamma)\mathbb{KU}_j^{(s)}(\gamma) - \mathbb{KU}_n^{(s)}(\gamma)\mathbb{KU}_{j+1}^{(s)}(\gamma) = q^{sn}(\gamma)\left[\Phi U_{j-n}^{(s)}(\gamma) + q^s(\gamma)\boldsymbol{\omega} V_{s(j-n)}(\gamma)\right].$$

Remark 2. *For $\gamma = 1$, higher-order generalized Fibonacci hybrid numbers have the following character:*

$$\begin{aligned} C\left(\mathbb{KU}_n^{(s)}\right) &= \left(U_n^{(s)}\right)^2 + \left(U_{n+1}^{(s)}\right)^2 - 2U_{n+1}^{(s)}U_{n+2}^{(s)} - \left(U_{n+3}^{(s)}\right)^2 \\ &< \left(U_n^{(s)}\right)^2 + \left(U_{n+1}^{(s)}\right)^2 - 2\left(U_{n+1}^{(s)}\right)^2 - \left(U_{n+3}^{(s)}\right)^2 \\ &= \left(U_n^{(s)}\right)^2 - \left(U_{n+1}^{(s)}\right)^2 - \left(U_{n+3}^{(s)}\right)^2 < 0. \end{aligned} \tag{17}$$

Thus, from (17), the higher-order generalized Fibonacci hybrid numbers are space-like.

3. Matrix Representations for Higher-Order Generalized Fibonacci Hybrinomials

In this part of the our paper, we define three different matrices as follows: for $n \in \mathbb{N}$,

$$\mathbf{A}_n^{(s)} = \begin{pmatrix} \mathbb{KU}_{n+1}^{(s)}(\gamma) & \mathbb{KU}_n^{(s)}(\gamma) \\ \mathbb{KU}_n^{(s)}(\gamma) & \mathbb{KU}_{n-1}^{(s)}(\gamma) \end{pmatrix}, \tag{18}$$

$$\mathbf{Q}^{(s)} = \begin{pmatrix} V_s(\gamma) & -q^s(\gamma) \\ 1 & 0 \end{pmatrix}, \tag{19}$$

$$\mathbf{R}_n^{(s)} = \begin{pmatrix} U_{n+1}^{(s)}(\gamma) & U_n^{(s)}(\gamma) \\ U_n^{(s)}(\gamma) & U_{n-1}^{(s)}(\gamma) \end{pmatrix}, \tag{20}$$

where $\mathbb{KU}_n^{(s)}(\gamma)$, $V_s(\gamma)$, and $U_n^{(s)}(\gamma)$ are higher-order generalized Fibonacci hybrinomials, generalized Lucas polynomials, and higher-order generalized Fibonacci polynomials, respectively. Based on these matrices, we will obtain some identities for higher-order generalized Fibonacci hybrinomials.

Lemma 2. *Let $n \geq 1$ be an integer. Then, we have*

$$\mathbf{A}_n^{(s)} = \mathbf{Q}^{(s)}\mathbf{A}_{n-1}^{(s)}.$$

Proof. Applying matrix multiplication and (15) yields

$$\begin{aligned} \mathbf{Q}^{(s)}\mathbf{A}_{n-1}^{(s)} &= \begin{pmatrix} V_s(\gamma) & -q^s(\gamma) \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \mathbb{KU}_n^{(s)}(\gamma) & \mathbb{KU}_{n-1}^{(s)}(\gamma) \\ \mathbb{KU}_{n-1}^{(s)}(\gamma) & \mathbb{KU}_{n-2}^{(s)}(\gamma) \end{pmatrix} \\ &= \begin{pmatrix} V_s(\gamma)\mathbb{KU}_n^{(s)}(\gamma) - q^s(\gamma)\mathbb{KU}_{n-1}^{(s)}(\gamma) & V_s(\gamma)\mathbb{KU}_{n-1}^{(s)}(\gamma) - q^s(\gamma)\mathbb{KU}_{n-2}^{(s)}(\gamma) \\ \mathbb{KU}_n^{(s)}(\gamma) & \mathbb{KU}_{n-1}^{(s)}(\gamma) \end{pmatrix} \\ &= \begin{pmatrix} \mathbb{KU}_{n+1}^{(s)}(\gamma) & \mathbb{KU}_n^{(s)}(\gamma) \\ \mathbb{KU}_n^{(s)}(\gamma) & \mathbb{KU}_{n-1}^{(s)}(\gamma) \end{pmatrix} = \mathbf{A}_n^{(s)}. \end{aligned}$$

The proof is complete. □

Lemma 3. *Let $n \geq 1$ be an integer. Then, we have*

$$\mathbf{R}_n^{(s)} = \mathbf{Q}^{(s)}\mathbf{R}_{n-1}^{(s)}.$$

Proof. By using matrix multiplication and (7), we have

$$
\begin{aligned}
\mathbf{Q}^{(s)}\mathbf{R}_{n-1}^{(s)} &= \begin{pmatrix} V_s(\gamma) & -q^s(\gamma) \\ 1 & 0 \end{pmatrix}\begin{pmatrix} U_n^{(s)}(\gamma) & U_{n-1}^{(s)}(\gamma) \\ U_{n-1}^{(s)}(\gamma) & U_{n-2}^{(s)}(\gamma) \end{pmatrix} \\
&= \begin{pmatrix} V_s(\gamma)U_n^{(s)}(\gamma) - q^s(\gamma)U_{n-1}^{(s)}(\gamma) & V_s(\gamma)U_{n-1}^{(s)}(\gamma) - q^s(\gamma)U_{n-2}^{(s)}(\gamma) \\ U_n^{(s)}(\gamma) & U_{n-1}^{(s)}(\gamma) \end{pmatrix} \\
&= \begin{pmatrix} U_{n+1}^{(s)}(\gamma) & U_n^{(s)}(\gamma) \\ U_n^{(s)}(\gamma) & U_{n-1}^{(s)}(\gamma) \end{pmatrix} = \mathbf{R}_n^{(s)}.
\end{aligned}
$$

The proof is complete. □

Lemma 4. *Let $n \geq 1$ be an integer. Then, we have*

$$
\mathbf{A}_n^{(s)} = \left(\mathbf{Q}^{(s)}\right)^n \mathbf{A}_0^{(s)}. \tag{21}
$$

Proof. We will verify that (21) is true by induction on n. Since

$$
\begin{aligned}
\mathbf{Q}^{(s)}\mathbf{A}_0^{(s)} &= \begin{pmatrix} V_s(\gamma) & -q^s(\gamma) \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \mathbb{K}\mathrm{U}_1^{(s)}(\gamma) & \mathbb{K}\mathrm{U}_0^{(s)}(\gamma) \\ \mathbb{K}\mathrm{U}_0^{(s)}(\gamma) & \mathbb{K}\mathrm{U}_{-1}^{(s)}(\gamma) \end{pmatrix} \\
&= \begin{pmatrix} V_s(\gamma)\mathbb{K}\mathrm{U}_1^{(s)}(\gamma) - q^s(\gamma)\mathbb{K}\mathrm{U}_0^{(s)}(\gamma) & V_s(\gamma)\mathbb{K}\mathrm{U}_0^{(s)}(\gamma) - q^s(\gamma)\mathbb{K}\mathrm{U}_{-1}^{(s)}(\gamma) \\ \mathbb{K}\mathrm{U}_1^{(s)}(\gamma) & \mathbb{K}\mathrm{U}_0^{(s)}(\gamma) \end{pmatrix} \\
&= \begin{pmatrix} \mathbb{K}\mathrm{U}_2^{(s)}(\gamma) & \mathbb{K}\mathrm{U}_1^{(s)}(\gamma) \\ \mathbb{K}\mathrm{U}_1^{(s)}(\gamma) & \mathbb{K}\mathrm{U}_0^{(s)}(\gamma) \end{pmatrix},
\end{aligned}
$$

the equality (21) holds for $n = 1$. Suppose that (21) is true for n. Then, by the inductive hypothesis, we have

$$
\begin{aligned}
\left(\mathbf{Q}^{(s)}\right)^{n+1}\mathbf{A}_0^{(s)} &= \begin{pmatrix} V_s(\gamma) & -q^s(\gamma) \\ 1 & 0 \end{pmatrix}^{n+1}\begin{pmatrix} \mathbb{K}\mathrm{U}_1^{(s)}(\gamma) & \mathbb{K}\mathrm{U}_0^{(s)}(\gamma) \\ \mathbb{K}\mathrm{U}_0^{(s)}(\gamma) & \mathbb{K}\mathrm{U}_{-1}^{(s)}(\gamma) \end{pmatrix} \\
&= \begin{pmatrix} V_s(\gamma) & -q^s(\gamma) \\ 1 & 0 \end{pmatrix}\begin{pmatrix} V_s(\gamma) & -q^s(\gamma) \\ 1 & 0 \end{pmatrix}^{n}\begin{pmatrix} \mathbb{K}\mathrm{U}_1^{(s)}(\gamma) & \mathbb{K}\mathrm{U}_0^{(s)}(\gamma) \\ \mathbb{K}\mathrm{U}_0^{(s)}(\gamma) & \mathbb{K}\mathrm{U}_{-1}^{(s)}(\gamma) \end{pmatrix} \\
&= \begin{pmatrix} V_s(\gamma) & -q^s(\gamma) \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \mathbb{K}\mathrm{U}_{n+1}^{(s)}(\gamma) & \mathbb{K}\mathrm{U}_n^{(s)}(\gamma) \\ \mathbb{K}\mathrm{U}_n^{(s)}(\gamma) & \mathbb{K}\mathrm{U}_{n-1}^{(s)}(\gamma) \end{pmatrix} \\
&= \begin{pmatrix} V_s(\gamma)\mathbb{K}\mathrm{U}_{n+1}^{(s)}(\gamma) - q^s(\gamma)\mathbb{K}\mathrm{U}_n^{(s)}(\gamma) & V_s(\gamma)\mathbb{K}\mathrm{U}_n^{(s)}(\gamma) - q^s(\gamma)\mathbb{K}\mathrm{U}_{n-1}^{(s)}(\gamma) \\ \mathbb{K}\mathrm{U}_{n+1}^{(s)}(\gamma) & \mathbb{K}\mathrm{U}_n^{(s)}(\gamma) \end{pmatrix} \\
&= \begin{pmatrix} \mathbb{K}\mathrm{U}_{n+2}^{(s)}(\gamma) & \mathbb{K}\mathrm{U}_{n+1}^{(s)}(\gamma) \\ \mathbb{K}\mathrm{U}_{n+1}^{(s)}(\gamma) & \mathbb{K}\mathrm{U}_n^{(s)}(\gamma) \end{pmatrix} = \mathbf{A}_{n+1}^{(s)},
\end{aligned}
$$

which shows that (21) is also true for $n + 1$. Therefore, by the principle of mathematical induction, (21) is true for all $n \in \mathbb{N}$. The proof is complete. □

Lemma 5. *Let $n \geq 1$ be an integer. Then, we have*

$$
\mathbf{R}_n^{(s)} = \left(\mathbf{Q}^{(s)}\right)^n \mathbf{R}_0^{(s)}. \tag{22}
$$

Proof. We will prove that (22) is true by induction on n. Since

$$\begin{aligned}
\mathbf{Q}^{(s)}\mathbf{R}_0^{(s)} &= \begin{pmatrix} V_s(\gamma) & -q^s(\gamma) \\ 1 & 0 \end{pmatrix}\begin{pmatrix} U_1^{(s)}(\gamma) & U_0^{(s)}(\gamma) \\ U_0^{(s)}(\gamma) & U_{-1}^{(s)}(\gamma) \end{pmatrix} \\
&= \begin{pmatrix} V_s(\gamma)U_1^{(s)}(\gamma) - q^s(\gamma)U_0^{(s)}(\gamma) & V_s(\gamma)U_0^{(s)}(\gamma) - q^s(\gamma)U_{-1}^{(s)}(\gamma) \\ U_1^{(s)}(\gamma) & U_0^{(s)}(\gamma) \end{pmatrix} \\
&= \begin{pmatrix} U_2^{(s)}(\gamma) & U_1^{(s)}(\gamma) \\ U_1^{(s)}(\gamma) & U_0^{(s)}(\gamma) \end{pmatrix} = \mathbf{R}_1^{(s)}.
\end{aligned}$$

we know that (22) holds for $n = 1$. Suppose that (22) is true for n. Then, by the inductive hypothesis, we get

$$\begin{aligned}
\left(\mathbf{Q}^{(s)}\right)^{n+1}\mathbf{R}_0^{(s)} &= \begin{pmatrix} V_s(\gamma) & -q^s(\gamma) \\ 1 & 0 \end{pmatrix}^{n+1}\begin{pmatrix} U_1^{(s)}(\gamma) & U_0^{(s)}(\gamma) \\ U_0^{(s)}(\gamma) & U_{-1}^{(s)}(\gamma) \end{pmatrix} \\
&= \begin{pmatrix} V_s(\gamma) & -q^s(\gamma) \\ 1 & 0 \end{pmatrix}\begin{pmatrix} V_s(\gamma) & -q^s(\gamma) \\ 1 & 0 \end{pmatrix}^{n}\begin{pmatrix} U_1^{(s)}(\gamma) & U_0^{(s)}(\gamma) \\ U_0^{(s)}(\gamma) & U_{-1}^{(s)}(\gamma) \end{pmatrix} \\
&= \begin{pmatrix} V_s(\gamma) & -q^s(\gamma) \\ 1 & 0 \end{pmatrix}\begin{pmatrix} U_{n+1}^{(s)}(\gamma) & U_n^{(s)}(\gamma) \\ U_n^{(s)}(\gamma) & U_{n-1}^{(s)}(\gamma) \end{pmatrix} \\
&= \begin{pmatrix} V_s(\gamma)U_{n+1}^{(s)}(\gamma) - q^s(\gamma)U_n^{(s)}(\gamma) & V_s(\gamma)U_n^{(s)}(\gamma) - q^s(\gamma)U_{n-1}^{(s)}(\gamma) \\ U_{n+1}^{(s)}(\gamma) & U_n^{(s)}(\gamma) \end{pmatrix} \\
&= \begin{pmatrix} U_{n+2}^{(s)}(\gamma) & U_{n+1}^{(s)}(\gamma) \\ U_{n+1}^{(s)}(\gamma) & U_n^{(s)}(\gamma) \end{pmatrix} = \mathbf{R}_{n+1}^{(s)},
\end{aligned}$$

which means that (22) is also true for $n + 1$. Therefore, by the principle of mathematical induction, (22) is true for all $n \in \mathbb{N}$. The proof is complete. □

Lemma 6. *Let $n \geq 1$ be an integer. Then, we have*

$$\left(\mathbf{Q}^{(s)}\right)^n = \begin{pmatrix} U_{n+1}^{(s)}(\gamma) & -q^s(\gamma)U_n^{(s)}(\gamma) \\ U_n^{(s)}(\gamma) & -q^s(\gamma)U_{n-1}^{(s)}(\gamma) \end{pmatrix}. \tag{23}$$

Proof. We will show that (23) is true by induction on n. Clearly, from (19), the equality (23) holds for $n = 1$. Assume that our assertion holds for n. Then, by the inductive hypothesis, we have

$$\begin{aligned}
\left(\mathbf{Q}^{(s)}\right)^{n+1} &= \begin{pmatrix} V_s(\gamma) & -q^s(\gamma) \\ 1 & 0 \end{pmatrix}\begin{pmatrix} U_{n+1}^{(s)}(\gamma) & -q^s(\gamma)U_n^{(s)}(\gamma) \\ U_n^{(s)}(\gamma) & -q^s(\gamma)U_{n-1}^{(s)}(\gamma) \end{pmatrix} \\
&= \begin{pmatrix} V_s(\gamma)U_{n+1}^{(s)}(\gamma) - q^s(\gamma)U_n^{(s)}(\gamma) & -q^s(\gamma)\left(V_s(\gamma)U_n^{(s)}(\gamma) - q^s(\gamma)U_{n-1}^{(s)}(\gamma)\right) \\ U_{n+1}^{(s)}(\gamma) & -q^s(\gamma)U_n^{(s)}(\gamma) \end{pmatrix} \\
&= \begin{pmatrix} U_{n+2}^{(s)}(\gamma) & -q^s(\gamma)U_{n+1}^{(s)}(\gamma) \\ U_{n+1}^{(s)}(\gamma) & -q^s(\gamma)U_n^{(s)}(\gamma) \end{pmatrix},
\end{aligned}$$

which shows (23) is also true for $n + 1$. Therefore, by the principle of mathematical induction, (23) is true for all $n \in \mathbb{N}$. □

Theorem 7. *Let $m, n \in \mathbb{N}$. Then,*

$$\mathbb{K}\mathbb{U}_{n+m}^{(s)}(\gamma) = U_{n+1}^{(s)}(\gamma)\mathbb{K}\mathbb{U}_m^{(s)}(\gamma) - q^{-s}(\gamma)U_n^{(s)}(\gamma)\mathbb{K}\mathbb{U}_{m-1}^{(s)}(\gamma).$$

Proof. From (18), we have

$$\mathbf{A}_{n+m}^{(s)} = \begin{pmatrix} \mathbb{K}\mathbb{U}_{n+m+1}^{(s)}(\gamma) & \mathbb{K}\mathbb{U}_{n+m}^{(s)}(\gamma) \\ \mathbb{K}\mathbb{U}_{n+m}^{(s)}(\gamma) & \mathbb{K}\mathbb{U}_{n+m-1}^{(s)}(\gamma) \end{pmatrix}.$$

By using Theorem 4 and Lemma 3, we have

$$\mathbf{A}_{n+m}^{(s)} = \left(\mathbf{Q}^{(s)}\right)^{n+m}\mathbf{A}_0^{(s)} = \left(\mathbf{Q}^{(s)}\right)^{n}\left(\mathbf{Q}^{(s)}\right)^{m}\mathbf{A}_0^{(s)} \tag{24}$$

and

$$\mathbf{R}_n^{(s)} = \left(\mathbf{Q}^{(s)}\right)^{n}\mathbf{R}_0^{(s)} \Rightarrow \left(\mathbf{Q}^{(s)}\right)^{n} = \mathbf{R}_n^{(s)}\left(\mathbf{R}_0^{(s)}\right)^{-1}. \tag{25}$$

Substituting (25) into (24), we get

$$\mathbf{A}_{n+m}^{(s)} = \left(\mathbf{Q}^{(s)}\right)^{n}\left(\mathbf{Q}^{(s)}\right)^{m}\mathbf{A}_0^{(s)} = \mathbf{R}_n^{(s)}\left(\mathbf{R}_0^{(s)}\right)^{-1}\mathbf{A}_m^{(s)}.$$

Then, we have

$$\begin{aligned}
\mathbf{A}_{n+m}^{(s)} &= \begin{pmatrix} U_{n+1}^{(s)}(\gamma) & U_{n}^{(s)}(\gamma) \\ U_{n}^{(s)}(\gamma) & U_{n-2}^{(s)}(\gamma) \end{pmatrix} \frac{1}{U_1^{(s)}(\gamma)U_{-1}^{(s)}(\gamma) - \left(U_0^{(s)}(\gamma)\right)^2} \begin{pmatrix} U_{-1}^{(s)}(\gamma) & -U_{0}^{(s)}(\gamma) \\ -U_{0}^{(s)}(\gamma) & U_{1}^{(s)}(\gamma) \end{pmatrix} \\
&\quad \times \begin{pmatrix} \mathbb{K}\mathbb{U}_{m+1}^{(s)}(\gamma) & \mathbb{K}\mathbb{U}_{m}^{(s)}(\gamma) \\ \mathbb{K}\mathbb{U}_{m}^{(s)}(\gamma) & \mathbb{K}\mathbb{U}_{m-1}^{(s)}(\gamma) \end{pmatrix} \\
&= \frac{1}{U_1^{(s)}(\gamma)U_{-1}^{(s)}(\gamma) - \left(U_0^{(s)}(\gamma)\right)^2} \begin{pmatrix} U_{n+1}^{(s)}(\gamma) & U_{n}^{(s)}(\gamma) \\ U_{n}^{(s)}(\gamma) & U_{n-2}^{(s)}(\gamma) \end{pmatrix} \begin{pmatrix} U_{-1}^{(s)}(\gamma) & -U_{0}^{(s)}(\gamma) \\ -U_{0}^{(s)}(\gamma) & U_{1}^{(s)}(\gamma) \end{pmatrix} \\
&\quad \times \begin{pmatrix} \mathbb{K}\mathbb{U}_{m+1}^{(s)}(\gamma) & \mathbb{K}\mathbb{U}_{m}^{(s)}(\gamma) \\ \mathbb{K}\mathbb{U}_{m}^{(s)}(\gamma) & \mathbb{K}\mathbb{U}_{m-1}^{(s)}(\gamma) \end{pmatrix}
\end{aligned}$$

Taking the first row's and second column's entry of each side from the above formula gives

$$\begin{aligned}
&\mathbb{K}\mathbb{U}_{n+m}^{(s)}(\gamma) \\
&= \frac{1}{U_1^{(s)}(\gamma)U_{-1}^{(s)}(\gamma) - \left(U_0^{(s)}(\gamma)\right)^2} \begin{pmatrix} \left(U_{n+1}^{(s)}(\gamma)U_{-1}^{(s)}(\gamma) - U_{n}^{(s)}(\gamma)U_{0}^{(s)}(\gamma)\right)\mathbb{K}\mathbb{U}_{m}^{(s)}(\gamma) \\ +\left(-U_{n+1}^{(s)}(\gamma)U_{0}^{(s)}(\gamma) + U_{n}^{(s)}(\gamma)U_{1}^{(s)}(\gamma)\right)\mathbb{K}\mathbb{U}_{m-1}^{(s)}(\gamma) \end{pmatrix} \\
&= \frac{1}{-q^s(\gamma)\left(U_1^{(s)}(\gamma)\right)^2} \begin{pmatrix} \left(-q^s(\gamma)U_{n+1}^{(s)}(\gamma)U_{1}^{(s)}(\gamma) - U_{n}^{(s)}(\gamma)U_{0}^{(s)}(\gamma)\right)\mathbb{K}\mathbb{U}_{m}^{(s)}(\gamma) \\ +\left(-U_{n+1}^{(s)}(\gamma)U_{0}^{(s)}(\gamma) + U_{n}^{(s)}(\gamma)U_{1}^{(s)}(\gamma)\right)\mathbb{K}\mathbb{U}_{m-1}^{(s)}(\gamma) \end{pmatrix} \\
&= \frac{1}{-q^s(\gamma)}\left(-q^s(\gamma)U_{n+1}^{(s)}(\gamma)\mathbb{K}\mathbb{U}_{m}^{(s)}(\gamma) + U_{n}^{(s)}(\gamma)\mathbb{K}\mathbb{U}_{m-1}^{(s)}(\gamma)\right) \\
&= U_{n+1}^{(s)}(\gamma)\mathbb{K}\mathbb{U}_{m}^{(s)}(\gamma) - q^{-s}(\gamma)U_{n}^{(s)}(\gamma)\mathbb{K}\mathbb{U}_{m-1}^{(s)}(\gamma).
\end{aligned}$$

The proof is complete. □

Theorem 8. *For any k, l, m, n integers for which $k + l = m + n$, we have*

$$U_k^{(s)}(\gamma)\mathbb{K}\mathbb{U}_l^{(s)}(\gamma) - U_m^{(s)}(\gamma)\mathbb{K}\mathbb{U}_n^{(s)}(\gamma) = -q^s(\gamma)\left(U_{m-1}^{(s)}(\gamma)\mathbb{K}\mathbb{U}_{n-1}^{(s)}(\gamma) - U_{k-1}^{(s)}(\gamma)\mathbb{K}\mathbb{U}_{l-1}^{(s)}(\gamma)\right). \tag{26}$$

Proof. From (21), we have

$$\left(\mathbf{Q}^{(s)}\right)^{k}\mathbf{A}_l^{(s)} = \left(\mathbf{Q}^{(s)}\right)^{k}\left(\mathbf{Q}^{(s)}\right)^{l}\mathbf{A}_0^{(s)} = \left(\mathbf{Q}^{(s)}\right)^{k+l}\mathbf{A}_0^{(s)}$$

$$= \left(\mathbf{Q}^{(s)}\right)^{m+n}\mathbf{A}_0^{(s)} = \left(\mathbf{Q}^{(s)}\right)^{m}\left(\mathbf{Q}^{(s)}\right)^{n}\mathbf{A}_0^{(s)} = \left(\mathbf{Q}^{(s)}\right)^{m}\mathbf{A}_n^{(s)}.$$

Namely, we have

$$\left(\mathbf{Q}^{(s)}\right)^{k}\mathbf{A}_l^{(s)} = \left(\mathbf{Q}^{(s)}\right)^{m}\mathbf{A}_n^{(s)}.$$

$$\begin{aligned}\left(\mathbf{Q}^{(s)}\right)^{k}\mathbf{A}_l^{(s)} &= \begin{pmatrix} V_s(\gamma) & -q^s(\gamma) \\ 1 & 0 \end{pmatrix}^k \begin{pmatrix} \mathbb{KU}_{l+1}^{(s)}(\gamma) & \mathbb{KU}_{l}^{(s)}(\gamma) \\ \mathbb{KU}_{l}^{(s)}(\gamma) & \mathbb{KU}_{l-1}^{(s)}(\gamma) \end{pmatrix} \\ &= \begin{pmatrix} U_{k+1}^{(s)}(\gamma) & -q^s(\gamma)U_{k}^{(s)}(\gamma) \\ U_{k}^{(s)}(\gamma) & -q^s(\gamma)U_{k-1}^{(s)}(\gamma) \end{pmatrix} \begin{pmatrix} \mathbb{KU}_{l+1}^{(s)}(\gamma) & \mathbb{KU}_{l}^{(s)}(\gamma) \\ \mathbb{KU}_{l}^{(s)}(\gamma) & \mathbb{KU}_{l-1}^{(s)}(\gamma) \end{pmatrix}. \end{aligned} \quad (27)$$

Similarly, $\left(\mathbf{Q}^{(s)}\right)^{m}\mathbf{A}_n^{(s)}$ is equal to

$$\begin{pmatrix} U_{m+1}^{(s)}(\gamma)\mathbb{KU}_{n+1}^{(s)}(\gamma) - q^s(\gamma)U_{m}^{(s)}(\gamma)\mathbb{KU}_{n}^{(s)}(\gamma) & U_{m+1}^{(s)}(\gamma)\mathbb{KU}_{n}^{(s)}(\gamma) - q^s(\gamma)U_{m}^{(s)}(\gamma)\mathbb{KU}_{n-1}^{(s)}(\gamma) \\ U_{m}^{(s)}(\gamma)\mathbb{KU}_{n+1}^{(s)}(\gamma) - q^s(\gamma)U_{m-1}^{(s)}(\gamma)\mathbb{KU}_{n}^{(s)}(\gamma) & U_{m}^{(s)}(\gamma)\mathbb{KU}_{n}^{(s)}(\gamma) - q^s(\gamma)U_{m-1}^{(s)}(\gamma)\mathbb{KU}_{n-1}^{(s)}(\gamma) \end{pmatrix}. \quad (28)$$

If we equate the second row and second column elements of matrices (27) and (28), we conclude that

$$U_{k}^{(s)}(\gamma)\mathbb{KU}_{l}^{(s)}(\gamma) - q^s(\gamma)U_{k-1}^{(s)}(\gamma)\mathbb{KU}_{l-1}^{(s)}(\gamma) = U_{m}^{(s)}(\gamma)\mathbb{KU}_{n}^{(s)}(\gamma) - q^s(\gamma)U_{m-1}^{(s)}(\gamma)\mathbb{KU}_{n-1}^{(s)}(\gamma)$$

and

$$U_{k}^{(s)}(\gamma)\mathbb{KU}_{l}^{(s)}(\gamma) - U_{m}^{(s)}(\gamma)\mathbb{KU}_{n}^{(s)}(\gamma) = -q^s(\gamma)\left(U_{m-1}^{(s)}(\gamma)\mathbb{KU}_{n-1}^{(s)}(\gamma) - U_{k-1}^{(s)}(\gamma)\mathbb{KU}_{l-1}^{(s)}(\gamma)\right).$$

So, Equation (26) is obtained. □

Theorem 9. *Let $n \in \mathbb{N}$. Then,*

$$\mathbb{KU}_{m+1}^{(s)}(\gamma)\mathbb{KU}_{m-1}^{(s)}(\gamma) - \left(\mathbb{KU}_{m}^{(s)}(\gamma)\right)^2 = \frac{q^{sn}(\gamma)}{(\eta^s(\gamma)-\theta^s(\gamma))^2}\left(-\Phi\frac{V_{2s}(\gamma)}{q^s(\gamma)} + \Omega^2\boldsymbol{\omega}V_s(\gamma) + 2\Phi\right) \quad (29)$$

and

$$\mathbb{KU}_{m-1}^{(s)}(\gamma)\mathbb{KU}_{m+1}^{(s)}(\gamma) - \left(\mathbb{KU}_{m}^{(s)}(\gamma)\right)^2 = \frac{q^{sn}(\gamma)}{(\eta^s(\gamma)-\theta^s(\gamma))^2}\left(-\Phi\frac{V_{2s}(\gamma)}{q^s(\gamma)} - \Omega^2\boldsymbol{\omega}V_s(\gamma) + 2\Phi\right). \quad (30)$$

Proof. If we take the determinant of both sides of Equation (21) and use the multiplication from above to down below rule for the determinant $\mathbf{A}_n^{(s)}$ [24] (p. 141), we find that

$$\begin{aligned} & q^{sn}(\gamma)\left(\mathbb{KU}_{1}^{(s)}(\gamma)\mathbb{KU}_{-1}^{(s)}(\gamma) - \left(\mathbb{KU}_{0}^{(s)}(\gamma)\right)^2\right) \\ =\ & \frac{q^{sn}(\gamma)}{(\eta^s(\gamma)-\theta^s(\gamma))^2}\left(\begin{array}{c} \left(\widetilde{\eta}(\gamma)(\eta^s(\gamma)) - \widetilde{\theta}(\gamma)(\theta^s(\gamma))\right)\left(\widetilde{\eta}(\gamma)(\eta^s(\gamma))^{-1} - \widetilde{\theta}(\gamma)(\theta^s(\gamma))^{-1}\right) \\ -\left(\widetilde{\eta}(\gamma) - \widetilde{\theta}(\gamma)\right)^2 \end{array}\right) \\ =\ & \frac{q^{sn}(\gamma)}{(\eta^s(\gamma)-\theta^s(\gamma))^2}\left(\begin{array}{c} \widetilde{\eta}^2(\gamma) + \widetilde{\theta}^2(\gamma) \\ -\widetilde{\eta}(\gamma)\widetilde{\theta}(\gamma)\eta^s(\gamma)(\theta^s(\gamma))^{-1} - \widetilde{\theta}(\gamma)\widetilde{\eta}(\gamma)\theta^s(\gamma)(\eta^s(\gamma))^{-1} \\ -\widetilde{\eta}^2(\gamma) + \widetilde{\eta}(\gamma)\widetilde{\theta}(\gamma) + \widetilde{\theta}(\gamma)\widetilde{\eta}(\gamma) - \widetilde{\theta}^2(\gamma) \end{array}\right) \\ =\ & \frac{q^{sn}(\gamma)}{(\eta^s(\gamma)-\theta^s(\gamma))^2}\left(\begin{array}{c} -\widetilde{\eta}(\gamma)\widetilde{\theta}(\gamma)\eta^s(\gamma)(\theta^s(\gamma))^{-1} - \widetilde{\theta}(\gamma)\widetilde{\eta}(\gamma)\theta^s(\gamma)(\eta^s(\gamma))^{-1} \\ +\widetilde{\eta}(\gamma)\widetilde{\theta}(\gamma) + \widetilde{\theta}(\gamma)\widetilde{\eta}(\gamma) \end{array}\right). \end{aligned}$$

By virtue of (10) and (11), we deduce

$$
\begin{aligned}
&= \frac{q^{sn}(\gamma)}{(\eta^s(\gamma)-\theta^s(\gamma))^2}\begin{pmatrix} -\widetilde{\eta}(\gamma)\widetilde{\theta}(\gamma)\eta^s(\gamma)(\theta^s(\gamma))^{-1} - \widetilde{\theta}(\gamma)\widetilde{\eta}(\gamma)\theta^s(\gamma)(\eta^s(\gamma))^{-1} \\ +\Phi - \Omega q^s(\gamma)\boldsymbol{\omega} + \Phi + \Omega q^s(\gamma)\boldsymbol{\omega} \end{pmatrix} \\
&= \frac{q^{sn}(\gamma)}{(\eta^s(\gamma)-\theta^s(\gamma))^2}\left(-\widetilde{\eta}(\gamma)\widetilde{\theta}(\gamma)\frac{\eta^s(\gamma)}{\theta^s(\gamma)} - \widetilde{\theta}(\gamma)\widetilde{\eta}(\gamma)\frac{\theta^s(\gamma)}{\eta^s(\gamma)} + 2\Phi\right) \\
&= \frac{q^{sn}(\gamma)}{(\eta^s(\gamma)-\theta^s(\gamma))^2}\left(-(\Phi - \Omega q^s(\gamma)\boldsymbol{\omega})\frac{\eta^s(\gamma)}{\theta^s(\gamma)} - (\Phi + \Omega q^s(\gamma)\boldsymbol{\omega})\frac{\theta^s(\gamma)}{\eta^s(\gamma)} + 2\Phi\right) \\
&= \frac{q^{sn}(\gamma)}{(\eta^s(\gamma)-\theta^s(\gamma))^2}\left(-\Phi\frac{\eta^s(\gamma)}{\theta^s(\gamma)} - \Phi\frac{\theta^s(\gamma)}{\eta^s(\gamma)} + \Omega q^s(\gamma)\boldsymbol{\omega}\frac{\eta^s(\gamma)}{\theta^s(\gamma)} - \Omega q^s(\gamma)\boldsymbol{\omega}\frac{\theta^s(\gamma)}{\eta^s(\gamma)} + 2\Phi\right) \\
&= \frac{q^{sn}(\gamma)}{(\eta^s(\gamma)-\theta^s(\gamma))^2}\left(-\Phi\frac{\eta^{2s}(\gamma)+\theta^{2s}(\gamma)}{q^s(\gamma)} + \Omega q^s(\gamma)\boldsymbol{\omega}\frac{\eta^{2s}(\gamma)-\theta^{2s}(\gamma)}{q^s(\gamma)} + 2\Phi\right) \\
&= \frac{q^{sn}(\gamma)}{(\eta^s(\gamma)-\theta^s(\gamma))^2}\left(-\Phi\frac{\eta^{2s}(\gamma)+\theta^{2s}(\gamma)}{q^s(\gamma)} + \Omega\boldsymbol{\omega}(\eta^s(\gamma)-\theta^s(\gamma))(\eta^s(\gamma)+\theta^s(\gamma)) + 2\Phi\right) \\
&= \frac{q^{sn}(\gamma)}{(\eta^s(\gamma)-\theta^s(\gamma))^2}\left(-\Phi\frac{\eta^{2s}(\gamma)+\theta^{2s}(\gamma)}{q^s(\gamma)} + \Omega^2\boldsymbol{\omega}V_s(\gamma) + 2\Phi\right) \\
&= \frac{q^{sn}(\gamma)}{(\eta^s(\gamma)-\theta^s(\gamma))^2}\left(-\Phi\frac{V_{2s}(\gamma)}{q^s(\gamma)} + \Omega^2\boldsymbol{\omega}V_s(\gamma) + 2\Phi\right).
\end{aligned}
$$

Hence, we show the assertion (29). Using the multiplication from down below to the above rule for the determinant $\mathbf{A}_n^{(s)}$, (30) can be proved similarly. □

4. Conclusions and Discussion

In this paper, we have defined and studied a different generalization of Fibonacci hybrid numbers and polynomials, which resulted in not only some existing families of hybrid numbers and polynomials in the literature but also some new families of hybrid numbers and polynomials according to the values of the parameters. In the last section, we have obtained some identities for higher-order generalized Fibonacci hybrid polynomials by using some matrices of a special type. Therefore, our results generalize some previous papers [16,17,19]. For future research, in view of this paper, researchers can define many hypercomplex polynomials using other higher-order polynomial families.

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