mathematics

Article

Fixed Point Results in Modified Intuitionistic Fuzzy Soft Metric Spaces with Application

Vishal Gupta [1,†], Aanchal Gondhi [2,†] and Rahul Shukla [3,*]

1 Department of Mathematics, MMEC, Maharishi Markandeshwar (Deemed to be University), Mullana 133207, Haryana, India; vishal.gmn@gmail.com or vgupta@mmumullana.org
2 Department of Mathematics, M.D.S.D. College, Ambala City 134003, Haryana, India; aanchalgondhi@gmail.com or dr.aanchalgondhi@gmail.com
3 Department of Mathematical Sciences and Computing, Walter Sisulu University, Mthatha 5117, South Africa
* Correspondence: rshukla.vnit@gmail.com or rshukla@wsu.ac.za
† These authors contributed equally to this work.

Abstract: This paper establishes a new type of space, modified intuitionistic fuzzy soft metric space (MIFSMS). Basic properties and topological structures are defined in the setting of this new notion with valid examples. Moreover, we have given some new results along with suitable examples to show their validity. An application for finding the solution of an integral equation is also given by utilizing our newly developed results.

Keywords: soft set; modified intuitionistic fuzzy metric space; Cauchy sequence; fixed point

MSC: 54H25; 47H10

check for updates

Citation: Gupta, V.; Gondhi, A.; Shukla, R. Fixed Point Results in Modified Intuitionistic Fuzzy Soft Metric Spaces with Application. *Mathematics* **2024**, *12*, 1154. https://doi.org/10.3390/math12081154

Academic Editor: Wei-Shih Du

Received: 23 February 2024
Revised: 29 March 2024
Accepted: 7 April 2024
Published: 11 April 2024

1. Introduction

The concept of vagueness came into existence when it was difficult for the interval mathematics to solve problems with different uncertainties. To tackle with such issues, Zadeh [1] gave fuzzy sets (FS) that were very well defined by using the indicator function. Many properties and important results are given in [1] that build a new field for the researchers to explore. Thereafter, K.T. Atanassov [2] presented an intuitionistic fuzzy set (IFS) consisting of membership and non membership functions as well.

Experiments solving issues in economics, engineering, environmental science, sociology, medical science and many other fields deal with the complex problems of modeling uncertain data. While some mathematical theories such as "probability theory, fuzzy set theory, rough set theory, vague set theory and the interval mathematics" are practical in defining uncertainty, each of these concepts has their own drawbacks. Further, in case of data containing parameters, Molodtsov [3] gave the concept of soft sets to deal with the uncertainties. Soft sets have applications in several fields including the "smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability theory and measurement theory". Especially, it has been very well applied to soft decision making problems. Das and Samanta [4–6] applied the concept of soft sets to metric spaces (MS) and hence presented Soft Metric Spaces (SMS) using soft points of soft sets. Maji et al. [7] in 2001 introduced Fuzzy Soft Sets. Beaula and Gunaseli [8] applied the MS concept to Fuzzy Soft Sets and hence introduced Fuzzy Soft Metric Spaces (FSMS) using a fuzzy soft point and defined some of its characteristics. See also [9–11]. Saadiat et al. [12] gave a vital concept of Modified Intuitionistic Fuzzy Metric Spaces (MIFMS) by availing the continuous t-representable norm.

2. Preliminaries

In the given section, χ denotes the universe, Ω depicts the parameter set, $\bar{\Omega}$ represents the absolute soft set, and $SP(\bar{\chi})$ denotes the set consisting of all the soft points of $\bar{\chi}$.

Definition 1 ([3]). *A 2-tuple (Δ, Ω) depicts a soft set on a universe χ where Ω represents the parameter set and Δ denotes the map from Ω to power set of χ, i.e., $\Delta : \Omega \to P(\chi)$.*

Definition 2 ([7]). *A soft set (Δ, Ω) is an absolute soft set if $\Delta(\iota) = \chi$ for every $\iota \in \Omega$.*

Definition 3 ([7]). *A soft point is a soft set (Δ, Ω) if $\Delta(\iota) = \{\kappa\}$ for any $\kappa \in \chi$ and $\Delta(\lambda) = \phi$ for $\lambda \in \chi / \{\iota\}$.*

Definition 4 ([4,5]). *A 3-tuple $(\bar{\chi}, \bar{\rho}, \Omega)$ denotes a SMS, where $\bar{\rho} : SP(\bar{\chi}) \times SP(\bar{\chi}) \to R(\Omega)$ is the soft metric and $R(\Omega)$ is the set having non-negative soft real numbers with $\bar{\rho}$ satisfying the given assertions for all $\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \bar{\lambda}_{e_3} \in SP(\bar{\chi})$:*

(i) $\bar{\rho}(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}) > \bar{0}$;
(ii) $\bar{\rho}(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}) = \bar{0}$ *iff* $\bar{\iota}_{e_1} = \bar{\kappa}_{e_2}$;
(iii) $\bar{\rho}(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}) = \bar{\rho}(\bar{\kappa}_{e_2}, \bar{\iota}_{e_1})$;
(iv) $\bar{\rho}(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}) \leq \bar{\rho}(\bar{\iota}_{e_1}, \bar{\lambda}_{e_3}) + \bar{\rho}(\bar{\lambda}_{e_3}, \bar{\kappa}_{e_2})$.

Definition 5 ([7]). *A 2-tuple (Δ, Ω) over a universe χ is a fuzzy soft set, where Ω represents the parameter set and Δ is the map from Ω to $F(\chi)$ which is the set having fuzzy subsets in universe χ, i.e., $\Delta : \Omega \to F(\chi)$.*

Definition 6 ([8]). *A 3-tuple $(\bar{\chi}, \Delta, *)$ is a soft fuzzy metric space (SFMS), where SFM on $\bar{\chi}$ is given by map $\Delta : SP(\bar{\chi}) \times SP(\bar{\chi}) \times (0, \infty) \to [0, 1]$ satisfying the below assertions for all $\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \bar{\lambda}_{e_3} \in SP(\bar{\chi})$ and $\rho, \varrho > 0$:*

(i) $\Delta(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \varrho) > 0$;
(ii) $\Delta(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \varrho) = 1$ *iff* $\bar{\iota}_{e_1} = \bar{\kappa}_{e_2}$;
(iii) $\Delta(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \varrho) = \Delta(\bar{\kappa}_{e_2}, \bar{\iota}_{e_1}, \varrho)$;
(iv) $\Delta(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \varrho + \rho) \geq \Delta(\bar{\iota}_{e_1}, \bar{\lambda}_{e_3}, \varrho) * \Delta(\bar{\lambda}_{e_3}, \bar{\kappa}_{e_2}, \rho)$;
(v) $\Delta(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, .) : (0, \infty) \to [0, 1]$ *is continuous.*

Definition 7 ([12]). *A 3-tuple $(\chi, M_{\omega,\psi}, \tau)$ is a MIFMS, where χ is an arbitrary set, ω, ψ depicts the fuzzy sets from $\chi^2 \times (0, \infty)$ to $[0, 1]$ asserting $\omega(\iota, \kappa, \varrho) + \psi(\iota, \kappa, \varrho) \leq 1$ for every $\iota, \kappa \in \chi$ and $\varrho > 0$, continuous t-representable norm is denoted by τ and $M_{\omega,\psi}$ depicts a map $\chi^2 \times (0, \infty) \to L^*$ that satisfies the below assertions for every $\iota, \kappa, \lambda \in \chi$ and $\rho, \varrho > 0$:*

(i) $M_{\omega,\psi}(\iota, \kappa, \varrho) >_{L^*} 0_{L^*}$;
(ii) $M_{\omega,\psi}(\iota, \kappa, \varrho) = 1_{L^*}$ *iff* $\iota = \kappa$;
(iii) $M_{\omega,\psi}(\iota, \kappa, \varrho) = M_{\omega,\psi}(\kappa, \iota, \varrho)$;
(iv) $M_{\omega,\psi}(\iota, \kappa, \varrho + \rho) \geq_{L^*} \tau(M_{\omega,\psi}(\iota, \lambda, \varrho), M_{\omega,\psi}(\lambda, \kappa, \rho))$;
(v) $M_{\omega,\psi}(\iota, \kappa, .) : (0, \infty) \to L^*$ *is continuous.*

Here, MIFM $M_{\omega,\psi}$ is given as

$$M_{\omega,\psi}(\iota, \kappa, \varrho) = (\omega(\iota, \kappa, \varrho), \psi(\iota, \kappa, \varrho)).$$

3. Modified Intuitionistic Fuzzy Soft Metric Space

Definition 8 ([13]). *A map Θ on L^*, $\Theta : (L^*)^2 \to L^*$ represents a triangular norm $(t - norm)$ if it satisfies the below assertions:*

(i) $\Theta(\iota, 1_{L^*}) = \iota$, *for all* $\iota \in L^*$;
(ii) $\Theta(\iota, \kappa) = \Theta(\kappa, \iota)$, *for all* $(\iota, \kappa) \in (L^*)^2$;
(iii) $\Theta(\iota, \Theta(\kappa, \lambda)) = \Theta(\Theta(\iota, \kappa), \lambda)$, *for all* $(\iota, \kappa, \lambda) \in (L^*)^3$;
(iv) $\iota \leq \iota'$ *and* $\kappa \leq \kappa' \Rightarrow \Theta(\iota, \kappa) \leq_{L^*} \Theta(\iota', \kappa')$, *for all* $(\iota, \iota', \kappa, \kappa') \in (L^*)^4$.

Definition 9 ([13]). *A continuous t-representable norm is a continuous $t-norm$ Θ on L^* if it implies the existence of a $t-conorm$ $\diamond$ on $[0,1]$ which is continuous so that*

$$\Theta(\iota, \kappa) = (\iota_1 * \kappa_1, \iota_2 \diamond \kappa_2)$$

for every $\iota = (\iota_1, \iota_2)$, $\kappa = (\kappa_1, \kappa_2) \in L^*$.

Definition 10. *A 3-tuple $(\bar{\chi}, M_{\omega,\psi}, \Theta)$ is a MIFSMS, where $\bar{\chi}$ is any set, ω and ψ are SFS, Θ denotes a t-representable norm possessing continuity and $M_{\omega,\psi}$ represents a map from $SP(\bar{\chi}) \times SP(\bar{\chi}) \times (0,\infty)$ to L^* fulfilling the below assertions for all $\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \bar{\lambda}_{e_3} \in SP(\bar{\chi})$ and $\rho, \varrho > 0$:*

(i) $M_{\omega,\psi}(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \varrho) >_{L^*} 0_{L^*}$;
(ii) $M_{\omega,\psi}(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \varrho) = 1_{L^*}$ *iff* $\bar{\iota}_{e_1} = \bar{\kappa}_{e_2}$;
(iii) $M_{\omega,\psi}(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \varrho) = M_{\omega,\psi}(\bar{\kappa}_{e_2}, \bar{\iota}_{e_1}, \varrho)$;
(iv) $M_{\omega,\psi}(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \varrho + \rho) \geq_{L^*} \Theta(M_{\omega,\psi}(\bar{\iota}_{e_1}, \bar{\lambda}_{e_3}, \varrho), M_{\omega,\psi}(\bar{\lambda}_{e_3}, \bar{\kappa}_{e_2}, \rho))$;
(v) $M_{\omega,\psi}(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, .) : (0,\infty) \to L^*$ *is continuous.*

Then, $M_{\omega,\psi}$ is MIFSM on $\bar{\chi}$ where level of closeness and non closeness between $\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}$ w.r.t. ϱ is depicted by the functions $\omega(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \varrho)$ and $\psi(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \varrho)$ respectively and metric $M_{\omega,\psi}$ is given as

$$M_{\omega,\psi}(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \varrho) = (\omega(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \varrho), \psi(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \varrho)).$$

Remark 1. *The function $\omega(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \varrho)$ is increasing and the function $\psi(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \varrho)$ is decreasing in a MIFSMS, for every $\bar{\iota}_{e_1}, \bar{\kappa}_{e_2} \in SP(\bar{\chi})$.*

Example 1. *Take $(\bar{\chi}, d)$ a SMS and ω, ψ soft fuzzy sets on $SP(\bar{\chi}) \times SP(\bar{\chi}) \times (0,\infty)$ given as below:*

$$\begin{aligned} M_{\omega,\psi}(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \varrho) &= (\omega(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \varrho), \psi(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \varrho)) \\ &= (\frac{h\varrho^n}{h\varrho^n + md(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2})}, \frac{md(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2})}{h\varrho^n + md(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2})}) \end{aligned}$$

for all $h, \varrho, m, n \in R^+$; $\Theta(\bar{r}, \bar{s}) = (\bar{r}_1\bar{s}_1, min(\bar{r}_2 + \bar{s}_2, 1))$, where $\bar{r} = (\bar{r}_1, \bar{r}_2)$ and $\bar{s} = (\bar{s}_1, \bar{s}_2) \in L^*$. Then, 3-tuple $(\bar{\chi}, M_{\omega,\psi}, \Theta)$ is a MIFSMS.

Example 2. *Consider $\chi = N$. Define $\Theta(\bar{\jmath}, \bar{\ell}) = (max\{0, \bar{\jmath}_1 + \bar{\ell}_1 - 1\}, \bar{\jmath}_2 + \bar{\ell}_2 - \bar{\jmath}_2\bar{\ell}_2)$, where $\bar{\jmath} = (\bar{\jmath}_1, \bar{\jmath}_2)$ and $\bar{\ell} = (\bar{\ell}_1, \bar{\ell}_2) \in L^*$. Take ω, ψ be soft fuzzy sets on $SP(\bar{\chi}) \times SP(\bar{\chi}) \times (0,\infty)$ given as $M_{\omega,\psi}(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \varrho) = (\omega(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \varrho), \psi(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \varrho))$*

$$M_{\omega,\psi}(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \varrho) = \begin{cases} (\frac{\bar{\iota}_{e_1}}{\bar{\kappa}_{e_2}}, \frac{\bar{\kappa}_{e_2} - \bar{\iota}_{e_1}}{\bar{\kappa}_{e_2}}) & if\, \bar{\iota}_{e_1} \leq \bar{\kappa}_{e_2} \\ (\frac{\bar{\kappa}_{e_2}}{\bar{\iota}_{e_1}}, \frac{\bar{\iota}_{e_1} - \bar{\kappa}_{e_2}}{\bar{\iota}_{e_1}}) & if\, \bar{\kappa}_{e_2} \leq \bar{\iota}_{e_1} \end{cases}$$

for all $\bar{\iota}_{e_1}, \bar{\kappa}_{e_2} \in SP(\bar{\chi})$ and $\varrho > 0$. Then, 3-tuple $(\bar{\chi}, M_{\omega,\psi}, \Theta)$ is also a MIFSMS.

Lemma 1. *Let $(\bar{\chi}, M_{\omega,\psi}, \tau)$ be MIFSMS. Then, $M_{\omega,\psi}(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \varrho)$ is increasing with respect to $\varrho > 0$ in $(L^*, \leq_{L^*})$ for every $\bar{\iota}_{e_1}, \bar{\kappa}_{e_2} \in SP(\bar{\chi})$.*

Definition 11. *Let $(\bar{\chi}, M_{\omega,\psi}, \tau)$ be MIFSMS. $M_{\omega,\psi}$ possesses continuity on $SP(\bar{\chi}) \times SP(\bar{\chi}) \times (0,\infty)$ if*

$$\lim_{n\to\infty} M_{\omega,\psi}(\bar{\iota}_{e_n}, \bar{\kappa}_{e_n}, \varrho_n) = M_{\omega,\psi}(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \varrho)$$

where sequence $\{(\bar{\iota}_{e_n}, \bar{\kappa}_{e_n}, \varrho_n)\}$ converges to $(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \varrho)$.

Lemma 2. *For* $(\bar{\chi}, M_{\omega,\psi}, \tau)$ *be a MIFSMS, then* $M_{\omega,\psi}$ *possesses continuity on* $SP(\bar{\chi}) \times SP(\bar{\chi}) \times (0, \infty)$.

4. Results and Discussion

Definition 12. *An open ball in a MIFSMS* $(\bar{\chi}, M_{\omega,\psi}, \tau)$ *is depicted by* $B(\bar{\iota}_{e_1}, \epsilon, \varrho)$ *having center* $\bar{\iota}_{e_1} \in \bar{\chi}$ *and radius* $0 < \epsilon < 1$ *for any* $\varrho > 0$ *and is given as,*

$$B(\bar{\iota}_{e_1}, \epsilon, \varrho) = \{\bar{\kappa}_{e_2} \in \bar{\chi} : M_{\omega,\psi}(\bar{\iota}_{e_1}, \bar{\kappa}_{e_2}, \varrho) >_{L^*} (N_s(\epsilon), \epsilon)\}.$$

Theorem 3. *Every open ball* $B(\bar{u}_{e_1}, \epsilon, \varrho)$ *in MIFSMS is an open set.*

Proof. Take $\bar{v}_{e_2} \in B(\bar{u}_{e_1}, \epsilon, \varrho)$, then $M_{\omega,\psi}(\bar{u}_{e_1}, \bar{v}_{e_2}, \varrho) >_{L^*} (N_s(\epsilon), \epsilon)$, so

$$\omega(\bar{u}_{e_1}, \bar{v}_{e_2}, \varrho) > 1 - \epsilon, \psi(\bar{u}_{e_1},, \bar{v}_{e_2}, \varrho) < \epsilon.$$

Since $\omega(\bar{u}_{e_1}, \bar{v}_{e_2}, \varrho) > 1 - \epsilon$, there exists $\varrho_o \in (0, \varrho)$ so that $\omega(\bar{u}_{e_1}, \bar{v}_{e_2}, \varrho_o) > 1 - \epsilon$ and $\psi(\bar{u}_{e_1}, \bar{v}_{e_2}, \varrho_o) < \epsilon$.

Put $\epsilon_o = \omega(\bar{u}_{e_1}, \bar{v}_{e_2}, \varrho_o)$.

Since $\epsilon_o > 1 - \epsilon$, then there exists $\eta \in (0,1)$ so that $\epsilon_0 > 1 - \eta > 1 - \epsilon$.

Now, for given ϵ and η such that $\epsilon_o > 1 - \eta$, there exist ϵ_1,$\epsilon_2 \in (0,1)$ such that $\tau(\epsilon_o^{'}, \epsilon_1^{'}) >_{L^*} (N_s(\eta), \eta)$, where $\epsilon_o^{'} = (\epsilon_o, 1 - \epsilon_o)$ and $\epsilon_1^{'} = (\epsilon_1, 1 - \epsilon_2)$.

Let ϵ_3 =max$\{\epsilon_1, \epsilon_2\}$ and open ball $B(\bar{v}_{e_2}, 1 - \epsilon_3, \varrho - \varrho_o)$.

Claim that $B(\bar{v}_{e_2}, 1 - \epsilon_3, \varrho - \varrho_o) \subset B(\bar{u}_{e_1}, \epsilon, \varrho)$.

Consider $\bar{w}_{e_3} \in B(\bar{v}_{e_2}, 1 - \epsilon_3, \varrho - \varrho_o)$, then $M_{\omega,\psi}(\bar{v}_{e_2}, \bar{w}_{e_3}, \varrho - \varrho_o) >_{L^*} (N_s(1 - \epsilon_3), \epsilon_3)$, so

$$\omega(\bar{v}_{e_2}, \bar{w}_{e_3}, \varrho - \varrho_o) > \epsilon_3, \psi(\bar{v}_{e_2}, \bar{w}_{e_3}, \varrho - \varrho_o) < 1 - \epsilon_3.$$

Now, $M_{\omega,\psi}(\bar{u}_{e_1}, \bar{w}_{e_3}, \varrho) \geq_{L^*} \tau(M_{\omega,\psi}(\bar{u}_{e_1}, \bar{v}_{e_2}, \varrho_o), M_{\omega,\psi}(\bar{v}_{e_2}, \bar{w}_{e_3}, \varrho - \varrho_o)) \geq_{L^*} (N_s(\epsilon), \epsilon)$.

Thus, $\bar{w}_{e_3} \in B(\bar{u}_{e_1}, \epsilon, \varrho)$. Hence, $B(\bar{v}_{e_2}, 1 - \epsilon_3, \varrho - \varrho_o) \subset B(\bar{u}_{e_1}, \epsilon, \varrho)$. □

Remark 2. *The topology generated by* $M_{\omega,\psi}$ *on* $\bar{\chi}$ *in MIFSMS* $(\bar{\chi}, M_{\omega,\psi}, \tau)$ *is given as*

$$\tau_{\omega,\psi}=\{Y \subseteq \bar{\chi}\text{: for every } \bar{u}_{e_1} \in Y\text{, there exist } \varrho > 0 \text{ and } \epsilon \in (0,1) \text{ so that } B(\bar{u}_{e_1}, \epsilon, \varrho) \subseteq Y\}.$$

Theorem 4. *If* $(\bar{\chi}, M_{\omega,\psi}, \tau)$ *is a MIFSMS, then it is a Hausdorff space.*

Proof. Given that $(\bar{\chi}, M_{\omega,\psi}, \tau)$ is MIFSMS, consider $\bar{u}_{e_1}, \bar{v}_{e_2} \in \bar{\chi}$ be two distinct points, then

$$0_{L^*} <_{L^*} M_{\omega,\psi}(\bar{u}_{e_1}, \bar{v}_{e_2}, \varrho) <_{L^*} 1_{L^*}.$$

Let $s_1 = \omega(\bar{u}_{e_1}, \bar{v}_{e_2}, \varrho)$, $s_2 = \psi(\bar{u}_{e_1}, \bar{v}_{e_2}, \varrho)$ and s =max$\{s_1, 1 - s_2\}$.

For each $s_0 \in (s, 1)$, there exist s_3, s_4 such that $\tau((s_3, 1 - s_4), (s_3, 1 - s_4)) \geq_{L^*} (N_s(1 - s_0), s_0)$.

Let $s_5 = \max\{s_3, s_4\}$.

Consider two MIFSOBs $B(\bar{u}_{e_1}, 1 - s_5, \frac{\varrho}{2})$ and $B(\bar{v}_{e_2}, 1 - s_5, \frac{\varrho}{2})$.

Claim that $B(\bar{u}_{e_1}, 1 - s_5, \frac{t}{2}) \cap B(\bar{v}_{e_2}, 1 - s_5, \frac{\varrho}{2}) = \phi$.

Let $\bar{w}_{e_3} \in B(\bar{u}_{e_1}, 1 - s_5, \frac{\varrho}{2}) \cap B(\bar{v}_{e_2}, 1 - s_5, \frac{\varrho}{2})$.

$$\begin{aligned}(s_1, s_2) &= M_{\omega,\psi}(\bar{u}_{e_1}, \bar{v}_{e_2}, \varrho) \\ &\geq_{L^*} \tau(M_{\omega,\psi}(\bar{u}_{e_1}, \bar{w}_{e_3}, \frac{\varrho}{2}), M_{\omega,\psi}(\bar{w}_{e_3}, \bar{v}_{e_2}, \frac{\varrho}{2})) \\ &\geq_{L^*} \tau((s_5, 1 - s_5), (s_5, 1 - s_5)) \geq_{L^*} (N_s(1 - s_0), s_0) \\ &>_{L^*} (s_1, s_2),\end{aligned}$$

which is contrary. So $(\bar{\chi}, M_{\omega,\psi}, \tau)$ is a Hausdorff space. □

Definition 13. *A subset Y of $\bar{\chi}$ in a MIFSMS $(\bar{X}, M_{\omega,\psi}, \tau)$ is called IF-bounded if it implies the existence of $\varrho > 0$ and $0 < \epsilon < 1$ so that $M_{\omega,\psi}(\bar{u}_{e_1}, \bar{v}_{e_2}, t) >_{L^*} (N_s(\epsilon), \epsilon)$ for each $\bar{u}_{e_1}, \bar{v}_{e_2} \in Y$.*

Theorem 5. *If $(\bar{\chi}, M_{\omega,\psi}, \tau)$ is MIFSMS and $Y \subseteq \bar{\chi}$ is compact, then Y is IF-bounded.*

Proof. Consider $\varrho > 0$ and $0 < \epsilon < 1$.

Let $\{B(\bar{u}_{e_1}, \epsilon, t) : \bar{u}_{e_1} \in Y\}$ be an open cover of Y.

As Y is given to be compact, there exist $\bar{u}_{e_1}$, $\bar{u}_{e_2}$,...,$\bar{u}_{e_n} \in Y$ such that $Y \subseteq \cup_{i=1}^{n} B(\bar{u}_{e_i}, \epsilon, t)$.

Let $\bar{u}_{e_1}, \bar{v}_{e_2} \in Y$, then $\bar{u}_{e_1} \in B(\bar{u}_{e_i}, \epsilon, \varrho)$ and $\bar{v}_{e_2} \in B(\bar{u}_{e_j}, \epsilon, \varrho)$ for some *i*,*j*; then

$$M_{\omega,\psi}(\bar{u}_{e_1}, \bar{u}_{e_i}, \varrho) >_{L^*} (N_s(\epsilon), \epsilon), M_{\omega,\psi}(\bar{v}_{e_2}, \bar{u}_{e_j}, \varrho) >_{L^*} (N_s(\epsilon), \epsilon).$$

Let $\alpha = max\{\omega(\bar{u}_{e_i}, \bar{u}_{e_j}, \varrho) : 1 \leq i, j \leq p\}$ and $\beta = max\{\psi(\bar{u}_{e_i}, \bar{u}_{e_j}, \varrho) : 1 \leq i, j \leq p\}$.

Then, $\alpha, \beta > 0$.

Now,

$$\begin{aligned} M_{\omega,\psi}(\bar{u}_{e_1}, \bar{v}_{e_2}, 3\varrho) &>_{L^*} \tau^2(M_{\omega,\psi}(\bar{u}_{e_1}, \bar{u}_{e_i}, \varrho), M_{\omega,\psi}(\bar{u}_{e_i}, \bar{u}_{e_j}, \varrho), M_{\omega,\psi}(\bar{u}_{e_j}, \bar{v}_{e_2}, \varrho)) \\ &\geq_{L^*} \tau^2((1-\epsilon, \epsilon), (\alpha, \beta), (1-\epsilon, \epsilon)) \\ &>_{L^*} (N_s(\eta_1), N_s(\eta_2)) \end{aligned}$$

for some $0 < \eta_1, \eta_2 < 1$.

Let $\eta = max\{\eta_1, \eta_2\}$ and $\varrho' = 3\varrho$, then $M_{\omega,\psi}(\bar{u}_{e_1}, \bar{v}_{e_2}, \varrho') >_{L^*} (N_s(\eta), \eta)$ for all $\bar{u}_{e_1}, \bar{v}_{e_2} \in Y$.

Hence, Y is IF-bounded. □

Theorem 6. *If $(\bar{\chi}, M_{\omega,\psi}, \tau)$ is a MIFSMS and $\tau_{\omega,\psi}$ is a topology on $\bar{\chi}$, then $\bar{u}_{e_n} \to \bar{u}_{e_1}$ if*

$$lt_{n\to\infty} M_{\omega,\psi}(\bar{u}_{e_n}, \bar{u}_{e_1}, \varrho) = 1_{L^*}$$

for $\bar{u}_{e_n}$ in $\bar{\chi}$.

Proof. Take $\varrho > 0$.

Consider $\bar{u}_{e_n} \to \bar{u}_{e_1}$, then there exists $n_0 \in N$ so that $\bar{u}_{e_n} \in B(\bar{u}_{e_1}, \epsilon, \varrho)$ for all $n \geq n_0, \epsilon \in (0,1)$; then, $M_{\omega,\psi}(\bar{u}_{e_n}, \bar{u}_{e_1}, \varrho) >_{L^*} (N_s(\epsilon), \epsilon)$.

Hence, $lt_{n\to\infty} M_{\omega,\psi}(\bar{u}_{e_n}, \bar{u}_{e_1}, \varrho) = 1_{L^*}$.

Conversely, let $lt_{n\to\infty} M_{\omega,\psi}(\bar{u}_{e_n}, \bar{u}_{e_1}, \varrho) = 1_{L^*}$, thus for $\epsilon \in (0,1)$, there exists $n_0 \in N$ satisfying $M_{\omega,\psi}(\bar{u}_{e_n}, \bar{u}_{e_1}, \varrho) >_{L^*} (N_s(\epsilon), \epsilon)$ for each $n \geq n_0$.

Thus, $\bar{u}_{e_n} \in B(\bar{u}_{e_1}, \epsilon, \varrho)$ where $n \geq n_0$.

Hence, $\bar{u}_{e_n} \to \bar{u}_{e_1}$. □

Definition 14. *Consider $(\bar{\chi}, M_{\omega,\psi}, \tau)$ to be a MIFSMS and $\{\bar{u}_{e_n}\}$ a sequence in $\bar{\chi}$, then*

1. *The sequence is Cauchy iff for every $\varrho > 0$, which implies the existence of $\delta_o \in N$ that satisfies*

$$lt_{n_o\to\infty} M_{\omega,\psi}(\bar{u}_{e_n}, \bar{u}_{e_{n+m}}, t) = 1_{L^*}$$

   where $n, m \geq \delta_o$.
2. *The sequence converges to $\bar{u}$ iff for every $\varrho > 0$*

$$lt_{n\to\infty} M_{\omega,\psi}(\bar{u}_{e_n}, \bar{u}, \varrho) = 1_{L^*}$$

Definition 15. *A MIFSMS $(\bar{\chi}, M_{\omega,\psi}, \tau)$ is complete iff every Cauchy sequence converges in it.*

Theorem 7. *If any Cauchy sequence in MIFSMS $(\bar{\chi}, M_{\omega,\psi}, \tau)$ has a subsequence that converges in it, then it is a complete space.*

Proof. Consider $\{\bar{u}_{e_n}\}$ be any sequence which is Cauchy and $\{\bar{u}_{e_{n_i}}\}$ be any of its subsequence converging to $\bar{u}$.

Claim that $\bar{u}_{e_n} \to \bar{u}$.

Take $\varrho > 0$ and $\epsilon \in (0,1)$.

Consider $s \in (0,1)$ such that

$$\tau((1-s,s),(1-s,s)) \geq_{L^*} (N_s(\epsilon),\epsilon).$$

Since sequence $\{u_{e_n}\}$ is given as Cauchy, there exist $e_{n_o} \in N$ such that

$$M_{\omega,\psi}(\bar{u}_{e_m},\bar{u}_{e_n},\frac{\varrho}{2}) >_{L^*} (N_s(s),s)$$

for all $e_m, e_n \geq e_{n_o}$.

Since $\bar{u}_{e_{n_i}} \to \bar{u}$, there exist positive integer e_{i_p} such that $e_{i_p} > e_{n_o}$,

$$M_{\omega,\psi}(\bar{u}_{e_i},\bar{u},\frac{\varrho}{2}) >_{L^*} (N_s(s),s).$$

For, if $e_n \geq e_{n_o}$, we have

$$\begin{aligned} M_{\omega,\psi}(\bar{u}_{e_n},\bar{u},\varrho) &\geq_{L^*} \tau(M_{\omega,\psi}(\bar{u}_{e_n},\bar{u}_{e_{i_p}},\frac{\varrho}{2}),M_{\omega,\psi}(\bar{u}_{e_{i_p}},\bar{u},\frac{\varrho}{2})) \\ &>_{L^*} \tau((1-s,s),(1-s,s)) \\ &\geq_{L^*} (N_s(\epsilon),\epsilon). \end{aligned}$$

Thus, $\bar{u}_{e_{n_i}} \to \bar{u}$ and $(\bar{\chi}, M_{\omega,\psi}, \tau)$ is a complete space. □

5. Fixed Point Theorems

In this section, we have extended Gregori-Sapena's [14] and Zikic's fixed point Theorem [15] to MIFSMS.

Definition 16. *A sequence $\{t_n\}$ is known as s-increasing if there exists $n_o \in N$ such that*

$$t_n + 1 \leq t_{n+1},$$

for all $n \geq n_o$.

Theorem 8. *Consider $(\bar{\chi}, M_{\omega,\psi}, \tau)$ be complete MIFSMS, so that for every s-increasing sequence $\{\varrho_n\}$ and arbitrary $\bar{u}_{e_1}, \bar{v}_{e_2} \in SP(\bar{\chi})$, (1) holds*

$$lt_{n\to\infty}\prod_{i=1}^{n} M_{\omega,\psi}(\bar{u}_{e_1},\bar{v}_{e_2},\varrho_i) = 1_{L^*}. \tag{1}$$

Consider $h \in (0,1)$ and $S : SP(\bar{\chi}) \to SP(\bar{\chi})$ be any map that satisfies

$$M_{\omega,\psi}(S\bar{u}_{e_1},S\bar{v}_{e_2},h\varrho) \geq_{L^*} M_{\omega,\psi}(\bar{u}_{e_1},\bar{v}_{e_2},\varrho)$$

for each $\bar{u}_{e_1}, \bar{v}_{e_2} \in SP(\bar{\chi})$. Then, S possesses a fixed point which is unique as well.

Proof. Consider $\bar{u} \in SP(\bar{\chi})$ and let $\bar{u}_{e_n} = S^n(\bar{u})$, $n \in N$. Thus,

$$M_{\omega,\psi}(\bar{u}_{e_1},\bar{u}_{e_2},\varrho) = M_{\omega,\psi}(S\bar{u},S^2\bar{u},\varrho) \geq_{L^*} M_{\omega,\psi}(\bar{u},S\bar{u},\frac{\varrho}{h}) = M_{\omega,\psi}(\bar{u},\bar{u}_{e_1},\frac{\varrho}{h}). \tag{2}$$

By induction, we have

$$M_{\omega,\psi}(\bar{u}_{e_n},\bar{u}_{e_{n+1}},\varrho) \geq_{L^*} M_{\omega,\psi}(\bar{u},\bar{u}_{e_1},\frac{\varrho}{h^n}). \tag{3}$$

Consider $\varrho > 0$. Now, for $m, n \in N$, take $n < m$; considering $r_i > 0$, $i = n, \ldots, m-1$ that satisfies $r_n + r_{n+1} + \cdots + r_{m-1} \leq 1$, we have

$$\begin{aligned} M_{\omega,\psi}(\bar{u}_{e_n}, \bar{u}_{e_m}, \varrho) \geq_{L^*} & \tau^{m-n-2}(M_{\omega,\psi}(\bar{u}_{e_n}, \bar{u}_{e_{n+1}}, r_n\varrho) \\ & , \ldots, M_{\omega,\psi}(\bar{u}_{e_{m-1}}, \bar{u}_{e_m}, r_{m-1}\varrho)) \\ & \geq_{L^*} \tau^{m-n-2}(M_{\omega,\psi}(\bar{u}, \bar{u}_{e_1}, \frac{r_n\varrho}{h^n}), \\ & \ldots, M_{\omega,\psi}(\bar{u}, \bar{u}_{e_1}, \frac{r_{m-1}\varrho}{h^{m-1}})). \end{aligned} \tag{4}$$

Considering $r_p = \dfrac{1}{p(p+1)}$, $p = n, \ldots, m-1$, we get

$$\begin{aligned} M_{\omega,\psi}(\bar{u}_{e_n}, \bar{u}_{e_m}, \varrho) \geq_{L^*} \tau^{m-n-2}(M_{\omega,\psi}(\bar{u}, \bar{u}_{e_1}, \frac{\varrho}{n(n+1)h^n}), \ldots, \\ M_{\omega,\psi}(\bar{u}, \bar{u}_{e_1}, \frac{\varrho}{(m-1)mh^{m-1}})). \end{aligned} \tag{5}$$

Now, define $\varrho_s = \dfrac{\varrho}{s(s+1)h^s}$. It is trivial that $\varrho_{s+1} - \varrho_s \to \infty$ as $s \to \infty$, thus $\{\varrho_s\}$ is an s-increasing sequence. So, we have

$$lt_{n\to\infty} \prod_{n=m}^{\infty} M_{\omega,\psi}(\bar{u}, \bar{u}_{e_1}, \frac{\varrho}{n(n+1)h^n}) = 1_{L^*}. \tag{6}$$

Equations (5) and (6) implies $lt_{n\to\infty} M_{\omega,\psi}(\bar{u}_{e_n}, \bar{u}_{e_m}, \varrho) = 1_{L^*}$ for $n < m$. Thus, sequence $\{\bar{u}_{e_n}\}$ is Cauchy. As $\bar{\chi}$ is complete, there exist $\bar{v} \in SP(\bar{\chi})$ so that $\bar{u}_{e_n} \to \bar{v}$. Claim that S possesses $\bar{v}$ as its fixed point.

$$\begin{aligned} M_{\omega,\psi}(S\bar{v}, \bar{v}, \varrho) &\geq_{L^*} \tau(lt_{n\to\infty} M_{\omega,\psi}(S\bar{v}, S\bar{u}_{e_n}, \frac{\varrho}{2}), lt_{n\to\infty} M_{\omega,\psi}(\bar{u}_{e_{n+1}}, \bar{v}, \frac{\varrho}{2})) \\ &\geq_{L^*} \tau(lt_{n\to\infty} M_{\omega,\psi}(\bar{v}, \bar{u}_{e_n}, \frac{\varrho}{2}), lt_{n\to\infty} M_{\omega,\psi}(\bar{u}_{e_{n+1}}, \bar{v}, \frac{\varrho}{2})) = 1_{L^*}. \end{aligned} \tag{7}$$

Thus, $M_{\omega,\psi}(S\bar{v}, \bar{v}, \varrho) = 1_{L^*}$, that implies $S(\bar{v}) = \bar{v}$.

Uniqueness: Consider $\bar{\varpi} \in SP(\bar{X})$ to be any other fixed point of S so that $S(\bar{\varpi}) = \bar{\varpi}$. Thus, we have

$$\begin{aligned} 1_{L^*} &\geq_{L^*} M_{\omega,\psi}(\bar{v}, \bar{\varpi}, t) = M_{\omega,\psi}(S(\bar{v}), S(\bar{\varpi}), \varrho) \\ &\geq_{L^*} M_{\omega,\psi}(\bar{v}, \bar{\varpi}, \frac{\varrho}{h}) = M_{\omega,\psi}(S(\bar{v}), S(\bar{\varpi}), \frac{\varrho}{h}) \\ &\geq_{L^*} M_{\omega,\psi}(\bar{v}, \bar{\varpi}, \frac{\varrho}{h^2}) = M_{\omega,\psi}(S(\bar{v}), S(\bar{\varpi}), \frac{\varrho}{h^2}) \\ &\cdots \\ &\geq_{L^*} lt_{n\to\infty} M_{\omega,\psi}(\bar{v}, \bar{\varpi}, \frac{\varrho}{h^n}) \\ &= 1_{L^*}. \end{aligned} \tag{8}$$

Hence, $M_{\omega,\psi}(\bar{v}, \bar{\varpi}, \varrho) = 1_{L^*}$, that implies $\bar{v} = \bar{\varpi}$. □

Example 3. *Take $\bar{\chi} = [0, 2]$ and define $\Theta(\bar{\jmath}, \bar{\ell}) = (\bar{\jmath}_1\bar{\ell}_1, max\{\bar{\jmath}_2, \bar{\ell}_2\})$. Take ω, ψ soft fuzzy sets on $SP(\bar{\chi}) \times SP(\bar{\chi}) \times (0, \infty)$ given as $M_{\omega,\psi}(\bar{v}, \bar{\varpi}, \varrho) = (e^{-(\frac{max\{\bar{v},\bar{\varpi}\}}{\varrho})}, 1 - e^{-(\frac{max\{\bar{v},\bar{\varpi}\}}{\varrho})})$, for all $\bar{v}, \bar{\varpi} \in$*

$SP(\bar{\chi})$, $\varrho > 0$. Then, $(\bar{\chi}, M_{\omega,\psi}, \Theta)$ is a complete MIFSMS. Define a self map $T : SP(\bar{\chi}) \to SP(\bar{\chi})$ so that

$$T(\bar{\iota}) = \begin{cases} 0 & if\, \bar{\iota} = 1 \\ \frac{\bar{\iota}}{2} & if\, \bar{\iota} \in [0,1) \\ \frac{\bar{\iota}}{4} & if\, \bar{\iota} \in (1,2]. \end{cases}$$

Then, T satisfies Theorem 8 and possesses 0 as its unique fixed point.

Lemma 9. *If $G : (0,\infty) \to [0,1]$ is an increasing function, then it satisfies*

$$lt_{n\to\infty} \prod_{i=n}^{\infty} G(\rho_o^i) = 0 \Rightarrow lt_{n\to\infty} \prod_{i=n}^{\infty} G(\rho^i) = 0 \tag{9}$$

for all $\rho \in (0,1)$ and $\prod$ is taken as co-norm $\diamond$.

Proof. *Case I* Consider $\rho < \rho_o$. Now, $\rho^i < \rho_o^i$ for $i \in N$, in view of the fact that G is increasing $G(\rho) \le G(\rho_o)$. So, $\prod_{i=n}^{\infty} G(\rho^i) \le \prod_{i=n}^{\infty} G(\rho_o^i)$, where $n \in N$. Hence, the proof is complete.

Case II Consider $\rho \ge \rho_o$. Suppose $\rho = \sqrt{\rho_o}$, then

$$\prod_{i=2p'}^{\infty} G(\rho^i) = [\prod_{i=p'}^{\infty} G(\rho^{2i})] \diamond [\prod_{i=p'}^{\infty} G(\rho^{2i+1})] \le [\prod_{i=p'}^{\infty} G(\rho_o^i)] \diamond [\prod_{i=p'}^{\infty} G(\rho_o^i)]. \tag{10}$$

Thus, we have $lt_{p'\to\infty} \prod_{i=2p'}^{\infty} G(\rho^i) \le 0 \diamond 0 = 0$. Furthermore, $lt_{p'\to\infty} \prod_{i=2p'+1}^{\infty} G(\rho^i) \le lt_{p'\to\infty} \prod_{i=2p'+2}^{\infty} G(\rho^i) = 0$ that implies $lt_{p'\to\infty} \prod_{i=p'}^{\infty} G(\rho^i) = 0$ if $\rho = \sqrt{\rho_o}$. As G is increasing it can be verified easily that $lt_{p'\to\infty} \prod_{i=p'}^{\infty} G(\rho^i) = 0$ if $\rho < \sqrt{\rho_o}$.

Now, if $\rho > \rho_o$, that implies the existence of $p' \in N$ so that $\rho < \rho_o^{(1/2)^{p'}}$ and on repeating the above process p'-times, we obtain $lt_{p'\to\infty} \prod_{i=p'}^{\infty} G(\rho^i) = 0$. □

Lemma 10. *If $H : (0,\infty) \to [0,1]$ is a decreasing function, then it satisfies*

$$lt_{n\to\infty} \prod_{i=n}^{\infty} H(\rho_o^i) = 1 \Rightarrow lt_{n\to\infty} \prod_{i=n}^{\infty} H(\rho^i) = 1 \tag{11}$$

for all $\rho \in (0,1)$ and $\prod$ is taken as norm $*$.

Proof. The above can be easily proved on the similar lines of Lemma 9. □

Lemma 11. *The sequence $\bar{u}_{e_n} = S^n(\bar{u}_{e_1})$ is a Cauchy sequence.*

Proof. Consider $G(\bar{u}) = \psi(\bar{u}_{e_1}, S(\bar{u}_{e_1}), \dfrac{1}{\bar{u}})$ and $H(\bar{u}) = \omega(\bar{u}_{e_1}, S(\bar{u}_{e_1}), \dfrac{1}{\bar{u}})$, then maps G and H are increasing and decreasing, respectively, from $(0,\infty)$ to $[0,1]$. Consider $h < \rho < 1$, then by Lemmas 9 and 10, we have

$$lt_{n\to\infty} \prod_{i=n}^{\infty} M_{\omega,\psi}(\bar{u}_{e_1}, S\bar{u}_{e_1}, \frac{1}{(h/\rho)^i}) = 1_{L^*}. \tag{12}$$

As $\rho < 1$, $\sum_{n=1}^{\infty} \rho^n < \infty$, so for every $\epsilon_o > 0$ there exist n_o so that $\sum_{n=n_o}^{\infty} \rho^n < \epsilon_o$. Now, if $\rho > n > n_o$ and $t > \epsilon_o$, we have

$$\begin{aligned} M_{\omega,\psi}(\bar{u}_{e_n}, \bar{u}_{e_p}, t) \geq_{L^*} M_{\omega,\psi}(\bar{u}_{e_n}, \bar{u}_{e_p}, \epsilon_o) &\geq_{L^*} \prod_{i=n}^{p-1} M_{\omega,\psi}(\bar{u}_{e_i}, \bar{u}_{e_{i-1}}, \rho^i) \\ \geq_{L^*} \prod_{i=n}^{p-1} M_{\omega,\psi}(\bar{u}_{e_1}, S\bar{u}_{e_1}, \frac{\rho^i}{h^i}) &= \prod_{i=n}^{p-1} M_{\omega,\psi}(\bar{u}_{e_1}, S\bar{u}_{e_1}, \frac{1}{(h/\rho)^i}). \end{aligned} \tag{13}$$

Thus, from Equation (12), we have $lt_{n\to\infty} M_{\omega,\psi}(\bar{u}_{e_n}, S\bar{u}_{e_p}, t) = 1_{L^*}$ where $p > n$. Thus, $\{\bar{u}_{e_n}\}$ is Cauchy. □

Theorem 12. *Consider $(\bar{\chi}, M_{\omega,\psi}, \tau)$ to be complete MIFSMS, so that for some $\rho_o \in (0,1)$ and $\bar{u}_{e_o} \in SP(\bar{\chi})$, (14) holds*

$$lt_{n\to\infty} \prod_{i=n}^{\infty} M_{\omega,\psi}(\bar{u}_{e_o}, S\bar{u}_{e_o}, \frac{1}{\rho_o^i}) = 1_{L^*}. \tag{14}$$

Consider $h \in (0,1)$ and $S : SP(\bar{\chi}) \to SP(\bar{\chi})$ that satisfies

$$M_{\omega,\psi}(S\bar{u}_{e_1}, S\bar{v}_{e_2}, ht) \geq_{L^*} M_{\omega,\psi}(\bar{u}_{e_1}, \bar{v}_{e_2}, t)$$

for each $\bar{u}_{e_1}, \bar{v}_{e_2} \in SP(\bar{\chi})$. Then, S possesses a fixed point which is unique as well.

Proof. We will be proving Theorem 12 by the above lemmas.

As $(\bar{\chi}, M_{\omega,\psi}, \tau)$ is complete MIFSMS, there exists $\bar{v} \in SP(\bar{\chi})$ so that $lt_{n\to\infty} \bar{u}_{e_n} = \bar{v}$. Now, it can be easily proven that S possesses $\bar{v}$ as its fixed point which is unique as well by the similar argument as used in Theorem 8. This completes the proof of Theorem 12. □

Example 4. *Take $\bar{\chi} = [0,1] \cap Q$ and define $\Theta(\bar{\jmath}, \bar{\ell}) = (max\{\bar{\jmath}_1 + \bar{\ell}_1 - 1, 0\}, min\{\bar{\jmath}_2 + \bar{\ell}_2, 1\})$. Take ω, ψ soft fuzzy sets on $SP(\bar{\chi}) \times SP(\bar{\chi}) \times (0,\infty)$ given as $M_{\omega,\psi}(\bar{v}, \bar{\varpi}, \varrho) = (1 - \frac{max\{\bar{v},\bar{\varpi}\}}{1+\varrho}, \frac{max\{\bar{v},\bar{\varpi}\}}{1+\varrho})$, for all $\bar{v}, \bar{\varpi} \in SP(\bar{\chi})$, $\varrho > 0$. Then, $(\bar{\chi}, M_{\omega,\psi}, \Theta)$ is a complete MIFSMS. Define a self map $T : SP(\bar{\chi}) \to SP(\bar{\chi})$ so that*

$$T(\bar{\imath}) = \begin{cases} \frac{\bar{\imath}}{4} & if\, \bar{\imath} \in [0, \frac{1}{2}] \cap Q \\ \frac{\bar{\imath}}{2} & if\, \bar{\imath} \in (\frac{1}{2}, 1] \cap Q. \end{cases}$$

Then, T satisfies Theorem 12 and possesses 0 as its unique fixed point.

6. Application

Now, we are giving an application of Theorem 8 in solving integral equation.

Consider the following integral equation:

$$\bar{\imath}_e(r) = \int_0^r K(r, s, \bar{\imath}_e(s)) ds \tag{15}$$

for all $r \in [0, I]$, where $I > 0$ and $K \in C([0,I] \times [0,I] \times R, R)$. Consider $\bar{\chi} = C([0,I], R)$ be the space consisting of continuous functions on $[0, I]$ with the norm $\| \bar{\imath}_e \| = sup_{r\in[0,I]} \mid \bar{\imath}_e(r) \mid$, where $\bar{\imath}_e \in \bar{\chi}$ and the induced soft metric is defined as $\mu(\bar{\imath}_e, \bar{\kappa}_e) = sup_{r\in[0,I]} \mid \bar{\imath}_e(r) - \bar{\kappa}_e(r) \mid$, for all $\bar{\imath}_e, \bar{\kappa}_e \in \bar{\chi}$. Let the MIFSM $M_{\omega,\psi}$ be defined as

$$M_{\omega,\psi}(\bar{\imath}_e, \bar{\kappa}_e, \varrho) = (\frac{\varrho}{\varrho + sup_{r\in[0,I]} \mid \bar{\imath}_e(r) - \bar{\kappa}_e(r) \mid}, \frac{sup_{r\in[0,I]} \mid \bar{\imath}_e(r) - \bar{\kappa}_e(r) \mid}{\varrho + sup_{r\in[0,I]} \mid \bar{\imath}_e(r) - \bar{\kappa}_e(r) \mid}),$$

where $\varrho, m, n \in R^+$ and $\tau(\bar{u}, \bar{v}) = (\bar{u}_1\bar{v}_1, min(\bar{u}_2 + \bar{v}_2, 1))$. Then, $(\bar{X}, M_{\omega,\psi}, \tau)$ is a complete MIFSMS.

Now, claim the existence of a solution of (15).

Let function K satisfy the following conditions:

(i) $K(r, s, \bar{\iota}_e(s)) \geq 0$, for all $r, s \in [0, I]$ and $\bar{\iota}_e \in \bar{\chi}$;

(ii) There exist $\lambda > 0$ so that

$$\mid K(r, s, \bar{\iota}_e(s)) - K(r, s, \bar{\kappa}_e(s)) \mid \leq \lambda sup_{r\in[0,I]} \mid \bar{\iota}_e(r) - \bar{\kappa}_e(r) \mid,$$

for all $r, s \in [0, I]$;

(iii) There exist $h \in (0, 1)$ so that $\lambda r < h$.

Consider $\{t_s\}$ to be any s-increasing sequence, so that $t_{s+1} - t_s \to \infty$ as $s \to \infty$, thus $lt_{n\to\infty} \prod_{i=1}^{n} M_{\omega,\psi}(\bar{\iota}_e, \bar{\kappa}_e, t_i) = 1_{L^*}$.

Define a self map $S : \bar{\chi} \to \bar{\chi}$ as

$$S(\bar{\iota}_e)(\jmath) = \int_0^{\jmath} K(\jmath, \ell, \bar{\iota}_e(\ell))d\ell,$$

then, we have

$$S(\bar{\iota}_e)(\jmath) - S(\bar{\kappa}_e)(\jmath) = \int_0^{\jmath} K(\jmath, \ell, \bar{\iota}_e(\ell)) - K(\jmath, \ell, \bar{\kappa}_e(\ell))d\ell,$$

thus

$$\begin{aligned}
\mid S(\bar{\iota}_e)(\jmath) - S(\bar{\kappa}_e)(\jmath) \mid &= \mid \int_0^{\jmath} K(\jmath, \ell, \bar{\iota}_e(\ell)) - K(\jmath, \ell, \bar{\kappa}_e(\ell))d\ell \mid \\
&\leq \int_0^{\jmath} \mid K(\jmath, \ell, \bar{\iota}_e(\ell)) - K(\jmath, \ell, \bar{\kappa}_e(\ell)) \mid d\ell \\
&\leq \lambda \jmath sup_{\jmath\in[0,I]} \mid \bar{\iota}_e(\jmath) - \bar{\kappa}_e(\jmath) \mid \\
&< h sup_{\jmath\in[0,I]} \mid \bar{\iota}_e(\jmath) - \bar{\kappa}_e(\jmath) \mid .
\end{aligned}$$

Now, we have

$$\begin{aligned}
M_{\omega,\psi}(S\bar{\iota}_e(\jmath), S\bar{\kappa}_e(\jmath), h\varrho) &= \left(\frac{h\varrho}{h\varrho + sup_{\jmath\in[0,I]} \mid S\bar{\iota}_e(\jmath) - S\bar{\kappa}_e(\jmath) \mid}, \frac{sup_{\jmath\in[0,I]} \mid S\bar{\iota}_e(\jmath) - S\bar{\kappa}_e(\jmath) \mid}{h\varrho + sup_{\jmath\in[0,I]} \mid S\bar{\iota}_e(\jmath) - S\bar{\kappa}_e(\jmath) \mid}\right) \\
&= \left(\frac{\varrho}{\varrho + \frac{sup_{\jmath\in[0,I]}|S\bar{\iota}_e(\jmath) - S\bar{\kappa}_e(\jmath)|}{h}}, \frac{\frac{sup_{\jmath\in[0,I]}|S\bar{\iota}_e(\jmath) - S\bar{\kappa}_e(\jmath)|}{h}}{\varrho + \frac{sup_{\jmath\in[0,I]}|S\bar{\iota}_e(\jmath) - S\bar{\kappa}_e(\jmath)|}{h}}\right) \\
&\geq_{L^*} \left(\frac{\varrho}{\varrho + sup_{\jmath\in[0,I]} \mid \bar{\iota}_e(\jmath) - \bar{\kappa}_e(\jmath) \mid}, \frac{sup_{\jmath\in[0,I]} \mid \bar{\iota}_e(\jmath) - \bar{\kappa}_e(\jmath) \mid}{\varrho + sup_{\jmath\in[0,I]} \mid \bar{\iota}_e(\jmath) - \bar{\kappa}_e(\jmath) \mid}\right) \\
&= M_{\omega,\psi}(\bar{\iota}_e(\jmath), \bar{\kappa}_e(\jmath), \varrho).
\end{aligned}$$

Thus, $M_{\omega,\psi}(S\bar{\iota}_e(\jmath), S\bar{\kappa}_e(\jmath), h\varrho) \geq_{L^*} M_{\omega,\psi}(\bar{\iota}_e(\jmath), \bar{\kappa}_e(\jmath), \varrho)$. Therefore, every assertion of Theorem 8 holds. Hence, S possesses a fixed point $\bar{\zeta} \in \bar{\chi}$ which is unique as well so that $S\bar{\zeta} = \bar{\zeta}$, thus $\bar{\zeta} \in C([0, I], R)$ satisfies integral Equation (15).

7. Conclusions

We have defined basic notions of MIFSMS in this paper. Some Theorems of MIFMS have been broadened in MIFSMS. FPT's are also proven in our new space along with an application to the integral equation.

8. Discussion

The new results and examples formulated in this work lay the foundation of new results in the future. Moreover, to prove the validity of new results, an application is given in solving the integral equation.

Author Contributions: All the authors contributed equally to the preparation of this paper. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Acknowledgments: The authors thank all reviewers for their useful remarks which made our paper complete and significant.

Conflicts of Interest: The authors declare that there is no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

MIFSMS	Modified Intuitionistic Fuzzy Soft Metric Spaces
FS	Fuzzy Sets
IFS	Intuitionistic Fuzzy Set
MS	Metric Space
SMS	Soft Metric Spaces
FSMS	Fuzzy Soft Metric Spaces
MIFMS	Modified Intuitionistic Fuzzy Metric Spaces

References

1. Zadeh, L.A. Fuzzy Sets. *Fuzzy Sets Inf. Control* **1965**, *8*, 338–353.
2. Atanassov, K.T. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96.
3. Molodtsov, D. Soft set theory-First results. *Comput. Math. Appl.* **1999**, *37*, 19–31.
4. Das, S.; Samanta, S.K. Soft real sets, soft real numbers and their properties. *J. Fuzzy Math.* **2012**, *20*, 551–576.
5. Das, S.; Samanta, S.K. On soft metric spaces. *J. Fuzzy Math.* **2013**, *21*, 707–734.
6. Das, S.; Samanta, S.K. Soft metric. *Ann. Fuzzy Math. Inform* **2013**, *6*, 77–94.
7. Maji, P.K.; Biswas, R.; Roy, A.R. Fuzzy Soft Set. *J. Fuzzy Math.* **2001**, *9*, 589–602.
8. Beaula, T.; Gunaseeli, C. On fuzzy soft metric spaces. *Malaya J. Math.* **2014**, *2*, 197–202.
9. Shukla, S.; Dubey, N.; Shukla, R.; Mezník, I. Coincidence point of Edelstein type mappings in fuzzy metric spaces and application to the stability of dynamic markets. *Axioms* **2023**, *12*, 854.
10. Gupta, V.; Saini, R.K.; Kanwar, A. Some Common Coupled Fixed Point Results on Modified Intuitionistic Fuzzy Metric Spaces. *Procedia Comput. Sci.* **2016**, *79*, 32–40.
11. Gupta, V.; Gondhi, A. Fixed points of weakly compatible maps on modified intuitionistic fuzzy soft metric spaces. *Int. J. Syst. Assur. Eng. Manag.* **2021**, *13*, 1232–1238.
12. Saadati, R.; Sedgi, S.; Shobe, N. Modified intuitionistic fuzzy metric spaces and fixed point theorems. *Chaos Solitons Fractals* **2005**, *38*, 36–47.
13. Deschrijver, G.; Cornelis, C.; Kerre, E.E. On the representation of intuitionistic fuzzy t-norms and t-conorms. *IEEE Trans. Fuzzy Syst.* **2004**, *12*, 45–61.
14. Gregori, V.; Sapena, A. On fixed-point theorems in fuzzy metric spaces. *Fuzzy Sets Syst.* **2002**, *125*, 245–252.
15. Zikic, T. On fixed point theorems of Gregori and Sapena. *Fuzzy Sets Syst.* **2004**, *144*, 421–429.