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The Explicit Form of the Unitary Representation of the Poincaré Group for Vector-Valued Wave Functions (Massive and Massless), with Applications to Photon Localization and Position Operators

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Abstract: We geometrically derive the explicit form of the unitary representation of the Poincaré group for vector-valued wave functions and use it to apply speed-of-light boosts to a simple polarization basis to end up with a Hawton–Baylis photon position operator with commuting components. We give explicit formulas for other photon boost eigenmodes. We investigate the underlying affine connections on the light cone in momentum space and find that while the Pryce connection is metric semi-symmetric, the flat Hawton–Baylis connection is not semi-symmetric. Finally, we discuss the localizability of photon states on closed loops and show that photon states on the circle, both unnormalized improper states and finite-norm wave packet smeared-over washer-like regions are strictly localized not only with respect to Hawton–Baylis operators with commuting components but also with respect to the noncommutative Jauch–Piron–Amrein POV measure.

Keywords: photon wave function; unitary representation; Poincaré group; boost eigenmodes; position operator; POV measure; Riemannian metric; affine connection; semi-symmetric metric connection; loop-localized photon states

MSC: 81V99

1. Introduction

In classical relativistic field theory, the electric and magnetic field strengths are combined into a closed two-form F over the Minkowski spacetime. Since F is closed, $dF = 0$, it admits a four-vector potential A (a 1-form), so that $F = dA$. While in classical theory A is convenient, but not strictly necessary, when we move to quantum theory A is almost unavoidable. There, when describing the interaction of the electromagnetic field with charged matter (particles or fields), A has a simple geometrical interpretation as a one-form connection in a complex line bundle, and F becomes a curvature of this connection (physicists sometimes like to use the term "a non-integrable phase factor" for the fact that we are dealing with a connection form of a non-zero curvature). This simple and beautiful geometrical picture becomes, however, somewhat lost when we move to the quantum theory of an electromagnetic field itself (both first- and second-quantized). (A promising new way of keeping the geometry alive in quantum field theory (within the algebraic framework) was also suggested by D. Buchholz et al.—cf. [1] and references therein.) Photons, the quanta of the electromagnetic field, are treated there as relativistic elementary particles and as such are described by irreducible unitary representations of the Poincaré group. The whole machinery of Lie algebras, Casimir operators, Lie groups, induced representations, and "little groups" is brought forward, and geometry is almost completely forgotten. Physicists construct one-particle Hilbert spaces and multi-particle Fock spaces (while the first quantization is a "miracle", the second quantization is a functor), moving into the algebra of operators, and this is a whole new world, with little place for differential geometry. V.S. Varadarajan in his classic monograph *Geometry of Quantum Theory*, chapter "Representations in vector bundles and wave equations", comes very close to fulfilling this

task; unfortunately, when it comes to photons, the chapter ends with the sentence "We do not get into these ideas here." ([2], p. 371).

The group-theoretical analysis of elementary relativistic quantum systems led to the concept of imprimitivity systems, developed by G.W. Mackey (cf., e.g., ([2], Ch. VI) and references therein), and to the associated concept of the localization of elementary quantum particles. A.S. Wightman [3] applied these concepts to the study of the localizability of quantum mechanical systems and came to a conclusion confirming the previous analysis of T.D. Newton and E.P Wigner [4], namely, that photons (as well as other particles of rest mass zero and helicity ≥ 1) are covariantly non-localizable in the strict sense of an imprimitivity system based on the 3-d Euclidean group acting on $\mathbb{R}^3$.

J.M. Jauch and C. Piron [5] developed a concept of "weak localizability", replacing a projection-valued measure with POV (positive operator-valued) measures, and A.O. Amrein [6] proved that there exist photon states which are strictly POV-localized in arbitrarily small regions of space, while, more recently, I. and Z. Bialynicki-Birula [7] argued that photons cannot be sharply localized because of a kind of complementarity between magnetic and electric energy localization.

Closely related to the problem of a photon's localization is the problem of the existence of the photon position operator **Q**. The problem is not exactly the same, since a given vector-valued operator may have infinitely many representations in terms of POV measures (except when Q_is commute, and then there is a distinguished projection-valued spectral measure). It is known [8] that the standard requirements of the covariance with respect to the Euclidean group and inversions lead to a unique **Q**—known as the Pryce photon position operator (the proof of uniqueness provided in this reference has a hole, as it requires an additional restriction on the form of the operator. But the hole in the proof can be completed, and no extra assumptions are, in fact, necessary). The trouble is that the components Q_i do not commute, which makes the simple probabilistic interpretation for the photon's localization problem impossible (while Q_i admits a natural decomposition with respect to a POV measure, K. Kraus [9] showed that there exists more than one such measure, so the question appears: which one of them is more natural than others, and why?). B.S. Skagerstam [10] interpreted the noncommutativity of the components of the Pryce operators Q_i^{PR} in terms of the curvature of a connection in a photon's momentum space (Skagerstam is using Jackiw's concept of "three-cocycle" [11], developed before under a different name (generalized imprimitivity system) by the present author [12]), and applied it to the derivation of the Berry phase for a photon [13]. The same idea was discussed before by I. and Z. Bialynicki-Birula [14], except that in their paper the same connection was derived independently of the photon's position operator question.

Abandoning the requirement of a covariance with respect to the 3D rotation group opens the way towards the construction of a huge family of "position operators" for photons, including those with commuting components and therefore admitting a unique spectral decomposition. M. Hawton [15] started a whole series of works in this direction. The commuting of the components of the position operator implies a flat (curvature-zero) connection on the positive light cone in momentum space. Connections of this type (no curvature, but torsion) were long ago investigated by A. Staruszkiewicz [16,17], who required that the parallel transport preserves the natural (degenerate) metric, the volume form, and that the connection is semi-symmetric (a constraint on the torsion). Hawton and Baylis [18] investigated a particular photon position operator **Q**$^{\text{HB}}$ with commuting components, the Hawton–Baylis operator, with axial symmetry. (If the Berry phase is related to the non-zero curvature of the connection responsible for the parallel transport, as it is usually assumed, then any such position operator must lead to a vanishing Berry phase.) Recently, these ideas were further developed by Dobrski et al. [19,20].

The present work started with the realization that there is an apparent discrepancy between the geometrical picture of the photon wave function as a section of the tangent or cotangent bundle (whenever a (non-degenerate) metric is available, there is no need to distinguish between tangent and cotangent bundles. The distinction between the two

becomes relevant only in premetric formulations of electrodynamics—cf. [21] and references therein) and the way the Lorentz boosts act on vector-valued functions within the unitary representation of the Poincaré group considered in all papers dealing with the photon wave function and with photon position operators (for the photon wave function, see, e.g., [22], while for photon position operators discussed in a similar context as that taken in the present paper, see [20] and references therein). In the present paper, we start with a geometrical description of massive vector fields transforming naturally under the Poincaré group. We work in momentum space (a good introduction to a photon's wave mechanics, in both momentum and position space, as well as to the photon's localization problem, can be found in the review article [23] and in the monograph [24] by Ole Keller) and discuss the natural unitary representation acting on the Hilbert space of sections of the tangent bundle of the positive mass hyperboloid which is square-integrable with respect to the natural Lorentz-invariant measure. The natural Riemannian metric appearing there becomes degenerate in the limit $m = 0$. For $m > 0$, taking the positive square root of this metric, we split the tangent space at each point into two mutually orthogonal parts: the longitudinal-transversal split with respect to the mass-independent standard Euclidean metric. This enables us to twist the unitary representation along the longitudinal part. We obtain an explicit form of the so-obtained irreducible unitary representation and then take the limit $m = 0$, which now is finite (though not irreducible). Only then, the Lie algebra of the Poincaré group acquires the standard form.

We then use the explicit form of the boost unitary operators to obtain a rather unexpected result: the polarization basis used for constructing the teleparallel connection by Hawton and Baylis [18] can be obtained by taking the speed-of-light limit in the z-direction of simple Hertz-type potentials $\mathbf{e}_1 \sim \mathbf{p} \times \mathbf{w}$ and $\mathbf{e}_2 \sim \mathbf{p} \times (\mathbf{p} \times \mathbf{w})$, $\mathbf{w} = (0, 0, 1)$. We then discuss the Hawton–Baylis connection and the associated photon position operator with commuting components and compare it to the Pryce connection with non-zero curvature and torsion. In particular, we find that the Pryce connection is metric semi-symmetric, while the Hawton–Baylis connection does not have the semi-symmetry property.

In the last part of this paper, we analyze photon states $\mathbf{f}_\ell$ localized on loops ℓ in a photon's position space. They are given as simple superpositions of plane waves localized at the points of the loop. For the particular case when the loop is a circle on a $z = 0$ plane, we show that $\mathbf{f}_l$ are localized on the circle not only with respect to the Jauch–Piron–Amrein POV measure $F(\Delta)$ (what was known before) but also (which came as a surprise) with respect to $\mathbf{Q}^{\mathrm{HB}}$.

We construct simple wave packets made of these circle states that provide normalized photon states localized in arbitrarily small washer-like regions of space, again both with respect to $\mathbf{Q}^{\mathrm{HB}}$ and $F(\Delta)$.

Notation 1. *We work in the momentum representation, spacetime signature $(- + + +)$. Coordinates $(p^0, p^1, p^2, p^3) = (p^0, \mathbf{p})$, $p^2 = \mathbf{p}^2 - (p^0)^2$. We will write $\mathbb{R}^3$ to denote the three-dimensional real vector space of the momentum vectors $\mathbf{p}$. Greek indices μ, ν, ρ, σ run from 0 to 3. Latin indices i, j, k, l run from 1 to 3. We use the Greek letter α, also running from 1 to 3, for numbering the basis vectors in momentum space. Summation over repeated indices is implied. Only positive energies, $p^0 > 0$, are used. We will use the notation $\boldsymbol{\pi}$ to denote the dimensionless unit vector in the direction of the momentum $\mathbf{p}$: $\boldsymbol{\pi} = \mathbf{p}/|\mathbf{p}|$.*

2. Massive Vector Field

For $m > 0$, we denote by V_m^+ the hyperboloid $p^2 + m^2 = 0$, $p^0 > 0$, i.e.,

$$p^0 = +\sqrt{\mathbf{p}^2 + m^2}. \tag{1}$$

Later on, we will be interested in the limit $m \to 0$. The mass hyperboloid V_m^+ is globally parametrized by $\mathbf{p} \in \mathbb{R}^3$. In the limit $m \to 0$, V_m^+ becomes the positive cone V_0^+, and we remove the origin $\mathbf{p} = 0$.

2.1. The Tangent Bundle

Let TV_m^+ be the tangent bundle of V_m^+. Let $p^\mu(t)$ be a (differentiable) path in V_m^+, with $(p^\mu(0)) = (p^0 = \sqrt{\mathbf{p}^2 + m^2}, \mathbf{p})$. For each t, we have

$$p^0(t)^2 = \mathbf{p}(t)^2 + m^2. \tag{2}$$

Differentiating at $t = 0$ and setting $v^\mu = (dp^\mu(t)/dt)|_{t=0}$, we obtain

$$v^0 p^0 = \mathbf{v} \cdot \mathbf{p}, \tag{3}$$

Therefore, in TV_m^+, at $\mathbf{p}$, we can use only the coordinates v^i, the coordinate v^0 being given by

$$v^0(\mathbf{p}) = \frac{\mathbf{v} \cdot \mathbf{p}}{p^0}. \tag{4}$$

2.2. The Action of the Lorentz Group

Let $\eta = (\eta_{\mu\nu})$ be the matrix $\eta = \mathrm{diag}(-1, +1, +1, +1)$. The inverse matrix $\eta^{-1} = (\eta^{\mu\nu})$ has the same matrix elements. Indices are raised and lowered with the matrices η^{-1} and η, respectively. In particular, $p_i = p^i$ and $p_0 = -p^0$ (we will never use p_0).

Let $\mathcal{L}^\uparrow$ be the orthochronous Lorentz group, that is, the group of those 4×4 real matrices $L = (L^\mu{}_\nu)$ satisfying $\eta^{-1} L^T \eta = L^{-1}$, $L^0{}_0 > 0$.

The group $\mathcal{L}^\uparrow$ acts on V_m^+ via $p \mapsto Lp$, $(Lp)^\mu = L^\mu{}_\nu p^\nu$. On the mass hyperboloid V_m^+, p^0 is determined by $\mathbf{p}$. Therefore, on V_m^+, we can write $(Lp)^i = L^i{}_j p^j + L^i{}_0 p^0$, where $p^0 = \sqrt{\mathbf{p}^2 + m^2}$. Let $\mathbf{p} \mapsto L\mathbf{p}$ denote this action:

$$(L\mathbf{p})^i = L^i{}_j\, p^j + (L^i)_0\, p^0. \tag{5}$$

It induces the action on the tangent bundle TV_m^+ as follows.

If v^μ is a vector tangent to V_m^+ at $\mathbf{p}$, then $(Lv)^\mu = L^\mu{}_\nu v^\nu$ is tangent to V_m^+ at Lp. Now, using Equation (4), we obtain

$$(Lv)^i = (L^i{}_j + \frac{L^i{}_0 p_j}{p^0})v^j. \tag{6}$$

We set

$$\tilde{D}_{(m)}{}^i{}_j(L, \mathbf{p}) = L^i{}_j + \frac{L^i{}_0 p_j}{p^0}. \tag{7}$$

The wave functions of the massive spin-1 particle are sections of the tangent bundle TV_m^+. We denote by $\mathrm{Sec}(TV_m^+)$ the space of these sections. If $\mathbf{p} \mapsto \mathbf{f}(\mathbf{p})$ is in $\mathrm{Sec}(TV_m^+)$, then $L\mathbf{f}$ is defined through the formula

$$(L\mathbf{f})(L\mathbf{p}) = L(\mathbf{f}(\mathbf{p})), \tag{8}$$

or, in coordinates:

$$(L\mathbf{f})^i(L\mathbf{p}) = \tilde{D}_{(m)}{}^i{}_j(L, \mathbf{p}) f^j(\mathbf{p}). \tag{9}$$

Replacing $L\mathbf{p}$ with $\mathbf{p}$ and $\mathbf{p}$ with $L^{-1}\mathbf{p}$, we obtain

$$(L\mathbf{f})^i(\mathbf{p}) = \tilde{D}_{(m)}{}^i{}_j(L, L^{-1}\mathbf{p}) f^j(L^{-1}\mathbf{p}). \tag{10}$$

Formula (10) defines the natural linear action of $\mathcal{L}^\uparrow$ on $\mathrm{Sec}(TV_m^+)$.

It can be verified by a direct calculation that the matrices $\tilde{D}_{(m)}(L, \mathbf{p})$ satisfy the following "cocycle relations":

$$\tilde{D}_{(m)}(L_1 L_2, \mathbf{p}) = \tilde{D}_{(m)}(L_1, L_2\mathbf{p}) \tilde{D}_{(m)}(L_2, \mathbf{p}), \tag{11}$$

which is equivalent to

$$(L_1 L_2)\mathbf{f} = L_1(L_2\mathbf{f}), \tag{12}$$

which means that we have a linear representation of the group $\mathcal{L}^{\uparrow}$ on $\mathrm{Sec}(TV_m^+)$. So far, no restrictions on sections $\mathbf{p} \mapsto f(\mathbf{p})$ are needed. The minimal assumption is that they are (Borel) measurable. Notice also that $f(\mathbf{p})$ can be assumed to be real. As long as we are interested only in pure Lorentz transformations, and not in spacetime translations, there is no need to complexify TV_m^+.

2.3. Riemannian Metric on V_m^+

For $m > 0$, the fibers of TV_m^+ carry a natural Lorentz-invariant Riemannian structure: the scalar product $v \cdot v' = \mathbf{v} \cdot \mathbf{v}' - v^0 v'^0$ is evidently Lorentz-invariant. Substituting the expression (4), we obtain the coordinate expression for the Riemannian metric $g_{(m)ij}(\mathbf{p})$

$$g_{(m)ij}(\mathbf{p}) = \delta_{ij} - \frac{p_i p_j}{|\mathbf{p}|^2 + m^2}. \tag{13}$$

Its inverse is given by

$$g_{(m)}^{ij} = \delta^{ij} + \frac{p^i p^j}{m^2}. \tag{14}$$

Since the flat metric η is invariant under the linear action of the Lorentz group, the induced metric g_m is invariant under the induced action. We have

$$\tilde{D_{(m)}}(L, \mathbf{p})^T g_{(m)}(L\mathbf{p}) \tilde{D_{(m)}}(L, \mathbf{p}) = g_{(m)}(\mathbf{p}). \tag{15}$$

In what follows, we will need the positive square root of $g_{(m)}$. Since, as long as $m > 0$, $g_{(m)}$ is positive definite, there exists a unique positive definite square root $h_{(m)} = (h^{(m)i}{}_j)$ of $g_{(m)}$. It can be verified that $h_{(m)}$ is given by the following explicit expression:

$$h_{(m)}{}^i{}_j(\mathbf{p}) = \delta_j^i + \lambda(\mathbf{p}) \pi^i \pi_j, \tag{16}$$

$$\lambda(\mathbf{p}) = \frac{m}{p^0} - 1. \tag{17}$$

We will also need its inverse $(h_{(m)})^{-1}$, which is given by

$$h_{(m)}{}^{-1}{}^i{}_j(\mathbf{p}) = \delta_j^i + \mu(\mathbf{p}) \pi^i \pi_j, \tag{18}$$

$$\mu(\mathbf{p}) = \frac{p^0}{m} - 1. \tag{19}$$

Remark 1. *We have two scalar products in $\mathbb{R}^3_{\mathbf{p}}$, the standard, Euclidean one, δ_{ij}, and the one determined by the metric $g_{(m)ij}$. In the following, whenever we write the dot product or raise or lower the space index $(i, j, \dots)$, we will always use the standard Euclidean metric.*

2.4. The Hilbert Space and Unitary Representation

We have

$$\sqrt{\det g_{(m)}} = \frac{m}{p^0}, \tag{20}$$

Therefore, since $g_{(m)}$ is Lorentz-invariant, d^3p/p^0 is a Lorentz-invariant measure on V_m^+. We define the Hilbert space $\tilde{\mathcal{H}}_m$ as the Hilbert space of sections $\mathbf{f}(\mathbf{p})$ of TV_m^+ which are square-integrable with respect to the scalar product

$$(\mathbf{f}, \mathbf{f}')_m = \int g_{(m)ij}(\mathbf{p}) \bar{f}^i(\mathbf{p}) f'^j(\mathbf{p}) \frac{d^3p}{p^0}. \tag{21}$$

By construction, the formula

$$(\tilde{U}(L)\mathbf{f})(\mathbf{p}) \doteq (L\mathbf{f})(\mathbf{p}) = D_{(m)}^{\sim}(L, L^{-1}\mathbf{p})\mathbf{f}(L^{-1}\mathbf{p}) \tag{22}$$

defines a unitary representation of $\mathcal{L}^{\uparrow}$ on $\tilde{\mathcal{H}}_m$.

The unitary representations of $\mathcal{L}^{\uparrow}$ for different values of m have the same form but act in Hilbert spaces with different scalar products in the fibers. In order to be able to take the limit $m = 0$, it is convenient to use just one standard fiber scalar product, independent of the value of m, but make the form of the representation m-dependent. To this end, let $\mathcal{H}_m$ be the Hilbert space of sections of TV_m^+ which are square-integrable with respect to the standard Hermitian scalar product

$$< \mathbf{f}, \mathbf{f}' >_m = \int \bar{\mathbf{f}}(\mathbf{p}) \cdot \mathbf{f}'(\mathbf{p}) \frac{d^3 p}{p^0}, \tag{23}$$

where $\bar{\mathbf{f}}$ denotes the complex conjugate (not needed if $\mathbf{f}$ is real). Then, the map

$$h_{(m)} : \mathbf{f}(\mathbf{p}) \mapsto h_{(m)}(\mathbf{p})\mathbf{f}(\mathbf{p}), \tag{24}$$

where $h_{(m)}$ is given by Equation (16), is an isometry from $\tilde{\mathcal{H}}_m$ to $\mathcal{H}_m$. Correspondingly, we have a unitary representation $L \mapsto U_m(L) = h_{(m)} \circ \tilde{U}(L) \circ h_{(m)}^{-1}$ of $\mathcal{L}^{\uparrow}$ on $\mathcal{H}_m$. It then follows from the definition that $U(L)$ can be written as

$$(U_m(L)f)(\mathbf{p}) = D_m(L, L^{-1}\mathbf{p})f(L^{-1}\mathbf{p}), \tag{25}$$

where

$$D_m(L, \mathbf{p}) = h_{(m)}(L\mathbf{p})D_{(m)}^{\sim}(L, \mathbf{p})h_{(m)}^{-1}(\mathbf{p}). \tag{26}$$

By using Equation (15) and the definition of $h_{(m)}$, we find that the matrices $D_m(L, \mathbf{p})$ are orthogonal, which makes the property of the unitarity of $U_m(L)$ evident (using the terminology of ([2], p. 175), one says that the cocycles $D_{(m)}^{\sim}$ and D_m are strictly cohomologous).

The Longitudinal-Transversal Split

Assuming $\mathbf{p} \neq 0$, the eigenvalue equation $g_{(m)}\mathbf{f} = \lambda \mathbf{f}$ for the real symmetric matrix $g_{(m)}$ reads

$$f^i(\mathbf{p}) - \frac{\mathbf{p} \cdot \mathbf{f}(\mathbf{p})}{|\mathbf{p}|^2 + m^2}p^i = \lambda f^i(\mathbf{p}),$$

i.e.,

$$(1 - \lambda)f^i(\mathbf{p}) = \frac{\mathbf{p} \cdot \mathbf{f}(\mathbf{p})}{|\mathbf{p}|^2 + m^2}p^i. \tag{27}$$

Thus, either $\lambda = 1$, and then $\mathbf{p} \cdot \mathbf{f}(\mathbf{p}) = 0$, or $\lambda \neq 1$, and then $\mathbf{f}(\mathbf{p})$ is proportional to $\mathbf{p}$. For each $\mathbf{p} \neq 0$, let $\mathcal{P}_0(\mathbf{p})$ be the orthogonal projection (in $\mathbb{R}^3$ endowed with the Euclidean metric δ_{ij}) on the one-dimensional subspace consisting of vectors proportional to $\mathbf{p}$:

$$\mathcal{P}_0(\mathbf{p}) \mathbf{f}(\mathbf{p}) = (\pi \cdot \mathbf{f}(\mathbf{p}))\pi. \tag{28}$$

Then, $\mathcal{P}_0(\mathbf{p})$ projects onto the eigenspace of $g_{(m)}(\mathbf{p})$ belonging to the eigenvalue $m^2 / (|\mathbf{p}|^2 + m^2)$. Let $\mathcal{P}_1(\mathbf{p}) = I - \mathcal{P}_0(\mathbf{p})$ be the orthogonal projection on the complementary subspace of vectors corresponding to the eigenvalue 1:

$$\mathcal{P}_1(\mathbf{p})\mathbf{f}(\mathbf{p}) = \mathbf{f}(\mathbf{p}) - - - (\pi \cdot \mathbf{f}(\mathbf{p}))\pi, \tag{29}$$

such that

$$g_{(m)}(\mathbf{p}) = \frac{m^2}{|\mathbf{p}|^2 + m^2}\mathcal{P}_0(\mathbf{p}) + \mathcal{P}_1(\mathbf{p}). \tag{30}$$

Then, we immediately obtain

$$h_{(m)}(\mathbf{p}) = \frac{m}{\sqrt{|\mathbf{p}|^2 + m^2}} \mathcal{P}_0(\mathbf{p}) + \mathcal{P}_1(\mathbf{p}), \tag{31}$$

and

$$h_{(m)}{}^{-1}(\mathbf{p}) = \frac{\sqrt{|\mathbf{p}|^2 + m^2}}{m} \mathcal{P}_0(\mathbf{p}) + \mathcal{P}_1(\mathbf{p}). \tag{32}$$

It should be noticed, however, that, as long as $m > 0$, the split above is not invariant under the action of the boosts of the Lorentz group on V_m^+.

2.5. The Limit $m = 0$

For $m = 0$, the matrices $D_{(m)}^{\sim}(\mathbf{p})$ are still given by Equation (7), but now $p^0 = |\mathbf{p}|$. It is less obvious that the matrices D_m given by Equation (26) remain finite in the zero-mass limit. In Proposition 1 below, we show that, for all L in $\mathcal{L}^{\uparrow}$ and $\mathbf{p}$ in V_m^+, the limit

$$D_0(L, \mathbf{p}) = \lim_{m=0} D_m(L, \mathbf{p}) \tag{33}$$

exists, and we provide its explicit form.

Proposition 1. $D_0(L, \mathbf{p})$ *is finite and it is given by the following explicit formula:*

$$D_0{}^i{}_j(L, \mathbf{p}) = L^i{}_j + L^i{}_0 \pi_j - \pi'^i L^0{}_j + \pi'^i \pi_j (1 - L^0{}_0), \tag{34}$$

where $\pi' = \frac{L\mathbf{p}}{|L\mathbf{p}|}$. For each $L \in \mathcal{L}^{\uparrow}$ and $\mathbf{p} \neq 0$, the matrix $D_0(L, \mathbf{p})$ is orthogonal:

$$D_0(L, \mathbf{p})^T = D_0(L, \mathbf{p})^{-1}. \tag{35}$$

The representation

$$(U_0(L)\mathbf{f})(\mathbf{p}) \doteq D_0(L, L^{-1}\mathbf{p})\mathbf{f}(L^{-1}\mathbf{p}) \tag{36}$$

of $\mathcal{L}^{\uparrow}$ on the Hilbert space $\mathcal{H}_0 = L^2(\mathbb{R}_{\mathbf{p}}^3, d^3p/|\mathbf{p}|) \otimes \mathbb{C}^3$ is unitary.

Proof. Substituting Equations (31) and (32) into Equation (26), we obtain for $D_m(L, \mathbf{p})$ the sum of four terms t_1, t_2, t_3, t_4, where

$$t_1(m) = \frac{\sqrt{|\mathbf{p}|^2 + m^2}}{\sqrt{|L\mathbf{p}|^2 + m^2}} \mathcal{P}_0(L\mathbf{p}) D_{(m)}^{\sim}(L, \mathbf{p}) \mathcal{P}_0(\mathbf{p}), \tag{37}$$

$$t_2(m) = \mathcal{P}_1(L\mathbf{p}) D_{(m)}^{\sim}(L, \mathbf{p}) \mathcal{P}_1(\mathbf{p}), \tag{38}$$

$$t_3(m) = \frac{m}{\sqrt{|L\mathbf{p}|^2 + m^2}} \mathcal{P}_0(L\mathbf{p}) D_{(m)}^{\sim}(L, \mathbf{p}) \mathcal{P}_1(\mathbf{p}), \tag{39}$$

$$t_4(m) = \frac{\sqrt{|\mathbf{p}|^2 + m^2}}{m} \mathcal{P}_1(L\mathbf{p}) D_{(m)}^{\sim}(L, \mathbf{p}) \mathcal{P}_0(\mathbf{p}). \tag{40}$$

Noticing that the orthogonal projections $\mathcal{P}_0$ and $\mathcal{P}_1$ do not depend on m, we see that $t_1(0)$ and $t_2(0)$ are finite and are given by

$$t_1(0) = \frac{|\mathbf{p}|}{|L\mathbf{p}|} \mathcal{P}_0(L\mathbf{p}) D_{(0)}^{\sim}(L, \mathbf{p}) \mathcal{P}_0(\mathbf{p}), \tag{41}$$

$$t_2(0) = \mathcal{P}_1(L\mathbf{p}) D_{(0)}^{\sim}(L, \mathbf{p}) \mathcal{P}_1(\mathbf{p}), \tag{42}$$

while the third term vanishes:

$$t_3(0) = 0. \tag{43}$$

We will now show that, surprisingly, the fourth term $t_4(m)$ also vanishes in the limit $m = 0$. To this end, we first consider the product

$$D\mathcal{P}_0(m) \doteq \tilde{D}_{(m)}(L, \mathbf{p})\mathcal{P}_0(\mathbf{p}) \tag{44}$$

of its last two factors. Using the definitions, we have

$$D\mathcal{P}_0(m)^i{}_k = \left(L^i{}_j + \frac{L^i{}_0 p_j}{p^0} \right) \frac{p^j p_k}{|\mathbf{p}|^2} = \left(\frac{L^i{}_j p^j}{|\mathbf{p}|^2} + \frac{L^i{}_0}{p^0} \right) p_k. \tag{45}$$

Setting $p' = Lp$, we have $L^i{}_j p^j = p'^i - L^i{}_0 p^0$; therefore,

$$D\mathcal{P}_0(m)^i{}_k = \frac{p'^i p_k}{|\mathbf{p}|^2} + L^i{}_0 \left(\frac{1}{p^0} - \frac{p^0}{|\mathbf{p}|^2} \right) p_k = \frac{p'^i p_k}{|\mathbf{p}|^2} - m^2 \frac{L^i{}_0 p_k}{p^0 |\mathbf{p}|^2}. \tag{46}$$

Now, the projection $\mathcal{P}_1(\mathbf{p}')$ vanishes on the first term $\frac{p'^i p_k}{|\mathbf{p}|^2}$, as the range of this matrix is in the longitudinal subspace at $\mathbf{p}'$, and $\frac{\sqrt{|\mathbf{p}|^2 + m^2}}{m}\mathcal{P}_1(L\mathbf{p})$ acting on the second term vanishes linearly in m. Therefore, $t_4(0) = 0$, and so

$$D_0(L, \mathbf{p}) = \frac{|\mathbf{p}|}{|\mathbf{p}'|}\mathcal{P}_0(\mathbf{p}')\tilde{D}_{(0)}(L, \mathbf{p})\mathcal{P}_0(\mathbf{p}) + \mathcal{P}_1(\mathbf{p}')\tilde{D}_{(0)}(L, \mathbf{p})\mathcal{P}_1(\mathbf{p}). \tag{47}$$

Since the sub-bundle of longitudinal vectors is invariant under the action of $\tilde{D}_{(0)}$, we have $\mathcal{P}_1(\mathbf{p}')\tilde{D}_{(0)}(L, \mathbf{p})\mathcal{P}_0(\mathbf{p}) = 0$ and $\mathcal{P}_0(\mathbf{p}')\tilde{D}_{(0)}(L, \mathbf{p})\mathcal{P}_0(\mathbf{p}) = \tilde{D}_{(0)}(L, \mathbf{p})\mathcal{P}_0(\mathbf{p})$. Therefore, Equation (47) simplifies to

$$D_0(L, \mathbf{p}) = \tilde{D}_{(0)}(L, \mathbf{p}) + \frac{|\mathbf{p}|}{|\mathbf{p}'|}\tilde{D}_{(0)}(L, \mathbf{p})\mathcal{P}_0(\mathbf{p}) - \mathcal{P}_0(\mathbf{p}')\tilde{D}_{(0)}(L, \mathbf{p}). \tag{48}$$

We have already calculated the matrix $\tilde{D}_{(0)}(L, \mathbf{p})\mathcal{P}_0(\mathbf{p})$—we just set $m = 0$ in Equation (46). Now, using the fact that $L^T \eta L = \eta$, a straightforward algebra gives us

$$(\mathcal{P}_0(\mathbf{p}')\tilde{D}_{(0)}(L, \mathbf{p}))^i{}_j = \pi'^i (L^0{}_j + \pi_j L^0{}_0), \tag{49}$$

and leads to Formula (34) in Proposition 1 (notice that $\tilde{D}_{(0)}(L, \mathbf{p})$ gives the first two terms of the right-hand side of (34)). The matrices $D_0(L, \mathbf{p})$ are orthogonal as limits of a continuous family of orthogonal matrices, and, since the measure d^3p/p^0 is Lorentz-invariant, the representation U_0 is unitary. □

We notice that for pure space rotations $L_i{}^0 = L_0{}^j = 0$, $L_0{}^0 = 1$, we have $D_{0i}{}^j = L_i{}^j$.

2.6. The Poincaré Group Lie Algebra

So far, we have only discussed the homogeneous transformations from the Poincaré group—the Lorentz transformations from $\mathcal{L}^{\uparrow}$. Now, we add translations. They are implemented by complex phase rotations:

$$(U_0(a)\mathbf{f})(\mathbf{p}) = e^{ia \cdot p}\mathbf{f}(\mathbf{p}) = e^{i(a^1 p^1 + a^2 p^2 + a^3 p^3 - a^0 p^0)}\mathbf{f}(\mathbf{p}). \tag{50}$$

These transformations are evidently unitary on $\mathcal{H}_0$. Having the unitary representation of the whole group, we will now calculate the self-adjoint infinitesimal generators. For translations, we define $P_\mu = -i(dU(a)/da^\mu)|_{a=0}$, obtaining

$$P^i\mathbf{f}(\mathbf{p}) = p^i\mathbf{f}(\mathbf{p}), \quad P^0\mathbf{f}(\mathbf{p}) = |\mathbf{p}|\mathbf{f}(\mathbf{p}). \tag{51}$$

The Lie algebra of the orthogonal group $SO(4)$ consists of antisymmetric matrices $\tilde{m}_{\mu\nu}$ given by

$$(\tilde{m}_{\mu\nu})^{\sigma\rho} = \delta^\sigma_\mu \delta^\rho_\nu - \delta^\sigma_\nu \delta^\rho_\mu. \tag{52}$$

The matrices $m^{\mu\nu} = \tilde{m}^{\mu\nu}\eta$ then form a basis in the Lie algebra of the Lorentz group

$$(m_{\mu\nu})^\sigma{}_\rho = \delta^\sigma_\mu \eta_{\nu\rho} - \delta^\sigma_\nu \eta_{\mu\rho}. \tag{53}$$

Defining $\mathbf{m} = (m^i)$ and $\mathbf{n} = (n^i)$ through $m^i = \frac{1}{2}\epsilon^{ijk}m_{jk}$, $n^i = m_{0i}$, we obtain the commutation relations

$$[m^i, m^j] = -\epsilon_{ijk}m^k,\ [m^i, n^j] = -\epsilon_{ijk}n^k,\ [n^i, n^j] = \epsilon_{ijk}m^k. \tag{54}$$

Then, the densely defined self-adjoint operators M^i, N^i are given by

$$M^i = -idU(\exp(t\,m^i))/dt|_{t=0},\ N^i = -idU(\exp(t\,n^i))/dt|_{t=0}. \tag{55}$$

Using our explicit formulas, we obtain:

$$\mathbf{M} = \mathbf{L} + \mathbf{s}, \tag{56}$$

$$\mathbf{N} = \mathbf{K} + \mathbf{k}, \tag{57}$$

where

$$\mathbf{L} = -i\,\mathbf{p} \times \partial/\partial\mathbf{p}, \quad \mathbf{K} = i|\mathbf{p}|\partial/\partial\mathbf{p}, \quad \mathbf{k} = \boldsymbol{\pi} \times \mathbf{s}, \tag{58}$$

$$\mathbf{s} = -i\mathbf{m}. \tag{59}$$

The commutation relations obtained from the definitions are the standard ones:

$$[N^i, P^j] = i\delta^{ij}P^0,\ [N^i, P^0] = iP^i,\ [M^i, P^j] = i\epsilon_{ijk}P^k,\ [M^i, P^0] = 0, \tag{60}$$

$$[N^i, N^j] = -i\epsilon_{ijk}M^k,\ [M^i, N^j] = i\epsilon_{ijk}N^k,\ [M^i, M^j] = i\epsilon_{ijk}M^k. \tag{61}$$

Replacing $\mathbf{M}$ by $\mathbf{L}$ and $\mathbf{N}$ by $\mathbf{K}$, we obtain the same commutation relations.

Remark 2. *The representation U_0 defined by (36) is unitary on the Hilbert space $\mathcal{H}_0$ of vector-valued functions which are square-integrable with respect to the scalar product*

$$< \mathbf{f}, \mathbf{f}' >_0 = \int \mathbf{f}^\dagger(\mathbf{p})\mathbf{f}'(\mathbf{p})\frac{d^3p}{|\mathbf{p}|}. \tag{62}$$

In Section 4, we will use a different scalar product using a non-Lorentz-invariant measure d^3p. To make the representation U_0 unitary with respect to this scalar product, we need to adjust D_0 by introducing an extra scaling factor and defining

$$D(L, \mathbf{p}) = \sqrt{\frac{|\mathbf{p}|}{|L\mathbf{p}|}}D_0(L, \mathbf{p}). \tag{63}$$

The expression for infinitesimal generators for this unitary representation remains the same, except that in the definition of $\mathbf{K}$ we must replace $i|\mathbf{p}|\partial/\partial\mathbf{p}$ with $i|\mathbf{p}|\partial/\partial\mathbf{p} + \frac{1}{2}\boldsymbol{\pi}$.

Notice, however, that with the scalar product (62), and with D_0 replaced by D, the transversal components of $\mathbf{f}(\mathbf{p})$ cannot be directly interpreted as tangent vectors, since they do not have the correct transformation properties under boosts.

The unitary space inversion operator Π and antiunitary time inversion operator Θ for this representation are given by

$$(\Pi\mathbf{f})(\mathbf{p}) = \mathbf{f}(-\mathbf{p}), \quad (\Theta\mathbf{f})(\mathbf{p}) = \mathbf{f}^*(-\mathbf{p}). \tag{64}$$

For a general representation of the Poincaré group, one defines the four-dimensional Pauli–Lubanski pseudovector W^μ as

$$W_\mu = \frac{1}{2}\epsilon_{\nu\rho\sigma\mu}\,P^\nu M^{\rho\sigma}. \tag{65}$$

We have

$$W^0 = \mathbf{P} \cdot \mathbf{M}, \ \mathbf{W} = P^0\,\mathbf{M} - \mathbf{P} \times \mathbf{N}. \tag{66}$$

It follows from the very definition that $\eta_{\mu\nu}P^\mu W^\nu = 0$. For mass-zero representations, P^μ is lightlike, and therefore W^μ is proportional to P^μ:

$$W^\mu = \Lambda P^\mu. \tag{67}$$

The proportionality operator Λ commutes with all the generators and is called the helicity operator (see, e.g., [25], p. 64). In our case, one finds that for the generators P^μ, $\mathbf{L}$, $\mathbf{K}$, we have $W^\mu = 0$, and therefore, $\Lambda = 0$, while for the generators P^μ, $\mathbf{M}$, $\mathbf{N}$, we have

$$\Lambda = \boldsymbol{\pi} \cdot \mathbf{M} = \boldsymbol{\pi} \cdot \mathbf{s}, \tag{68}$$

or explicitly, using Equation (59),

$$(\Lambda \mathbf{f})(\mathbf{p}) = i\boldsymbol{\pi} \times \mathbf{f}(\mathbf{p}). \tag{69}$$

The spectrum of Λ is discrete and consists of three points $\lambda = \pm 1$ and 0. Therefore, Λ^2 is a projection onto the subspace $\mathcal{H}_{ph}$ of $\mathcal{H}_0$. $\mathcal{H}_{ph}$ is a direct sum of eigenspaces $\mathcal{H}_\pm$ of Λ corresponding to eigenvalues $\lambda = \pm 1$. The photon states are represented by vectors in $\mathcal{H}_{ph}$. The orthogonal complement $\mathcal{H}_l$ of $\mathcal{H}_{ph}$ in $\mathcal{H}_0$ describes a spinless particle. It follows from these definitions that

$$\mathcal{H}_{ph} = \{\mathbf{f} \in \mathcal{H}_0 : \mathbf{p} \cdot \mathbf{f}(\mathbf{p}) = 0\} \tag{70}$$

and that

$$\mathcal{H}_l = \{\mathbf{f} \in \mathcal{H}_0 : \mathbf{f}(\mathbf{p}) = c(\mathbf{p})\mathbf{p}\} \tag{71}$$

for some scalar function $c(\mathbf{p})$. Since Λ commutes with all the generators of the Poincaré group, they leave the subspaces $\mathcal{H}_{ph}$ and $\mathcal{H}_l$ invariant.

3. An Application of the Explicit Form: Photon Polarization Vectors Boosted to the Speed of Light

3.1. Action of Pure Boosts

We are interested in one-parameter subgroups of Lorentz transformations $L(\mathbf{w}, s)$, $\mathbf{w} \in \mathbb{R}^3$, $\mathbf{w}^2 = 1$, $s \in \mathbb{R}$, defined by

$$L(\mathbf{w}, s) = \exp(-s\,\mathbf{w} \cdot \mathbf{n}) = \exp\left(s\begin{pmatrix} -w_1 & 0 & 0 & 0 \\ -w_2 & 0 & 0 & 0 \\ -w_3 & 0 & 0 & 0 \\ 0 & -w_1 & -w_2 & -w_3 \end{pmatrix}\right). \tag{72}$$

For the matrix $L(\mathbf{w}, s)^\mu{}_\nu$, we then obtain (the parameter s is known as *rapidity*, $s = \operatorname{atanh}(\beta)$, $\beta = v/c$)

$$\begin{aligned} L^0{}_i &= L^i{}_0 = -w_i \sinh(s), \\ L^i{}_j &= \delta^i_j + w^i w_j(\cosh(s) - 1), \\ L^0{}_0 &= \cosh(s). \end{aligned} \tag{73}$$

For a fixed $\mathbf{w}$, we have the group property:

$$L(\mathbf{w}, s_1)L(\mathbf{w}, s_2) = L(\mathbf{w}, s_1 + s_2). \tag{74}$$

In particular, we have $L(\mathbf{w}, s)^{-1} = L(\mathbf{w}, -s)$.

For some special sections $\mathbf{f}(\mathbf{p})$ of TV_0^+, we are interested in calculating the limits

$$\lim_{s \to \pm\infty} U_0(L(\mathbf{w}, s))\mathbf{f}. \tag{75}$$

Having in mind Equation (36), let

$$D_0'(\mathbf{w}, s, \mathbf{p}) \doteq D_0(L(\mathbf{w}, s), L(\mathbf{w}, s)^{-1}\mathbf{p}). \tag{76}$$

The following proposition shows that the limits

$$D_0'(\mathbf{w}, \mathbf{p})_\pm = \lim_{s \to \pm\infty} D_0'(\mathbf{w}, s, \mathbf{p}) \tag{77}$$

exist, and it provides their explicit form.

Proposition 2. *For all* $\mathbf{p}$ *not parallel to* $\mathbf{w}$, *we have*

$$D_0'(\mathbf{w}, \mathbf{p})_\pm = \mathbb{1} \pm \boldsymbol{\pi}\mathbf{w}^T - \frac{\mathbf{w}\mathbf{w}^T - (\boldsymbol{\pi} \cdot \mathbf{w})\boldsymbol{\pi}\mathbf{w}^T + \boldsymbol{\pi}\boldsymbol{\pi}^T \pm \mathbf{w}\boldsymbol{\pi}^T}{1 \pm \boldsymbol{\pi} \cdot \mathbf{w}}. \tag{78}$$

where for any two vectors $\mathbf{a}, \mathbf{b}$ *we denote by* $\mathbf{a}\mathbf{b}^T$ *the matrix with components* $a_i b^j$.

Proof. The proof is performed by using a straightforward (though tedious) algebra. From the definition of D_0' and Equation (34), we have the following explicit expression for $D_0(L, L^{-1}\mathbf{p})$:

$$D_0(L, L^{-1}\mathbf{p})^i{}_j(L, \mathbf{p}) = L^i{}_j + L^i{}_0\pi_j' - \pi^i L^0{}_j + \pi^i\pi_j'(1 - L^0{}_0), \tag{79}$$

where now $\boldsymbol{\pi}' = \frac{L^{-1}\mathbf{p}}{|L^{-1}\mathbf{p}|}$. Substituting $L(\mathbf{w}, s)$ for L, we obtain fractions containing $\cosh(s)$, $\sinh(s)$, and their squares and products. The terms with $\sinh(s)\cosh(s)$ cancel out, while the terms with $\cosh^2(s), \sinh^2(s)$ collect so that we can apply the identity $\cosh^2(s) - \sinh^2(s) = 1$. Dividing the numerators and denominators (which are now linear in $\cosh(s)$ and $\sinh(s)$) by $\cosh(s)$, we use the fact that $\lim_{s \to \pm\infty} \tanh(s) = \pm 1$ to obtain the result. $\quad\square$

3.2. The Polarization Basis

Using the Euclidean metric, we will silently rise and lower the space indices with the Kronecker deltas δ^{ij} and δ_{ij}.

Let $\mathbf{a}$ be a unit ($\mathbf{a} \cdot \mathbf{a} = 1$) vector in $\mathbb{R}^3$, and let $\mathbf{p}$ be a non-zero vector in $\mathbb{R}^3$, with $\mathbf{p}$ not parallel to $\mathbf{a}$. Define the following two vectors $e_1^{\mathbf{a}}(\mathbf{p}), e_2^{\mathbf{a}}(\mathbf{p})$ in $\mathbb{R}^3$ as (cf. Ref. [26], Equation (3.3))

$$e_1^{\mathbf{a}}(\mathbf{p}) = \frac{\mathbf{p} \times \mathbf{a}}{|\mathbf{p} \times \mathbf{a}|}, \tag{80}$$

$$e_2^{\mathbf{a}}(\mathbf{p}) = \frac{\mathbf{p} \times (\mathbf{p} \times \mathbf{a})}{|\mathbf{p} \times (\mathbf{p} \times \mathbf{a})|}. \tag{81}$$

Adding the third vector field $e_\parallel$ defined as

$$e_\parallel(\mathbf{p}) = \boldsymbol{\pi}, \tag{82}$$

we obtain an orthonormal basis, which can be considered as an orthonormal moving frame in the (real) tangent bundle of the positive light cone in momentum space. The sections $e_1^{\mathbf{a}}$ and $e_2^{\mathbf{a}}$ of the tangent bundle TV_0^+ become singular on the straight line determined by $\mathbf{a}$.

Remark 3. *One can check that $\Lambda e_{\parallel}(\mathbf{p}) = 0$ and that the states $e_1^{\mathbf{a}}(\mathbf{p}) \pm i e_2^{\mathbf{a}}(\mathbf{p})$ are eigenstates of the helicity operator to the eigenvalue ± 1:*

$$\Lambda(e_1^{\mathbf{a}} \pm i e_2^{\mathbf{a}}) = \pm(e_1^{\mathbf{a}} \pm i e_2^{\mathbf{a}}). \tag{83}$$

Proposition 3. *The limits $e_{\alpha}^{\mathbf{a}\pm} = \lim\limits_{s \to \pm\infty} U_0(L(\mathbf{w}, t)) \, e_{\alpha}^{\mathbf{a}}$, $\alpha = 1, 2$, exist, and they are given by*

$$e_1^{\mathbf{a}\pm} = \frac{\pm 1}{\sqrt{1 - (\mathbf{w} \cdot \mathbf{a})^2}} \left(\mathbf{w} \times \mathbf{a} - \frac{\boldsymbol{\pi} \cdot (\mathbf{w} \times \mathbf{a})}{1 \pm \boldsymbol{\pi} \cdot \mathbf{w}} (\boldsymbol{\pi} \pm \mathbf{w}) \right), \tag{84}$$

$$e_2^{\mathbf{a}\pm} = \frac{1}{\sqrt{1 - (\mathbf{w} \cdot \mathbf{a})^2}} \left(\mathbf{w} \times (\mathbf{w} \times \mathbf{a}) - \frac{\boldsymbol{\pi} \cdot (\mathbf{w} \times (\mathbf{w} \times \mathbf{a}))}{1 \pm \boldsymbol{\pi} \cdot \mathbf{w}} (\boldsymbol{\pi} \pm \mathbf{w}) \right). \tag{85}$$

Moreover, the vector fields $e_{\mathbf{a}}^{\alpha}$ are invariant under the boosts in the $\mathbf{w}$-direction. We have

$$(\mathbf{w} \cdot \mathbf{N}) \, e_{\alpha}^{\mathbf{a}} = 0, \tag{86}$$

where $\mathbf{N}$ is the boost generator (57).

Proof. We know from Proposition 2 that the limits $D_0'(\mathbf{w}, \mathbf{p})_{\pm}$ exist. On the other hand, using the definitions of $e_{\alpha}^{\mathbf{a}}$ and $L(\mathbf{w}, s)^{-1} = L(\mathbf{w}, -s)$, as well as the fact that $\lim\limits_{s \to \pm\infty} \tanh(s) = \pm 1$, we obtain

$$\lim_{s \to \pm\infty} e_1^{\mathbf{a}}(L(\mathbf{w}, s)^{-1} \mathbf{p}) = \frac{\pm \mathbf{w} \times \mathbf{a}}{\sqrt{1 - (\mathbf{w} \cdot \mathbf{a})^2}}, \tag{87}$$

$$\lim_{w \to \pm\infty} e_2^{\mathbf{a}}(L(\mathbf{w}, w)^{-1} \mathbf{p}) = \frac{\mathbf{w} \times (\mathbf{w} \times \mathbf{a})}{\sqrt{1 - (\mathbf{w} \cdot \mathbf{a})^2}}. \tag{88}$$

The result then follows by an application of the matrix of $D_0'(\mathbf{w}, \mathbf{p})_{\pm}$ given by Formula (78) to the vectors (87) and (88). The last statement of the proposition is the immediate consequence of the definitions. $\square$

Lorentz-Boost Eigenmodes

Let us take for $\mathbf{w}$ the vector

$$\mathbf{w} = (0, 0, 1),$$

(boost in the direction of the third axis), and for $\mathbf{a}$ the vector

$$\mathbf{a} = (0, 1, 0).$$

We consider the case of $\lim\limits_{w \to -\infty}$. Using the formulas (87) and (88), we obtain the following two mutually orthogonal unit vector fields:

$$e_1(\mathbf{p}) = \frac{1}{1 - \pi^3} \begin{pmatrix} 1 - \pi^3 - (\pi^1)^2 \\ -\pi^1 \pi^2 \\ (1 - \pi^3) \pi^1 \end{pmatrix}, \quad e_2(\mathbf{p}) = \frac{1}{1 - \pi^3} \begin{pmatrix} \pi^1 \pi^2 \\ -(1 - \pi^3) + (\pi^2)^2 \\ -(1 - \pi^3) \pi_2 \end{pmatrix}. \tag{89}$$

These two vector fields are transversal; they define a photon polarization basis (notice that while each of the original vector fields $e_1^{\mathbf{a}}, e_2^{\mathbf{a}}$ has two singular points on the unit sphere, in their speed-of-light limits these two singularities merge into just one—the minimum required by the "hairy ball theorem" of algebraic topology). Together with the third vector field

$$e_3(\mathbf{p}) = \boldsymbol{\pi}, \tag{90}$$

which is longitudinal and spans the helicity-zero subspace, they form an orthonormal basis in $T_{\mathbf{p}} V_0^+$—cf. Figure 1.
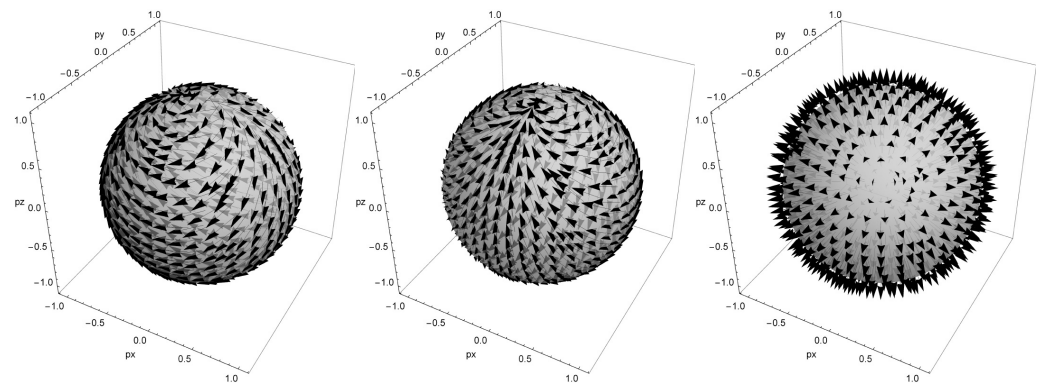
Figure 1. The vector fields $e_1(\mathbf{p}), e_2(\mathbf{p}), e_3(\mathbf{p})$ plotted at the unit sphere in momentum space.

It is easily verified that

$$M^3 e_1 = -i e_2, \ M^3 e_2 = i e_1, \ M^3 e_3 = 0, \tag{91}$$

and

$$N^3 e_1 = N^3 e_2 = N^3 e_3 = 0. \tag{92}$$

Remark 4. *The vector fields e_1, e_2 are real. Introducing their complex combinations*

$$e_\pm = \frac{1}{\sqrt{2}}(e_2 \pm i e_1), \tag{93}$$

we obtain two unit complex vector fields of definite helicity and the third component of angular momentum:

$$M^3 e_\pm = \mp e_\pm, \ \Lambda e_\pm = \pm e_\pm. \tag{94}$$

Evidently, we also have

$$N^3 e_\pm = 0. \tag{95}$$

The basis e_α is naturally embedded into a one-parameter family of Lorentz-boost eigenmodes (for the scalar wave equation, some of their properties are discussed in Ref. [27]) $e_\alpha^{(\lambda)}$ defined by the formula below:

$$e_\alpha^{(\lambda)}(\mathbf{p}) = \exp\left(i\lambda \log \sqrt{\frac{|\mathbf{p}| - p_3}{|\mathbf{p}| + p_3}}\right) e_\alpha(\mathbf{p}), \ (\alpha = 1, 2, 3), \ \lambda \in \mathbb{R}. \tag{96}$$

A straightforward verification shows that $e_\alpha^{(\lambda)}$ still satisfies Equation (91), but Equation (92) is replaced by

$$N^3 e_\alpha^{(\lambda)} = \lambda \, e_\alpha^{(\lambda)}. \tag{97}$$

Remark 5. *In fact, one can show that every solution of the eigenvalue problem*

$$N^3 f = \lambda f \tag{98}$$

is of the form

$$f(\mathbf{p}) = \sum_{\alpha=1}^{3} c_\alpha(p_1, p_2) e_\alpha^{(\lambda)}(\mathbf{p}). \tag{99}$$

3.3. The Teleparallel Connection

In the following, we will use the notation, assumptions, and results of Section 3.2.

3.3.1. Stereographic Coordinates on the Light Cone

We will use the moving frame $e_\alpha(\mathbf{p})$, ($\alpha = 1, 2, 3$) on V_0^+ to define a teleparallel affine connection on V_0^+. The frame has a singularity at the points $\mathbf{p} = (0, 0, t)$, $t > 0$, and a natural rotational covariance with respect to rotations around the third axis. It takes especially simple form in a coordinate system $X = (x, y, \omega)$ using the stereographic projection from the unit sphere in momentum space (stereographic projection coordinates are also used, in a similar context, in Refs. [16,17], but the coordinates x, y in these papers are twice our x, y, as the projection plane in these papers is positioned at the bottom of the unit sphere, and not through the center).

We set

$$p^1 = \frac{2x\omega}{x^2 + y^2 + 1}, \tag{100}$$

$$p^2 = \frac{2y\omega}{x^2 + y^2 + 1}, \tag{101}$$

$$p^3 = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}\omega, \tag{102}$$

and the inverse transform

$$x = \frac{\pi^1}{1 - \pi^3}, \tag{103}$$

$$y = \frac{\pi^2}{1 - \pi^3}, \tag{104}$$

$$\omega = |\mathbf{p}|. \tag{105}$$

Given a coordinate system X^i and a vector ξ with coordinates ξ^i, we have the standard transformation law to another coordinate system $X^{i'}$:

$$\xi^{i'} = \frac{\partial X^{i'}}{\partial X^i}\xi^i. \tag{106}$$

Applying this law to the vectors e_1, e_2, e_3, we obtain their components in stereographic coordinates ω, x, y (surprisingly, they happen to essentially coincide with the basis $\vec{E}_1, \vec{E}_2, \vec{E}_3$ considered in (Ref. [19], Equation (2.42)): $e_1 = -\vec{E}_1$, $e_2 = -\vec{E}_2$, $e_3 = \vec{E}_3$. In fact, we have $e_1 = \widehat{\partial_x}$, $e_2 = -\widehat{\partial_y}$, $e_3 = \widehat{\partial_\omega} = \partial_\omega$, where the hat over a vector denotes the unit vector in its direction):

$$e_1 = \begin{pmatrix} \frac{1+x^2+y^2}{2\omega} \\ 0 \\ 0 \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 \\ -\frac{1+x^2+y^2}{2\omega} \\ 0 \end{pmatrix}, \ e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{107}$$

The Euclidean metric in stereographic coordinates has the form

$$g = (g_{ij}) = \begin{pmatrix} \frac{4\omega^2}{(1+x^2+y^2)^2} & 0 & 0 \\ 0 & \frac{4\omega^2}{(1+x^2+y^2)^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{108}$$

$$g^{-1} = (g^{ij}) = \begin{pmatrix} \frac{(1+x^2+y^2)^2}{4\omega^2} & 0 & 0 \\ 0 & \frac{(1+x^2+y^2)^2}{4\omega^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{109}$$

while the Lorentz-invariant degenerate metric $g_{0ij} = \delta_{ij} - \pi_i \pi_j$ is obtained from g_{ij} by replacing 1 with 0 in the right bottom corner. We can use the metric g_{ij} to obtain the corresponding dual basis e^1, e^2, e^3 in the cotangent bundle:

$$e^1 = \begin{pmatrix} \frac{2\omega}{1+x^2+y^2} \\ 0 \\ 0 \end{pmatrix}, e^2 = \begin{pmatrix} 0 \\ -\frac{2\omega}{1+x^2+y^2} \\ 0 \end{pmatrix}, e^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{110}$$

The Lorentz-invariant volume form $vol = d^3p/|\mathbf{p}|$ becomes

$$vol = \frac{4\omega}{(1+x^2+y^2)^2} \, d\omega \, dx \, dy. \tag{111}$$

3.3.2. The Connection Coefficients

The moving frame e_α defines a unique flat affine connection on V_0 in which the vector fields e_α are parallel:

$$\nabla_i e_\alpha^k \doteq \partial_i e_\alpha^k + \Gamma_{ij}^k e_\alpha^j = 0. \tag{112}$$

The coefficients Γ_{ij}^k of this teleparallel connection in the coordinate system $(X^i) = (x, y, \omega)$ are then given by

$$\Gamma_{ij}^k = e_\alpha^k \frac{\partial e_j^\alpha}{\partial X^i}, \tag{113}$$

where e^α is the dual basis; thus, $e_i{}^\alpha e^j{}_\alpha = \delta_i^j$.

A straightforward calculation then gives the following expressions:

$$\Gamma_1 = \frac{-2x}{a} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \Gamma_2 = \frac{-2y}{a} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \Gamma_3 = \frac{1}{\omega} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{114}$$

where $a = 1 + x^2 + y^2$. Our connection Γ has the properties $\nabla_i g = \nabla_i g_0 = 0$ and $\nabla_i vol = 0$; therefore, it should come under the scope of affine connections discussed by Staruszkiewicz in Refs. [16,17]. However, as we show in the paragraph below, it is not "semi-symmetric"—an extra condition imposed on the class of connections analyzed by Staruszkiewicz.

The connection Γ_{ij}^k given by Equation (114) is not semi-symmetric.

Let M be an n-dimensional manifold with a coordinate system x^i. A linear connection ∇ with connection coefficients Γ_{ij}^k is said to be semi-symmetric if its torsion tensor $T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k$ is of the form

$$T_{ij}^k = \delta_i^k \tau_j - \delta_j^k \tau_i, \tag{115}$$

τ being a 1-form. If that is the case, contracting the indices i, k, we obtain $\tau_j = \frac{1}{n-1} T_{ij}^i$. In our case, $n = 3$; therefore,

$$\tau_j = \frac{1}{2} T_{ij}^i. \tag{116}$$

For the connection given by Equation (114), we obtain, for instance, $\tau_2 = -y/(1+x^2+y^2)$. But for a semi-symmetric connection, we should have, for instance, $T_{12}^1 = \delta_1^1 \tau_2 - \delta_2^1 \tau_1 = \tau_2$, while from Equation (114) we have $T_{12}^1 = \Gamma_{12}^1 - \Gamma_{21}^1 = 0 - 0 = 0$. Thus, our connection is not semi-symmetric.

4. Photon Position Operator with Commuting Components and Axial Symmetry

To discuss photon localization, it is more convenient to work in a representation in which the scalar product in the Hilbert space of the sections of the tangent bundle TV_0^+ is given by the formula (cf. Remark 2)

$$(f, f') = \int_{\mathbb{R}^3} f(\mathbf{p})^\dagger f'(\mathbf{p}) d^3 p. \tag{117}$$

Since the measure d^3p is not Lorentz-invariant, the formula for the boost operator now obtains an extra term (compared to Equation (57)) and takes the form

$$\mathbf{N} = \mathbf{K} + \frac{i}{2}\pi + \mathbf{n}. \tag{118}$$

As a consequence, the sections $\mathbf{p} \mapsto e_\alpha(\mathbf{p})$ do not satisfy Equation (92) any longer—but the sections $\mathbf{p} \mapsto |\mathbf{p}|^{-1/2}e_\alpha(\mathbf{p})$ do.

In what follows, we will be using the scalar product (117). We will denote $\mathcal{H}$ as the corresponding Hilbert space: $\mathcal{H} = L^2(\mathbb{R}^3, d^3p) \otimes \mathbb{C}^3$.

The map $f \mapsto |\mathbf{p}|^{1/2}f$ is an isometry between the two Hilbert spaces, the space $\mathcal{H}$, with the scalar product defined with the measure d^3p, and $\mathcal{H}_0$, with the scalar product d^3p/p_0.

4.1. Position Operators as Covariant Derivatives

There is a straightforward relation between the position operator and a covariant derivative concept in vector bundles. Indeed, the main property required from any position operator Q_i is the property of satisfying the canonical commutation relations with the momentum operators P_i:

$$[Q_j, P^k] = i\delta_j^k I. \tag{119}$$

In our case, it implies that for any section $\mathbf{f}(\mathbf{p})$ of $T(V_0^+)$ and any scalar function $\phi(\mathbf{p})$, we have

$$-i\,Q_j(\phi\mathbf{f})(\mathbf{p}) = \frac{\partial\phi(\mathbf{p})}{\partial p_j}\,\mathbf{f}(\mathbf{p}) + \phi(\mathbf{p})(Q_j\mathbf{f})(\mathbf{p}). \tag{120}$$

Thus, the operators $-iQ_i$ have the Leibniz rule property, the main property defining a covariant derivative ∇_i in a vector bundle (cf., e.g., [28], p. 89, Equation (1.1)). For Q_i to be Hermitian, ∇_i must be anti-Hermitian, and for this to be the case, the (linear) connection determined by ∇_i should preserve the fiber scalar product $(\mathbf{f}(\mathbf{p}), \mathbf{f}'(\mathbf{p}))_{\mathbf{p}} = \mathbf{f}(\mathbf{p})^\dagger \mathbf{f}'(\mathbf{p})$. Then,

$$\mathbf{Q} = i\boldsymbol{\nabla}, \tag{121}$$

or, explicitly,

$$(Q_j\mathbf{f})^k(\mathbf{p}) = i\partial_j f^k(\mathbf{p}) + i\Gamma_{il}^k(\mathbf{p})f^l(\mathbf{p}). \tag{122}$$

4.2. The Pryce Connection and Operator—Geometric Construction

There is a standard construction in the differential geometry of vector bundles that results in a canonical connection adapted to a split of a trivial vector bundle into a direct sum of its two vector sub-bundles (cf., e.g., [29], p. 319, 4, [30], Exercise 10). We adapt this standard construction to our purpose as follows. First, using the global coordinates p^i, we realize the tangent bundle TV_0^+ as a trivial product bundle $V_0^+ \times \mathbb{R}^3$. In the trivial bundle, we have a canonical covariant derivative $\partial_i = \frac{\partial}{\partial p^i}$. Our trivial bundle is naturally split into a direct sum of the helicity-zero sub-bundle and the helicity-± 1 sub-bundle of photon states. Let $P(\mathbf{p})$ denote the orthogonal projection on the helicity-zero states

$$(P(\mathbf{p})f)^i(\mathbf{p}) = \pi^i \pi_j f^j(\mathbf{p}). \tag{123}$$

Then, $I - P$ is the orthogonal projection on the complementary sub-bundle of photon states. The natural covariant derivative adapted to this splitting is then defined by the formula (cf. [31], Equation (2); we use the label $^{\text{PR}}$ to mean either "Projection" or "Pryce")

$$\nabla_i^{\text{PR}} = Pd_iP + (I - P)\partial_i(I - P). \tag{124}$$

Using the idempotent property $P^2 = P$, we find a simpler form:

$$\nabla_i^{\mathrm{PR}} = \partial_i + [P, \partial_i(P)], \tag{125}$$

while substituting the explicit form (123) of P leads to

$$(\nabla_i^{\mathrm{PR}} \mathbf{f})^j = \partial_i f^j + \frac{1}{|\mathbf{p}|}\left(\pi^j \delta_{ik} - \pi_k \delta_i^j\right) f^k, \tag{126}$$

Thus, the corresponding connection coefficients are given by

$$\Gamma^{\mathrm{PR}}{}_{ik}^j = [P, \partial_i(P)]^j{}_k = \frac{1}{|\mathbf{p}|}\left(\pi^j \delta_{ik} - \pi_k \delta_i^j\right). \tag{127}$$

4.2.1. The Pryce Connection Is Metric Semi-Symmetric

A *metric connection* is semi-symmetric—cf., e.g., [32,33] and references therein—if its connection coefficients Γ_{ij}^k are of the form

$$\Gamma_{ik}^j = \bar{\Gamma}_{ik}^j + \delta_i^j \tau_k - g_{ik}\tau^j, \tag{128}$$

where $\bar{\Gamma}_{ik}^j$ are the coefficients of the Levi-Civita connection of the metric, and where τ_i and $\tau^i = g^{ij}\tau_j$ are the covariant and contravariant components of a vector field, respectively. In that case, the curvatures of the two connections are related by (for the curvature tensor components of a connection ∇, we use the convention $R_{ij}{}^k{}_l \xi^l = ((\nabla_i\nabla_j - \nabla_j\nabla_i)\xi)^k$)

$$R_{ij}{}^k{}_l = \bar{R}_{ij}{}^k{}_l + \delta_j^k \tau_{il} - \delta_i^k \tau_{jl} + \delta_{il}\tau_j^k - \delta_{jl}\tau_i^k, \tag{129}$$

where

$$\tau_{ij} = \nabla_i \tau_j - \tau_i \tau_j + \frac{1}{2}g_{ij}\tau^2, \tag{130}$$

and $\tau^2 = g^{ij}\tau_i\tau_j$.

In our case, $g_{ij} = \delta_{ij}$ and the connection coefficients $\bar{\Gamma}_{ik}^j$ are all zero. Comparing Equations (128) and (127), we can see that the connection ∇^{PR} is metric semi-symmetric, with

$$\tau_i = -\pi_i/|\mathbf{p}|,. \tag{131}$$

We then obtain

$$\pi_{ij} = \pi_{ji} = \frac{1}{|\mathbf{p}|^2}\left((\Sigma^2)_{ij} + \frac{1}{2}\delta_{ij}\right), \tag{132}$$

where

$$\Sigma_{ij} = -\epsilon_{ijk}\pi^k, \tag{133}$$

and the curvature tensor simplifies to

$$R_{ijkl}^{\mathrm{PR}} = \frac{1}{|\mathbf{p}|^2}\Sigma_{ij}\Sigma_{kl}. \tag{134}$$

From Equation (128), we obtain for the torsion

$$T^{\mathrm{PR}}{}_{ij}^k = \delta_i^k \tau_j - \delta_j^k \tau_i, \tag{135}$$

where τ_i is given by (131).

4.2.2. The Difference between the Teleparallel and Pryce Connections

The difference between two connections is a tensor. In our case, it is a matter of straightforward calculations to find this tensor for the teleparallel connection Γ defined by Equation (114) and the Pryce connection given by Equation (126). We obtain

$$\Gamma^k_{ij} - \Gamma^{\mathrm{PR}\,k}_{\phantom{\mathrm{PR}}ij} = b_i \Sigma^k{}_j, \tag{136}$$

where

$$\mathbf{b} = \frac{\boldsymbol{\pi} \times \mathbf{w}}{|\mathbf{p}| - \mathbf{p} \cdot \mathbf{w}}, \quad \mathbf{w} = (0,0,1). \tag{137}$$

4.2.3. The Pryce Operator

We now use Equations (121) and (124). Denoting

$$\mathbf{X} = i\frac{\partial}{\partial \mathbf{p}}, \tag{138}$$

the position operator corresponding to ∇^{PR} is

$$\mathbf{Q}^{\mathrm{PR}} \doteq i\boldsymbol{\nabla}^{\mathrm{PR}} = P\mathbf{X}P + (I - P)\mathbf{X}(I - P), \tag{139}$$

or, explicitly,

$$(\mathbf{Q}^{\mathrm{PR}}_j \mathbf{f})^k = i\partial_j f^k + \frac{i}{|\mathbf{p}|}\left(\pi^k \delta_{jl} - \pi_l \delta^k_j\right)f^l. \tag{140}$$

Using Equations (53) and (59), it can be also written as

$$\mathbf{Q}^{\mathrm{PR}} = \mathbf{X} + \frac{(\mathbf{p} \times \mathbf{s})}{|\mathbf{p}|^2}, \tag{141}$$

and, in this form, it is known as the Pryce position operator. The following alternative formula for this operator, expressing it in terms of the Poincaré group generators, is easily verified and is well known:

$$\mathbf{Q}^{\mathrm{PR}} = \frac{1}{2}\left(\frac{1}{P^0}\mathbf{N} + \mathbf{N}\frac{1}{P^0}\right), \tag{142}$$

where N^i are the boost generators (118). It should be mentioned that even though the bundle TV_0^+ is trivial, the non-triviality of its splitting defined by the helicity-related projection P is reflected in the fact that the adapted connection has a non-zero curvature, and, as a consequence, the components of the Pryce operator do not commute.

Remark 6. *The flat covariant derivative d does not mix the two photon ± 1 helicities. As a consequence, the Pryce operator commutes with the helicity operator Λ and not only with its square $\Lambda^2 = I - P$.*

Now, using Equation (134), we obtain

$$([Q_i, Q_j]\mathbf{f})^k = -\frac{1}{|\mathbf{p}|^2}\Sigma_{ij}\Sigma^k{}_l f^l, \tag{143}$$

which is usually written as (see, e.g., [34], Equation (5,9), p. 40) (in Ref. [34], helicity is defined with the opposite sign)

$$[Q_i, Q_j] = i\epsilon_{ijk}\frac{p_k}{|\mathbf{p}|^3}\Lambda, \tag{144}$$

where $\Lambda = i\Sigma$ is the helicity operator.

4.3. The Hawton–Baylis Operator

The teleparallel connection described in Section 3.3 leads to the Hawton–Baylis photon position operator Q_i^{HB} with commuting components

$$Q_i^{\mathrm{HB}} = i\nabla_i, \tag{145}$$

with the defining relations

$$\nabla_i e_\alpha = 0, \ (i = 1, 2, 3), \ (\alpha = 1, 2, 3), \tag{146}$$

where e_α are given by Equations (89) and (90). It follows from Equations (145) and (146) that the states $e_\alpha(\mathbf{p})$ are localized at $\mathbf{x} = 0$. To obtain states localized at any point $\mathbf{x} = \mathbf{a}$, we multiply these states by $\exp(-i\mathbf{p} \cdot \mathbf{a})$.

From Equations (136) and (137), we then obtain (cf. [18], Equation (53) and [19], Equation (2.49))

$$Q_i^{\mathrm{HB}}\mathbf{f} = Q_i^{\mathrm{PR}}\mathbf{f} + b_i \Lambda \mathbf{f}. \tag{147}$$

In particular,

$$Q_3^{\mathrm{HB}} = Q_3^{\mathrm{PR}}. \tag{148}$$

Using this last equality, together with Equations (142) and (118), we recover the fact, mentioned after Equation (118), that the states $\tilde{e}_\alpha = |\mathbf{p}|^{-1/2}e_\alpha$ are invariant under boosts in the direction of the third axis. Indeed, from the Leibniz property (120) and from (146), we obtain

$$Q_i^{\mathrm{HB}}\tilde{e}_\alpha = -\frac{i\pi_i}{2|\mathbf{p}|}\tilde{e}_\alpha. \tag{149}$$

Thus,

$$Q_3^{\mathrm{PR}}\tilde{e}_\alpha = -\frac{i\pi_3}{2|\mathbf{p}|}\tilde{e}_\alpha. \tag{150}$$

On the other hand, using (142) and the Lie algebra commutation relation $[N^3, P^0] = iP^3$, Q_3^{PR} can be written as

$$Q_3^{\mathrm{PR}} = \frac{1}{|\mathbf{p}|}N^3 - \frac{i\pi_3}{2|\mathbf{p}|}, \tag{151}$$

which, together with (150), leads to $(1/|\mathbf{p}|)N^3\tilde{e}_\alpha = 0$, and thus, $N^3\tilde{e}^\alpha = 0$. But this reasoning does not give us a clue as how the basis e_α can be obtained by taking the speed-of-light limit of the simple polarization basis (81). (A brief discussion of an application of the Hawton–Baylis photon position operator to optical beams (derived via Wigner's little group method) can be found in Ref. [35]; cf. also references therein.)

It is straightforward to verify that the operators $Q_i = Q_i^{\mathrm{HB}}$ have axial symmetry:

$$[M^3, Q_1] = iQ_2, \ [M^3, Q_2] = -iQ_1, \ [M^3, Q_3] = 0. \tag{152}$$

4.3.1. Photon States Localized on Circles

Since the three components of Q^{HB} commute, they can be simultaneously diagonalized, and the simultaneous eigenvalue equation $Q_i^{\mathrm{HB}}\mathbf{f} = q_i\mathbf{f}$ has three independent solutions:

$$\mathbf{f}_{\alpha,\mathbf{q}}(\mathbf{p}) = \exp(-i\mathbf{q} \cdot \mathbf{p})e_\alpha(\mathbf{p}), \ (\mathbf{q} \in \mathbb{R}^3, \alpha = 1, 2, 3). \tag{153}$$

The states $\mathbf{f}_{1,\mathbf{q}} \pm i\mathbf{f}_{2,\mathbf{q}}$ describe photons localized at $\mathbf{q} \in \mathbb{R}^3$ of helicity ± 1, and $\mathbf{f}_{3,\mathbf{q}}$ is of helicity 0. However, since the states $e_\alpha(\mathbf{p})$ have a rather complicated $\mathbf{p}$-dependence, and owing to the axial symmetry of Q^{HB}, it is natural to look for **simple axially symmetric** simultaneous eigenvalue equations for $Q = Q^{\mathrm{HB}}$:

$$M^3\, \mathbf{f} \;=\; 0, \tag{154}$$

$$Q_3\, \mathbf{f} \;=\; 0, \tag{155}$$

$$(Q_1^2 + Q_2^2)\, \mathbf{f} \;=\; R^2\, \mathbf{f}, \; R > 0. \tag{156}$$

These would give us states localized on the circle $x^2 + y^2 = R^2$, $z = 0$, in the Cartesian coordinates (x, y, z) in the photon's position space. To this end, it is convenient to introduce cylindrical coordinates (ρ, ϕ, p_3), $\rho = \sqrt{(p_1^2 + p_2^2)}$. One can then verify that states of the form

$$\mathbf{f}(\rho, \phi, p_z) = F(\rho) \begin{pmatrix} -\sin(\phi) \\ \cos(\phi) \\ 0 \end{pmatrix} \tag{157}$$

are simple solutions of Equations (154) and (155), while Equation (156) imposes a second-order ordinary differential equation on $F(\rho)$:

$$\rho^2 F''(\rho) + \rho F'(\rho) + (R^2 \rho^2 - 1)F(\rho) = 0. \tag{158}$$

with the following solution finite at the origin [36]:

$$F(\rho) = c J_1(R\rho), \tag{159}$$

where J_1 is the Bessel function of the first kind and c is a constant—see Figure 2.
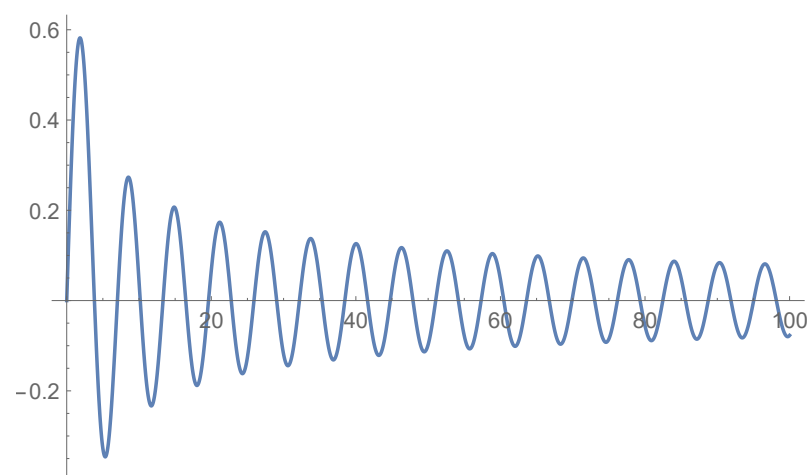
Figure 2. Bessel function $J_1(\rho)$.

For reasons that will be clear from the next paragraph, we choose for the constant c the value:

$$c = -iR. \tag{160}$$

Loop States

Let ℓ be a closed loop in $\mathbb{R}^3_{\mathbf{x}}$ defined by a function $\mathbf{r}(t)$, $\quad 0 \le t \le 2\pi$. Following Ref. [31], let us define the state $\mathbf{f}_\ell$ by

$$\mathbf{f}_\ell(\mathbf{p}) = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\mathbf{p}\cdot\mathbf{r}(t)}\, d\mathbf{r}(t). \tag{161}$$

Then, $\mathbf{f}_l(\mathbf{p})$ is a superposition of simultaneous eigenstates of X_i—see Equation (138)—$\mathbf{X}$-localized in $\mathbb{R}^3_{\mathbf{x}}$ at the points of the loop ℓ. Therefore, it is $\mathbf{X}$-localized on ℓ. The fact that the loop is closed, i.e., $\mathbf{r}(2\pi) = \mathbf{r}(0)$, implies that the state $\mathbf{f}_\ell(\mathbf{p})$ is an (improper) element of $\mathcal{H}_{ph}$, i.e., that

$$\mathbf{p} \cdot \mathbf{f}_\ell(\mathbf{p}) = 0. \tag{162}$$

Taking the Fourier transform $\mathbf{f}_\ell(\mathbf{x})$ of $\mathbf{f}_\ell(\mathbf{p})$, we obtain

$$\mathbf{f}_\ell(\mathbf{x}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}}^3 e^{i\mathbf{p}\mathbf{x}} \mathbf{f}_\ell(\mathbf{p}) d^3p = \frac{1}{2\pi} \int_0^{2\pi} \delta(\mathbf{x} - \mathbf{r}(t)) d\mathbf{r}(t), \tag{163}$$

and it is clear that $\mathbf{f}_\ell(\mathbf{x})$ has its support on the loop in the position coordinates space—it vanishes at all points $\mathbf{x}$ outside the loop. It is also clear that for two non-intersecting loops ℓ, ℓ' the states $\mathbf{f}_\ell$ and $\mathbf{f}_{\ell'}$ are orthogonal to each other. (Loops may form topologically inequivalent knots. In this respect, the loop states discussed above are similar to knotted solutions of Maxwell equations discussed in Ref. [26], Sec. 7.)

As an example, let us take for ℓ the circle in the (x, y) plane of radius R given by the parametric equations

$$x = R \cos t, \, y = R \sin t, \, z = 0. \tag{164}$$

A straightforward calculation in cylindrical coordinates (ρ, ϕ, p_4) (in momentum space) then leads to

$$\mathbf{f}_{x^2+y^2=R^2}(\mathbf{p}) = -i \frac{R}{\rho} J_1(R\rho) \begin{pmatrix} -p_2 \\ p_1 \\ 0 \end{pmatrix}. \tag{165}$$

where J_1 is the Bessel function and $\rho = \sqrt{(p^1)^2 + (p^2)^2}$, which coincides with the state $\mathbf{f}$ given by Formula (157).

Amrein's Washer Photon States

We use Equation (163) for a circle of radius R at $z = z_0$; thus,

$$\begin{aligned} x(t) &= R \cos t, \\ y(t) &= R \sin t, \\ z &= z_0. \end{aligned} \tag{166}$$

Thus,

$$\mathbf{f}_{R,z_0} = \frac{1}{2\pi} \int_0^{2\pi} \delta(x - R\cos t, y - R\sin t, z - z_0) \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} dt. \tag{167}$$

We will use cylindrical coordinates (r, ϕ, z) in position space. In these coordinates,

$$\delta(x - x', y - y', z - z') = \frac{1}{r} \delta(r - r') \delta(\phi - \phi') \delta(z - z'), \tag{168}$$

while

$$d^3x = r dr \, d\phi \, dz. \tag{169}$$

Thus,

$$\mathbf{f}_{R,z_0}(r, \phi, z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\delta(r - R)}{r} \delta(z - z_0) \delta(\phi - t) \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} dt. \tag{170}$$

After integrating over dt, we obtain

$$\mathbf{f}_{R,z_0}(r, \phi, z) = \frac{1}{2\pi} \frac{\delta(r - R)}{r} \delta(z - z_0) \begin{pmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{pmatrix}. \tag{171}$$

We now take a continuous superposition of these states for R varying between R_1 and $R_2 > R_1$ and z_0 varying between z_1 and $z_2 > z_1$. As a superposition of photon states, it

will still be a photon state. Let $\chi_{R_1,R_2}(r)$ be the function equal to 1 for $R_1 \leq r \leq R_2$ and zero otherwise, and, similarly, let χ_{z_1,z_2} be the function equal to 1 for $z_1 \leq z \leq z_2$ and zero otherwise. Taking the integral

$$\int_{-\infty}^{\infty} dz \int_0^{\infty} \chi_{R_1,R_2}(R)\chi_{z_1,z_2}(z)\mathbf{f}_{R,z_0}(r,\phi,z)\,dR\,dz \tag{172}$$

we obtain

$$\mathbf{f}_{R_1,R_2,z_1,z_2}(r,\phi,z) = \frac{1}{2\pi r}\chi_{R_1,R_2}(r)\chi_{z_1,z_2}(z))\begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}. \tag{173}$$

The function is evidently square-integrable with respect to $d^3x = rdr\,d\phi\,dz$, and its probability density is zero everywhere except for the bolt washer-like region $R_1 \leq r \leq R_2$, $z_1 \leq z \leq z_2$—see Figure 3.
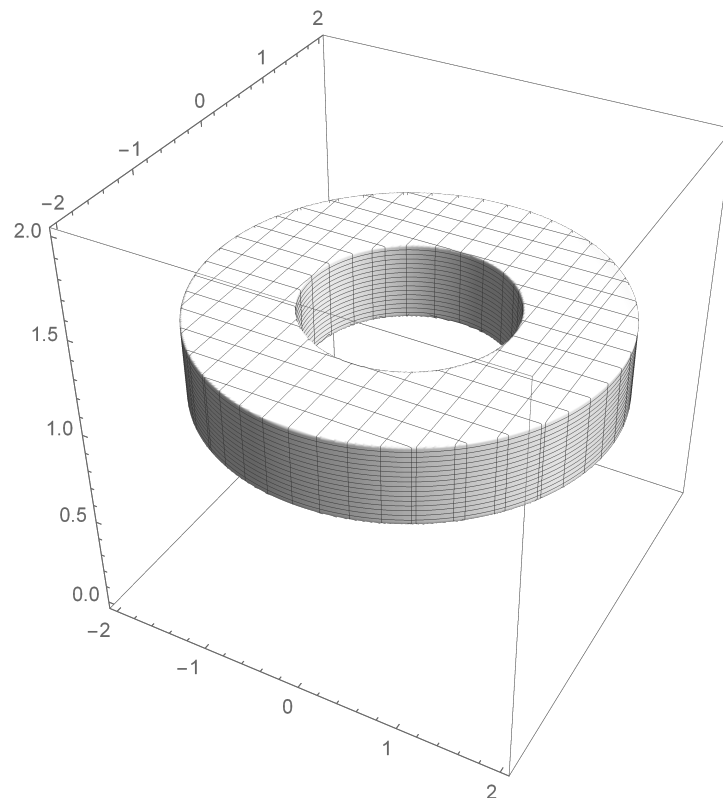
Figure 3. Amrein's washer state (171) is strictly (weakly) localized in the region $1\,\text{nm} \leq R \leq \sqrt{2}\,\text{nm}$, $1\,\text{nm} \leq z \leq 1.4\,\text{nm}$.

We call it Amrein's state, as the existence of such states was first proven in 1968 by A.O. Amrein [6]. These states are also strictly localized with respect to the commuting position operators Q_i^{HB}.

4.4. POV Measure Photon Localization

Every self-adjoint operator admits a spectral decomposition. Usually, we write it as:

$$A = \int \lambda dE(\lambda). \tag{174}$$

More generally, given a family of commuting observables, we have a spectral measure on the common spectrum of these observables. Here, we have operators $X_i = i\partial/\partial p^i$

defined on $\mathcal{H}$, with commuting components $[X_i, X_j] = 0$, and we have a unique spectral measure on $\mathbb{R}^3$ such that

$$X_i = \int_{\mathbb{R}^3} x_i dE(\mathbf{x}). \tag{175}$$

Then, for every Borel set $\Delta \subset \mathbb{R}^3$, the operator

$$E(\Delta) = \int_{\Delta} dE(\mathbf{x}) \tag{176}$$

is a projection operator on the subspace of states localized in Δ; for any two sets Δ and Δ', the operators $E(\Delta)$ and $E(\Delta')$ commute.

However, the operators X_i do not leave the subspace $\mathcal{H}_{ph}$ invariant; therefore, we introduce

$$Q_i^{\mathrm{PR}} = PX_iP + (I - P)X_i(I - P), \tag{177}$$

where $(I - P)$ is the orthogonal projection operator on $\mathcal{H}_{ph}$. Thus, we have

$$Q_i^{\mathrm{PR}} = \int_{\mathbb{R}^3} x_i dF(\mathbf{x}), \tag{178}$$

where

$$F(\Delta) = PE(\Delta)P + (I - P)E(\Delta)(I - P). \tag{179}$$

Now, $F(\Delta)$ are not projection operators any longer, and for different Δ they do not commute. Nevertheless, they are non-negative operators $0 \leq F(\Delta) \leq 1$ and

$$\int_{\mathbb{R}^3} dF(\mathbf{x}) = I, \tag{180}$$

Therefore, we have a POV—positive operator-valued—measure. Jauch and Piron [5] called a photon state $\mathbf{f}$ weakly localized in Δ if $F(\Delta)\mathbf{f} = \mathbf{f}$, and they conjectured the existence of such states. Amrein [6] provided a rigorous proof of their existence and showed how to construct them. Our formula (171) provides a rich explicit family of such states. Notice, however, that the circle states $\mathbf{f}_{x^2+y^2=R^2}(\mathbf{p})$, while satisfying

$$\left(Q_1^{\mathrm{HB}}\right)^2 + \left(Q_2^{\mathrm{HB}}\right)^2 \mathbf{f}_{x^2+y^2=R^2} = R^2 \mathbf{f}_{x^2+y^2=R^2}, \tag{181}$$

are not eigenstates of $(Q_1^{\mathrm{PR}})^2 + (Q_2^{\mathrm{PR}})^2$. Instead, for any real measurable function ϕ on $\mathbb{R}^+$, they are eigenstates of the eigenvalue $\phi(R^2)$ of the operator $Q_{\phi(x^2+y^2)}$ defined as

$$
\begin{aligned}
Q_{\phi(x^2+y^2)} &\doteq P\phi\left((X_1)^2 + (X_2)^2\right)P + (I - P)\phi\left((X_1)^2 + (X_2)^2\right)(I - P) \\
&= \int \phi(x^2 + y^2) \, dF(\mathbf{x}),
\end{aligned}
\tag{182}
$$

where $dF(\mathbf{x})$ is the POV measure defined by Equation (179) (for instance, in Ref [37], the square root function is proposed).

The circle-localized states $\mathbf{f}_{x^2+y^2=R^2}$ are superpositions of helicity-$+1$ and helicity--1 states, in agreement with Theorem 2 of Ref. [6]. Their projections on definite helicity subspaces, $+1$ or -1, are still circle-localized with respect to the Hawton–Baylis operator, but they are not weakly localized in the sense of the POV measure $F(\Delta)$—in agreement with Theorem 1 of Ref. [6].

5. Conclusions

In conclusion, using differential geometric structures on the mass hyperboloid and on the light cone in momentum space, we derived the explicit form of the unitary representation of the Poincaré group for helicity-zero and helicity-± 1 massless particles. Using this explicit form for a boost in the direction of the third axis, we have found that simple

photon polarization states based on Hertz-type potentials survive the light speed limit and generate a polarization basis used in the construction of photon position operators with commuting components. We have compared these operators, as well as the underlying affine connections, to the classical Pryce operator and connection, and we have found that the Pryce connection, Equation (125), non-flat but with rotational symmetry, is metric semi-symmetric, while the Hawton–Baylis connection, flat but only axially symmetric, Equation (114), does not have this property. We have constructed finite-norm photon states localized in bolt washer-like regions and proven that they are strictly localized in these regions with respect to Hawton–Baylis position operators and also with respect to the Jauch–Piron–Amrein POV measures. They are also *z*-localized (but not radius-localized) with respect to the Pryce photon position operator.

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Conflicts of Interest: The author declares no conflicts of interest.

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