mathematics

MDPI

Article

Saddle-Point Equilibrium Strategy for Linear Quadratic Uncertain Stochastic Hybrid Differential Games Based on Subadditive Measures

Zhifu Jia [1,†] and Cunlin Li [2,*,†]

[1] School of Sciences and Arts, Suqian University, Suqian 223800, China; jzflzbx@nuaa.edu.cn
[2] Ningxia Key Laboratory of Intelligent Information and Big Data Processing, Governance and Social Management Research Center of Northwest Ethnic Regions, North Minzu University, Yinchuan 750021, China
* Correspondence: 2000014@nmu.edu.cn
† These authors contributed equally to this work.

Abstract: This paper describes a kind of linear quadratic uncertain stochastic hybrid differential game system grounded in the framework of subadditive measures, in which the system dynamics are described by a hybrid differential equation with Wiener–Liu noise and the performance index function is quadratic. Firstly, we introduce the concept of hybrid differential games and establish the Max–Min Lemma for the two-player zero-sum game scenario. Next, we discuss the analysis of saddle-point equilibrium strategies for linear quadratic hybrid differential games, addressing both finite and infinite time horizons. Through the incorporation of a generalized Riccati differential equation (GRDE) and guided by the principles of the Itô–Liu formula, we prove that that solving the GRDE is crucial and serves as both a sufficient and necessary precondition for identifying equilibrium strategies within a finite horizon. In addition, we also acquire the explicit formulations of equilibrium strategies in closed forms, alongside determining the optimal values of the cost function. Through the adoption of a generalized Riccati equation (GRE) and applying a similar approach to that used for the finite horizon case, we establish that the ability to solve the GRE constitutes a sufficient criterion for the emergence of equilibrium strategies in scenarios extending over an infinite horizon. Moreover, we explore the dynamics of a resource extraction problem within a finite horizon and separately delve into an H_∞ control problem applicable to an infinite horizon. Finally, we present the conclusions.

Keywords: uncertain stochastic hybrid differential games; saddle-point equilibrium; generalized Riccati differential equation; generalized Riccati equation

MSC: 49L20; 49N90; 91A10

check for updates

Citation: Jia, Z.; Li, C. Saddle-Point Equilibrium Strategy for Linear Quadratic Uncertain Stochastic Hybrid Differential Games Based on Subadditive Measures. *Mathematics* **2024**, *12*, 1132. https://doi.org/10.3390/math12081132

Academic Editors: Mihail Ioan Abrudean and Vlad Muresan

Received: 28 March 2024
Revised: 7 April 2024
Accepted: 9 April 2024
Published: 9 April 2024

1. Introduction

Rufus Isaacs [1] pioneered differential game theory in 1965, and initially applied it to military strategies like pursuit and evasion. This sparked enduring interest in decision-making dynamics. In 1971, Friedman [2] established key theoretical foundations, revealing value and saddle points. Basar and Olsder [3] expanded on this, exploring noncooperative games, enriching our understanding of strategic dynamics. Later, Docker et al. [4] utilized mathematical tools to analyze equilibrium conditions, strengthening the field's mathematical underpinnings. Overall, research in differential game theory has led to significant advancements in understanding strategic decision making in dynamic systems.

Linear quadratic differential games, a subset of the broader field of differential games, attract both dynamic game theorists and economists exploring policy coordination, resource extraction, and capital accumulation due to their applicability to real-world scenarios and ability to model complex strategic interactions. In 1965, Ho et al. [5] explored pursuit-evasion games, a key example of linear quadratic differential games, revealing foundational insights into strategic dynamics within this framework. Building on this foundation, Starr and Ho in 1969 [6] established a crucial condition for closed-loop strategies' existence, essential for strategic decision making, utilizing the solution of the Riccati equation to provide valuable insights into dynamic optimal strategies. In 1970, Schmitendorf's work [7] illuminated a notable aspect of linear quadratic differential games, revealing that the existence of a closed-loop saddle point does not guarantee the presence of an open-loop saddle point, emphasizing the complexities inherent in strategic decision making within dynamic systems. In recent years, scholars have continued to delve into linear quadratic differential games, refining methodologies and uncovering deeper strategic insights, underscoring the enduring relevance and complexity of this specialized area within differential game theory (see, e.g., Bernhard [8], Delfour et al. [9] and Delfour [10]).

In real-world scenarios, the evolution of states in dynamic systems is frequently disrupted by environmental noise, which can infiltrate the state equation or affect players' observations of the system. When noise follows the Wiener process, stochastic differential equations become essential for characterizing system evolution, thus converting the differential game into a stochastic differential game. Similarly, when noise is shaped by the Liu process, reflecting the uncertainty tied to experts' belief degrees, uncertain differential equations offer a pivotal means of characterizing system evolution, leading to the emergence of uncertain differential games. Fleming's seminal contributions [11] in stochastic control paved the way for solving differential game scenarios with stochastic state dynamics. For instance, in 2006, Mou and Yong [12] utilized the Hilbert space method to examine open-loop strategies in stochastic linear quadratic differential games, while Sun and Yong [13] explored both open-loop and closed-loop saddle-point equilibriums in 2014. In uncertain settings, Zhu's introduction [14] of uncertain optimal control in 2010 facilitated the analysis of differential games with uncertain state dynamics. Yang and Gao [15,16] further advanced this field by proposing uncertain differential games incorporating Liu process noise, establishing conditions for the existence of a feedback Nash equilibrium in 2013 and delving into linear quadratic uncertain differential games in 2016. The integration of chance theory based on subadditive measures addresses systems where both uncertainty and randomness coexist, modeling noise through the Wiener–Liu process and system evolution through uncertain stochastic hybrid differential equations, thus transforming the differential game into a hybrid differential game. Liu's pioneering work [17] in 2013 laid the foundation for exploring uncertain stochastic hybrid systems [18,19] based on subadditive measures, followed by subsequent research by Fei et al. [20] in 2014, introducing uncertain stochastic hybrid optimal control and the equation of optimality via uncertain stochastic hybrid differential equations. With the help of uncertain stochastic optimal control, there has been a growing body of literature focusing on uncertain stochastic hybrid differential game systems [21–24]. Unlike the models and methods mentioned earlier, this work is the beginning of linear quadratic uncertain stochastic hybrid differential games in finite and infinite horizons.

The structure of this paper is organized as follows. Section 2 begins by recalling some basic results about the principle of optimality, the HJB equation and the feedback Nash equilibrium of hybrid differential games and so on, and then gives Max–Min Lemma of a two-player zero-sum game, which is essential for our analysis. Section 3 is devoted to the study of the saddle-point equilibrium strategy for linear quadratic hybrid differential games in continuous time. Section 4 presents a resource extraction problem and an H_∞ control problem. Finally, Section 5 presents the conclusions.

2. Preliminaries

For any $(k, z) \in [0, T] \times \Re^m$, $s \in [k, T]$, and any control variable $D(\cdot) \in \mathcal{U}$, where f, g, and h are the given functions, we consider the state equation,

$$\begin{cases} \mathrm{d}z(s) = f(s, z(s), D(s))ds + g(s, z(s), D(s))dW_s \\ \qquad\qquad +h(s, z(s), D(s))dC_s, z(T) = z. \end{cases}$$

The corresponding cost functional is

$$J(k,z,D) = E_{CH}\left[\int_s^T \Phi_1(s,z(s),D(s))ds + \Phi_2(T,z(T))\right],$$

where E_{CH} is the chance expectation, W_s is a Wiener process and C_s a Liu process. Define the value function $V(k,z)$ by $V(k,z) \triangleq \inf_{D\in\mathcal{U}} J(k,z,D)$.

Theorem 1 ([20]). *(Principle of optimality.) For any $(k,z) \in [0,T] \times \Re^p$, we have*

$$V(k,z) = \inf_{D\in\mathcal{U}} E_{CH}\left[\int_k^{\hat{k}} \Phi_1(s,z(s),D(s))ds + V(\hat{k},z(\hat{k}))\right], 0 \le k \le \hat{k} \le T.$$

Theorem 2 ([20]). *(HJB equation.) Let $C([0,T] \times \Re^p)$ denote all functions $V(k,z)$ on $[0,T] \times \Re^p$ that are continuously differentiable in time k and continuously twice differentiable in z. If $V(\cdot) \in C([0,T] \times \Re^p)$, define operators $L(D)V$ by*

$$\begin{aligned} L(D)V(k,z) \triangleq \frac{\partial V}{\partial k}(k,z) &+ \sum_{k=1}^{p} \frac{\partial V}{\partial z_k}(k,z) f_k(k,z,D) \\ &+ \frac{1}{2}\sum_{k=1}^{p}\sum_{l=1}^{p} \frac{\partial^2 V}{z_k z_l}(k,z)\sum_{i=1}^{m} g_{ki}(k,z,D) g_{li}(k,z,D). \end{aligned}$$

Then, V is a solution of the following terminal problem of an HJB equation

$$\inf_{D\in\mathcal{U}} \{L(D)V(k,z) + \Phi_1(k,z,D)\} = 0$$

with the terminal condition $V(T,z) = \Phi_2(T,z)$.

A differential game is a class of decision problems in which the evolution of the state is described by a differential equation. The players act throughout a time interval $[k_0, T]$ and aim to maximize their payoffs. In the general n-person differential game model, player i optimizes the objective

$$\sup_{u_i} \int_{k_0}^{T} R_i(k,z(k),u_1(k),u_2(k),\cdots,u_n(k))dk + \Theta_i(z(T)) \text{ for } i \in N = \{1,2,\cdots,n\} \quad (1)$$

subject to

$$dz(k) = f(k,z(k),u_1(k),u_2(k),\cdots,u_n(k))dk, z(k_0) = z_0, \quad (2)$$

where $T > k_0 \ge 0, z(k) \in \Re^m$ is the state variable, z_0 is the given initial state, $u_i \in U_i$ is the control variable of player i, and U_i is a compact metric space. The function $R_i(k,z,u_1,u_2,\cdots,u_n)$ is the transient payoff function of player i at time k, $\Theta_i(\cdot)$ is the terminal reward function of player i at terminal time T, and $f(k,z,u_1,u_2,\cdots,u_n)$ is a vector function. All functions mentioned are differentiable.

In the true essence of the game, the state evolution is inevitably disturbed by environmental noise. This noise may may occur directly within the state equations or indirectly through the players' observations of the system's condition. In a vector-valued n-person uncertain stochastic hybrid differential game model, player i optimizes the objective

$$\begin{aligned} \sup_{u_i} \mathrm{E}_{\mathrm{CH}}^{\mathrm{k}_0}\Bigg[\int_0^T & R_i[k,z(k),u_1(k),u_2(k),\ldots,u_n(k)]dk \\ &+ \Theta_i(z(T))\Bigg], i \in N = \{1,2,\ldots,n\}. \end{aligned} \quad (3)$$

A vector-valued hybrid differential equation, which delineates the evolution of the state and n objective functions (3), provides a more suitable framework for analyzing differential games with uncertain stochastic noise driven by the Wiener–Liu process:

$$\begin{aligned} dz(k) =& f[k, z(k), u_1(k), u_2(k), \cdots, u_n(k)]dk \\ &+ \sigma[k, z(k), u_1(k), u_2(k), \cdots, u_n(k)]d\mathcal{H}_k \\ z(0) =& z_0 \end{aligned} \tag{4}$$

where $E_{CH}^{k_0}$ represents the chance expectation operator performed at time k_0, $z(k) \in \Re^m$ represents the state variables, $u_i \in \Re^{l_i}$ is the control of player i, $\mathcal{H}_k = (W_k, C_k)$ is an l-dimensional Wiener–Liu process, similar to refs. [18,19], and z_0 is the given initial state.

For the subsequent analysis, the following assumptions are presented:

$$f : [0, T] \times \Re^m \times \Re^{l_1} \times \ldots \times \Re^{l_n} \to \Re^m$$

and

$$\sigma : [0, T] \times \Re^m \times \Re^{l_1} \times \ldots \times \Re^{l_n} \to \Re^m \times \Re^l$$

which satisfy the Lipschitz condition and linear growth condition and possess continuous partial derivatives.

For $k \in [k_0, T]$, the admissible feedback control denotes:

$$u_i(k) = u_i(k, z(k))$$

where

$$u_i(\cdot, \cdot) : [0, T] \times \Re^m \longrightarrow \Re^{l_i}.$$

Denote

$$u_{-i}(k, z) = \{u_1(k, z), u_2(k, z), \cdots, u_{i-1}(k, z), u_{i+1}(k, z), \cdots, u_n(k, z)\}.$$

A feedback Nash equilibrium of the hybrid differential game (3)–(4) can be defined as follows.

Definition 1. *A set of strategies $\{u_1^*(s, z), u_2^*(s, z), \cdots, u_n^*(s, z)\}$ is called a feedback Nash equilibrium for the n-person hybrid differential game (3)–(4), and $\{z^*(s), k \leq s \leq T\}$ is the corresponding state trajectory, if there exist real-valued functions $V^i(k, z) : [0, T] \times \Re^m \to \Re$, satisfying the following relations for each $i \in N$:*

$$\begin{aligned} V_i(k, z) &= E_{CH}^{k_0}[\int_{k_0}^{T} R_i(k, z^*(k), u_1^*(k), u_2^*(k), \cdots, u_n^*(k))dk + \Theta_i^*(z(T)) \\ &\geq E_{CH}^{k_0}[\int_{k_0}^{T} R_i(k, z^{[i]}(k), u_1^{[i]}(k), u_2^{[i]}(k), \cdots, u_n^{[i]}(k))dk + \Theta_i^{[i]}(z(T)) \\ &\quad \forall u_i(\cdot, \cdot) \in [k_0, T] \times \Re^m, z(\cdot) \in \Re^m; \\ V_i(T, z) &= \Theta_i(z(T)) \end{aligned}$$

where on the time interval $[k, T]$:

$$\begin{aligned} dz^*(s) =& f[s, z^*(s), u_i^*(s, z^*), u_{-i}^*(s, z^*)]ds \\ &+ g[s, z^*(s), u_i^*(s, z^*), u_{-i}^*(s, z^*)]d\mathcal{H}_s \\ z^*(k) =& z; \\ dz^{[i]}(s) =& f[s, z^{[i]}(s), u_i(s, z^{[i]}), u_{-i}^*(s, z^{[i]})]ds \\ &+ g[s, z^{[i]}(s), u_i(s, z^{[i]}), u_{-i}^*(s, z^{[i]})]d\mathcal{H}_s \\ z^{[i]}(k) =& z. \end{aligned}$$

Remark 1. *If an n-pair $u_i^*(s,z); i \in N$ establishes a feedback Nash equilibrium for an n-person differential game, as defined in Equations (3)–(4), over the duration $[k_0, T]$, then its restriction to the time interval $[k, T]$ also constitutes a feedback Nash equilibrium, just as Docker et al. [4] described. Importantly, the feedback Nash equilibrium depends solely on the the current state value $z(T)$ and time variable k, not on any prior history (including the initial state z_0).*

Next, we give sufficient conditions guaranteeing that $u_i(k,z); i \in N$ is a feedback Nash equilibrium for the game (3)–(4).

Lemma 1. *An n-tuple of strategies $\{u_i^*(k,z); i \in N\}$ provides a feedback Nash equilibrium to the n-player uncertain differential game (3)–(4) if there exist real-valued functions $V^i(k,z)$: $[0,T] \times \Re^m \to \Re, i \in N$, satisfying the partial differential equations:*

$$\begin{aligned}
-V_k^i(k,z) =& \sup_{u_i}\Big\{R_i[k,z(k),u_i(k,z),u_{-i}^*(k,z)] \\
&+\nabla_z V^i(k,z) f[k,z,u_i(k,z),u_{-i}^*(k,z)] + \frac{1}{2}\nabla_{zz}V^i(k,z)\sigma^\tau\sigma[k,z,u_i(k,z),u_{-i}^*(k,z)]\Big\} \\
=& R_i[k,z,u_i^*(k,z),u_{-i}^*(k,z)] \\
&+ \nabla_z V^i(k,z) f[k,z,u_i^*(k,z),u_{-i}^*(k,z)] + \frac{1}{2}\nabla_{zz}V^i(k,z)\sigma^\tau\sigma[k,z,u_i^*(k,x),u_{-i}^*(k,z)] \\
V^i(T,z) =& \Theta_i(z(T)).
\end{aligned}$$

Proof. The result can be readily derived from Theorem 2 and the definition of the feedback Nash equilibrium. By holding the strategies of all players fixed at their equilibrium choices, except for the ith player's, we transform the scenario into a hybrid optimal control problem as described by Theorem 2. □

Now, let us delve into The "Max–Min Lemma" of the two-player zero-sum hybrid differential game.

Lemma 2. *(Max–Min Lemma): A pair of strategies $\{u_i^*(k,z) \in \Re^{l_i}; i = 1,2\}$ provides a feedback Nash equilibrium solution (called a saddle-point Nash equilibrium) to the two-player zero-sum of the game (3)–(4) if there exists a real-valued function $V(k,z) : [0,T] \times \Re^m \to \Re$, satisfying the partial differential equations:*

$$\begin{aligned}
-V_k(k,z) =& \max_{u_1}\min_{u_2}\{R(k,z,u_1,u_2) \\
&+ \nabla_z V(k,z) f(k,z,u_1,u_2)\} + \frac{1}{2}\sigma^\tau(k,z,u_1,u_2)\nabla_{zz}V(k,z)\sigma(k,z,u_1,u_2)\} \\
=& \min_{u_2}\max_{u_1}\{R(k,z,u_1,u_2) \\
&+ \nabla_z V(k,z) f(k,z,u_1,u_2)\} + \frac{1}{2}\sigma^\tau(k,z,u_1,u_2)\nabla_{zz}V(k,z)\sigma(k,z,u_1,u_2)\} \\
=& R(k,z,u_1^*,u_2^*) + \nabla_z V(k,z) f(k,z,u_1^*,u_2^*) + \frac{1}{2}\sigma^\tau(k,z,u_1^*,u_2^*)\nabla_{zz}V(k,z)\sigma(k,z,u_1^*,u_2^*) \\
V(T,z) =& \Theta(z(T)).
\end{aligned}$$

Proof. As a special case of Lemma 1, the result can be easily obtained by taking $n = 2, R_1(\cdot) = -R_2(\cdot) \doteq R(\cdot)$, and $\Theta_1(\cdot) = -\Theta_2(\cdot) \doteq \Theta(\cdot)$, in which case $V^1 = -V^2 \doteq V$ and the Max–Min Lemma is completed. □

3. Main Results

Notation: in the following, denote by $\Re^n$ the set of n-dimensional Euclidean spaces, $\Re^{m\times n}$ the set of all $m \times n$ matrices, S^n the set of all real symmetric $n \times n$ matrices, and $\tilde{S}^n$ the set of all positive definite $n \times n$ matrices. $P > 0$ denotes $P \in \tilde{S}^n$, P^τ denotes the transpose of a matrix or vector, $\dot{P} = \frac{dP}{dk}$, and k is the time. For a Hilbert space $\widetilde{H}$ and an interval I, let $L^{\infty}(I, \widetilde{H})$ be the space of all bounded and measurable functions from I to $\widetilde{H}$, that is, $f : I \to \widetilde{H}$, $L^{1,\infty} = \{P \in L^{\infty}(I, \widetilde{H}) | \dot{P} \in L^{\infty}(I, \widetilde{H})\}$, where $I = [s, T)$ or $[s, \infty)$. Now, we discuss two-player zero-sum hybrid differential games in finite and infinite horizons.

3.1. The Case of Finite Horizons

Fix $(s, z) \in [k_0, T] \times \Re^n$. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two standard independent Wiener–Liu processes in the chance space over $[s, T]$ with $\mathcal{H}_i(s) = 0$ almost surely. Let $U_i[s, T]$ be the set of $\Re^l$-valued square integrable processes adapted with the σ-field generated by $\mathcal{H}_i(\cdot), i = 1, 2$, respectively. Associated with each $(u_1, u_2) \in \mathcal{U}[s, T] \equiv U_1[s, T] \times U_2[s, T]$ is a quadratic cost $J(u_1, u_2)$. The performance index function is defined as:

$$J(u_1(\cdot), u_2(\cdot)) = E_{CH}\{z^\tau(T)Nz(T) + \int_s^T (z^\tau Gz + 2u^\tau{}_1 L_1 z + u^\tau{}_1 R_1 u_1 + 2u^\tau{}_2 L_2 z + u^\tau{}_2 R_2 u_2)dk\} \tag{5}$$

where z is the solution to the following equation

$$\begin{cases} dz = [Az + B_1u_1(k) + B_2u_2(k)]dk \\ \quad +[C_1z + D_1u_1(k)]d\mathcal{H}_1(k) \\ \quad +[C_2z + D_2u_2(k)]d\mathcal{H}_2(k) \\ \quad z(s) = z_0 \end{cases} \tag{6}$$

and $E\{\}$ symbolizes the chance expectation. $A, B_i, C_i, D_i, G, N, R_i$ and L_i are matrix functions, $i = 1, 2$.

Assumption 1. *We assume that $A, B_i, C_i, D_i, G, N, R_i$ and L_i in (5)–(6) satisfy*

$$\begin{cases} A, C_i \in L^{\infty}(I, \Re^{n\times n}); B_i, D_i, L_i^\tau \in L^{\infty}(I, \Re^{n\times l}), \\ R_i \in L^{\infty}(I, S^l), i = 1, 2; G \in L^{\infty}(I, S^n); N \in S^n. \end{cases}$$

Hybrid differential game problem 1: for the hybrid system described by (6), find the feasible control $(u_1{}^*(\cdot), u_2{}^*(\cdot)) \in \mathcal{U}[s, T]$ and ensure that the following holds:

$$J(u_1{}^*(\cdot), u_2(\cdot)) \le J(u_1{}^*(\cdot), u_2{}^*(\cdot)) \le J(u_1(\cdot), u_2{}^*(\cdot))$$

Theorem 3. *The two-player zero-sum linear quadratic hybrid differential game (5)–(6) has a saddle-point Nash equilibrium solution if the following differential Riccati equation has a solution $P(k) \in L^{1,\infty}(I, S^n)$*

$$\begin{cases} \dot{P} + A^\tau P + PA + \sum_{i=1}^{2} C_i{}^\tau PC_i + G - \sum_{i=1}^{2} S_i{}^\tau(P)R_i^{-1}(P)S_i(P) = 0 \\ R_1(P) > 0, R_2(P) < 0 \\ R_i(P)K_i(P) = S_i(P), i = 1,2 \\ P(T) = N \end{cases} \tag{7}$$

where $S_i(P)$, $R_i(P)$, and $K_i(P)$ are defined as

$$\begin{cases} S_i(P) = B_i^\tau P + D_i^\tau PC_i + L_i \\ R_i(P) = R_i + D_i^\tau PD_i \\ K_i(P) = R_i^{-1}(P)S_i(P), i = 1,2. \end{cases}$$

and the saddle point and the optimum value are

$$(u_1{}^*(\cdot), u_2{}^*(\cdot)) = (-K_1(P)z(\cdot), -K_2(P)z(\cdot))$$

and

$$J(u_1{}^*(\cdot), u_2{}^*(\cdot)) = z^\tau P(s)z.$$

Proof. Let $P(k) \in L^{1,\infty}(I, S^n)$ be the solution of Equation (7) and $z(k)$ be the solution of Equation (6) corresponding to control $(u_1(k), u_2(k))$. Using the fundamental theorem of calculus and Itô–Liu formula in ref. [20], applied to $z^\tau(k)p(k)z(k)$, we obtain

$$\begin{aligned} d[z^\tau pz] =&[z^\tau(\dot{P} + PA + A^\tau P + \sum_{i=1}^{2} C_i^\tau PC_i)z \\ &+ 2u_1^\tau(B_1^\tau P + D_1^\tau PC_1)z + u_1^\tau D_1^\tau PD_1u_1 + \\ &+ 2u_2^\tau(B_2^\tau P + D_2^\tau PC_2)z + u_2^\tau D_2^\tau PD_2u_2]dk \\ &+ \{\cdots\}dW_1 + \{\cdots\}dW_2 + \{\cdots\}dC_1 + \{\cdots\}dC_2. \end{aligned}$$

Taking integrations on $[s, T]$ and chance expectation, we obtain

$$\begin{aligned} E_{CH}[z^\tau(T)P(T)z(T)] =& z_0{}^\tau P(s)z_0 + E_{CH}\int_s^T z^\tau(\dot{P} + PA + A^\tau P + \sum_{i=1}^{2} C_i^\tau PC_i)zdk \\ &+ \sum_{i=1}^{2} E_{CH}\int_s^T [2u_i^\tau(B_i^\tau P + D_i^\tau PC_i)z + u_i^\tau D_i^\tau PD_iu_i]dk. \end{aligned} \tag{8}$$

Substituting (8) into (5), $J(u_1, u_2)$ can be be reduced to

$$\begin{aligned} J(u_1(\cdot), u_2(\cdot)) =& z_0{}^\tau P(s)z_0 + E_{CH}\{z^\tau(T)(N - P(T))z(T)\} \\ &+ E_{CH}\int_s^T \{[z^\tau(\dot{P} + PA + A^\tau P + \sum_{i=1}^{2} C_i^\tau PC_i + G)z]\}dk \\ &+ E_{CH}\int_s^T \sum_{i=1}^{2} \{[2u_i^\tau(B_i^\tau P + D_i^\tau PC_i + L_i)z \\ &+ u_i^\tau(R_i + D_i^\tau PD_i)u_i]\}dk \\ =& z_0{}^\tau P(s)z_0 + E_{CH}\int_s^T \{[z^\tau(\dot{P} + PA + A^\tau P + \sum_{i=1}^{2} C_i^\tau PC_i + G)z]dk \\ &+ \sum_{i=1}^{2}[2u_i^\tau S_i(P)z + u_i^\tau R_i(P)u_i]\}dk. \end{aligned}$$

While, by the following equality

$$\begin{aligned} &2u_i^\tau S_i(P)z + u_i^\tau R_i(P)u_i \\ =& u_i^\tau R_i(P)R_i(P)^{-1}R_i(P)u_i + 2u_i^\tau S_i(P)z \\ &+ z^\tau S_i^\tau(P)R_i^{-1}(P)S_i(P)z - z^\tau S_i^\tau(P)R_i^{-1}(P)S_i(P)z \\ =& [u_i + R_i^{-1}(P)S_i(P)z]^\tau R_i(P)[u_i + R_i^{-1}(P)S_i(P)z] \\ &- z^\tau S_i^\tau(P)R_i^{-1}(P)S_i(P)z. \end{aligned}$$

we can obtain that

$$\begin{aligned} J(u_1(\cdot), u_2(\cdot)) =& z_0{}^\tau P(s)z_0 + E_{CH}\int_s^T \{z^\tau[\dot{P} + PA + A^\tau P + \sum_{i=1}^{2} C_i^\tau PC_i + G - \\ &\sum_{i=1}^{2} S_i^\tau(P)R_i^{-1}(P)S_i(P)]zdk + \\ &+ \sum_{i=1}^{2} [u_i + R_i^{-1}(P)S_i(P)z]^\tau R_i(P)[u_i + R_i^{-1}(P)S_i(P)z]\}dk. \end{aligned}$$

So,

$$J(u_1(\cdot),u_2(\cdot)) = {z_0}^{\tau} P(s) z_0 + \sum_{i=1}^{2} E_{CH} \int_s^T [u_i + R_i^{-1}(P)S_i(P)z]^{\tau} R_i(P)[u_i + R_i^{-1}(P)S_i(P)z] dk.$$

The theorem is proved. □

Theorem 4. *If linear feedback controls* $({u_1}^*(\cdot), {u_2}^*(\cdot)) = (-K_1(P)z(\cdot), -K_2(P)z(\cdot))$ *are optimal for the two-player zero-sum linear quadratic hybrid differential game* (5)–(6)*, then the Riccati Equation* (7) *must have a solution* $P(\cdot)$*. Moreover,* $K_i(P) = R_i^{-1}(P)S_i(P)$

Proof. By Theorem (2), the value function $V(k,z)$ satisfies the HJB equation

$$\begin{aligned} V_k(k,z) &+ \min_{u_1} \max_{u_2} \{z^{\tau} G z + 2{u^{\tau}}_1 L_1 z + {u^{\tau}}_1 R_1 u_1 + 2{u^{\tau}}_2 L_2 z + {u^{\tau}}_2 R_2 u_2 \\ &+ [Az(T) + B_1 u_1(k) + B_2 u_2(k)]^{\tau} \cdot V_z(k,z) \\ &+ \frac{1}{2}[C_1 z(T) + D_1 u_1(k)]^{\tau} V_{zz}(k,z)[C_1 z(T) + D_1 u_1(k)] \\ &+ \frac{1}{2}[C_2 z(T) + D_2 u_2(k)]^{\tau} V_{zz}(k,z)[C_2 z(T) + D_2 u_2(k)]\} \\ &= 0. \end{aligned} \tag{9}$$

Take the following value equation

$$V(k,z) = z^{\tau} P z. \tag{10}$$

So, substituting (10) into (9), and by the assumption that $({u_1}^*(\cdot), {u_2}^*(\cdot)) = (-K_1(P)z(\cdot), -K_2(P)z(\cdot))$, we can obtain that

$$\begin{aligned} 0 =& z^{\tau}(\dot{P} + PA + A^{\tau}P + \sum_{i=1}^{2} C_i^{\tau} P C_i + G)z \\ &+ \min_{u_1} \max_{u_2} \{2u_i^{\tau} S_i(P) z + u_i^{\tau} R_i(P) u_i\} \\ =& z^{\tau}(\dot{P} + PA + A^{\tau}P + \sum_{i=1}^{2} C_i^{\tau} P C_i + G)z \\ &+ \min_{u_1} \max_{u_2} \{[u_i + R_i^{-1}(P)S_i(P)z]^{\tau} R_i(P)[u_i + R_i^{-1}(P)S_i(P)z] \\ &- z^{\tau} \sum_{i=1}^{2} S_i^{\tau}(P) R_i^{-1}(P) S_i(P) z\} \\ =& z^{\tau}(\dot{P} + PA + A^{\tau}P + \sum_{i=1}^{2} C_i^{\tau} P C_i + G - \sum_{i=1}^{2} S_i^{\tau}(P) R_i^{-1}(P) S_i(P))z. \end{aligned}$$

So, we obtain that $K_i(P) = R_i^{-1}(P)S_i(P)$ and

$$\dot{P} + A^{\tau}P + PA + \sum_{i=1}^{2} {C_i}^{\tau} P C_i + G - \sum_{i=1}^{2} {S_i}^{\tau}(P) R_i^{-1}(P) S_i(P) = 0.$$

The proof is complete. □

3.2. The Case of Infinite Horizons

Fix $(s,z) \in [k_0,\infty) \times \Re^n$. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two standard independent Wiener–Liu processes in a chance space over $[s,\infty)$ with $\mathcal{H}_i(s) = 0$ almost surely. Let $U_i[s,\infty)$ be the set of $\Re^l$-valued square integrable processes (denote by $L_i^2(\Re^l)$) adapted with the σ-

field generated by $\mathcal{H}_i(\cdot), i = 1, 2$, respectively. Associated with each $(u_1, u_2) \in \mathcal{U}[s, \infty) \equiv U_1[s, \infty) \times U_2[s, \infty)$ is a quadratic cost $J_\infty(u_1, u_2)$. The performance index function is defined as:

$$J_\infty(u_1(\cdot), u_2(\cdot)) = E_{CH}\{\int_s^\infty [(z^\tau Gz + 2u^\tau{}_1 L_1 z + u^\tau{}_1 R_1 u_1 + 2u^\tau{}_2 L_2 z + u^\tau{}_2 R_2 u_2)]dk\} \tag{11}$$

where z is the solution to the following equation

$$\begin{cases} dz = [Az + B_1u_1(k) + B_2u_2(k)]dk \\ \quad +[C_1z + D_1u_1(k)]d\mathcal{H}_1(k) \\ \quad +[C_2z + D_2u_2(k)]d\mathcal{H}_2(k) \\ \quad z(s) = z_0 \end{cases} \tag{12}$$

and $E\{\}$ represents the expectation of the enclosed uncertain random variable. A, B_i, C_i, D_i, G, R_i and L_i are matrix functions, $i = 1, 2$. and they also satisfy Assumption 1.

Hybrid differential game problem 2: for the hybrid system described by the given Formula (11), find the feasible control $(u_1{}^*(\cdot), u_2{}^*(\cdot)) \in \mathcal{U}[s, \infty)$ and ensure that the following holds:

$$J_\infty(u_1{}^*(\cdot), u_2(\cdot)) \le J_\infty(u_1{}^*(\cdot), u_2{}^*(\cdot)) \le J_\infty(u_1(\cdot), u_2{}^*(\cdot)).$$

For this problem, we introduce the following generalized Riccati equation (GRE)

$$\begin{cases} A^\tau P + PA + \sum_{i=1}^{2} C_i{}^\tau PC_i + G - \sum_{i=1}^{2} S_i{}^\tau(P)R_i^{-1}(P)S_i(P) = 0 \\ R_1(P) > 0, R_2(P) < 0 \\ R_i(P)K_i(P) = S_i(P), i = 1,2 \end{cases} \tag{13}$$

where $P \in S^n$ is an unknown matrix, and $S_i(P)$, $R_i(P)$, and $K_i(P)$ are defined as

$$\begin{cases} S_i(P) = B_i^\tau P + D_i^\tau PC_i + L_i \\ R_i(P) = R_i + D_i^\tau PD_i \\ K_i(P) = R_i^{-1}(P)S_i(P), i = 1,2. \end{cases}$$

Since we are considering the hybrid differential game problem in an infinite horizon, we need the concept of mean-square stabilizability.

Definition 2. *(i) An open-loop control $(u_1(\cdot), u_2(\cdot)) \in L_1^2(\Re^l) \times L_2^2(\Re^l)$ is called (mean-square) stabilizing if the corresponding state of (12) with the initial state $z \in \Re^n$ satisfies $\lim_{k\to\infty} E\{z^\tau(k)z(k)\} = 0$. A feedback control $(u_1(\cdot), u_2(\cdot)) = (-K_1z(\cdot), -K_2z(\cdot))$, where $K_1, K_2 \in \Re^{l\times n}$ is a constant matrix, is called stabilizing if for every initial state $z \in \Re^n$, the solution of the following equation*

$$\begin{cases} dz(k) = (A - B_1K_1 - B_2K_2)z(k)dk \\ \quad +(C_1 - D_1K_1)z(k)d\mathcal{H}_1(k) \\ \quad +(C_2 - D_2K_2)z(k)d\mathcal{H}_2(k) \\ \quad z(s) = z_0 \end{cases} \tag{14}$$

satisfies $\lim_{k\to\infty} E_{CH}\{z^\tau(k)z(k)\} = 0$.

(ii) The system (12) is called (mean-square) stablizable if there exists a mean-square stabilizing feedback control of the form $(u_1(\cdot), u_2(\cdot)) = (-K_1z(\cdot), -K_2z(\cdot))$, where K_1, K_2 is a constant matrix.

Mean-square stabilizability is pivotal in this paper. We now introduce the equivalent conditions for verifying stabilizability, both analytically and computationally.

Lemma 3. *$(u_1(\cdot), u_2(\cdot)) = (-K_1 z(\cdot), -K_2 z(\cdot))$ is mean-square stabilizable if and only if there exist a matrix K and $U \in S^n$, $U > 0$ such that*

$$\begin{aligned}&(A - B_1K_1 - B_2K_2)U^\tau + U(A - B_1K_1 - B_2K_2)\\&+\sum_{i=1}^{2}(C_1 - D_1K_1)^\tau U(C_1 - D_1K_1) < 0.\end{aligned} \tag{15}$$

Proof. For any $n \times n$ matrix K, define an operator $\Phi : S^n \to S^n$ by

$$\Phi(U) = (A + BK)U + U(A + BK)^\tau + (C + DK)U(C + DK)^\tau$$

If $z(\cdot)$ satisfies the feedback Equation (14) (under the feedback gain K), then by Itô–Liu's formula, the matrix $Z(k) = E[z(k)z(k)^\tau]$ satisfies the differential matrix system $\dot{Z}(k) = \Phi(z(k))$. Applying the result in ref. [25], we have the equivalence between the mean-square stabilizability and (15). □

Remark 2. *Lemma 3 gives the equivalent conditions of mean-square stabilizability, which provides a theoretical basis for the hypothesis of Theorem 5.*

Theorem 5. *Suppose the system (14) is mean-square stabilizable. If the GRE (13) exists and $P^* \in S^n$, then hybrid differential game problem 2 is solvable; moreover, the saddle point and the optimum value of the performance index function are*

$$({u_1}^*(\cdot), {u_2}^*(\cdot)) = (-K_1(P^*)z(\cdot), -K_2(P^*)z(\cdot))$$

and

$$J({u_1}^*(\cdot), {u_2}^*(\cdot)) = z^\tau P^*(s)z,$$

that is, $\forall (u_1(\cdot), u_2(\cdot))$ satisfies $({u_1}^*(\cdot), u_2(\cdot)), (u_1(\cdot), {u_2}^*(\cdot)) \in \mathcal{U}[s, \infty)$. Then, we have

$$J_\infty({u_1}^*(\cdot), u_2(\cdot)) \le J_\infty({u_1}^*(\cdot), {u_2}^*(\cdot)) \le J_\infty(u_1(\cdot), {u_2}^*(\cdot))$$

Proof. Suppose $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}[s, \infty)$, $z(\cdot)$ is the solution of (14). For $\forall\, T > s$, denote

$$J(u_1(\cdot), u_2(\cdot)) = E_{CH} \int_s^T (z^\tau Gz + 2u^\tau{}_1 L_1 z + u^\tau{}_1 R_1 u_1 + 2u^\tau{}_2 L_2 z + u^\tau{}_2 R_2 u_2)dk.$$

Similar to the proof of Theorem 3, we have

$$\begin{aligned}J_T(u_1(\cdot), u_2(\cdot)) =& z_0^\tau P^* z_0 - E_{CH}\{z^\tau(T)P^*z(T)\} + E_{CH}\int_s^T \{z^\tau\{P^*A + A^\tau P^*\\&+\sum_{i=1}^{2}[C_i^\tau P^* C_i - S_i^\tau(P^*)R_i^{-1}(P^*)S_i(P^*)]z + G\\&+\sum_{i=1}^{2}[z^\tau S_i^\tau(P^*)R_i^{-1}(P^*)S_i(P^*)z + 2z^\tau S_i(P^*)u_i + u_i^\tau R_i(P^*)u_i]\}dk\\=& z_0^\tau P^* z_0 - E_{CH}\{z^\tau(T)P^*z(T)\}\\&+\sum_{i=1}^{2} E_{CH}\int_s^T [u_i + K_i(P^*)z]^\tau \times R_i(P^*)[u_i + K_i(P^*)z]dk.\end{aligned}$$

Because $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}[s, \infty)$, then $\lim\limits_{k\to\infty} E\{z^\tau(k)P^*z(k)\} = 0$. Therefore,

$$J_\infty(u_1(\cdot), u_2(\cdot)) = z_0^\tau P^* z_0 + \sum_{i=1}^{2} E_{CH}\int_s^\infty [u_i + K_i(P^*)z]^\tau \times R_i(P^*)[u_i + K_i(P^*)z]dk.$$

The proof is complete. □

Remark 3. *Unlike refs. [8–10] concerning stochastic differential games and refs. [15,16], the models in this paper have a wider range of applications and cover these models as well.*

4. Applications

4.1. Uncertain Stochastic Resource Extraction Game

The resource extraction problem is a classic issue in economics, involving economic agents (such as firms or countries) exploiting natural resources. Under certain assumptions, this problem typically aligns with differential game theory. For instance, Jørgensen and Yeung [26] explored a stochastic differential game model applied to a common-property fishery. Similarly, Yang and Gao [16] examined an uncertain differential game model for resource extraction. Assuming the resource dynamics are governed by a hybrid differential equation driven by a Wiener–Liu process, we can then consider studying a uncertain stochastic hybrid differential game model of resource extraction using chance theory.

Consider two companies (resource extractors) that are exploiting a renewable resource (such as fish stocks). The lease for this resource extraction starts at time 0 and ends at time T, where $T > 0$. Let $u_i(T)$ represent the amount of resources extracted by company i at time k, with $i = 1, 2$, where each extractor can control their own extraction quantity. Let $z(k)$ represent the size of the resource stock at time k, with $z(k) > 0$, and the equation of resource dynamics is

$$dz(k) = [mz(k) - u_1(k) - u_2(k)]dk + z(k)d\mathcal{H}_k, \quad z(0) = z_0$$

where $m > 0$ represents the growth rate of resources, with the initial state z_0 being provided. The hybrid process $\mathcal{H}_k$ is a one-dimensional Wiener–Liu process that is defined in a chance space

$$(\Gamma \times \Omega, \mathcal{L} \otimes \mathcal{F}, \mathcal{M} \times P).$$

The performance index function is

$$J(u_1, u_2) = E_{CH}\left\{\int_0^T \left[z(k)^2 - u_1(k)^2 + u_2(k)^2\right]dk + z(T)^2\right\}.$$

Extractor 1 endeavors to maximize the value of the performance index function; on the other hand, Extractor 2 is determined to minimize this value.

Drawing upon Theorems 3 and 4, we arrive at the subsequent Riccati differential equation:

$$\begin{cases} \dot{P} + 2mP + 3 = 0 \\ P(T) = 1. \end{cases}$$

Thus,

$$P(k) = \left[(2m+3)e^{2m(T-k)} - 3\right] \Big/ (2m).$$

The saddle-point Nash equilibria are

$$\begin{cases} u_1^*(k) = -\frac{1}{2m}\left[(2m+3)e^{2m(T-k)} - 3\right]z(k) \\ u_2^*(k) = \frac{1}{2m}\left[(2m+3)e^{2m(T-k)} - 3\right]z(k). \end{cases}$$

The optimum value of the performance index function is

$$J(u_1^*, u_2^*) = \frac{1}{2m}\left[(2m+3)e^{2mT} - 3\right]z_0^2.$$

By taking $m = 1$, we can obtain the dynamic change curve of the resource stock $z(k)$, equilibrium strategy (u_1^*, u_2^*) and equilibrium value $J(k, z)$ in Figure 1.

(**a**) (**b**)

Figure 1. Dynamic change curve of the resource stock $z(k)$, equilibrium strategy (u_1^*, u_2^*) of (**a**) and equilibrium value $J(k, z)$ of (**b**).

4.2. Uncertain Stochastic H_∞ Control

Now, we apply the previous developed theory to solve some problems related to uncertain stochastic H_∞ control.

$$dz(k) = [Az(k) + B_1 v(k) + B_2 u(k)]dk + Cz(k)d\mathcal{H}_k \tag{16}$$

with the cost functional

$$J(u, v; z_0) = E_{CH}\{\int_0^\infty [z(k)^T Dz(k) - \gamma^2 |v(k)|^2]dk\} \tag{17}$$

where the hybrid process $\mathcal{H}_k$ is a one-dimensional Wiener–Liu process that is defined in a chance space

$$(\mathit{\Gamma} \times \mathit{\Omega}, \mathcal{L} \otimes \mathcal{F}, \mathcal{M} \times P).$$

In Equations (16) and (17), $z(k) \in \Re^n$ is the state vector, $u(k) \in \Re^{m2}$ is the input control and $v(k) \in \Re^{m1}$ is the vector of the exogenous disturbances, and in Equation (17), $J(u, v; z_0)$ represents H_∞ constraints. The infinite-horizon uncertain stochastic H_∞ control of system (16) is parallel with Definition 2 in [27], which can be described as follows.

Definition 3. *For a given disturbance attenuation level $\gamma > 0$, we can find $u^*(k) \times v^*(k) \in L^2([0, \infty), \Re^{m2}) \times L^2([0, \infty), \Re^{m1})$, such that*

(i) $u^*(k)$ stabilizes system (16) internally; i.e., when $v(k) = 0, u = u^*$, the state trajectory of Equation (16) with any initial value $z_0 \in \Re^n$ satisfies

$$\lim_{k \to \infty} E_{CH}[z^T(k) z(k)] = 0.$$

(ii) $|L_{u^*}|_\infty < \infty$ with

$$|L_{u^*}|_\infty = \sup_{v \in L_F^2([0,\infty), R^{m1}), v \neq 0, u = u^*, z_0 = 0} \frac{\{E_{CH} \int_0^\infty z(k)^T Dz(k)dk\}^{1/2}}{\{E_{CH} \int_0^\infty |v(k)|^2 dk\}^{1/2}}$$

In essence, the H_∞ control issue, as outlined by Equations (16) and (17), seeks to identify control u^* that ensures $J(u^*) < 0$ in the face of any exogenous disturbances $v(t)$. Following the insights in [27], if we conceptualize $u(t)$ and $v(t)$ in the uncertain stochastic H_∞ control scenario as dual strategies employed by players P_1 and P_2 through a game theory lens, this H_∞ control challenge transforms into resolving an uncertain stochastic game dilemma. Consequently, it is acknowledged that the infinite-time horizon uncertain stochastic H_∞ control issue yields a solution pair. Clearly, the pair (u^*, v^*) represents the equilibrium strategy of the saddle point such that

$$J(u^*, v) \leq J(u^*, v^*) \leq J(u, v^*).$$

According to Theorem 5, the following Theorem 6 can be obtained directly.

Theorem 6. *For system (16), uncertain stochastic control has a pair of solutions (u^*, v^*), with $u^* = K_2 z(k)$ and $v^* = K_1 z(k)$, if the following coupled uncertain stochastic algebraic Riccati equation*

$$PA + A^T P + C^T PC + D - PBR^{-1}B^T P = 0$$

with

$$B = (B_1, B_2), R = \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & I \end{bmatrix}$$

has a solution $P \in S^n$, where

$$\begin{cases} K_1 = \gamma^{-2} B_1^T P \\ K_2 = -B_2^T P \end{cases}$$

In this scenario, u^* serves as an H_∞ control for the system outlined in Equation (16), while v^* acts as the associated worst-case disturbance.

Example 1. *Consider system (16) with the coefficients as follows:*

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}, C = \begin{bmatrix} -2 & -4 \\ 3 & -1 \end{bmatrix}, B_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, D = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

Setting $\gamma = 3$ and solving (16) by using the LMI toolbox, we have

$$P = \begin{bmatrix} -0.1236 & -0.0615 \\ -0.0615 & -0.1935 \end{bmatrix}, K_1 = \begin{bmatrix} 0.0069 \\ -0.0147 \end{bmatrix}, K_2 = \begin{bmatrix} 0.0622 \\ -0.1321 \end{bmatrix}.$$

Therefore, H_∞ control is given by

$$\begin{cases} u(k) = 0.0622 z_1(k) - 0.1321 z_2(k) \\ v(k) = 0.0069 z_1(k) - 0.0147 z_2(k) \end{cases}$$

Furthermore, we can obtain the dynamic change curve of system state $(z_1(k), z_2(k))$ and control inputs $(u(k), v(k))$ in Figure 2.

(**a**) (**b**)

Figure 2. Dynamic change curve of system state $(z_1(k), z_2(k))$ of (**a**) and control inputs $(u(k), v(k))$ of (**b**).

5. Conclusions

This study aims to propose a new type of uncertain stochastic hybrid differential game. The main contribution of this study is the development of a saddle-point equilibrium strategy for linear quadratic uncertain stochastic two-player zero-sum hybrid differential games over both finite and infinite horizons. In future work, we may consider uncertain stochastic hybrid differential games with jumps.

Author Contributions: Conceptualization, Z.J. and C.L.; software, C.L.; writing—original draft preparation, Z.J.; writing—review and editing, Z.J. and C.L.; supervision, C.L. All authors have read and agreed to the published version of the manuscript.

Funding: The authors were supported partially by the Suqian Sci & Tech Program (Grant No. K202332) and the Startup Foundation for Newly Recruited Employees (Grant No. 2024XRC004); in part by the Major Projects of North Minzu University under Grant ZDZX201805; and in part by the Governance and Social Management Research Center, Northwest Ethnic Regions and First-Class Disciplines Foundation of Ningxia, under Grant NXYLXK2017B09.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Isaacs, R. *Differential Games*; Wiley: New York, NY, USA, 1965.
2. Friedman, A. *Differential Games*; Wiley: New York, NY, USA, 1971.
3. Basar, T.; Olsder, G. *Dynamic Noncooperative Game Theory*; Academic Press: New York, NY, USA, 1991.
4. Dockner, E.; Jorgensen, S.; Long, N.; Sorger, G. *Differential Games in Economics and Management Science*; Cambridge University Press: Cambridge, UK, 2000.
5. Ho, Y.; Bryson, A.; Baron, S. Differential games and optimal pursuit-evasion strategies. *IEEE Trans. Autom. Control* **1965**, *10*, 385–389. [CrossRef]
6. Starr, A.W.; Ho, Y.C. Nonzero-sum differential games. *J. Optim. Theory Appl.* **1969**, *3*, 184–206. [CrossRef]
7. Schmitendorf, W.E. Existence of optimal open-loop strategies for a class of differential games. *J. Optim. Theory Appl.* **1970**, *5*, 363–375. [CrossRef]
8. Bernhard, P. Linear-quadratic, two-person, zero-sum differential games: Necessary and sufficient conditions. *J. Optim. Theory Appl.* **1979**, *27*, 51–69. [CrossRef]
9. Delfour, M.C. Linear quadratic differential games: Saddle point and Riccati differential equations. *SIAM J. Control Optim.* **2007**, *46*, 750–774. [CrossRef]
10. Delfour, M.C.; Sbarba, O.D. Linear quadratic differential games: Closed loop saddle points. *SIAM J. Control Optim.* **2009**, *47*, 3138–3166. [CrossRef]
11. Fleming, W.H. Optimal continuous-parameter stochastic control. *SIAM Rev.* **1969**, *11*, 470–509. [CrossRef]
12. Mou, L.; Yong, J. Two-person zero-sum linear quadratic stochastic differential games by a Hilbert space method. *J. Ind. Manag. Optim.* **2006**, *2*, 95–117. [CrossRef]
13. Sun, J.; Yong, J. Linear quadratic stochastic differential games: Open loop and closed-loop saddle point. *SIAM J. Control Optim.* **2014**, *52*, 4082–4121. [CrossRef]
14. Zhu, Y. Uncertain optimal control with application to a portfolio selection model. *Cybern. Syst.* **2010**, *41*, 535–547. [CrossRef]
15. Yang, X.; Gao, J. Uncertain differential games with application to capitalism. *J. Uncertain. Anal. Appl.* **2013**, *1*, 17. [CrossRef]
16. Yang, X.; Gao, J. Linear quadratic uncertain differential game with application to resource extraction problem. *IEEE Trans. Fuzzy Syst.* **2016**, *24*, 819–826. [CrossRef]
17. Liu, Y. Uncertain random variables: A mixture of uncertainty and randomness. *Soft Comput.* **2013**, *17*, 625–634. [CrossRef]
18. Jia, Z.; Li, C. Almost sure exponential stability of uncertain stochastic Hopfield neural networks based on subadditive measures. *Mathematics* **2023**, *11*, 3110. [CrossRef]
19. Jia, Z.; Liu, X. Uncertain stochastic hybrid age-dependent population equation based on subadditive measure: Existence, uniqueness and exponential stability. *Symmetry* **2023**, *15*, 1512. [CrossRef]
20. Fei, W. Optimal control of uncertain stochastic systems with markovian switching and its applications to portfolio decisions. *Cybern. Syst.* **2014**, *45*, 69–88. [CrossRef]
21. Chen, X.; Zhu, Y.; Park, J.H. Two person games for uncertain random singular dynamic systems. *IET Control Theory Appl.* **2022**, *17*, 542–558. [CrossRef]
22. Chen, X.; Tian, C.; Jin, T. Optimal control and zero-sum game subject to multifactor uncertain random systems with jump. *Optimization* **2023**, 1–42. [CrossRef]
23. Jia, Z.; Liu, X. Uncertain stochastic hybrid differential game system with V-n jumps: Saddle point equilibrium, strategies and application to advertising duopoly game. *Chaos Solitons Fract.* **2023**, *171*, 113490. [CrossRef]
24. Chen, X.; Zhu, Y. Optimal control and zero-sum game subject to differential equations with Liu processes and random matrices. *Opt. Control Appl. Meth.* **2024**, 1–28. [CrossRef]

25. Ghaoui, L.; Rami, M. Robust state-feedback stabilization of jump linear systems via LMIs. *Int. J. Robust Nonlinear Contr.* **1996**, *6*, 1015–1022. [CrossRef]
26. Jørgensen, S. Yeung, D.W. Stochastic differential game model of a common property fishery. *J. Optim. Theory Appl.* **1996**, *99*, 391–403.
27. Zhang, W.; Chen, B. State feedback H_∞ control for a class of nonlinear stochastic systems. *Siam J. Control Optim.* **2006**, *44*, 1973–1991. [CrossRef]