mathematics

Article

Existence of Periodic Solutions for Second-Order Ordinary *p*-Laplacian Systems

Shaomin Wang [1,*], Cunji Yang [1] and Guozhi Cha [2,3]

[1] School of Mathematics and Computer, Dali University, Dali 671003, China; kmycj@126.com
[2] School of Engineering, Dali University, Dali 671003, China; chagz@dali.edu.cn
[3] Pen-Tung Sah Institute of Micro-Nano Science and Technology, Xiamen University, Xiamen 361102, China
* Correspondence: smw2023@dali.edu.cn

Abstract: In this paper, we study the variational principle and the existence of periodic solutions for a new class of second-order ordinary p-Laplacian systems. The variational principle is given by making use of two methods. We obtain three existence theorems of periodic solutions to this problem on various sufficient conditions on the potential function $F(t,x)$ or nonlinearity $\nabla F(t,x)$. Four examples are presented to illustrate the feasibility and effectiveness of our results.

Keywords: ordinary p-Laplacian system; the variational principle; periodic solutions; the least action principle; saddle point theorem

MSC: 35A15; 34A12; 34K13

Citation: Wang, S.; Yang, C.; Cha, G. Existence of Periodic Solutions for Second-Order Ordinary *p*-Laplacian Systems. *Mathematics* **2024**, *12*, 1131. https://doi.org/10.3390/math12081131

Academic Editors: Ravi P. Agarwal and Maria Alessandra Ragusa

Received: 12 March 2024
Revised: 31 March 2024
Accepted: 1 April 2024
Published: 9 April 2024

1. Introduction

Consider the following second-order ordinary p-Laplacian system

$$\begin{cases} \frac{d}{dt}(|\dot{u}(t)|^{p-2}\dot{u}(t)) + g(t)|\dot{u}(t)|^{p-2}\dot{u}(t) - A(t)|u(t)|^{p-2}u(t) + \nabla F(t,u(t)) = 0, \text{a.e } t \in [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1}$$

where we can see $p > 1$, $T > 0$, $g \in L^1(0,T;R)$, and where $A(t) = [a_{ij}(t)]$ is a positivedefinite symmetric $N \times N$ matrix-valued function defined in $[0,T]$ with $a_{ij} \in C(0,T)$. Moreover, $F : [0,T] \times R^N \to R$ satisfies the following assumption:

(H_0) $F(t,x)$ is measurable in t for every $x \in R^N$ and continuously differentiable in x for a.e $t \in [0,T]$, and there exist $a \in C(R^+, R^+)$ and $b \in L^1(0,T;R^+)$ such that

$$|F(t,x)| \le a(|x|)b(t),\ |\nabla F(t,x)| \le a(|x|)b(t),\ x \in R^N, \text{a.e } \ t \in [0,T].$$

when $p = 2$, $g(t) \equiv 0$ and $A(t) = 0$, the second-order ordinary p-Laplacian system of form (1) becomes the following second-order Hamiltonian system

$$\begin{cases} \ddot{u}(t) + \nabla F(t,u(t)) = 0, \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases} \tag{2}$$

Over the last few decades, system (2) has been studied by using the variational method. The existence and multiplicity of periodic solutions to problem (2) were obtained on various hypotheses on the potential function $F(t,x)$ or nonlinearity $\nabla F(t,x)$ (see, Refs. [1–6]).

Given that $p = 2$, the second-order ordinary p-Laplacian system of form (1) reduces to the following second-order damped vibration system

$$\begin{cases} \ddot{u}(t) + g(t)\dot{u}(t) - A(t)u(t) + \nabla F(t,u(t)) = 0, \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases} \tag{3}$$

Wu, Chen and Teng [7] provided the variational principle of system (3). At the same time, some existence results of system (3) were obtained by using critical point theorem.

Taking $g(t) \equiv 0$ and $A(t) = 0$ in problem (1), many scholars have studied the following second-order ordinary p-Laplacian system

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(|\dot{u}(t)|^{p-2}\dot{u}(t)) = \nabla F(t, u(t)), \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases} \tag{4}$$

A lot of important existence and multiplicity results of periodic solutions to problem (4) have been obtained by using the critical point theory (see, Refs. [8–12]). In particular, Lv [10] gave the following existence theorem of periodic solutions to problem (4).

Theorem 1. *Let $F(t,x) = F_1(t,x) + F_2(x)$, where F_1 and F_2 satisfy assumption (H_0) and the following conditions:*

(i) *There exist $k, m \in L^1(0,T;R^+)$ and $\gamma \in [0, p-1)$ such that*

$$|\nabla F_1(t,x)| \le k(t)|x|^{\gamma} + m(t)$$

for all $x \in R^N$ and a.e $t \in [0,T]$;

(ii) *There exist constants $0 \le r_1 < \frac{1}{T^p}$ and $r_2 \in [0, +\infty)$ such that*

$$(\nabla F_2(x) - \nabla F_2(y), x - y) \ge -r_1|x-y|^p - r_2|x-y|$$

for all $x \in R^N$ and a.e $t \in [0,T]$;

(iii) $\frac{1}{|x|^{\gamma q}}\int_0^T F(t,x)\mathrm{d}t \to +\infty$ *as $|x| \to +\infty$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

Then, problem (4) has at least one solution which minimizes φ on $W_T^{1,p}$.

In addition, taking $g(t) \equiv 0$ in problem (1), then one has

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(|\dot{u}(t)|^{p-2}\dot{u}(t)) - A(t)|u(t)|^{p-2}u(t) + \nabla F(t, u(t)) = 0, \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases} \tag{5}$$

Zhang and Tang [13] studied the more general second-order ordinary p-Laplacian system of form (5). Some existence theorems of periodic solutions to problem (5) were obtained by using minimax methods in critical point theory.

In the paper, inspired by the results of [7,10,13], we study the second-order ordinary p-Laplacian system of form (1) under the conditions $p > 1$, $g(t) \neq 0$ and, $A(t) \neq 0$. Obviously, the second-order ordinary p-Laplacian system of form (1) is more general than systems (2)–(5). Namely, systems (2)–(5) are only special cases of system (1). As far as we know, there is no relevant research or any results by using the variational method to study the ordinary p-Laplacian system of form (1). Therefore, system (1) is proved to be a more general new system. The remaining part of the paper is as follows. We first study the variational principle of problem (1) by two methods in Section 2. Furthermore, as an application, we obtain three existence theorems for problem (1) by using the critical point theorem in Section 3. Finally, four examples are given to illustrate our results in Section 4. Our results generalize existing relevant conclusions.

For convenience, the meanings of the main symbols used in the paper are as follows: $a = \max\limits_{i,j=1,\cdots,N}\{a_{ij}\}$ where $a_{ij} = \max\limits_{t\in[0,T]}\{|a_{ij}(t)|\}$, $d_1 = \max\limits_{t\in[0,T]} \mathrm{e}^{Q(t)}$, $d_2 = \min\limits_{t\in[0,T]} \mathrm{e}^{Q(t)}$, $\|\widetilde{u}\|_{L^p} = (\int_0^T |\widetilde{u}(t)|^p \mathrm{d}t)^{\frac{1}{p}}$, $\|\dot{u}\|_{L^p} = (\int_0^T |\dot{u}(t)|^p \mathrm{d}t)^{\frac{1}{p}}$, $\|u\|_\infty = \max\limits_{t\in[0,T]}|u(t)|$, and various positive constants as C_i $(i = 1, 2, \cdots)$.

2. The Variational Principle

In this section, we will obtain the variational principle of problem (1) by two methods.

$W_T^{1,p}$ is a Sobolev space defined by $W_T^{1,p} = \{u : [0,T] \to R^N | u$ is absolutely continuous, $u(0) = u(T)$ and $\dot{u} \in L^p([0,T]; R^N)\}$ with the norm

$$\|u\| = (\int_0^T |u(t)|^p \mathrm{d}t + \int_0^T |\dot{u}(t)|^p \mathrm{d}t)^{\frac{1}{p}},\ u \in W_T^{1,p}.$$

Let $\overline{u} = \frac{1}{T}\int_0^T u(t)\mathrm{d}t$ and $\widetilde{u}(t) = u(t) - \overline{u}$ for any $u \in W_T^{1,p}$. Then, one has $W_T^{1,p} = \widetilde{W}_T^{1,p} \oplus R^N$, where $\widetilde{W}_T^{1,p} = \left\{ u \in W_T^{1,p} | \overline{u} = 0 \right\}$. Using the technique of [14] (Proposition 1.3), one has

$$\|\widetilde{u}\|_{L^p} \leq T \|\dot{u}\|_{L^p} \text{ (Wirtinger's inequality)}$$

and

$$\|\widetilde{u}\|_\infty \leq T^{\frac{1}{q}} \|\dot{u}\|_{L^p} \text{ (Sobolev's inequality)}$$

for all $u \in W_T^{1,p}$, where $\frac{1}{p} + \frac{1}{q} = 1$ (see, Ref. [10]).

Lemma 1. ([14] (Proposition 1.1)). *There exists $c > 0$ such that, if $u \in W_T^{1,p}$, then $\|u\|_\infty \leq c\|u\|$. Moreover, if $\int_0^T u(t)\mathrm{d}t = 0$, then $\|u\|_\infty \leq c\|\dot{u}\|_{L^p}$.*

Lemma 2. ([14] (Fundamental Lemma)). *Let $u, v \in L^1(0,T;R^N)$. If for every $f \in C_T^\infty$, $\int_0^T (u(t), f'(t))\mathrm{d}t = -\int_0^T (v(t), f(t))\mathrm{d}t$, then $\int_0^T v(s)\mathrm{d}s = 0$ and there exists $c \in R^N$ such that $u(t) = \int_0^t v(s)\mathrm{d}s + c$ a.e on [0,T].* By Lemma 1, there is a constant $C_0 > 0$ such that

$$\|u\|_\infty \leq C_0\|u\|, \forall u \in W_T^{1,p}. \tag{6}$$

Method 1. We first offer an expression for the functional $I(u)$ and further prove that one solution to problem (1) is the critical point of the functional $I(u)$ in the sense of a weak derivative.

Define the functional $I(u)$ on $W_T^{1,p}$ by

$$I(u) = \frac{1}{p}\int_0^T \mathrm{e}^{Q(t)}|\dot{u}(t)|^p \mathrm{d}t + \frac{1}{p}\int_0^T \mathrm{e}^{Q(t)}(A(t)|u(t)|^{p-2}u(t), u(t))\mathrm{d}t - \int_0^T \mathrm{e}^{Q(t)}F(t, u(t))\mathrm{d}t,$$

where $Q(t) = \int_0^t g(s)\mathrm{d}s$. We know that $I(u)$ is continuously differentiable and the weak lower semi-continuity on $W_T^{1,p}$. The detailed proof is similar to the corresponding parts in [14] (pp. 10–11) and

$$(I'(u), v) = \int_0^T \mathrm{e}^{Q(t)}(|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{v}(t))\mathrm{d}t + \int_0^T \mathrm{e}^{Q(t)}(A(t)|u(t)|^{p-2}u(t), v(t))\mathrm{d}t$$
$$-\int_0^T \mathrm{e}^{Q(t)}(\nabla F(t, u(t)), v(t))\mathrm{d}t,\ u, v \in W_T^{1,p}.$$

Theorem 2. *If $u \in W_T^{1,p}$ is a solution to $I'(u) = 0$ (i.e., u is a critical point of $I(u)$), then u is a solution to problem (1).*

Proof. As $I'(u) = 0$, then

$$0 = (I'(u), v) = \int_0^T \mathrm{e}^{Q(t)}(|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{v}(t))\mathrm{d}t + \int_0^T \mathrm{e}^{Q(t)}(A(t)|u(t)|^{p-2}u(t), v(t))\mathrm{d}t$$
$$-\int_0^T \mathrm{e}^{Q(t)}(\nabla F(t, u(t)), v(t))\mathrm{d}t,$$

for all $u, v \in W_T^{1,p}$. That is,

$$\int_0^T (\mathrm{e}^{Q(t)}|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{v}(t))\mathrm{d}t = -\int_0^T \mathrm{e}^{Q(t)}(A(t)|u(t)|^{p-2}u(t) - \nabla F(t, u(t)), v(t))\mathrm{d}t,$$

for all $v \in W_T^{1,p}$. By Lemma 2, it can be seen that $\mathrm{e}^{Q(t)}|\dot{u}(t)|^{p-2}\dot{u}(t)$ has a weak derivative, and

$$(\mathrm{e}^{Q(t)}|\dot{u}(t)|^{p-2}\dot{u}(t))\prime = \mathrm{e}^{Q(t)}(A(t)|u(t)|^{p-2}u(t) - \nabla F(t, u(t)), \text{ a.e } t \in [0, T].$$

Hence, u satisfies the following equation

$$(|\dot{u}(t)|^{p-2}\dot{u}(t))\prime + g(t)|\dot{u}(t)|^{p-2}\dot{u}(t) - A(t)|u(t)|^{p-2}u(t) + \nabla F(t, u(t) = 0, \text{ a.e } t \in [0, T].$$

Then, u is a solution to problem (1). This completes the proof.

Method 2. By the semi-inverse method [15], we can obtain the variational principle of problem (1). Its derivation process is as follows.

The problem (5) has the following variational principle:

$$\varphi_1(u) = \frac{1}{p}\int_0^T |\dot{u}(t)|^p \mathrm{d}t + \frac{1}{p}\int_0^T (A(t)|u(t)|^{p-2}u(t), u(t))\mathrm{d}t - \int_0^T F(t, u(t))\mathrm{d}t.$$

To acquire the variational principle for problem (1), we introduce an integrating factor $f(t)$ and consider the following integral:

$$\varphi(u) = \int_0^T \left\{ f(t)[\frac{1}{p}|\dot{u}(t)|^p + \frac{1}{p}(A(t)|u(t)|^{p-2}u(t), u(t)) - F(t, u(t))] + L(u, u_t, u_{tt}, \cdots)\right\}\mathrm{d}t, \tag{7}$$

where L is an unknown function of u and/or its derivatives. The Euler–Lagrange equation of Equation (7) is

$$-f(t)\nabla F(t, u(t)) + f(t)A(t)|u(t)|^{p-2}u(t) - (f(t)|\dot{u}(t)|^{p-2}\dot{u}(t))'_t + \frac{\delta L}{\delta u} = 0, \tag{8}$$

where $\frac{\delta L}{\delta u}$ is called the variational derivative [16,17] and is defined as

$$\frac{\delta L}{\delta u} = \frac{\partial L}{\partial u} - \frac{\partial}{\partial t}\frac{\partial L}{\partial u_t} + \frac{\partial^2}{\partial t^2}\frac{\partial L}{\partial u_{tt}} - \cdots.$$

We simplify (8) as follows

$$(|\dot{u}(t)|^{p-2}\dot{u}(t))\prime + \frac{f'}{f}(|\dot{u}(t)|^{p-2}\dot{u}(t)) = -\nabla F(t, u(t)) + A(t)|u(t)|^{p-2}u(t) + \frac{1}{f}\frac{\delta L}{\delta u}. \tag{9}$$

By comparison between Equation (9) and problem (1), we set

$$\frac{f'}{f} = g(t), \ \frac{\delta L}{\delta u} = 0.$$

Hence, we have

$$f = \exp\int_0^t g(s)\mathrm{d}s = \mathrm{e}^{Q(t)}, \ L = 0.$$

Consequently, we obtain the energy functional for problem (1) (i.e., the variational principle of problem (1)), which is

$$\varphi(u) = \int_0^T \mathrm{e}^{Q(t)}\left[\frac{1}{p}|\dot{u}(t)|^p + \frac{1}{p}(A(t)|u(t)|^{p-2}u(t), u(t)) - F(t, u(t))\right]\mathrm{d}t.$$

Obviously, $I(u) = \varphi(u)$.

3. Existence of Solutions for the Ordinary p-Laplacian System

Lemma 3. ([10] (Lemma 2.1)). *In Sobolev space* $W_T^{1,p}$, *for* $u \in W_T^{1,p}$, $\|u\| \to +\infty$ *if and only if* $(|\overline{u}|^p + \|\dot{u}\|_{L^p}^p)^{\frac{1}{p}} \to +\infty$.

Theorem 3. *Let* $F(t,x) = F_1(t,x) + F_2(t,x)$, *and suppose that* $F_1(t,x)$ *and* $F_2(t,x)$ *satisfy assumption* (H_0). *If the following conditions hold:*

(H_1) *There exists a function* $h_1 \in C([0,+\infty);\ [0,+\infty))$ *with the properties:*

(i) $h_1(s) \le h_1(t) \quad \forall s \le t,\ s,t \in [0,+\infty)$,

(ii) $h_1(s+t) \le h_1(s) + h_1(t) \quad \forall s,t \in [0,+\infty)$,

(iii) $h_1(t) \to +\infty$ *as* $t \to +\infty$.

Moreover, there exist $r \in L^1(0,T;R^+)$, $K_1 > 0$ *and* $\alpha \in [0,p)$ *such that*

$$F_1(t,x) \le h_1(|x|) + r(t), \forall \quad x \in R^N, \text{a.e } t \in [0,T]$$

and

$$\limsup_{s \to +\infty} \frac{h_1(s)}{s^\alpha} \le K_1;$$

(H_2) *There exist* $m_1,\ m_2 \in L^1(0,T;R^+)$, $K_2 > 0$, $\beta \in [0,p-1)$ *and a function* $h_2 \in C([0,+\infty);\ [0,+\infty))$ *which satisfies the conditions (i)–(iii) such that*

$$|\nabla F_2(t,x)| \le m_1(t)h_2(|x|) + m_2(t), \ \forall \quad x \in R^N, \ \text{a.e } t \in [0,T]$$

and

$$\limsup_{s \to +\infty} \frac{h_2(s)}{s^\beta} \le K_2;$$

$$(H_3)(A(t)|x|^{p-2}x, x) \ge \frac{1}{2}|x|^p, \ \forall \quad x \in R^N, \ \text{a.e } t \in [0,T];$$

$$(H_4) \lim_{|x| \to +\infty} \frac{1}{|x|^{q\beta}} \int_0^T \mathrm{e}^{Q(t)}(F_2(t,x) + h_1(|x|))\mathrm{d}t = -\infty, \text{where } \frac{1}{p} + \frac{1}{q} = 1.$$

Then, problem (1) has no less than one solution on $W_T^{1,p}$.

Proof. By (H_1), for all $u \in W_T^{1,p}$, we have

$$\begin{aligned}
\int_0^T \mathrm{e}^{Q(t)} F_1(t,u(t))\mathrm{d}t &\le \int_0^T \mathrm{e}^{Q(t)}(h_1(|u(t)|) + r(t))\mathrm{d}t \\
&\le \int_0^T \mathrm{e}^{Q(t)} h_1(|\overline{u}| + \|\widetilde{u}\|_\infty)\mathrm{d}t + \int_0^T \mathrm{e}^{Q(t)} r(t)\mathrm{d}t \\
&\le \int_0^T \mathrm{e}^{Q(t)} h_1(|\overline{u}|)\mathrm{d}t + \int_0^T \mathrm{e}^{Q(t)} h_1(\|\widetilde{u}\|_\infty)\mathrm{d}t + \int_0^T \mathrm{e}^{Q(t)} r(t)\mathrm{d}t \\
&\le \int_0^T \mathrm{e}^{Q(t)} h_1(|\overline{u}|)\mathrm{d}t + Td_1(K_1\| \widetilde{u}\|_\infty^\alpha + C_1) + \int_0^T \mathrm{e}^{Q(t)} r(t)\mathrm{d}t \\
&\le \int_0^T \mathrm{e}^{Q(t)} h_1(|\overline{u}|)\mathrm{d}t + T^{1+\frac{\alpha}{q}} d_1 K_1 \|\dot{u}\|_{L^p}^\alpha + C_2.
\end{aligned} \tag{10}$$

By (H_2), Sobolev's inequality and ε-Young's inequality, one has

$$
\begin{aligned}
&|\int_0^T e^{Q(t)}(F_2(t,u(t)) - F_2(t,\overline{u}))dt| \\
&= |\int_0^T e^{Q(t)} \int_0^1 (\nabla F_2(t, \overline{u} + s\widetilde{u}(t)), \widetilde{u}(t))\, dsdt| \\
&\le \int_0^T e^{Q(t)} m_1(t)(h_2(|\overline{u}|)|\widetilde{u}(t)| + h_2(|\widetilde{u}(t)|)|\widetilde{u}(t)|)dt + \int_0^T e^{Q(t)} m_2(t)|\widetilde{u}(t)|\, dt \\
&\le \int_0^T e^{Q(t)} m_1(t)\, dt\ \ h_2(|\overline{u}|)\|\widetilde{u}\|_\infty + \int_0^T e^{Q(t)} m_1(t)\, dt\, h_2(\|\widetilde{u}\|_\infty)\|\widetilde{u}\|_\infty + \int_0^T e^{Q(t)} m_2(t)\, dt\|\ \widetilde{u}\|_\infty \\
&\le \varepsilon\|\widetilde{u}\|_\infty^p + \frac{\varepsilon^{\frac{-q}{p}}}{q} h_2^q(|\overline{u}|)(\int_0^T e^{Q(t)} m_1(t)\, dt)^q + \int_0^T e^{Q(t)} m_1(t)\, dt (K_2\|\widetilde{u}\|_\infty^\beta + C_3)\ \|\ \widetilde{u}\|_\infty \\
&+ \int_0^T e^{Q(t)} m_2(t)\, dt \quad \|\widetilde{u}\|_\infty \\
&\le \varepsilon\|\widetilde{u}\|_\infty^p + C_4|\overline{u}|^{q\beta} + C_5 + \int_0^T e^{Q(t)} m_1(t)\, dt(K_2\|\widetilde{u}\|_\infty^\beta + C_3)\ \|\widetilde{u}\|_\infty + \int_0^T e^{Q(t)} m_2(t)\, dt\ \|\ \widetilde{u}\,\|_\infty \\
&\le \varepsilon T^{\frac{p}{q}}\|\dot{u}\|_{L^p}^p + C_6\|\dot{u}\|_{L^p}^{\beta+1} + C_7\|\dot{u}\|_{L^p} + C_4|\overline{u}|^{q\beta} + C_5.
\end{aligned}
\tag{11}
$$

for all $u \in W_T^{1,p}$, where $\varepsilon T^{\frac{p}{q}} < \frac{d_2}{2p}$.

Thus, by (10), (11) and (H_3), we obtain

$$
\begin{aligned}
I(u) &= \tfrac{1}{p}\int_0^T e^{Q(t)}|\dot{u}(t)|^p dt + \tfrac{1}{p}\int_0^T e^{Q(t)}(A(t)|u(t)|^{p-2}u(t), u(t))dt - \int_0^T e^{Q(t)} F(t,u(t))dt \\
&= \tfrac{1}{p}\int_0^T e^{Q(t)}|\dot{u}(t)|^p dt + \tfrac{1}{p}\int_0^T e^{Q(t)}(A(t)|u(t)|^{p-2}u(t), u(t))dt - \int_0^T e^{Q(t)} F_1(t,u(t))\, dt \\
&- \int_0^T e^{Q(t)}(F_2(t,u(t) - F_2(t,\overline{u}))dt - \int_0^T e^{Q(t)} F_2(t,\overline{u})\, dt \\
&\ge \tfrac{d_2}{p}\int_0^T |\dot{u}(t)|^p dt + \tfrac{d_2}{2p}\int_0^T |u(t)|^p dt - \int_0^T e^{Q(t)} h_1(|\overline{u}|)dt - T^{1+\frac{\alpha}{q}} d_1 K_1\|\dot{u}\|_{L^p}^\alpha - C_2 \\
&- \varepsilon T^{\frac{p}{q}}\|\dot{u}\|_{L^p}^p - C_6\|\dot{u}\|_{L^p}^{\beta+1} - C_7\|\dot{u}\|_{L^p} - C_4|\overline{u}|^{q\beta} - C_5 - \int_0^T e^{Q(t)} F_2(t,\overline{u})\, dt \\
&= \tfrac{d_2}{2p}\|u\|^p + (\tfrac{d_2}{2p} - \varepsilon T^{\frac{p}{q}})\|\dot{u}\|_{L^p}^p - T^{1+\frac{\alpha}{q}} d_1 K_1\|\dot{u}\|_{L^p}^\alpha - C_6\|\dot{u}\|_{L^p}^{\beta+1} - C_7\|\dot{u}\|_{L^p} \\
&- |\overline{u}|^{q\beta}(\tfrac{1}{|\overline{u}|^{q\beta}}\int_0^T e^{Q(t)}(F_2(t,\overline{u}) + h_1(|\overline{u}|))dt + C_4) - C_8
\end{aligned}
\tag{12}
$$

for all $u \in W_T^{1,p}$. Since $\alpha \in [0,p)$, $\beta \in [0,p-1)$ and Lemma 3, we have $I(u) \to +\infty$ as $\|u\| \to +\infty$ by (H_4) and (12). Namely, $I(u)$ is coercive. Therefore, by using the least action principle [14], problem (1) has no less than one solution on $W_T^{1,p}$.

Theorem 4. *Let $F(t,x) = F_1(t,x) + F_2(x)$, suppose that $F_1(t,x)$ and $F_2(x)$ satisfy assumption (H_0). If (H_3) and the following conditions hold:*

(H_5) There exist $r_1, r_2 \in L^1(0,T;R^+)$ and $\alpha \in [0,p-1)$ such that

$$|\nabla F_1(t,x)| \le r_1(t)|x|^\alpha + r_2(t),\ \forall\ \ x \in R^N,\ \text{a.e}\ t \in [0,T];$$

(H_6) There exist $0 < K < \frac{d_2}{2d_1 p T^{\frac{p}{q}+1}}$ and an increasing function $h \in C([0,+\infty);\ [0,+\infty))$ such that

$$(\nabla F_2(x) - \nabla F_2(y), x-y) \le h(|x-y|),\ \forall\ \ x \in R^N,\ \text{a.e}\ t \in [0,T]$$

and

$$\limsup_{s\to+\infty} \frac{h(s)}{s^p} \le K;$$

$$(H_7) \lim_{|x|\to+\infty} \frac{1}{|x|^{\alpha q}} \int_0^T e^{Q(t)} F(t,x)dt = -\infty,\ \text{where}\ \frac{1}{p} + \frac{1}{q} = 1$$

Then, problem (1) has no less than one solution on $W_T^{1,p}$.

Proof. By (H_5), ε-Young's inequality and Sobolev's inequality, we obtain

$$
\begin{aligned}
&|\int_0^T \mathrm{e}^{Q(t)}(F_2(t,u(t)) - F_2(t,\overline{u}))\mathrm{d}t| \\
&= |\int_0^T \mathrm{e}^{Q(t)} \int_0^1 (\nabla F_1(t,\overline{u}+s\widetilde{u}(t)),\widetilde{u}(t))\mathrm{d}s\mathrm{d}t| \\
&\leq \int_0^T \int_0^1 \mathrm{e}^{Q(t)} r_1(t)|\overline{u}+s\widetilde{u}(t)|^{\alpha}\cdot|\widetilde{u}(t)| \ \mathrm{d}s\mathrm{d}t + \int_0^T\int_0^1 \mathrm{e}^{Q(t)} r_2(t)|\widetilde{u}(t)| \ \mathrm{d}s\mathrm{d}t \\
&\leq 2^{\alpha}|\overline{u}|^{\alpha}\|\widetilde{u}\|_{\infty}\int_0^T \mathrm{e}^{Q(t)} r_1(t)\mathrm{d}t + 2^{\alpha}\|\widetilde{u}\|_{\infty}^{\alpha+1}\int_0^T \mathrm{e}^{Q(t)} r_1(t)\mathrm{d}t + \|\widetilde{u}\|_{\infty}\int_0^T \mathrm{e}^{Q(t)} r_2(t)\mathrm{d}t \\
&\leq \varepsilon\|\widetilde{u}\|_{\infty}^{p} + \frac{\varepsilon^{\frac{-q}{p}}}{q}2^{\alpha q}|\overline{u}|^{\alpha q}\left(\int_0^T \mathrm{e}^{Q(t)} r_1(t)\ \mathrm{d}t\right)^{q} + 2^{\alpha}\int_0^T \mathrm{e}^{Q(t)} r_1(t)\mathrm{d}t\| \ \widetilde{u}\ \|_{\infty}^{\alpha+1} + \int_0^T \mathrm{e}^{Q(t)} r_2(t)\mathrm{d}t\| \ \widetilde{u}\ \|_{\infty} \\
&\leq \varepsilon T^{\frac{p}{q}}\|\dot{u}\|_{L^p}^{p} + C_9\|\dot{u}\|_{L^p}^{\alpha+1} + C_{10}\|\dot{u}\|_{L^p} + C_{11}|\overline{u}|^{\alpha q}
\end{aligned}
\tag{13}
$$

for all $u \in W_T^{1,p}$, where $\varepsilon T^{\frac{p}{q}} < \frac{d_2}{2p} - d_1 K T^{\frac{p}{q}+1}$.

By (H_6) and Sobolev's inequality, we obtain

$$
\begin{aligned}
\int_0^T \mathrm{e}^{Q(t)}[F_2(u(t)) - F_2(\overline{u})]\mathrm{d}t &= \int_0^T \mathrm{e}^{Q(t)}\int_0^1 (\nabla F_2(\overline{u}+s\widetilde{u}(t)),\widetilde{u}(t))\mathrm{d}s\mathrm{d}t \\
&= \int_0^T \mathrm{e}^{Q(t)}\int_0^1 (\nabla F_2(\overline{u}+s\widetilde{u}(t)) - \nabla F_2(\overline{u}),\widetilde{u}(t))\mathrm{d}s\mathrm{d}t \\
&\leq \int_0^T \mathrm{e}^{Q(t)}\int_0^1 \frac{1}{s}h(|s\widetilde{u}(t)|)\mathrm{d}s\mathrm{d}t \\
&\leq \int_0^T \mathrm{e}^{Q(t)}\int_0^1 K s^{p-1}\|\widetilde{u}\|_{\infty}^{p}\mathrm{d}s\mathrm{d}t + C_{12} \\
&\leq d_1 K T^{\frac{p}{q}+1}\|\dot{u}\|_{L^p}^{p} + C_{12}.
\end{aligned}
\tag{14}
$$

Thus, by (13), (14) and (H_3), we have

$$
\begin{aligned}
I(u) &= \frac{1}{p}\int_0^T \mathrm{e}^{Q(t)}|\dot{u}(t)|^p\mathrm{d}t + \frac{1}{p}\int_0^T \mathrm{e}^{Q(t)}(A(t)|u(t)|^{p-2}u(t),u(t))\mathrm{d}t - \int_0^T \mathrm{e}^{Q(t)}F(t,u(t))\mathrm{d}t \\
&= \frac{1}{p}\int_0^T \mathrm{e}^{Q(t)}|\dot{u}(t)|^p\mathrm{d}t + \frac{1}{p}\int_0^T \mathrm{e}^{Q(t)}(A(t)|u(t)|^{p-2}u(t),u(t))\mathrm{d}t \\
&\quad -\int_0^T \mathrm{e}^{Q(t)}[F_1(t,u(t)) - F_1(t,\overline{u})]\mathrm{d}t - \int_0^T \mathrm{e}^{Q(t)}(F_2(u(t)) - F_2(\overline{u}))\mathrm{d}t - \int_0^T \mathrm{e}^{Q(t)}F(t,\overline{u})\ \mathrm{d}t \\
&\geq \frac{d_2}{p}\int_0^T|\dot{u}(t)|^p\mathrm{d}t + \frac{d_2}{2p}\int_0^T|u(t)|^p\mathrm{d}t - \varepsilon T^{\frac{p}{q}}\|\dot{u}\|_{L^p}^{p} - C_9\|\dot{u}\|_{L^p}^{\alpha+1} - C_{10}\|\dot{u}\|_{L^p} \\
&\quad -C_{11}|\overline{u}|^{\alpha q} - d_1 K T^{\frac{p}{q}+1}\|\dot{u}\|_{L^p}^{p} - C_{12} - \int_0^T \mathrm{e}^{Q(t)}F(t,\overline{u})\ \mathrm{d}t \\
&= \frac{d_2}{2p}\|u\|^p + \left(\frac{d_2}{2p} - d_1 K T^{\frac{p}{q}+1} - \varepsilon T^{\frac{p}{q}}\right)\|\dot{u}\|_{L^p}^{p} - C_9\|\dot{u}\|_{L^p}^{\alpha+1} - C_{10}\|\dot{u}\|_{L^p} \\
&\quad -|\overline{u}|^{\alpha q}\left(\frac{1}{|\overline{u}|^{\alpha q}}\int_0^T \mathrm{e}^{Q(t)}(F(t,\overline{u})\mathrm{d}t + C_{11}\right) - C_{12} \text{ for all } u \in W_T^{1,p}.
\end{aligned}
\tag{15}
$$

As $\alpha \in [0, p-1)$ and Lemma 3, we have $I(u) \to +\infty$ as $\|u\| \to +\infty$ by (H_7) and (15). Namely, $I(u)$ is coercive. Hence, by the least action principle [14], problem (1) has no less than one solution on $W_T^{1,p}$.

Remark 1. *In a sense, the conditions "$(\nabla F_2(x) - \nabla F_2(y), \quad x-y) \leq h(|x-y|)$" and "$\limsup_{s\to+\infty}\frac{h(s)}{s^p} \leq K$" in Theorem 4 are weaker than the condition "$(\nabla F_2(x) - \nabla F_2(y), x-y) \geq -r_1|x-y|^p - r_2|x-y|$" in Theorem 1, so that Theorem 4 generalizes Theorem 1 even in the case of $g(t) \equiv 0$ and $A(t) = 0$. For another, the difference between Theorem 3 and Theorem 4 lies in the different decomposition formula of $F(t,x)$, and as an auxiliary function is used, the condition satisfied by $F_1(t,x)$ in Theorem 3 is weaker than that satisfied by $F_1(t,x)$ in Theorem 4.*

Theorem 5. *Let $F : [0,T] \times R^N \to R$ satisfy assumption (H_0). If the following conditions hold:*

(H_8) $|x|^p \leq (A(t)|x|^{p-2}x, x), \forall\ x \in R^N$, a.e $t \in [0,T]$;

(H_9) *There exist $M_1 > 0$ and $h \in C([0,+\infty);\ [0,+\infty))$ with $\lim_{s\to+\infty} h(s) = +\infty$, and $\frac{h(s)}{s^p}$ is non-increasing in s for all $s \in R^+$ such that*

$$
(\nabla F(t,x),x) - pF(t,x) \geq h(|x|)\frac{F(t,x)}{|x|^p}
$$

and

$$
F(t,x) > 0
$$

for all $|x| \geq M_1$ *and* a.e $t \in [0,T]$.

$$(H_{10})\ \frac{d_1 aN}{p} < \liminf_{|x|\to+\infty} \frac{e^{Q(t)}F(t,x)}{|x|^p} \leq \limsup_{|x|\to+\infty} \frac{e^{Q(t)}F(t,x)}{|x|^p} < \frac{d_2}{p} + \frac{d_2}{pT^p},\ \text{a.e } t \in [0,T].$$

Then, problem (1) has no less than one solution on $W_T^{1,p}$.

To prove our Theorem 5, we need the following result.

Lemma 4. *Suppose* $F(t,x)$ *satisfies assumption* (H_0) *and the conditions* (H_8), (H_9) *of Theorem 5. Then, the functional* $I(u)$ *satisfies the condition (C), that is, for every sequence* $\{u_n\} \subset W_T^{1,p}$, $\{u_n\}$ *has a convergent subsequence if* $I(u_n)$ *is bounded and* $\lim_{n\to\infty}(1+\|u_n\|)\|I'(u_n)\| = 0$.

Proof. Suppose $\{u_n\} \subset W_T^{1,p}$, $I(u_n)$ is bounded and $\lim_{n\to\infty}(1+\|u_n\|)\|I'(u_n)\| = 0$. Then, there exists a constant $L > 0$ such that

$$|I(u_n)| \leq L, (1+\|u_n\|)\|I'(u_n)\| \leq L, \forall\, n \in Z^+. \tag{16}$$

Since $[0,T] = \{t \mid |u_n(t)| \geq M_1, t \in [0,T]\} \cup \{t \mid |u_n(t)| < M_1, t \in [0,T]\}$, we can set $E_n = \{t \mid |u_n(t)| \geq M_1,\ t \in [0,T]\}$. From (16), (H_9) and assumption (H_0), we obtain

$$\begin{aligned}(p+1)L &\geq pI(u_n) - (I'(u_n), u_n)\\ &= \int_0^T e^{Q(t)}[(\nabla F(t,u_n), u_n) - pF(t,u_n)]dt\\ &= \int_{E_n} e^{Q(t)}[(\nabla F(t,u_n), u_n) - pF(t,u_n)]\, dt + \int_{[0,T]\setminus E_n} e^{Q(t)}[(\nabla F(t,u_n), u_n) - pF(t,u_n)]dt\\ &\geq \int_{E_n} h(|u_n|)\frac{e^{Q(t)}F(t,u_n)}{|u_n|^p}\ dt - d_1\int_0^T \gamma_1(t)\, dt,\ \forall\, n \in Z^+,\end{aligned} \tag{17}$$

where $\gamma_1(t) = (p+M_1)\max_{|u_n|\leq M_1} a(|u_n|)b(t) \geq 0$. We have

$$\int_{E_n} h(\|u_n\|_\infty)\frac{e^{Q(t)}F(t,u_n)}{\|u_n\|_\infty{}^p}\ dt \leq \int_{E_n} h(|u_n|)\frac{e^{Q(t)}F(t,u_n)}{|u_n|^p}\ dt < C_{13}. \tag{18}$$

By (6) and (18), one has

$$\int_{E_n} e^{Q(t)}F(t,u_n)\ dt < C_{13}\frac{C_0^P\|u_n\|^P}{h(C_0\|u_n\|)}.$$

Then, we obtain

$$|\int_0^T e^{Q(t)}F(t,u_n)\, dt| \leq |\int_{E_n} e^{Q(t)}F(t,u_n)\, dt| + |\int_{[0,T]\setminus E_n} e^{Q(t)}F(t,u_n)\, dt| \leq C_{13}\frac{C_0^P\|u_n\|^P}{h(C_0\|u_n\|)} + d_1\int_0^T \gamma_2(t)\, dt, \tag{19}$$

where $\gamma_2(t) = \max_{|u_n|\leq M_1} a(|u_n|)b(t)$.

Hence, by (H_8), (H_9) and (19), we have

$$\begin{aligned}L &\geq I(u_n)\\ &= \frac{1}{p}\int_0^T e^{Q(t)}|\dot{u}_n(t)|^p dt + \frac{1}{p}\int_0^T e^{Q(t)}(A(t)|u_n(t)|^{p-2}u_n(t), u_n(t))dt - \int_0^T e^{Q(t)}F(t,u_n(t))dt\\ &\geq \frac{1}{p}d_2\|u_n\|^p - C_{13}\frac{C_0^P\|u_n\|^P}{h(C_0\|u_n\|)} - d_1\int_0^T \gamma_2(t)\, dt\\ &\geq (\frac{d_2}{p} - \frac{C_{13}C_0^P}{h(C_0\|u_n\|)})\|u_n\|^p - d_1\int_0^T \gamma_2(t)\, dt.\end{aligned} \tag{20}$$

Since $\lim_{\|u_n\|\to+\infty} \frac{d_2}{p} - \frac{C_{13}C_0^P}{h(C_0\|u_n\|)} = \frac{d_2}{p} > 0$, we can confirm $\{u_n\}$ is bounded. Or else, we suppose $\|u_n\| \to +\infty$ as $n \to +\infty$ and obtain a contradiction by (20). In the same manner [14] (Proposition 4.3), it can be induced that $\{u_n\}$ has a convergent subsequence. Hence, $I(u)$ satisfies the condition (C).

Lemma 5. ([14] (Theorem 4.7)). *Let X be a Banach space and let $\varphi \in C^1(X,R)$. Assume that X splits into a direct sum of closed subspaces $X = X^- \oplus X^+$ with $\dim X^- < \infty$ and $\sup_{S_R^-}\varphi < \inf_{X^+}\varphi$, where $S_R^- = \{u \in X^- : |u| = R\}$. Let $B_R^- = \{u \in X^- : |u| \le R\}$, $M = \{g \in C(B_R^-, X) : g(s) = s\ \text{ if } s \in S_R^-\}$ and $c = \inf_{g\in M}\max_{s\in B_R^-}\varphi(g(s))$. Then, if φ satisfies the $(PS)_c$-condition, c is a critical value of φ.*

Now, we provide the proof of Theorem 5.

Proof of Theorem 5. As shown in [18], the deformation lemma is proved to be true by replacing the (PS) condition with the weaker condition (C), and it can further demonstrate that Lemma 5 holds true under the condition (C). Hence, by Lemma 5, we only need to prove

$$(l_1)\; I(u) \to +\infty \ \text{ as } \|u\| \to +\infty,\ \ u \in \widetilde{W}_T^{1,p};$$
$$(l_2)\; I(u) \to -\infty \ \text{ as } |u| \to +\infty,\ \ u \in R^N.$$

Now, we prove (l_1). By (H_{10}), for

$$\varepsilon = \frac{d_2}{p} + \frac{d_2}{pT^p} - \sup_{t\in[0,T]}\limsup_{|x|\to+\infty}\frac{e^{Q(t)}F(t,x)}{|x|^p} > 0,$$

there exists $M_2 > 0$ such that

$$e^{Q(t)}F(t,x) \le (\frac{d_2}{p} + \frac{d_2}{pT^p} - \varepsilon)|x|^p,\ \forall\, |x| \ge M_2,\ \text{a.e } t \in [0,T]. \tag{21}$$

It can be induced from (21) and assumption (H_0) that

$$e^{Q(t)}F(t,x) \le (\frac{d_2}{p} + \frac{d_2}{pT^p} - \varepsilon)|x|^p + d_1\gamma_3(t),\ \forall\, x \in R^N,\ \text{a.e } t \in [0,T], \tag{22}$$

where $\gamma_3(t) = \max_{|x|\le M_2} a(|x|)b(t) \ge 0$.

For $u \in \widetilde{W}_T^{1,p}$, by (H_8) and (22), we have

$$\begin{aligned} I(u) &= \tfrac{1}{p}\int_0^T e^{Q(t)}|\dot{u}(t)|^p dt + \tfrac{1}{p}\int_0^T e^{Q(t)}(A(t)|u(t)|^{p-2}u(t),u(t))dt - \int_0^T e^{Q(t)}F(t,u(t))dt \\ &\ge \tfrac{d_2}{p}\int_0^T |\dot{u}(t)|^p dt + \tfrac{d_2}{p}\int_0^T |u(t)|^p dt - \int_0^T e^{Q(t)}F(t,u(t))dt \\ &\ge \tfrac{d_2}{p}\int_0^T |\dot{u}(t)|^p dt + \tfrac{d_2}{p}\int_0^T |u(t)|^p dt - \int_0^T (\tfrac{d_2}{p} + \tfrac{d_2}{pT^p} - \varepsilon)|u(t)|^p\,dt - d_1\int_0^T \gamma_3(t)\,dt \\ &= \tfrac{d_2}{p}\int_0^T |\dot{u}(t)|^p dt - (\tfrac{d_2}{pT^p} - \varepsilon)\int_0^T |u(t)|^p\,dt - d_1\int_0^T \gamma_3(t)\,dt. \end{aligned} \tag{23}$$

Next, we discuss two cases:
Case 1. When $\frac{d_2}{pT^p} - \varepsilon > 0$, by (23) and Wirtinger's inequality, one has

$$I(u) \ge \left[\frac{d_2}{p} - (\frac{d_2}{pT^p} - \varepsilon)T^p\right]\|\dot{u}\|_{L^p}^p - d_1\int_0^T \gamma_3(t)\,dt = \varepsilon T^p\|\dot{u}\|_{L^p}^p - d_1\int_0^T \gamma_3(t)\,dt. \tag{24}$$

Case 2. When $\frac{d_2}{pT^p} - \varepsilon \le 0$, by (23), we have

$$I(u) \ge \frac{d_2}{p}\|\dot{u}\|_{L^p}^p - d_1\int_0^T \gamma_3(t)\,dt. \tag{25}$$

By Wirtinger's inequality, we know that

$$\|u\| \to +\infty \Leftrightarrow \|\dot{u}\|_{L^p} \to +\infty,\ \ u \in \widetilde{W}_T^{1,p}. \tag{26}$$

Thus, by (24)–(26), (l_1) is proved.
Now, we prove (l_2).
For $u \in R^N$, we obtain

$$\begin{aligned} I(u) &= \frac{1}{p}\int_0^T e^{Q(t)}(A(t)|u|^{p-2}u,u)\,dt - \int_0^T e^{Q(t)}F(t,u)\,dt \\ &\le \frac{1}{p}d_1aNT|u|^p - \int_0^T e^{Q(t)}F(t,u)\,dt \\ &= -|u|^p\Big(\int_0^T e^{Q(t)}\frac{F(t,u)}{|u|^p} - \frac{d_1aN}{p}dt\Big). \end{aligned} \tag{27}$$

Therefore, by (H_{10}), (27) and Fatou's Lemma, we have

$$I(u) \to -\infty\,,\ \text{as}\ |u| \to +\infty,\ u \in R^N.$$

Thus, (l_2) is proved.

4. Examples

Now, we provide four examples of potential function $F(t,x)$ and matrix $A(t)$ to illustrate the effectiveness of Theorems 3, 4 and 5, respectively.

Example 1. Let $p = \frac{7}{2}$, $\alpha = 3$ and $\beta = 1$. Moreover, we can put $F_1(t,x) = \frac{|x|}{\ln(100+x^2)} + r(t)$, $F_2(t,x) = -(T-t)|x|^{\frac{3}{2}}$, $h_1(|x|) = \frac{|x|}{\ln(100+x^2)}$, $h_2(|x|) = |x|^{\frac{1}{2}}$, $A(t) = \mathrm{diag}(\frac{3}{2} + \sin\omega\ t, \cdots, \frac{3}{2} + \sin\omega\ t)$, $\omega = \frac{2\pi}{T}$.

Therefore, we have

(1) $h_1(s+t) = \frac{s+t}{\ln(100+(s+t)^2)} \le \frac{s}{\ln(100+s^2)} + \frac{t}{\ln(100+t^2)} = h_1(s) + h_1(t)$; $F_1(t,x) \le h_1(|x|) + r(t)$; $\limsup\limits_{s\to+\infty}\frac{h_1(s)}{s^\alpha} = \limsup\limits_{s\to+\infty}\frac{1}{s^2\ \ln(100+s^2)} = 0 \le K_1$;
(2) $h_2(s+t) = (s+t)^{\frac{1}{2}} \le s^{\frac{1}{2}} + t^{\frac{1}{2}} = h_2(s) + h_2(t)$; $|\nabla F_2(t,x)| = \frac{3}{2}|T-t||x|^{\frac{1}{2}} \le m_1(t)h_2(|x|) + m_2(t)$;
(3) $(A(t)|x|^{p-2}x,x) = (\frac{3}{2} + \sin\omega\ t)|x|^p \ge \frac{1}{2}|x|^p$;
(4) $\lim\limits_{|x|\to+\infty}\frac{1}{|x|^{q\beta}}\int_0^T e^{Q(t)}(F_2(t,x) + h_1(|x|))dt$

$= \lim\limits_{|x|\to+\infty}\frac{1}{|x|^{\frac{7}{5}}}\int_0^T -e^{Q(t)}((T-t)|x|^{\frac{3}{2}} + \frac{|x|}{\ln(100+x^2)})dt = -\infty.$

Then, (H_1), (H_2), (H_3) and (H_4) are true. By Theorem 3, problem (1) has no less than one solution on $W_T^{1,p}$.

Example 2. Let $p = \frac{7}{2}$, $\alpha = 3$ and $\beta = 1$. We can also put $F_1(t,x) = \frac{|x|}{\ln(100+x^2)} + r(t)$, $F_2(t,x) = -(T-t)|x|^2$, $h_1(|x|) = \frac{|x|}{\ln(100+x^2)}$, $h_2(|x|) = |x|$, $A(t) = \mathrm{diag}(\frac{3}{2} + \cos\omega\ t, \cdots, \frac{3}{2} + \cos\omega\ t)$, $\omega = \frac{2\pi}{T}$.

It can be seen from the derivation process of Example 1 that (H_1), (H_2), (H_3) and (H_4) are true. By Theorem 3, problem (1) has no less than one solution on $W_T^{1,p}$.

Example 3. Let $p = 2$ and $\alpha = \frac{1}{2}$. We can also choose $F_1(t,x) = (T-t)|x|^{\frac{3}{2}}$, $F_2(x) = -|x|^2$, $h(|x-y|) = K|x-y|^p(0 < K < \frac{d_2}{2d_1pT^{\frac{p}{q}}})$, $A(t) = \mathrm{diag}(\frac{3}{2} + \sin\omega\ t, \cdots, \frac{3}{2} + \sin\omega\ t)$, $\omega = \frac{2\pi}{T}$.

Then, we have

(1) $|\nabla F_1(t,x)| = \frac{3}{2}|T-t| \cdot |x|^{\frac{1}{2}} \le r_1(t)|x|^\alpha + r_2(t)$;

(2) $(\nabla F_2(x) - \nabla F_2(y), x-y) = -2|x-y|^2 \leq h(|x-y|); \limsup\limits_{s\to+\infty} \frac{h(s)}{s^p} = \limsup\limits_{s\to+\infty} \frac{Ks^p}{s^p} \leq K;$

(3) $\lim\limits_{|x|\to+\infty} \frac{1}{|x|^{\alpha q}} \int_0^T \mathrm{e}^{Q(t)} F(t,x)\mathrm{d}t = \lim\limits_{|x|\to+\infty} \frac{1}{|x|} \int_0^T \mathrm{e}^{Q(t)}((T-t)|x|^{\frac{3}{2}} - |x|^2)\mathrm{d}t = -\infty.$

We can know that (H_3), (H_5), (H_6) and (H_7) hold. By Theorem 4, problem (1) has no less than one solution on $W_T^{1,p}$.

Example 4. Let $p = 2$. We can choose $A(t) = \mathrm{diag}(2+\sin\omega\, \mathrm{t}, \cdots,\ 2+\sin\omega\, \mathrm{t})$, $F(t,x) = \lambda(2+\sin\omega t)(|x|^2 - \ln(1+|x|^2))$, $h(|x|) = \frac{|x|^2\ln(1+|x|^2) - \frac{|x|^4}{1+|x|^2}}{3\lambda(|x|^2 - \ln(1+|x|^2))}$, where $\frac{d_1 aN}{2d_2} < \lambda < \frac{d_2}{6d_1}(1+\frac{1}{T^2})$.

Hence, we obtained the following results:

(1) $|x|^2 \leq (A(t)x, x);$

(2) $(\nabla F(t,x), x) - 2F(t,x) \geq \frac{|x|^2\ln(1+|x|^2) - \frac{|x|^4}{1+|x|^2}}{3\lambda(|x|^2 - \ln(1+|x|^2))} \cdot \frac{F(t,x)}{|x|^2} = h(|x|)\frac{F(t,x)}{|x|^2};$

(3) $\lim\limits_{|x|\to+\infty} h(|x|) = \lim\limits_{|x|\to+\infty} \frac{|x|^2\ln(1+|x|^2) - \frac{|x|^4}{1+|x|^2}}{3\lambda(|x|^2 - \ln(1+|x|^2))} = +\infty;$

(4) $\frac{h(s)}{s^2}$ is non-increasing on $(0, +\infty)$;

(5) $\lim\limits_{|x|\to+\infty} \frac{\mathrm{e}^{Q(t)}F(t,x)}{|x|^2} = \mathrm{e}^{Q(t)} \lim\limits_{|x|\to+\infty} \frac{\lambda(2+\sin\omega t)(|x|^2 - \ln(1+|x|^2))}{|x|^2} = e^{Q(t)}\lambda(2+\sin\omega t),$

then

$$\frac{d_1 aN}{2} < \lambda d_2 \leq \liminf_{|x|\to+\infty} \frac{\mathrm{e}^{Q(t)}F(t,x)}{|x|^2} \leq \limsup_{|x|\to+\infty} \frac{\mathrm{e}^{Q(t)}F(t,x)}{|x|^2} \leq 3\lambda d_1 < \frac{d_2}{2}(1+\frac{1}{T^2}).$$

Therefore, we can know that (H_8), (H_9) and (H_{10}) hold. By Theorem 5, problem (1) has no less than one solution on $W_T^{1,p}$.

5. Conclusions

In this paper, we have studied a class of ordinary p-Laplacian systems by using the variational method. We have obtained the variational principle and the existence of periodic solutions to this system. We can further explore system (1) by using the critical point theorem in the future.

Author Contributions: Conceptualization, S.W.; Methodology, S.W.; Formal analysis, S.W., C.Y. and G.C.; Writing—original draft, S.W. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported in part by the Joint Special Fund Project for Basic Research of Local Undergraduate Universities in Yunnan Province (No.202101BA070001-219) and Foundation of Dali University (No.KY2319101540).

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare that they have no competing interests.

References

1. Bonanno, G.; Livrea, R.; Schechter, M. Some notes on a superlinear second order Hamiltonian system. *Manuscripta Math.* **2017**, *154*, 59–77. [CrossRef]
2. Wang, Z.Y.; Zhang, J.H. New existence results on periodic solutions of non-autonomous second order Hamiltonian systems. *Appl. Math. Lett.* **2018**, *79*, 43–50. [CrossRef]
3. Ye, Y.W.; Tang, C.L. Existence and multiplicity of periodic solutions for some second order Hamiltonian systems. *Bull. Belg. Math. Soc. Simon Stevin* **2014**, *21*, 613–633. [CrossRef]
4. Jiang, Q.; Tang, C.L. Periodic and subharmonic solutions of a class of subquadratic second-order Hamiltonian systems. *J. Math. Anal. Appl.* **2007**, *328*, 380–389. [CrossRef]

5. Ye, Y.W.; Liu, S. Notes on Multiple Periodic Solutions for Second Order Hamiltonian Systems. *Qual. Theory Dyn. Syst.* **2022**, *21*, 141. [CrossRef]
6. Tao, Z.L.; Tang, C.L. Periodic and subharmonic solutions of second-order Hamiltonian systems. *J. Math. Anal. Appl.* **2004**, *293*, 435–445. [CrossRef]
7. Wu, X.; Chen, S.X.; Teng, K.M. On variational methods for a class of damped vibration problems. *Nonlinear Anal-Theor.* **2008**, *68*, 1432–1441. [CrossRef]
8. Lv, X.; Lu, S.P.; Yan, P. Periodic solutions of non-autonomous ordinary p-Laplacian systems. *J. Appl. Math. Comput.* **2011**, *35*, 11–18. [CrossRef]
9. Liao, K.; Tang, C.L. Existence and multiplicity of periodic solutions for the ordinary p-Laplacian systems. *J. Appl. Math. Comput.* **2011**, *35*, 395–406. [CrossRef]
10. Lv, X. Existence of periodic solutions for a class of second-order p-Laplacian systems. *Appl. Math. Comput.* **2018**, *338*, 515–519. [CrossRef]
11. Li, C.; Agarwal, R.; Tang, C.L. Ininitely many periodic solutions for ordinary p-Laplacian systems. *Adv. Nonlinear Anal.* **2015**, *4*, 251–261. [CrossRef]
12. Ge, X.C. Homoclinic solutions for ordinary p-Laplacian systems with local super-p linear conditions. *Bull. Belg. Math. Soc. Simon Stevin* **2022**, *29*, 235–248. [CrossRef]
13. Zhang, Q.F.; Tang, X.H. On the existence of infinitely many periodic solutions for second –order ordinary p-Laplacian system. *Bull. Belg. Math. Soc. Simon Stevin* **2012**, *19*, 121–136. [CrossRef]
14. Mawhin, J.; Willem, M. *Critical Point Theory and Hamiltonian Systems*; Springer: New York, NY, USA, 1989.
15. He, J.H. Variational principles for some nonlinear partial differential equations with variable coefficients. *Chaos Solitons Fractals* **2004**, *19*, 847–851. [CrossRef]
16. He, J.H. Semi-inverse method of establishing generalized variational principles for fluid mechanics with emphasis on turbomachinery aerodynamics. *Int. J. Turbo Jet Eng.* **1997**, *14*, 23–28. [CrossRef]
17. He, J.H. A classical variational model for micropolar elastodynamics. *Int. J. Nonlinear Sci. Numer. Simulat.* **2000**, *1*, 133–138. [CrossRef]
18. Bartolo, P.; Benci, V.; Fortunato, D. Abstract critical point theorems and applications to some nonlinear problems with "strong" resonance at infinity. *Nonlinear Anal.* **1983**, *7*, 981–1012. [CrossRef]