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$(\mathscr{X},\mathscr{Y})$-Gorenstein Categories, Associated (Global) Homological Dimensions and Applications to Relative Foxby Classes

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Abstract: Recently, Gorenstein dimensions relative to a semidualizing module have been the subject of numerous studies with interesting extensions of the classical homological dimensions. Although all these studies share the same direction, a common basis, and similar final goals, there is no common framework encompassing them as parts of a whole, progressing, on different fronts, towards the same end. We provide this general and global framework in the context of abelian categories, standardizing terminology and notation: we establish a general context by defining Gorenstein categories relative to two classes of objects ($(\mathscr{X},\mathscr{Y})$-Gorenstein categories, denoted $\mathscr{G}(\mathscr{X},\mathscr{Y})$), and carry out a study of the homological dimensions associated with them. We prove, under some mild standard conditions, the corresponding version of the Comparison Lemma that ensures the consistency of a homological-dimension theory. We show that Ext functors can be used as tools to compute these $\mathscr{G}(\mathscr{X},\mathscr{Y})$-dimensions, and we compare the dimensions obtained using the classes $G(\mathscr{X})$ with those computed using $\mathscr{G}(\mathscr{X},\mathscr{Y})$. We also initiate a research of the global dimensions obtained with these classes $\mathscr{G}(\mathscr{X},\mathscr{Y})$ and find conditions for them to be finite. Finally, we show that these classes of Gorenstein objects are closely and interestingly related to the Foxby classes induced by a pair of functors. Namely, we prove that the Auslander and Bass classes are indeed $\mathscr{G}(\mathscr{X},\mathscr{Y})$ categories for some specific classes $\mathscr{X}$ and $\mathscr{Y}$.

Keywords: Gorenstein categories; Gorenstein injective module; Gorenstein projective module; Foxby classes

MSC: 16E30; 18G25

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1. Introduction

Over the past years, the investigation of (co)homology with respect to various classes of modules has emerged as a dynamic field of algebra, captivating the dedication of numerous mathematicians.

Within the realm of (co)homology calculations for different classes of modules, the significance of Gorenstein projective modules and Gorenstein injective modules (originally introduced by Enochs and Jenda in [1]) has been notably profound. The study of these module classes has played a pivotal role in advancing relative homological algebra, giving rise to a considerable number of generalizations as a testament to their importance.

One of the highly promising extensions in this context is the class of Gorenstein projective (injective) modules with respect to a semidualizing module, known as the class of G_C-projective (injective) modules. Notable research exploring the homological dimensions associated with these module classes has been conducted by Holm and Jørgensen in [2], as well as by White in [3], and by Huang, Liu zand Xu in [4]. These studies serve as noteworthy examples of the advancements made in understanding the properties of these generalized classes.

It is important to highlight that semidualizing modules possess rather stringent properties, as the condition $\mathrm{End}_R(C) \cong R$ renders C almost projective-like. Consequently, methods have been explored to weaken these requirements on C without forfeiting the essential characteristics that make it an ideal candidate for establishing a satisfactory relative (co)homology theory. These newly studied modules were termed weakly Wakamatsu tilting modules or simply w-tilting modules.

On the other hand, within the context of abelian category $\mathscr{A}$, the exploration of Gorenstein categories with respect to a specific full subcategory $\mathscr{X}$ of $\mathscr{A}$, known as the categories $\mathscr{G}(\mathscr{X})$, has already been investigated by Sather-Wagstaff, Sharif, and White in [5], or in more recent works, the same types of studies were carried out, for example, by Huang in [6] or by Wu and Gao in [7]. This implies that, even though not explicitly stated, the notion of $\mathscr{X}$-Gorenstein objects has already been treated in the literature, extending the original concepts of Gorenstein projective and Gorenstein injective modules into the realm of categorical frameworks.

In contrast to classical Gorenstein projective (injective) modules, G_C-projective (injective) modules require the definition of two module classes (either projective modules and $\mathrm{Add}_R(C)$ or injective modules and $\mathcal{P}roj_R(C)$), whereas the latter only necessitate one class (either projective modules or injective modules). This distinction prompted the natural inclination to explore the potential of studying Gorenstein objects relative to two distinct classes of objects (which, to some extent, expand upon the properties of a w-tilting module) within an abelian category. The objective is to construct a theory of relative Gorenstein-like objects within a categorical framework, which is the primary focus of this paper.

The inception of this idea dates back to 2016 during the development of Enrique Duarte's master's thesis. It is important to acknowledge that as our work was reaching an advanced stage, we discovered a paper that explores a similar concept (Zhao and Sun's paper [8]), where the authors introduced the notion of $(\mathscr{X},\mathscr{Y})$-Gorenstein objects. However, Zhao and Sun's ultimate objective and methods differed from the ones presented in our work, which led us to further develop our study and strategy.

2. Preliminaries

Throughout the paper, and unless otherwise specified, $\mathscr{A}$ will always be an abelian category with enough injectives, $\mathcal{I}nj(\mathscr{A})$ (respectively, $\mathcal{P}roj(\mathscr{A})$) will denote the class of all injective (respectively, projective) objects of $\mathscr{A}$, and $\mathscr{X}$ and $\mathscr{Y}$ will be two full subcategories of $\mathscr{A}$, closed under isomorphisms and containing the zero object.

We will occasionally use certain results of [8], and it is worth noting that, although the authors require the classes $\mathscr{X}$ and $\mathscr{Y}$ to be closed under direct summands, the proofs of [8] (Corollary 3.8, Theorem 4.2 and Proposition 4.8) do not require this assumption.

We start by recalling some well-known concepts. Given an object M of $\mathscr{A}$, an $\mathscr{X}$-precover of M is a morphism $\varphi : X \to M$ with $X \in \mathscr{X}$ such that for any other X' of $\mathscr{X}$, the morphism of abelian groups $\mathrm{Hom}_{\mathscr{A}}(X', \varphi)$ is surjective. If, furthermore, any solution of the equation $\mathrm{Hom}_{\mathscr{A}}(X, \varphi)(g) = \varphi$ is an automorphism, then φ is said to be an $\mathscr{X}$-cover. An $\mathscr{X}$-precover φ is said to be special if it is an epimorphism and $\mathrm{Ext}^1_{\mathscr{A}}(X', \mathrm{Ker}\,\varphi) = 0$ for all $X' \in \mathscr{X}$. Pre-envelopes, special pre-envelopes, and envelopes are defined dually.

The right orthogonal class of any class $\mathscr{C}$, denoted $\mathscr{C}^{\perp}$, is defined as the class of all objects A of $\mathscr{A}$ such that $\mathrm{Ext}^{\geq 1}_{\mathscr{A}}(C, A) = 0$ for every object C of $\mathscr{C}$. Similarly, the left orthogonal class of $\mathscr{C}$, denoted ${}^{\perp}\mathscr{C}$, is defined as the class of all those objects A such that $\mathrm{Ext}^{\geq 1}_{\mathscr{A}}(A, C) = 0$ for every C of $\mathscr{C}$. Given a class $\mathscr{D}$, if $\mathscr{C} \subseteq {}^{\perp}\mathscr{D}$ (equivalently, $\mathscr{D} \subseteq \mathscr{C}^{\perp}$), we write $\mathscr{C} \perp \mathscr{D}$. If $\mathscr{C} \perp \mathscr{C}$, $\mathscr{C}$ is called self-orthogonal.

An $\mathscr{X}$-resolution of an object M of $\mathscr{A}$ is a complex (not necessarily exact)

$$\mathbf{X} = \cdots \to X_1 \to X_0 \to M \to 0,$$

which is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X}, -)$-exact (that is, $\mathrm{Hom}_{\mathscr{A}}(X, \mathbf{X})$ is an exact complex for every $X \in \mathscr{X}$), and such that $X_i \in \mathscr{X}\ \forall i \geq 0$. It is immediate to see that M having an $\mathscr{X}$-resolution is equivalent to M having a $\mathscr{X}$-precover whose kernel has an $\mathscr{X}$-precover and so on. $\mathscr{X}$-coresolutions can be defined dually.

The full subcategory of $\mathscr{A}$ consisting of all objects that have $\mathscr{X}$-resolutions ($\mathscr{X}$-coresolutions) will be denoted as $\mathrm{res}(\mathscr{X})$ ($\mathrm{cores}(\mathscr{X})$), and the full subcategory of $\mathscr{A}$ consisting of all objects that admit an exact $\mathscr{X}$-resolution (an exact $\mathscr{X}$-coresolution) will be denoted by $\widehat{\mathrm{res}}(\mathscr{X})$ ($\widehat{\mathrm{cores}}(\mathscr{X})$).

If $0 \to A \to B \to C \to 0$ and $0 \to B \to D \to E \to 0$ are two exact sequences in $\mathscr{A}$, the pushout diagram of $B \to C$ and $B \to D$ gives the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & D \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\
 & & E \arrow[r, equal] \arrow[d] & E \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Throughout the paper, we will need results concerning the Hom-exactness of pullback and pushout diagrams. We think that these kinds of results should be known but we have not found them in the literature, so we will state what we need in the following four lemmas, avoiding giving the proofs since they are straightforward.

Lemma 1. *Let $0 \to A \to B \to C \to 0$ and $0 \to B \to D \to E \to 0$ be two exact and $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X}, -)$-exact ($\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{X})$-exact) sequences in $\mathscr{A}$. Then, the commutative pushout diagram*

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & D \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\
 & & E \arrow[r, equal] \arrow[d] & E \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

has exact and $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X}, -)$-exact ($\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{X})$-exact) rows and columns.

Lemma 2. *Let $0 \to A \to B \to C \to 0$ and $0 \to A \to D \to E \to 0$ be two exact and $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X}, -)$-exact ($\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{X})$-exact) sequences in $\mathscr{A}$. Then, the commutative pushout diagram*

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & D \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
 & E \arrow[r, equal] \arrow[d] & E \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}
$$

has exact and $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X}, -)$*-exact (*$\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{X})$*-exact) rows and columns.*

These two results have corresponding duals relative to pullbacks.

Lemma 3. *Let* $0 \to A \to B \to C \to 0$ *and* $0 \to D \to E \to B \to 0$ *be two exact and* $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X}, -)$*-exact (*$\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{X})$*-exact) sequences in* $\mathscr{A}$*. Then, the commutative pullback diagram*

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & D \arrow[r, equal] \arrow[d] & D \arrow[d] & & \\
0 \arrow[r] & F \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}
$$

has exact and $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X}, -)$*-exact (*$\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{X})$*-exact) rows and columns.*

Lemma 4. *Let* $0 \to A \to B \to C \to 0$ *and* $0 \to D \to E \to C \to 0$ *be two exact and* $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X}, -)$*-exact (*$\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{X})$*-exact) sequences in* $\mathscr{A}$*. Then, the commutative pullback diagram*

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & D \arrow[r, equal] \arrow[d] & D \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}
$$

has exact and $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X}, -)$*-exact (*$\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{X})$*-exact) rows and columns.*

Let us now recall some fact related to the exactness of the mapping cone associated with a morphism of complexes. We state the result without proof since it is standard and well-known.

Lemma 5. *Suppose the diagram*

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r, "\delta_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \arrow[r] & A_1 \arrow[r, "\delta_1"] \arrow[d, "f_1"] & A_0 \arrow[r] \arrow[d, "f_0"] & 0 \\
0 \arrow[r] & B_n \arrow[r, "d_n"'] & B_{n-1} \arrow[r] & \cdots \arrow[r] & B_1 \arrow[r, "d_1"'] & B_0 \arrow[r] & 0
\end{tikzcd}$$

is commutative with exact rows.

Let $\kappa_{A_i} : A_i \to B_{i+1} \oplus A_i$ and $\kappa_{B_{i+1}} : B_{i+1} \to B_{i+1} \oplus A_i$ be the canonical monomorphisms, and $\pi_{A_i} : B_{i+1} \oplus A_i \to A_i$ and $\pi_{B_{i+1}} : B_{i+1} \oplus A_i \to B_{i+1}$ be the canonical projections.

For any integer number i with $1 < i \leqslant n$, let $\Delta_i : B_i \oplus A_{i-1} \to B_{i-1} \oplus A_{i-2}$ be the morphism given by $\Delta_i = (-\kappa_{A_{i-2}}\delta_{i-1} + \kappa_{B_{i-1}} f_{i-1})\pi_{A_{i-1}} + \kappa_{B_{i-1}} d_i \pi_{B_i}$.

Finally, let $\Delta_1 : B_1 \oplus A_0 \to B_0$ be given by $\Delta_1 = f_0\pi_{A_0} + d_1\pi_{B_1}$ and $\Delta_{n+1} : A_n \to B_n \oplus A_{n-1}$ be given by $\Delta_{n+1} = -\kappa_{A_{n-1}}\delta_n + \kappa_{B_n} f_n$.

The following statements hold.

1. *If f_n is an isomorphism, then the sequence*

$$0 \to A_{n-1} \xrightarrow{\varphi} B_{n-1} \oplus A_{n-2} \xrightarrow{\Delta_{n-1}} \cdots \to B_1 \oplus A_0 \xrightarrow{\Delta_1} B_0 \to 0$$

   is exact, where $\varphi = -k_{A_{n-2}}\delta_{n-1} + k_{B_{n-1}} f_{n-1}$.
2. *If f_0 is an isomorphism, then the sequence*

$$0 \to A_n \xrightarrow{\Delta_{n+1}} B_n \oplus A_{n-1} \xrightarrow{\Delta_n} \cdots \to B_2 \oplus A_1 \xrightarrow{\rho} B_1 \to 0$$

   is exact, where $\rho = f_1\pi_{A_1} + d_2\pi_{B_2}$.

We now turn our attention to $\mathrm{Ext}_{\mathscr{A}}$ functors. The following result is an extension of the classical shift property of $\mathrm{Ext}_{\mathscr{A}}$ functors.

Lemma 6. *If $\mathscr{Y} \perp \mathscr{X}$ and*

$$0 \to A \to X_{n-1} \to \cdots \to X_1 \to X_0 \to B \to 0$$

is an exact sequence with each $X_i \in \mathscr{X}$, then $\mathrm{Ext}^k_{\mathscr{A}}(Y, B) \cong \mathrm{Ext}^{k+n}_{\mathscr{A}}(Y, A)$ for every $Y \in \mathscr{Y}$.

Dually, we have the following.

Lemma 7. *If $\mathscr{X} \perp \mathscr{Y}$ and*

$$0 \to A \to X_{n-1} \to \cdots \to X_1 \to X_0 \to B \to 0$$

is an exact sequence with each $X_i \in \mathscr{X}$, then $\mathrm{Ext}^k_{\mathscr{A}}(A, Y) \cong \mathrm{Ext}^{k+n}_{\mathscr{A}}(B, Y)$ for every $Y \in \mathscr{Y}$.

We will finish this section by giving a version of the Comparison Lemma that will later give consistency to the definition of dimensions.

Theorem 1. *Suppose that $\mathscr{X}$ is self-orthogonal and closed under finite direct sums and kernels of epimorphisms. If*

$$0 \longrightarrow A_n \xrightarrow{f_n} X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} M \longrightarrow 0$$

and

$$\begin{tikzcd} 0 \arrow[r] & B_n \arrow[r, "g_n"] & X'_{n-1} \arrow[r] & \cdots \arrow[r] & X'_1 \arrow[r, "g_1"] & X'_0 \arrow[r, "g_0"] & M \arrow[r] & 0 \end{tikzcd}$$

are any two exact sequences with $X_i, X'_i \in \mathscr{X}$ *for every* $i \in \{0, 1, \ldots, n-1\}$, *then*

$$A_n \in \mathscr{X} \Leftrightarrow B_n \in \mathscr{X}.$$

Proof. Call $A_i = \operatorname{Ker} f_{i-1}$ and $B_i = \operatorname{Ker} g_{i-1}$ for every $i \in \{1, 2, \ldots, n-1\}$.

If $A_n \in \mathscr{X}$, then by Lemma 6 we have

$$\operatorname{Ext}^k_{\mathscr{A}}(X, A_i) \cong \operatorname{Ext}^{k+n-i}_{\mathscr{A}}(X, A_n) = 0 \; \forall X \in \mathscr{X}, \; \forall k > 0, \; \forall i \in \{1, 2, \ldots, n-1\}.$$

Then, the exact sequence

$$0 \to A_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to M \to 0$$

is $\operatorname{Hom}_{\mathscr{A}}(\mathscr{X}, -)$-exact. Therefore, there exists $h_0 : X'_0 \to X_0$ such that $g_0 = f_0 h_0$, that is, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
\cdots \arrow[r] & X'_1 \arrow[rr, "g_1"] \arrow[dr, "v'_1"] & & X'_0 \arrow[r, "g_0"] \arrow[dd, "h_0"] & M \arrow[r] \arrow[dd, equal] & 0 \\
B_2 \arrow[ur, "u'_2"] & & B_1 \arrow[ur, "u'_1"] \arrow[dd, dashed, "\lambda_1"] & & & \\
\cdots \arrow[r] & X_1 \arrow[rr, "f_1"] \arrow[dr, "v_1"] & & X_0 \arrow[r, "f_0"] & M \arrow[r] & 0 \\
A_2 \arrow[ur, "u_2"] & & A_1 \arrow[ur, "u_1"] & & &
\end{tikzcd}$$

Now, since $0 = g_0 u'_1 = f_0 h_0 u'_1$, there is a unique $\lambda_1 : B_1 \to A_1$ such that $u_1 \lambda_1 = h_0 u'_1$. Therefore, since $\operatorname{Ext}^1_{\mathscr{A}}(X'_1, A_2) = 0$, there exists $h_1 : X'_1 \to X_1$ such that $v_1 h_1 = \lambda_1 v'_1$ and then $f_1 h_1 = h_0 g_1$.

We can repeat this argument until we reach the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & B_n \arrow[r] \arrow[d] & X'_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & X'_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A_n \arrow[r] & X_{n-1} \arrow[r] & \cdots \arrow[r] & X_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Applying then Lemma 5 we obtain the exact sequence

$$0 \to B_n \to A_n \oplus X'_{n-1} \to X_{n-1} \oplus X'_{n-2} \to \cdots \to X_1 \oplus X'_0 \to X_0 \to 0,$$

in which all objects are in $\mathscr{X}$ except, perhaps, B_n. But $\mathscr{X}$ is closed under kernels of epimorphisms so indeed we have that $B_n \in \mathscr{X}$.

The converse holds using the same arguments. □

The dual result says the following.

Theorem 2. *Suppose that* $\mathscr{X}$ *is self-orthogonal and closed under finite direct sums and cokernels of monomorphisms, and that* $0 \to M \to X_0 \to X_1 \to \cdots \to X_{n-1} \to A_n \to 0$ *and* $0 \to M \to X'_0 \to X'_1 \to \cdots \to X'_{n-1} \to B_n \to 0$ *are any two exact sequences with* $X_i, X'_i \in \mathscr{X}$ *for every* $i \in \{0, 1, \ldots, n-1\}$. *Then,*

$$A_n \in \mathscr{X} \Leftrightarrow B_n \in \mathscr{X}.$$

3. $(\mathscr{X},\mathscr{Y})$-Gorenstein Subcategories

In this section we will study $(\mathscr{X},\mathscr{Y})$-Gorenstein objects as a natural generalization of Gorenstein projective and Gorenstein injective objects.

Even in module categories, where the existence of enough projectives is always guaranteed, the study of objects that generalize Gorenstein projective modules (generalizations with respect to a semidualizing or w-tilting module as indicated in the Introduction) has proven to be of extraordinary interest and to have a more than remarkable impact. For example, the dimensions provided by these generalizations are a refinement of those provided by the classical Gorenstein classes. Thus, an extension of this theory to contexts where projectives may not be enough to generate, acquires even greater importance. But such a treatment requires laying the groundwork and establishing a solid terminological and result base framework.

We will investigate the structure of $(\mathscr{X},\mathscr{Y})$-Gorenstein objects and introduce the concepts of G-compatible and G-perfect pairs $(\mathscr{X},\mathscr{Y})$ for which the subcategory of all $(\mathscr{X},\mathscr{Y})$-Gorenstein objects verifies the usual desired properties of classes used to compute (co)homology.

We start by recalling the definition of $(\mathscr{X},\mathscr{Y})$-Gorenstein objects.

Definition 1 ([8] Definition 3.1)**.** *An object M is called $(\mathscr{X},\mathscr{Y})$-Gorenstein if there exists a $\operatorname{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact, $\operatorname{Hom}_{\mathscr{A}}(-,\mathscr{Y})$-exact, and exact sequence*

$$\cdots \to X_1 \to X_0 \to Y^0 \to Y^1 \to \cdots$$

with $X_i \in \mathscr{X}$, $Y^i \in \mathscr{Y}$ for all $i \geq 0$ and $M = \operatorname{Im}(X_0 \to Y^0)$.

An exact sequence of this type is called a complete $(\mathscr{X},\mathscr{Y})$-resolution of M.

The full subcategory of $\mathscr{A}$ consisting of all $(\mathscr{X},\mathscr{Y})$-Gorenstein objects will be denoted by $\mathscr{G}(\mathscr{X},\mathscr{Y})$. Set $\mathscr{G}(\mathscr{X}) = \mathscr{G}(\mathscr{X},\mathscr{X})$.

Since the subcategories $\mathscr{X}$ and $\mathscr{Y}$ contain the zero object, then $\mathscr{G}(\mathscr{X},\mathscr{Y})$ contains the zero object. Furthermore, the subcategory $\mathscr{G}(\mathscr{X},\mathscr{Y})$ is closed under isomorphisms.

From the definition, by abusing the language, we have:

1. $\mathscr{X} \cap \mathscr{Y} \subseteq \mathscr{G}(\mathscr{X},\mathscr{Y})$.
2. If $\mathscr{X} \perp \mathscr{Y}$, then $(\widehat{\operatorname{res}}(\mathscr{X}) \cap \mathscr{Y} \cap {}^{\perp}\mathscr{Y}) \subseteq \mathscr{G}(\mathscr{X},\mathscr{Y})$.
3. If $\mathscr{X} \perp \mathscr{Y}$, then $(\mathscr{X} \cap \mathscr{X}^{\perp} \cap \widehat{\operatorname{cores}}(\mathscr{Y})) \subseteq \mathscr{G}(\mathscr{X},\mathscr{Y})$.

This subcategory generalizes many well-known categories:

1. $\mathscr{G}(\mathcal{P}roj(\mathscr{A}),\mathcal{P}roj(\mathscr{A})) = G\mathcal{P}roj(\mathscr{A})$ is the subcategory of Gorenstein projective objects and $\mathscr{G}(\mathcal{I}nj(\mathscr{A}),\mathcal{I}nj(\mathscr{A})) = G\mathcal{I}nj(\mathscr{A})$ is the subcategory of Gorenstein injective objects, studied, for example, in [1].
2. In the category R-Mod, we have that the category of G_C-projective modules, studied for example in [4], is $\mathscr{G}(\mathcal{P}roj(R),\operatorname{Add}_R(C))$.

The following characterization of $(\mathscr{X},\mathscr{Y})$-Gorenstein objects can be observed by employing conventional arguments.

Proposition 1. *Let $\mathscr{X}$ and $\mathscr{Y}$ be such that $\mathscr{X} \perp \mathscr{Y}$. Then, M is $(\mathscr{X},\mathscr{Y})$-Gorenstein if and only if M admits an exact $\mathscr{X}$-resolution, an exact $\mathscr{Y}$-coresolution and $M \in \mathscr{X}^{\perp} \cap {}^{\perp}\mathscr{Y}$.*

In the following two propositions we investigate when the category of all $(\mathscr{X},\mathscr{Y})$-Gorenstein objects is closed under kernels of epimorphisms and cokernels of monomorphisms.

Proposition 2. *Let $\mathscr{X}$ and $\mathscr{Y}$ be such that $\mathscr{X} \perp \mathscr{X}$ and $\mathscr{X} \perp \mathscr{Y}$. If $\mathscr{X}$ and $\mathscr{Y}$ are closed under finite direct sums and $\mathscr{X}$ is closed under kernels of epimorphisms, then $\mathscr{G}(\mathscr{X},\mathscr{Y})$ is closed under kernels of epimorphisms.*

Proof. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence in $\mathscr{A}$ where $M, M'' \in \mathscr{G}(\mathscr{X}, \mathscr{Y})$. Then, $M, M'' \in \mathscr{X}^{\perp} \cap {}^{\perp}\mathscr{Y}$ (see Proposition 1). Given $Y \in \mathscr{Y}$, from the long exact homology sequence, it follows that $\mathrm{Ext}^{\geqslant 1}_{\mathscr{A}}(M', Y) = 0$. Thus, $M' \in {}^{\perp}\mathscr{Y}$.

Let us prove that M' admits an exact $\mathscr{Y}$-coresolution. For example, let

$$0 \to M \to Y^0 \to Y^1 \to \cdots$$

be an exact $\mathscr{Y}$-coresolution of M and call $C^0 = \mathrm{Im}(Y^0 \to Y^1)$. Then, by Lemma 1 we can obtain the following pushout diagram with all rows and all columns exact and $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{Y})$-exact

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M' \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & M' \arrow[r] & Y^0 \arrow[r, dashed] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\
 & & C^0 \arrow[r, equal] \arrow[d] & C^0 \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Since the sequence $0 \to M'' \to P \to C^0 \to 0$ is $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{Y})$-exact and $M'', C^0 \in \widehat{\mathrm{cores}}(\mathscr{Y})$, by the Horseshoe Lemma, we have $P \in \widehat{\mathrm{cores}}(\mathscr{Y})$. Hence, the exact $\mathscr{Y}$-coresolution of M' can be constructed by combining the short exact sequence $0 \to M' \to Y^0 \to P \to 0$ and an exact $\mathscr{Y}$-coresolution of P.

We now prove that M' admits an exact $\mathscr{X}$-resolution.

Since $M'' \in \mathscr{G}(\mathscr{X}, \mathscr{Y})$, we can obtain a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X}, -)$-exact exact sequence

$$0 \to K''_0 \to X''_0 \to M'' \to 0$$

where $X''_0 \in \mathscr{X}$ and $K''_0 \in \mathscr{X}^{\perp} \cap \widehat{\mathrm{res}}(\mathscr{X})$. Then, we consider the following pullback diagram with all rows and columns exact

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K''_0 \arrow[r, equal] \arrow[d] & K''_0 \arrow[d] & \\
0 \arrow[r] & M' \arrow[r] \arrow[d, equal] & P \arrow[r, dashed] \arrow[d, dashed] & X''_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

From the middle vertical sequence and Horseshoe Lemma, we obtain $P \in \widehat{\mathrm{res}}(\mathscr{X}) \cap \mathscr{X}^{\perp}$. Then, there is an exact sequence $0 \to K_0' \to X_0' \to P \to 0$ where $K_0' \in \mathscr{X}^{\perp} \cap \widehat{\mathrm{res}}(\mathscr{X})$. We can obtain the pullback diagram with all rows and columns exact

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K_0' \arrow[r, equal] \arrow[d] & K_0' \arrow[d] & & \\
0 \arrow[r] & P' \arrow[r, dashed] \arrow[d, dashed] & X_0' \arrow[r] \arrow[d] & X_0'' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M' \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & X_0'' \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Since $\mathscr{X}$ is closed under kernels of epimorphisms, $P' \in \mathscr{X}$ and then the exact $\mathscr{X}$-resolution of M' can be constructed by combining the short exact sequence $0 \to K_0' \to P' \to M' \to 0$ and an exact $\mathscr{X}$-resolution of K_0'.

The exactness of the left vertical sequence gives $M' \in \mathscr{X}^{\perp}$.

Hence, $M' \in \mathscr{G}(\mathscr{X}, \mathscr{Y})$ by Proposition 1. □

Dually.

Proposition 3. *Let $\mathscr{X}$ and $\mathscr{Y}$ be such that $\mathscr{Y} \perp \mathscr{Y}$ and $\mathscr{X} \perp \mathscr{Y}$. If $\mathscr{X}$ and $\mathscr{Y}$ are closed under finite direct sums and $\mathscr{Y}$ is closed under cokernels of monomorphisms, then $\mathscr{G}(\mathscr{X}, \mathscr{Y})$ is closed under cokernels of monomorphisms.*

We will study when the subcategories $\mathscr{X}$ and $\mathscr{Y}$ are "included" in $\mathscr{G}(\mathscr{X}, \mathscr{Y})$ since this fact increases the interest of this class: $\mathscr{X}$ and $\mathscr{Y}$ being included in $\mathscr{G}(\mathscr{X}, \mathscr{Y})$ means that the $\mathscr{G}(\mathscr{X}, \mathscr{Y})$-dimension gives a finer measurement than the $\mathscr{X}$-dimension and $\mathscr{Y}$-dimension, and therefore the $\mathscr{G}(\mathscr{X}, \mathscr{Y})$-dimension provides more accurate results than the other two dimensions.

Proposition 4. *Let $\mathscr{X}$ and $\mathscr{Y}$ be such that $\mathscr{X} \perp \mathscr{Y}$. Then, $\mathscr{X} \subseteq \mathscr{G}(\mathscr{X}, \mathscr{Y})$ if and only if $\mathscr{X} \subseteq \widehat{\mathrm{cores}}(\mathscr{Y})$ and $\mathscr{X} \perp \mathscr{X}$.*

Proof. Assume $\mathscr{X} \subseteq \widehat{\mathrm{cores}}(\mathscr{Y})$ and $\mathscr{X} \perp \mathscr{X}$. Given $X \in \mathscr{X}$, it is clear that $X \in \mathscr{X}^{\perp} \cap {}^{\perp}\mathscr{Y}$ and $X \in \widehat{\mathrm{res}}(\mathscr{X})$, and by hypothesis $X \in \widehat{\mathrm{cores}}(\mathscr{Y})$, then $X \in \mathscr{G}(\mathscr{X}, \mathscr{Y})$ by Proposition 1.

Conversely, it is sufficient to note that $\mathscr{G}(\mathscr{X}, \mathscr{Y}) \subseteq \widehat{\mathrm{cores}}(\mathscr{Y}) \cap \mathscr{X}^{\perp}$ (see Proposition 1). □

Dually, we have the following.

Proposition 5. *Let $\mathscr{X}$ and $\mathscr{Y}$ be such that $\mathscr{X} \perp \mathscr{Y}$. Then, $\mathscr{Y} \subseteq \mathscr{G}(\mathscr{X}, \mathscr{Y})$ if and only if $\mathscr{Y} \subseteq \widehat{\mathrm{res}}(\mathscr{X})$ and $\mathscr{Y} \perp \mathscr{Y}$.*

Building upon Propositions 4 and 5, we now introduce the concepts of G-compatibility.

Definition 2. *We say that the pair $(\mathscr{X}, \mathscr{Y})$ is left G-compatible when $\mathscr{X}$ and $\mathscr{Y}$ are closed under finite direct sums, $\mathscr{X} \perp \mathscr{Y}$, $\mathscr{X} \perp \mathscr{X}$ and $\mathscr{X} \subseteq \widehat{\mathrm{cores}}(\mathscr{Y})$.*

Dually, we say that the pair $(\mathscr{X}, \mathscr{Y})$ is right G-compatible if $\mathscr{X}$ and $\mathscr{Y}$ are closed under finite direct sums, $\mathscr{X} \perp \mathscr{Y}$, $\mathscr{Y} \perp \mathscr{Y}$ and $\mathscr{Y} \subseteq \widehat{\mathrm{res}}(\mathscr{X})$.

And we say that the pair $(\mathscr{X}, \mathscr{Y})$ is G-compatible if it is both left and right G-compatible.

The following result shows that if $(\mathscr{X}, \mathscr{Y})$ is left G-compatible, then every kernel of any exact $\mathscr{X}$-resolution of a $(\mathscr{X}, \mathscr{Y})$-Gorenstein object is $(\mathscr{X}, \mathscr{Y})$-Gorenstein.

Proposition 6. *Let $\mathscr{X}$ and $\mathscr{Y}$ be such that $(\mathscr{X}, \mathscr{Y})$ is left G-compatible. Then, every kernel of any exact $\mathscr{X}$-resolution of a $(\mathscr{X}, \mathscr{Y})$-Gorenstein object is $(\mathscr{X}, \mathscr{Y})$-Gorenstein.*

Proof. Let M be a $(\mathscr{X}, \mathscr{Y})$-Gorenstein object and $\cdots \to X_1 \to X_0 \to M \to 0$ be an exact $\mathscr{X}$-resolution of M. If we call $K_0 = \mathrm{Ker}(X_0 \to M)$, then we have the exact, $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X}, -)$-exact, and $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{Y})$-exact sequence $0 \to K_0 \to X_0 \to M \to 0$. Given $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$, from the long exact sequences of homology associated with the functors $\mathrm{Hom}_{\mathscr{A}}(X, -)$ and $\mathrm{Hom}_{\mathscr{A}}(-, Y)$, it follows that $K_0 \in \mathscr{X}^{\perp} \cap {}^{\perp}\mathscr{Y}$.

Let us prove that K_0 admits an exact $\mathscr{Y}$-coresolution.

Since $\mathscr{X} \subseteq \widehat{\mathrm{cores}}(\mathscr{Y})$, there is $0 \to X_0 \to Y_1 \to C_1 \to 0$ where $Y_1 \in \mathscr{Y}$ and $C_1 \in \widehat{\mathrm{cores}}(\mathscr{Y})$ and we consider the following pushout diagram, given by Lemma 1, with all rows and columns exact and $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{Y})$-exact

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_0 \arrow[r] \arrow[d, equal] & X_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & K_0 \arrow[r] & Y_1 \arrow[r, dashed] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\
 & & C_1 \arrow[r, equal] \arrow[d] & C_1 \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

By the right vertical sequence and Horseshoe Lemma, we have $P \in \widehat{\mathrm{cores}}(\mathscr{Y})$. Then, the exact $\mathscr{Y}$-coresolution of K_0 can be constructed by combining the short exact sequence and an exact $\mathscr{Y}$-coresolution of P. Thus, K_0 is $(\mathscr{X}, \mathscr{Y})$-Gorenstein by Proposition 1. $\square$

Dually.

Proposition 7. *Let $\mathscr{X}$ and $\mathscr{Y}$ be such that the pair $(\mathscr{X}, \mathscr{Y})$ is right G-compatible. Then, every cokernel of any exact $\mathscr{Y}$-coresolution of a $(\mathscr{X}, \mathscr{Y})$-Gorenstein object is $(\mathscr{X}, \mathscr{Y})$-Gorenstein.*

In the next result we prove that when the pair $(\mathscr{X}, \mathscr{Y})$ is G-compatible, the iterative construction process of $(\mathscr{X}, \mathscr{Y})$-Gorenstein objects reaches stability in the second iteration. This means that by constructing Gorenstein objects relative to the classes $\mathscr{X}$ and $\mathscr{Y}$, further construction of Gorenstein objects relative to $\mathscr{G}(\mathscr{X}, \mathscr{Y})$ does not yield any new entities.

Theorem 3. *If $(\mathscr{X}, \mathscr{Y})$ is G-compatible, then $\mathscr{G}(\mathscr{G}(\mathscr{X}, \mathscr{Y})) = \mathscr{G}(\mathscr{X}, \mathscr{Y})$.*

Proof. To see that $\mathscr{G}(\mathscr{X}, \mathscr{Y}) \subseteq \mathscr{G}(\mathscr{G}(\mathscr{X}, \mathscr{Y}))$, take any $M \in \mathscr{G}(\mathscr{X}, \mathscr{Y})$ and consider the disc complex

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{\mathrm{id}_M} M \longrightarrow 0 \longrightarrow \cdots$$

Conversely, given any $M \in \mathscr{G}(\mathscr{G}(\mathscr{X},\mathscr{Y}))$, there is a $\operatorname{Hom}_{\mathscr{A}}(\mathscr{G}(\mathscr{X},\mathscr{Y}),-)$-exact, $\operatorname{Hom}_{\mathscr{A}}(-,\mathscr{G}(\mathscr{X},\mathscr{Y}))$-exact, and exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & G_1 \arrow[rr] \arrow[dr] & & G_0 \arrow[rr] \arrow[dr] & & G^0 \arrow[rr] \arrow[dr] & & G^1 \arrow[r] & \cdots \\
& & M_1 \arrow[ur] & & M \arrow[ur] & & M^1 \arrow[ur] & &
\end{tikzcd} \tag{1}$$

where $G_i, G^i \in \mathscr{G}(\mathscr{X},\mathscr{Y})$, $M_i = \operatorname{Im}(G_i \to G_{i-1})$, $M^i = \operatorname{Im}(G^{i-1} \to G^i)$ and $M = \operatorname{Im}(G_0 \to G^0)$.

Since $\mathscr{X} \subseteq \mathscr{G}(\mathscr{X},\mathscr{Y})$ and the sequence (1) is $\operatorname{Hom}_{\mathscr{A}}(\mathscr{G}(\mathscr{X},\mathscr{Y}),-)$-exact, the sequence (1) is $\operatorname{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact. With the same reasoning, the sequence (1) is also $\operatorname{Hom}_{\mathscr{A}}(-,\mathscr{Y})$-exact. Then, $M \in \mathscr{X}^{\perp} \cap {}^{\perp}\mathscr{Y}$.

Let us prove that $M \in \widehat{\operatorname{res}}(\mathscr{X})$ (dually $M \in \widehat{\operatorname{cores}}(\mathscr{Y})$). Since $G_0 \in \mathscr{G}(\mathscr{X},\mathscr{Y})$, there is an exact sequence $0 \to K_0 \to X_0 \to G_0 \to 0$ where $X_0 \in \mathscr{X}$ and $K_0 \in \mathscr{G}(\mathscr{X},\mathscr{Y})$ (by Proposition 6). Then, by Lemma 3 we can obtain the pullback diagram with all rows and all columns being exact and $\operatorname{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & & \\
& K_0 \arrow[r, equal] \arrow[d] & K_0 \arrow[d] & & \\
0 \arrow[r] & P_0 \arrow[r, dashed] \arrow[d, dashed] & X_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
& 0 & 0 & &
\end{tikzcd}$$

With $0 \to K_0 \to P_0 \to M_1 \to 0$ and $0 \to M_2 \to G_1 \to M_1 \to 0$, we obtain, by Lemma 4, the pullback diagram with all rows and all columns exact and $\operatorname{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & \\
& & M_2 \arrow[r, equal] \arrow[d] & M_2 \arrow[d] & \\
0 \arrow[r] & K_0 \arrow[r] \arrow[d, equal] & D_0 \arrow[r, dashed] \arrow[d, dashed] & G_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_0 \arrow[r] & P_0 \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & 0 \\
& & 0 & 0 &
\end{tikzcd}$$

Since $G_1, K_0 \in \mathscr{G}(\mathscr{X},\mathscr{Y})$, by [8] (Corollary 3.8), we have $D_0 \in \mathscr{G}(\mathscr{X},\mathscr{Y})$, then there is a $\operatorname{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact exact sequence $0 \to K_1 \to X_1 \to D_0 \to 0$ where $X_1 \in \mathscr{X}$ and $K_1 \in \mathscr{G}(\mathscr{X},\mathscr{Y})$. Then, by Lemma 3 we obtain the pullback diagram with all rows and all columns exact and $\operatorname{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K_1 \arrow[r, equal] \arrow[d] & K_1 \arrow[d] & & \\
0 \arrow[r] & P_1 \arrow[r, dashed] \arrow[d, dashed] & X_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M_2 \arrow[r] \arrow[d] & D_0 \arrow[r] \arrow[d] & P_0 \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Hence, combining the $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact exact sequences $0 \to P_1 \to X_1 \to P_0 \to 0$ and $0 \to P_0 \to X_0 \to M \to 0$, we obtain the $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact exact sequence $0 \to P_1 \to X_1 \to X_0 \to M \to 0$. If we continue this process indefinitely, we obtain an exact $\mathscr{X}$-resolution of M. $\square$

It is worth noting that in [8] (Theorem 4.2), it was proved that the class $\mathscr{G}(\mathscr{X},\mathscr{Y})$ coincides with the class of all objects $M \in \mathscr{A}$ such that there exists an $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact, $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{Y})$-exact, and exact sequence $\cdots \to A_1 \to A_0 \to A^0 \to A^1 \to \cdots$ in $\mathscr{A}$ with all $A_i, A^j \in \mathscr{G}(\mathscr{X},\mathscr{Y})$ and $M = \mathrm{Im}(A_0 \to A^0)$. This establishes another perspective on the stability of $\mathscr{G}(\mathscr{X},\mathscr{Y})$ objects after iterations, extending the concept introduced in [4] (Theorem 2.9), while Proposition 3 generalizes the context elucidated in [5] (Corollary 4.10) by providing a broader framework for examining the relationship between $\mathscr{G}(\mathscr{X},\mathscr{Y})$ and $\mathscr{G}(\mathscr{G}(\mathscr{X},\mathscr{Y}))$.

To conclude this section, we introduce the concept of G-perfect pairs and present a study in Theorem 4 regarding the $(\mathscr{X},\mathscr{Y})$-Gorenstein nature of kernels in a $(\mathscr{X},\mathscr{Y})$-Gorenstein resolution. This finding carries substantial importance in the computation of $(\mathscr{X},\mathscr{Y})$-Gorenstein dimensions and sets the stage for further exploration in the upcoming sections.

Definition 3. *We say that the pair $(\mathscr{X},\mathscr{Y})$ is left G-perfect if $(\mathscr{X},\mathscr{Y})$ is G-compatible and $\mathscr{X}$ is closed under kernels of epimorphisms.*

Dually, we say that the pair $(\mathscr{X},\mathscr{Y})$ is right G-perfect if $(\mathscr{X},\mathscr{Y})$ is G-compatible and $\mathscr{Y}$ is closed under cokernels of monomorphisms.

And we say that the pair $(\mathscr{X},\mathscr{Y})$ is G-perfect if it is both left and right G-perfect.

The following lemma will be useful in the proof of Theorem 4.

Lemma 8. *Let $\mathscr{X}$ and $\mathscr{Y}$ be such that the pair $(\mathscr{X},\mathscr{Y})$ is left G-compatible. Given an exact sequence*

$$0 \to A_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

with all $G_i \in \mathscr{G}(\mathscr{X},\mathscr{Y})$, there exist an exact sequence

$$0 \to K_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to M \to 0$$

with all $X_i \in \mathscr{X}$, and a morphism of complexes

$$\begin{tikzcd}
0 \arrow[r] & K_n \arrow[r] \arrow[d] & X_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & X_1 \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A_n \arrow[r] & G_{n-1} \arrow[r] & \cdots \arrow[r] & G_1 \arrow[r] & G_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Proof. We call $A_i = \mathrm{Im}(G_i \to G_{i-1})$ for all $i \in \{1, 2, \cdots, n-1\}$ and consider the exact sequence $0 \to A_1 \to G_0 \to M \to 0$. Since $G_0 \in \mathscr{G}(\mathscr{X}, \mathscr{Y})$, there is an exact sequence $0 \to G_0' \to X_0 \to G_0 \to 0$ where $X_0 \in \mathscr{X}$ and $G_0' \in \mathscr{G}(\mathscr{X}, \mathscr{Y})$ (see Proposition 6). Then, we can obtain the pullback diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & G_0' \arrow[r, equal] \arrow[d] & G_0' \arrow[d] & & \\
0 \arrow[r] & P_1 \arrow[r, dashed] \arrow[d, dashed] & X_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

and hence, we have the following commutative diagram with exact rows

$$\begin{tikzcd}
 & & & & X_0 \arrow[r] \arrow[ddd] & M \arrow[r] \arrow[ddd, equal] & 0 \\
 & & & P_1 \arrow[ur] \arrow[ddd] & & & \\
 & & 0 \arrow[ur] & & & & \\
\cdots \arrow[r] & G_1 \arrow[rrr] \arrow[drr] & & & G_0 \arrow[r] & M \arrow[r] & 0 \\
 & & & A_1 \arrow[ur] \arrow[dr] & & & \\
 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Now, we consider the pullback diagram with exact rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & G_0' \arrow[r, equal] \arrow[d] & G_0' \arrow[d] & \\
0 \arrow[r] & A_2 \arrow[r] \arrow[d, equal] & D_1 \arrow[r, dashed] \arrow[d, dashed] & P_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_2 \arrow[r] & G_1 \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since $\mathscr{G}(\mathscr{X},\mathscr{Y})$ is closed under extensions (see [8] [Corollary 3.8]), $D_1 \in \mathscr{G}(\mathscr{X},\mathscr{Y})$ and then there is an exact sequence $0 \to G_1' \to X_1 \to D_1 \to 0$ where $X_1 \in \mathscr{X}$ and $G_1' \in \mathscr{G}(\mathscr{X},\mathscr{Y})$. Then, we can obtain the pullback diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & G_1' \arrow[r, equal] \arrow[d] & G_1' \arrow[d] & & \\
0 \arrow[r] & P_2 \arrow[r, dashed] \arrow[d, dashed] & X_1 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A_2 \arrow[r] \arrow[d] & D_1 \arrow[r] \arrow[d] & P_1 \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Thus, we have the following commutative diagram:

$$\begin{tikzcd}
P_2 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & P_1 \arrow[d, equal] \\
A_2 \arrow[r] \arrow[d, equal] & D_1 \arrow[r] \arrow[d] & P_1 \arrow[d] \\
A_2 \arrow[r] & G_1 \arrow[r] & A_1
\end{tikzcd}$$

From which we construct the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & & & & X_1 \arrow[r] \arrow[ddd] & X_0 \arrow[r] \arrow[ddd] & M \arrow[r] \arrow[ddd, equal] & 0 \\
 & & & P_2 \arrow[ur] \arrow[dd] & & & & \\
 & & 0 \arrow[ur] & & & & & \\
\cdots \arrow[r] & G_2 \arrow[rrr] \arrow[drr] & & & G_1 \arrow[r] & G_0 \arrow[r] & M \arrow[r] & 0 \\
 & & & A_2 \arrow[ur] \arrow[dr] & & & & \\
 & & 0 \arrow[ur] & & 0 & & &
\end{tikzcd}$$

If we continue this process until step n, we obtain the desired exact sequence together with the morphism of complexes. □

Dually, we have the following.

Lemma 9. *Let $\mathscr{X}$ and $\mathscr{Y}$ be such that the pair $(\mathscr{X},\mathscr{Y})$ is right G-compatible. Given an exact sequence*

$$0 \to M \to G_0 \to G_1 \to \cdots \to G_{n-1} \to A_n \to 0$$

with all $G_i \in \mathscr{G}(\mathscr{X},\mathscr{Y})$, there exist an exact sequence

$$0 \to M \to Y_0 \to Y_1 \to \cdots \to Y_{n-1} \to K_n \to 0$$

with all $Y_i \in \mathscr{Y}$, and a morphism of complexes

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & G_0 \arrow[r] \arrow[d] & G_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & G_{n-1} \arrow[r] \arrow[d] & A_n \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M \arrow[r] & Y_0 \arrow[r] & Y_1 \arrow[r] & \cdots \arrow[r] & Y_{n-1} \arrow[r] & K_n \arrow[r] & 0 \end{tikzcd}$$

Theorem 4. *Let $\mathscr{X}$ and $\mathscr{Y}$ be such that the pair $(\mathscr{X},\mathscr{Y})$ is left G-perfect. Given the exact sequences*

$$0 \to A_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$
$$0 \to B_n \to H_{n-1} \to \cdots \to H_1 \to H_0 \to M \to 0$$

with all $G_i, H_i \in \mathscr{G}(\mathscr{X},\mathscr{Y})$, then

$$A_n \in \mathscr{G}(\mathscr{X},\mathscr{Y}) \;\Leftrightarrow\; B_n \in \mathscr{G}(\mathscr{X},\mathscr{Y}).$$

Proof. Assume $A_n \in \mathscr{G}(\mathscr{X},\mathscr{Y}) \subseteq \mathscr{X}^{\perp}$. Since $G_i \in \mathscr{G}(\mathscr{X},\mathscr{Y}) \subseteq \mathscr{X}^{\perp}$, if we call $A_i = \mathrm{Im}(G_i \to G_{i-1})$ then $\mathrm{Ext}^1_{\mathscr{A}}(X, A_i) \cong \mathrm{Ext}^{1+n-i}_{\mathscr{A}}(X, A_n) = 0$ for all $X \in \mathscr{X}$. Then, the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A_n \arrow[r, "d_n"] & G_{n-1} \arrow[r] & \cdots \arrow[r] & G_1 \arrow[r, "d_1"] & G_0 \arrow[r, "d_0"] & M \arrow[r] & 0 \end{tikzcd} \tag{2}$$

is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact.

On the other hand, applying Lemma 8 there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K_n \arrow[r, "f_n"] \arrow[d] & X_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & X_1 \arrow[r, "f_1"] \arrow[d] & X_0 \arrow[r, "f_0"] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B_n \arrow[r] & H_{n-1} \arrow[r] & \cdots \arrow[r] & H_1 \arrow[r] & H_0 \arrow[r] & M \arrow[r] & 0 \end{tikzcd} \tag{3}$$

where $X_i \in \mathscr{X}$ for all $i = 0, 1, \ldots, n-1$.

Since the sequence (2) is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact, we can construct the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K_n \arrow[r, "f_n"] \arrow[d, "h_n"] & X_{n-1} \arrow[r] \arrow[d, "h_{n-1}"] & \cdots \arrow[r] & X_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & X_0 \arrow[r, "f_0"] \arrow[d, "h_0"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A_n \arrow[r, "d_n"] & G_{n-1} \arrow[r] & \cdots \arrow[r] & G_1 \arrow[r, "d_1"] & G_0 \arrow[r, "d_0"] & M \arrow[r] & 0 \end{tikzcd}$$

whose mapping cone,

$$0 \to K_n \to A_n \oplus X_{n-1} \to \cdots \to G_1 \oplus X_0 \to G_0 \to 0,$$

is exact. Since $\mathscr{X} \subseteq \mathscr{G}(\mathscr{X},\mathscr{Y})$ and $\mathscr{G}(\mathscr{X},\mathscr{Y})$ is closed under kernels of epimorphisms (see Proposition 2), we have $K_n \in \mathscr{G}(\mathscr{X},\mathscr{Y})$.

The mapping cone of the morphism of complexes (3) is also exact

$$0 \to K_n \to B_n \oplus X_{n-1} \to \cdots \to H_1 \oplus X_0 \to H_0 \to 0.$$

In this sequence, all objects are $(\mathscr{X},\mathscr{Y})$-Gorenstein except, perhaps, $B_n \oplus X_{n-1}$. Since $\mathscr{G}(\mathscr{X},\mathscr{Y})$ is closed under kernels of epimorphisms, we have an exact sequence

$$0 \to K_n \to B_n \oplus X_{n-1} \to G \to 0$$

with $G \in \mathscr{G}(\mathscr{X},\mathscr{Y})$. Since $\mathscr{G}(\mathscr{X},\mathscr{Y})$ is closed under extensions we obtain $B_n \oplus X_{n-1} \in \mathscr{G}(\mathscr{X},\mathscr{Y})$, so $B_n \in \mathscr{G}(\mathscr{X},\mathscr{Y})$ since $\mathscr{G}(\mathscr{X},\mathscr{Y})$ is closed under direct summands (see [8] Proposition 4.8). $\square$

Dually, we have the following.

Theorem 5. *Let $\mathscr{X}$ and $\mathscr{Y}$ be such that the pair $(\mathscr{X},\mathscr{Y})$ is right G-perfect. Given the exact sequences*

$$0 \to M \to G_0 \to G_1 \to \cdots \to G_{n-1} \to A_n \to 0$$
$$0 \to M \to H_0 \to H_1 \to \cdots \to H_{n-1} \to B_n \to 0$$

with all $G_i, H_i \in \mathscr{G}(\mathscr{X},\mathscr{Y})$, then

$$A_n \in \mathscr{G}(\mathscr{X},\mathscr{Y}) \quad \Leftrightarrow \quad B_n \in \mathscr{G}(\mathscr{X},\mathscr{Y}).$$

4. $\mathscr{G}(\mathscr{X},\mathscr{Y})$-Projective Dimension

Throughout this section we assume that the pair $(\mathscr{X},\mathscr{Y})$ is left G-perfect. Our primary focus in this section is to explore the relative homological dimensions induced by the subcategories $\mathscr{G}(\mathscr{X},\mathscr{Y})$. We will establish that the $\mathscr{G}(\mathscr{X},\mathscr{Y})$-projective dimension of an object (in the finite case) is completely determined by the minimum degree of the $\mathrm{Ext}_{\mathscr{A}}$ functor from which all higher degrees vanish. In simpler terms, it will be the smallest natural number n for which $\mathrm{Ext}^k_{\mathscr{A}} = 0$ holds for all $k \geq n+1$. Additionally, we will investigate the conditions under which the $\mathscr{X}$-projective dimension coincides with the $\mathscr{G}(\mathscr{X},\mathscr{Y})$-projective dimension.

We start by recalling the definition of the projective dimension relative to any subcategory.

Definition 4. *Let $\mathcal{T}$ be a subcategory of $\mathscr{A}$. An object M in $\mathscr{A}$ is said to have $\mathcal{T}$-projective dimension less than or equal to an integer n, $\mathcal{T}\text{-}\mathrm{pd}(M) \leqslant n$, if there is an exact sequence*

$$0 \to T_n \to T_{n-1} \to \cdots \to T_1 \to T_0 \to M \to 0$$

with $T_i \in \mathcal{T}$ for every $i \in \{0,1,\ldots,n\}$. If no such finite sequence exists, then $\mathcal{T}\text{-}\mathrm{pd}(M) = \infty$; otherwise, $\mathcal{T}\text{-}\mathrm{pd}(M) = n$ if n is the least non-negative integer for which such a sequence exists.

The $\mathcal{T}$-injective dimension of M, denoted $\mathcal{T}\text{-}\mathrm{id}(M)$, is defined dually.

Note that $\mathcal{P}roj\text{-}\mathrm{pd}$ (respectively, $\mathcal{I}nj\text{-}\mathrm{id}$) coincides with the classical projective dimension, pd (respectively, injective dimension, id).

Proposition 8. *Given M in $\mathscr{A}$, if $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\mathrm{pd}(M)$ is finite, then M admits an exact $\mathscr{X}$-resolution. Moreover, $M \in \mathscr{X}^{\perp}$.*

Proof. By definition there is an exact sequence

$$0 \to G_n \to \cdots \to G_0 \to M \to 0$$

where $G_i \in \mathscr{G}(\mathscr{X},\mathscr{Y})$ for all $i = 0,1,\ldots,n$. Applying Lemma 8 we have the exact sequence

$$0 \to K_n \to X_{n-1} \to \cdots \to X_0 \to M \to 0$$

with $X_i \in \mathscr{X}$. Then, $K_n \in \mathscr{G}(\mathscr{X},\mathscr{Y})$ by Theorem 4. Furthermore, this sequence is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact since, if we call $K_i = \mathrm{Im}(X_i \to X_{i-1})$, we have $\mathrm{Ext}^k_{\mathscr{A}}(X,K_i) \cong \mathrm{Ext}^{n-i+k}_{\mathscr{A}}(X,K_n) = 0$ for all $X \in \mathscr{X}$.

Now, combining the above sequence with an exact $\mathscr{X}$-resolution of K_n, we have an exact $\mathscr{X}$-resolution of M.

To see that $M \in \mathscr{X}^{\perp}$ just use that

$$\mathrm{Ext}^k_{\mathscr{A}}(X,M) \cong \mathrm{Ext}^{k+n}_{\mathscr{A}}(X,K_n) = 0$$

for all $k > 0$ and all $X \in \mathscr{X}$. □

Proposition 9. *The following conditions are equivalent for an object M and a non-negative integer n.*

1. $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\mathrm{pd}(M) \leqslant n$.
2. *There is an exact sequence $0 \to M \to P \to G \to 0$, where G is $(\mathscr{X},\mathscr{Y})$-Gorenstein and P admits an exact $\mathscr{Y}$-resolution of length n.*
3. *There is an exact sequence $0 \to M \to P \to G \to 0$, where G is $(\mathscr{X},\mathscr{Y})$-Gorenstein and there exists an exact sequence $0 \to Y_n \to \cdots \to Y_0 \to P \to 0$ with every $Y_i \in \mathscr{Y}$.*

Proof. For 1. ⇒ 2., assume $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\mathrm{pd}(M) \leqslant n$ and proceed by induction on n. The case $n = 0$ follows from Proposition 7. If $n \geq 1$, there exists an exact sequence

$$0 \to G_n \to \cdots \to G_0 \to M \to 0.$$

We decompose this sequence into two exact sequences

$$0 \to K \to G_0 \to M \to 0,$$

$$0 \to G_n \to \cdots \to G_1 \to K \to 0,$$

and see that $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\mathrm{pd}(K) \leqslant n-1$, so by induction, there is an exact sequence

$$0 \to K \to P' \to G' \to 0,$$

where $G' \in \mathscr{G}(\mathscr{X},\mathscr{Y})$ and P' admits an exact $\mathscr{Y}$-resolution of length $n-1$,

$$0 \to Y_{n-1} \to \cdots \to Y_0 \to P' \to 0.$$

Consider the following pushout diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d, dashed] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P' \arrow[r, dashed] \arrow[d] & D \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & G' \arrow[r, equal] \arrow[d] & G' \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

By the middle vertical sequence, $D \in \mathscr{G}(\mathscr{X},\mathscr{Y})$ ([8] Corollary 3.8). Then, there exists an exact sequence $0 \to D \to Y \to G'' \to 0$, where $Y \in \mathscr{Y}$ and $G'' \in \mathscr{G}(\mathscr{X},\mathscr{Y})$. Let us see that if we consider the pushout diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & P' \arrow[r] \arrow[d, equal] & D \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & P' \arrow[r] & Y \arrow[r, dashed] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\
 & & G'' \arrow[r, equal] \arrow[d] & G'' \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

the right vertical sequence gives the desired sequence.

We only need to show that P has an exact $\mathscr{Y}$-resolution of length n. But we have the exact $\mathscr{Y}$-resolution

$$0 \to Y_{n-1} \to \cdots \to Y_0 \to P' \to 0,$$

so the sequence

$$0 \to Y_{n-1} \to \cdots \to Y_0 \to Y \to P \to 0 \tag{4}$$

is exact, and since $\mathscr{Y}$ is self-orthogonal, we have

$$\mathrm{Ext}^1_{\mathscr{A}}(Y', P') \cong \mathrm{Ext}^n_{\mathscr{A}}(Y', Y_{n-1}) = 0 \quad \forall Y' \in \mathscr{Y},$$

so (4) is an exact $\mathscr{Y}$-resolution of P of length n.

2. $\Rightarrow$ 3. Clear.

To prove 3. $\Rightarrow$ 1. choose any exact sequence of length n,

$$0 \to Y_n \to \cdots \to Y_0 \to P \to 0$$

with all $Y_i \in \mathscr{Y}$ and call $K = \mathrm{Ker}(Y_0 \to P)$. Then, we can consider the pullback diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K \arrow[r, equal] \arrow[d] & K \arrow[d] & & \\
0 \arrow[r] & P' \arrow[r, dashed] \arrow[d, dashed] & Y_0 \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & G \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

By the middle horizontal sequence, $P' \in \mathscr{G}(\mathscr{X}, \mathscr{Y})$ (Propositions 2 and 5). Therefore, we obtain the result by combining the left vertical sequence and the exact sequence $0 \to Y_n \to \cdots \to Y_1 \to K \to 0$. $\square$

Theorem 6. *For any non-negative integer n, if $\mathscr{G}(\mathscr{X}, \mathscr{Y})$-$\mathrm{pd}(M) \leqslant n$, then there is an exact sequence $0 \to P \to G \to M \to 0$, where G is $(\mathscr{X}, \mathscr{Y})$-Gorenstein and P admits an exact $\mathscr{Y}$-resolution of length $n-1$. Moreover, $P \in \mathscr{G}(\mathscr{X}, \mathscr{Y})^{\perp}$.*

Proof. By definition, there is an exact sequence $0 \to N \to G_0 \to M \to 0$, where $G_0 \in \mathscr{G}(\mathscr{X},\mathscr{Y})$ and $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\operatorname{pd}(N) \leqslant n-1$. By Proposition 9, there is an exact sequence $0 \to N \to P \to G' \to 0$, where $G' \in \mathscr{G}(\mathscr{X},\mathscr{Y})$ and P admits an exact $\mathscr{Y}$-resolution

$$0 \to Y_{n-1} \to \cdots \to Y_0 \to P \to 0.$$

Consider the following pushout diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & N \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d, dashed] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P \arrow[r, dashed] \arrow[d] & G \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & G' \arrow[r, equal] \arrow[d] & G' \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

By the middle vertical sequence, $G \in \mathscr{G}(\mathscr{X},\mathscr{Y})$.

Finally, since $\mathscr{Y}$ is self-orthogonal, we have, for every $A \in \mathscr{G}(\mathscr{X},\mathscr{Y})$,

$$\operatorname{Ext}^k_{\mathscr{A}}(A,P) \cong \operatorname{Ext}^{k+n-1}_{\mathscr{A}}(A,Y_{n-1}) = 0,$$

which means that $P \in \mathscr{G}(\mathscr{X},\mathscr{Y})^{\perp}$. $\square$

Corollary 1. *Every object of finite $\mathscr{G}(\mathscr{X},\mathscr{Y})$-projective dimension has a special $\mathscr{G}(\mathscr{X},\mathscr{Y})$-precover.*

As a consequence of Theorem 6, we see that when $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\operatorname{pd}(M)$ is finite, it can be computed using exact $\mathscr{G}(\mathscr{X},\mathscr{Y})$-resolutions of M.

Corollary 2. *If M is any object of finite $\mathscr{G}(\mathscr{X},\mathscr{Y})$-projective dimension, then $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\operatorname{pd}(M) \leqslant n$ if and only if M has an exact left $\mathscr{G}(\mathscr{X},\mathscr{Y})$-resolution of length less than or equal to n.*

Proposition 10. *If there is a $\operatorname{Hom}_{\mathscr{A}}(-,\mathscr{Y})$-exact exact sequence $0 \to G' \to G \to M \to 0$ with $G, G' \in \mathscr{G}(\mathscr{X},\mathscr{Y})$, then $M \in \mathscr{G}(\mathscr{X},\mathscr{Y})$.*

Proof. Let Y be any object of $\mathscr{Y}$. From the long exact sequence of homology it follows that $\operatorname{Ext}^k_{\mathscr{A}}(M,Y) = 0$ for all $k > 0$.

By definition $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\operatorname{pd}(M) \leqslant 1$, so applying Theorem 6 there is an exact sequence $0 \to Y \to G'' \to M \to 0$ where $Y \in \mathscr{Y}$ and $G'' \in \mathscr{G}(\mathscr{X},\mathscr{Y})$. Since $\operatorname{Ext}^1_{\mathscr{A}}(M,Y) = 0$, this sequence splits, so M is a direct summand of G'' and therefore is $(\mathscr{X},\mathscr{Y})$-Gorenstein (see [8] Proposition 4.8). $\square$

Theorem 7. *For an object M of finite $\mathscr{G}(\mathscr{X},\mathscr{Y})$-projective dimension and an integer $n \geq 0$ the following conditions are equivalent:*

1. $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\operatorname{pd}(M) \leqslant n$.
2. $\operatorname{Ext}^i_{\mathscr{A}}(M,Y) = 0$ *for all $i > n$ and all $Y \in \mathscr{Y}$.*
3. *For every exact sequence*

$$0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0,$$

if each G_i is $(\mathscr{X},\mathscr{Y})$-Gorenstein, then so is K_n.

Proof. By Theorem 4 it is clear that 1. $\Leftrightarrow$ 3. To show 1. $\Rightarrow$ 2. we just use $\mathrm{Ext}^{n+i}_{\mathscr{A}}(M,Y) \cong \mathrm{Ext}^i_{\mathscr{A}}(G_n,Y) = 0\ \forall i > 0$, being

$$0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$$

an exact sequence in which each G_i is $(\mathscr{X},\mathscr{Y})$-Gorenstein.

2. $\Rightarrow$ 1. Since the dimension of M is finite, there is an exact sequence

$$0 \to G_m \to \cdots \to G_0 \to M \to 0$$

where $G_0,\dots,G_m \in \mathscr{G}(\mathscr{X},\mathscr{Y})$. If $m \leqslant n$ we are done so we suppose $m > n$ and consider $K_n = \mathrm{Ker}(G_{n-1} \to G_{n-2})$. Then, we have the exact sequence

$$0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$$

and therefore $\mathrm{Ext}^i_{\mathscr{A}}(K_n,Y) \cong \mathrm{Ext}^{i+n}_{\mathscr{A}}(M,Y) = 0$ for all $i > 0$ and all $Y \in \mathscr{Y}$.

On the other hand, we have that K_n is of finite dimension, so there is an exact sequence

$$0 \to G'_s \to \cdots \to G'_0 \to K_n \to 0,$$

where $G'_0,\dots,G'_s \in \mathscr{G}(\mathscr{X},\mathscr{Y})$. We decompose it into short exact sequences, $0 \to C'_j \to G'_{j-1} \to C'_{j-1} \to 0$ for $j = 1,\dots,s$, where $C'_s = G'_s$ and $C'_0 = K_n$, which are $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{Y})$-exact since

$$\mathrm{Ext}^1_{\mathscr{A}}(C'_{j-1},Y) \cong \mathrm{Ext}^j_{\mathscr{A}}(K_n,Y) = 0$$

for all $j = 1,\cdots s$ and all $Y \in \mathscr{Y}$. Thus, Proposition 10 can be applied successively to conclude that $C'_{s-1},\dots,C'_0$ are $(\mathscr{X},\mathscr{Y})$-Gorenstein. In particular, $K_n = C'_0$ is $(\mathscr{X},\mathscr{Y})$-Gorenstein. $\square$

Corollary 3. *If M is of finite $\mathscr{G}(\mathscr{X},\mathscr{Y})$-projective dimension, then*

$$\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\mathrm{pd}(M) = \sup\{i \in \mathbb{N} :\ \mathrm{Ext}^i_{\mathscr{A}}(M,Y) \neq 0 \ \textit{for some}\ \ Y \in \mathscr{Y}\}.$$

Proposition 11. *Given a short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathscr{A}$, if any two of M', M or M'' have finite $\mathscr{G}(\mathscr{X},\mathscr{Y})$-projective dimension, then so has the third. Moreover, we have the following:*

1. $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\mathrm{pd}(M') \leq \max\{\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\mathrm{pd}(M), \mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\mathrm{pd}(M'') - 1\}$.
2. $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\mathrm{pd}(M) \leq \max\{\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\mathrm{pd}(M'), \mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\mathrm{pd}(M'')\}$.
3. $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\mathrm{pd}(M'') \leq \max\{\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\mathrm{pd}(M') + 1, \mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\mathrm{pd}(M)\}$.

Proof. We already know (Proposition 8) that having finite $\mathscr{G}(\mathscr{X},\mathscr{Y})$-projective dimension implies having an exact $\mathscr{X}$-resolution. Let us first show that if any two of M, M' or M'' have finite $\mathscr{G}(\mathscr{X},\mathscr{Y})$-projective dimension, the third have an exact $\mathscr{X}$-resolution.

- If M' and M'' have finite dimension, then $M' \in \mathscr{X}^{\perp}$ by Proposition 8, so the sequence

  $$0 \to M' \to M \to M'' \to 0$$

  is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact and then, since M' and M'' both have exact $\mathscr{X}$-resolutions, M also does by the Horseshoe Lemma.
- If M' and M have finite dimension, then the sequence

  $$0 \to M' \to M \to M'' \to 0$$

is again $\mathrm{Hom}(\mathscr{X}, -)$-exact. Now, M has an exact $\mathscr{X}$-resolution, so there is an exact sequence $0 \to K_1 \to X_0 \to M \to 0$ with $X_0 \in \mathscr{X}$ and $K_1 \in \mathscr{X}^{\perp} \cap \widehat{\mathrm{res}}(\mathscr{X})$. Then, all rows and columns of the pullback diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K_1 \arrow[r, equal] \arrow[d] & K_1 \arrow[d] & & \\
0 \arrow[r] & P \arrow[r, dashed] \arrow[d, dashed] & X_0 \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M'' \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

are $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X}, -)$-exact and $P \in \mathscr{X}^{\perp}$ since $K_1, M' \in \mathscr{X}^{\perp}$. Applying the Horseshoe Lemma, we obtain an exact $\mathscr{X}$-resolution of P, and therefore M'' has an exact $\mathscr{X}$ resolution too.

- If M and M'' have finite dimension, then $M, M'' \in \mathscr{X}^{\perp}$ and both admit exact $\mathscr{X}$-resolutions, so there is an exact sequence

$$0 \to K_1'' \to X_0'' \to M'' \to 0$$

with $X_0'' \in \mathscr{X}$ and $K_1'' \in \mathscr{X}^{\perp} \cap \widehat{\mathrm{res}}(\mathscr{X})$. Consider then the pullback diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K_1'' \arrow[r, equal] \arrow[d] & K_1'' \arrow[d] & \\
0 \arrow[r] & M' \arrow[r] \arrow[d, equal] & P \arrow[r, dashed] \arrow[d, dashed] & X_0'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since $K_1'', M \in \mathscr{X}^{\perp}$, we have $P \in \mathscr{X}^{\perp}$, and since the sequence $0 \to K_1'' \to P \to M \to 0$ is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X}, -)$-exact, we can apply the Horseshoe Lemma to see that $P \in \widehat{\mathrm{res}}(\mathscr{X})$. Therefore, there is an exact (and $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X}, -)$-exact) sequence

$$0 \to K_1' \to X \to P \to 0$$

with $X \in \mathscr{X}$ and $K_1' \in \mathscr{X}^{\perp} \cap \widehat{\mathrm{res}}(\mathscr{X})$. We consider the following pullback diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K'_1 \arrow[r, equal] \arrow[d] & K'_1 \arrow[d] & & \\
0 \arrow[r] & X'_0 \arrow[r, dashed] \arrow[d, dashed] & X \arrow[r] \arrow[d] & X''_0 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M' \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & X''_0 \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Since $\mathscr{X}$ is closed under kernels of epimorphisms we have that $X'_0 \in \mathscr{X}$, and then, by the first column, we obtain $M' \in \mathscr{X}^{\perp} \cap \widehat{\operatorname{res}}(\mathscr{X})$.

Therefore, we have seen that, in any case, the sequence

$$0 \to M' \to M \to M'' \to 0$$

is $\operatorname{Hom}_{\mathscr{A}}(\mathscr{X}, -)$-exact and that the objects M', M, and M'' admit exact $\mathscr{X}$-resolutions. Then, we consider the diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & X'_1 \arrow[r] \arrow[d] & X'_1 \oplus X''_1 \arrow[r] \arrow[d] & X''_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X'_0 \arrow[r] \arrow[d] & X'_0 \oplus X''_0 \arrow[r] \arrow[d] & X''_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Let n be the maximum of the known dimensions and consider

$$\begin{aligned}
K'_n &= \operatorname{Ker}(X'_{n-1} \to X'_{n-2}), \\
K_n &= \operatorname{Ker}(X'_{n-1} \oplus X''_{n-1} \to X'_{n-2} \oplus X''_{n-2}), \textit{and} \\
K''_n &= \operatorname{Ker}(X''_{n-1} \to X''_{n-2}).
\end{aligned}$$

We obtain the exact sequence

$$0 \to K'_n \to K_n \to K''_n \to 0.$$

- If $\mathscr{G}(\mathscr{X}, \mathscr{Y})$-pd$(M') \leqslant n$ and $\mathscr{G}(\mathscr{X}, \mathscr{Y})$-pd$(M) \leqslant n$, by Theorem 7 we have $K'_n, K_n \in \mathscr{G}(\mathscr{X}, \mathscr{Y})$. Then, $\mathscr{G}(\mathscr{X}, \mathscr{Y})$- pd$(K''_n) \leqslant 1$ and therefore $\mathscr{G}(\mathscr{X}, \mathscr{Y})$-pd$(M'') \leqslant n + 1$.
- If $\mathscr{G}(\mathscr{X}, \mathscr{Y})$-pd$(M') \leqslant n$ and $\mathscr{G}(\mathscr{X}, \mathscr{Y})$-pd$(M'') \leqslant n$, then $K'_n, K''_n \in \mathscr{G}(\mathscr{X}, \mathscr{Y})$ and then $K_n \in \mathscr{G}(\mathscr{X}, \mathscr{Y})$ since $\mathscr{G}(\mathscr{X}, \mathscr{Y})$ is closed under extensions. Therefore, $\mathscr{G}(\mathscr{X}, \mathscr{Y})$-pd$(M) \leqslant n$.

- If $\mathscr{G}(\mathscr{X},\mathscr{Y})$-pd$(M) \leqslant n$ and $\mathscr{G}(\mathscr{X},\mathscr{Y})$-pd$(M'') \leqslant n$, we have $K_n, K''_n \in \mathscr{G}(\mathscr{X},\mathscr{Y})$. Then, $K'_n \in \mathscr{G}(\mathscr{X},\mathscr{Y})$ since $\mathscr{G}(\mathscr{X},\mathscr{Y})$ is closed under kernels of epimorphisms by Proposition 2 and therefore $\mathscr{G}(\mathscr{X},\mathscr{Y})$-pd$(M') \leqslant n$.

Now, after the first statement is proved, 1, 2 and 3 follow using standard arguments. □

Corollary 4. *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence where M', M, and M'' have finite $\mathscr{G}(\mathscr{X},\mathscr{Y})$-projective dimension, then the following statements hold:*

1. *If $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{- pd}(M) \neq \mathscr{G}(\mathscr{X},\mathscr{Y})\text{- pd}(M'')$, then*
$$\mathscr{G}(\mathscr{X},\mathscr{Y})\text{- pd}(M') = \max\{\mathscr{G}(\mathscr{X},\mathscr{Y})\text{- pd}(M), \mathscr{G}(\mathscr{X},\mathscr{Y})\text{- pd}(M'') - 1\}.$$
2. *If $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{- pd}(M'') \neq \mathscr{G}(\mathscr{X},\mathscr{Y})\text{- pd}(M') + 1$, then*
$$\mathscr{G}(\mathscr{X},\mathscr{Y})\text{- pd}(M) = \max\{\mathscr{G}(\mathscr{X},\mathscr{Y})\text{- pd}(M'), \mathscr{G}(\mathscr{X},\mathscr{Y})\text{- pd}(M'')\}.$$
3. *If $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{- pd}(M) \neq \mathscr{G}(\mathscr{X},\mathscr{Y})\text{- pd}(M')$, then*
$$\mathscr{G}(\mathscr{X},\mathscr{Y})\text{- pd}(M'') = \max\{\mathscr{G}(\mathscr{X},\mathscr{Y})\text{- pd}(M') + 1, \mathscr{G}(\mathscr{X},\mathscr{Y})\text{- pd}(M)\}.$$

Proof. Use Corollary 3. □

Applying Proposition 11 to a split short exact sequence
$$0 \to A \to A \oplus B \to B \to 0$$
we obtain that if $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{- pd}(A) \leqslant n$ and $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{- pd}(B) \leqslant n$, then also $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{- pd}(A \oplus B) \leqslant n$. The following proposition shows that if every object in $\mathscr{A}$ has an exact $\mathscr{X}$-resolution, then the converse is also true.

Proposition 12. *If every object in $\mathscr{A}$ has an exact $\mathscr{X}$-resolution, then for every two objects A and B of $\mathscr{A}$ and any integer $n \geq 0$, $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{- pd}(A \oplus B) \leqslant n$ if and only if $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{- pd}(A) \leqslant n$ and $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{- pd}(B) \leqslant n$.*

As a consequence,
$$\mathscr{G}(\mathscr{X},\mathscr{Y})\text{- pd}(A \oplus B) = \max\{\mathscr{G}(\mathscr{X},\mathscr{Y})\text{- pd}(A), \mathscr{G}(\mathscr{X},\mathscr{Y})\text{- pd}(B)\}.$$

Proof. Let $\cdots \to X_1 \to X_0 \to A \to 0$ and $\cdots \to X'_1 \to X'_0 \to B \to 0$ be exact $\mathscr{X}$-resolutions and call $K_n = \operatorname{Ker}(X_{n-1} \to X_{n-2})$ and $K'_n = \operatorname{Ker}(X'_{n-1} \to X'_{n-2})$.

The sequence
$$0 \to K_n \oplus K'_n \to X_{n-1} \oplus X'_{n-1} \to \cdots \to X_0 \oplus X'_0 \to A \oplus B \to 0$$
is then exact and all $X_i \oplus X'_i \in \mathscr{X} \subseteq \mathscr{G}(\mathscr{X},\mathscr{Y})$, so $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-pd}(A \oplus B) \leqslant n$ implies $K_n \oplus K'_n \in \mathscr{G}(\mathscr{X},\mathscr{Y})$. Then, by [8] (Proposition 4.8) we have $K_n, K'_n \in \mathscr{G}(\mathscr{X},\mathscr{Y})$ and so $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{- pd}(A) \leqslant n$ and $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{- pd}(B) \leqslant n$.

For the converse, we apply Proposition 11 to the short exact sequence $0 \to A \to A \oplus B \to B \to 0$. □

It is worth noting that when $\mathscr{X}$ is self-orthogonal, closed under finite direct sums and closed under kernels of epimorphisms, we observe that $\mathscr{G}(\mathscr{X})$ is left G-perfect. As a consequence, it satisfies all the previously discussed properties. In addition, we have the following.

Proposition 13. *Let $\mathscr{X}$ be such that $(\mathscr{X},\mathscr{X})$ is left G-perfect. Given any M in $\mathscr{A}$, if M has finite $\mathscr{X}$-projective dimension, then $\mathscr{X}\text{-pd}(M) = \mathscr{G}(\mathscr{X})\text{-pd}(M)$.*

Proof. Since $\mathscr{X} \subseteq \mathscr{G}(\mathscr{X})$, the inequality $\mathscr{G}(\mathscr{X})\text{-pd}(M) \leq \mathscr{X}\text{-pd}(M)$ always holds.

Conversely, let $n = \mathscr{X}\text{-pd}(M)$. Then, there is an exact sequence

$$0 \to X_n \to X_{n-1} \to \cdots \to X_0 \to M \to 0$$

with all $X_i \in \mathscr{X}$.

If $\mathscr{G}(\mathscr{X})\text{-pd}(M) < n$, then $K = \mathrm{Ker}(X_{n-2} \to X_{n-3}) \in \mathscr{G}(\mathscr{X})$ by Theorem 7 and then $K \in {}^{\perp}\mathscr{X}$. Thus, the exact sequence

$$0 \to X_n \to X_{n-1} \to K \to 0$$

is $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{X})$-exact and so it splits. Thus, $X_{n-1} \cong X_n \oplus K$ and we obtain that $K \in \mathscr{X}$ since $\mathscr{X}$ is closed under kernels of epimorphisms. But this means $\mathscr{X}\text{-pd}(M) < n$, a contradiction. □

Theorem 8. *Let $\mathscr{G}(\mathscr{X}) \subseteq \mathscr{G}(\mathscr{X}, \mathscr{Y})$ and $\mathscr{X} \subseteq \mathscr{Y}$. Then, for any object M in $\mathscr{A}$, the following assertions hold:*

1. *If $\mathscr{G}(\mathscr{X})\text{-pd}(M) < \infty$, then $\mathscr{G}(\mathscr{X}, \mathscr{Y})\text{-pd}(M) = \mathscr{G}(\mathscr{X})\text{-pd}(M)$.*
2. *If $\mathscr{X}\text{-pd}(M) < \infty$, then $\mathscr{G}(\mathscr{X}, \mathscr{Y})\text{-pd}(M) = \mathscr{G}(\mathscr{X})\text{-pd}(M) = \mathscr{X}\text{-pd}(M)$.*

Proof. Assertion 2. is a direct consequence of 1. and Proposition 13, so we just prove 1. But $\mathscr{G}(\mathscr{X}) \subseteq \mathscr{G}(\mathscr{X}, \mathscr{Y})$, so we always have $\mathscr{G}(\mathscr{X}, \mathscr{Y})\text{-pd}(M) \leqslant \mathscr{G}(\mathscr{X}) - \text{pd}(M)$. Thus, we only need to prove the inequality $\mathscr{G}(\mathscr{X})\text{-pd}(M) \leqslant \mathscr{G}(\mathscr{X}, \mathscr{Y})\text{-pd}(M)$. We use induction on $n = \mathscr{G}(\mathscr{X})\text{-pd}(M)$.

If $n = 0$, there is nothing to prove by hypothesis.

If $n = 1$, there is an exact sequence

$$0 \to X \to E \to M \to 0$$

with $X \in \mathscr{X}$ and $E \in \mathscr{G}(\mathscr{X})$ (see Theorem 6). If $\mathscr{G}(\mathscr{X}, \mathscr{Y})\text{-pd}(M) = 0$, then $M \in {}^{\perp}\mathscr{Y}$ by Proposition 1. Thus, the sequence splits since $\mathscr{X} \subseteq \mathscr{Y}$ by hypothesis. Hence, $\mathscr{G}(\mathscr{X})\text{-pd}(M) = 0$, a contradiction. Thus, $\mathscr{G}(\mathscr{X}, \mathscr{Y})\text{-pd}(M) = 1$.

Now, if $n > 1$, we know (Theorem 6) there is an exact sequence

$$0 \to K \to G \to M \to 0$$

where K has an exact $\mathscr{X}$-resolution of length $n - 1$ and $G \in \mathscr{G}(\mathscr{X})$. Then, $\mathscr{X}\text{-pd}(K) \leqslant n - 1$ and then $\mathscr{G}(\mathscr{X})\text{-pd}(K) \leqslant n - 1$ by Proposition 13. But $\mathscr{G}(\mathscr{X})\text{-pd}(M) = n$, so indeed $\mathscr{G}(\mathscr{X})\text{-pd}(K) = n - 1$ (see Theorem 7) and, by induction, $\mathscr{G}(\mathscr{X}, \mathscr{Y})\text{-pd}(K) = n - 1$. Therefore, we apply Corollary 4 to obtain

$$\mathscr{G}(\mathscr{X}, \mathscr{Y})\text{-pd}(M) = \max\{\mathscr{G}(\mathscr{X}, \mathscr{Y})\text{-pd}(G), \mathscr{G}(\mathscr{X}, \mathscr{Y})\text{-pd}(K) + 1\} = n.$$

□

All the results of this section have their dual version referring to the right $(\mathscr{X}, \mathscr{Y})$-Gorenstein dimension, whose proofs do not go beyond repeating the arguments by dualizing the corresponding ideas.

5. Global $(\mathscr{X}, \mathscr{Y})$-Gorenstein Dimension

The objective of this section is to investigate the conditions under which the global projective and injective $\mathscr{G}(\mathscr{X}, \mathscr{Y})$-dimensions of $\mathscr{A}$ are finite.

Definition 5. *Let $\mathscr{X}$ and $\mathscr{Y}$ be subcategories of an abelian category $\mathscr{A}$. The global $\mathscr{G}(\mathscr{X}, \mathscr{Y})$-projective dimension of $\mathscr{A}$, $\mathscr{G}(\mathscr{X}, \mathscr{Y})\text{-PD}(\mathscr{A})$ is defined as the supremum of the $\mathscr{G}(\mathscr{X}, \mathscr{Y})$-projective dimension of all objects of $\mathscr{A}$ or ∞ if there is no such supremum.*

Dually,

$$\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\operatorname{ID}(\mathscr{A}) = \sup\{\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\operatorname{id}(M)|\ M \in \mathscr{A}\} \text{ or } \infty.$$

Theorem 9. *Assume that $\mathscr{A}$ has arbitrary direct sums and let $(\mathscr{X},\mathscr{Y})$ be left G-perfect. Then, the following assertions are equivalent for any positive integer n:*

1. $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\operatorname{PD}(\mathscr{A}) \leqslant n$.
2. *The following statements hold:*
   (a) *$\mathscr{G}(\mathscr{X},\mathscr{Y})$ is closed under direct sums.*
   (b) *$\widehat{\operatorname{res}}(\mathscr{X}) = \mathscr{A}$ (every object of $\mathscr{A}$ admits an epic $\mathscr{X}$-precover).*
   (c) *$\mathscr{X} \subseteq \mathcal{P}roj(\mathscr{A})$.*
   (d) *$\operatorname{id}(Y) \leqslant n\ \forall Y \in \mathscr{Y}$.*
   (e) *$\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\operatorname{pd}(I) \leqslant n\ \forall I \in \mathcal{I}nj(\mathscr{A})$.*

Proof. 1. $\Rightarrow$ 2. (a) Let $G_i \in \mathscr{G}(\mathscr{X},\mathscr{Y})$ be a family of objects with $i \in I$. Then, for each i, $\operatorname{Ext}^k_{\mathscr{A}}(G_i, Y) = 0$ for any $k > 0$ and $Y \in \mathscr{Y}$, by Theorem 7, which implies $\operatorname{Ext}^k_{\mathscr{A}}(\oplus G_i, Y) \cong \prod \operatorname{Ext}^k_{\mathscr{A}}(G_i, Y) = 0$ for any $k > 0$ and $Y \in \mathscr{Y}$. Thus, applying Theorem 7, we obtain $\oplus G_i \in \mathscr{G}(\mathscr{X},\mathscr{Y})$.

(b) By Corollary 1, every object $M \in \mathscr{A}$ has a special $\mathscr{G}(\mathscr{X},\mathscr{Y})$-precover, that is, there exists an exact sequence $0 \to K \to G \to M \to 0$ with $G \in \mathscr{G}(\mathscr{X},\mathscr{Y})$ and $K \in \mathscr{G}(\mathscr{X},\mathscr{Y})^{\perp}$.

On the other hand, since $G \in \mathscr{G}(\mathscr{X},\mathscr{Y})$, by Proposition 6 there is an exact and $\operatorname{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact sequence $0 \to G' \to X \to G \to 0$ with $X \in \mathscr{X}$ and $G' \in \mathscr{G}(\mathscr{X},\mathscr{Y})$. Thus, by Lemma 3, all rows and columns of the pullback diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & G' \arrow[r, equal] \arrow[d] & G' \arrow[d] & & \\
0 \arrow[r] & F \arrow[r, dashed] \arrow[d, dashed] & X \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

are $\operatorname{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact, so the middle row represents an epic $\mathscr{X}$-precover of M. But $\mathscr{X}$ is self-orthogonal, so the middle row being $\operatorname{Hom}(\mathscr{X},-)$-exact and $\operatorname{Ext}^1_{\mathscr{A}}(X', X) = 0$ for any $X' \in \mathscr{X}$ imply $\operatorname{Ext}^1_{\mathscr{A}}(X', F) = 0$ for any $X' \in \mathscr{X}$, and then $X \to M$ is a special $\mathscr{X}$-precover.

To see (c), let $X \in \mathscr{X}$ and $M \in \mathscr{A}$. Since $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\operatorname{pd}(M) \leqslant n$, there is an exact sequence

$$0 \to G_n \to \cdots \to G_0 \to M \to 0$$

with $G_i \in \mathscr{G}(\mathscr{X},\mathscr{Y})$. Now, $\operatorname{Ext}^i_{\mathscr{A}}(X, M) \cong \operatorname{Ext}^{n+i}_{\mathscr{A}}(X, G_n) = 0$ for every i, so we have $X \in \mathcal{P}roj(\mathscr{A})$.

(d) Since $\mathscr{G}(\mathscr{X},\mathscr{Y})$-$\operatorname{pd}(M) \leqslant n$ for every M in $\mathscr{A}$, by Corollary 3, we have

$$\operatorname{Ext}^i_{\mathscr{A}}(M, Y) = 0,\ \ \forall i \geq n+1,\ \forall Y \in \mathscr{Y},\ \forall M \in \mathscr{A},$$

that is, $\operatorname{id}(Y) \leqslant n$, $\forall Y \in \mathscr{Y}$.

(e) Nothing to prove.

2. $\Rightarrow$ 1. Given any $M \in \mathscr{A}$, we consider an exact $\mathscr{X}$-resolution of M

$$\cdots \to X_1 \to X_0 \to M \to 0$$

and an injective resolution of M

$$0 \to M \to E_0 \to E_1 \to \cdots .$$

We decompose these exact sequences into short exact sequences for all $i \in \mathbb{N}$,

$$0 \to Q_i \to X_i \to Q_{i-1} \to 0,$$

$$0 \to L_{i-1} \to E_i \to L_i \to 0,$$

where $Q_i = \operatorname{Ker}(X_i \to X_{i-1})$ and $L_i = \operatorname{Im}(E_i \to E_{i+1})$. We take the direct sum of the first set of sequences,

$$0 \to \bigoplus_{i \in \mathbb{N}} Q_i \to \bigoplus_{i \in \mathbb{N}} X_i \to M \oplus \left(\bigoplus_{i \in \mathbb{N}} Q_i\right) \to 0,$$

and the direct product of the second set of sequences,

$$0 \to M \oplus \left(\prod_{i \in \mathbb{N}} L_i\right) \to \prod_{i \in \mathbb{N}} E_i \to \prod_{i \in \mathbb{N}} L_i \to 0.$$

Let $Q = \bigoplus_{i \in \mathbb{N}} Q_i$, $L = \prod_{i \in \mathbb{N}} L_i$, $G = \bigoplus_{i \in \mathbb{N}} X_i$ and $E = \prod_{i \in \mathbb{N}} E_i$ and consider the short exact sequence

$$0 \to M \oplus (Q \oplus L) \to G \oplus E \to M \oplus (Q \oplus L) \to 0.$$

Now, we know we can find an exact $\mathscr{X}$-resolution of $M \oplus Q \oplus L$:

$$\cdots \to X'_1 \to X'_0 \to M \oplus Q \oplus L \to 0.$$

Since $0 \to M \oplus (Q \oplus L) \to G \oplus E \to M \oplus (Q \oplus L) \to 0$ is $\operatorname{Hom}_{\mathscr{A}}(\mathscr{X}, -)$-exact ($\mathscr{X} \subseteq \mathcal{P}roj(\mathscr{A})$), by the Horseshoe Lemma, we have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & J_n \arrow[r] \arrow[d] & G' \arrow[r] \arrow[d] & J_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X'_{n-1} \arrow[r] \arrow[d] & X'_{n-1} \oplus X'_{n-1} \arrow[r] \arrow[d] & X'_{n-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & X'_0 \arrow[r] \arrow[d] & X'_0 \oplus X'_0 \arrow[r] \arrow[d] & X'_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \oplus Q \oplus L \arrow[r] \arrow[d] & G \oplus E \arrow[r] \arrow[d] & M \oplus Q \oplus L \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Since E is injective, we have $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\operatorname{pd}(E) \leqslant n$ by assumption, and since G is $(\mathscr{X},\mathscr{Y})$-Gorenstein ($\mathscr{G}(\mathscr{X},\mathscr{Y})$ is closed under taking direct sums), applying Proposition 11 we obtain $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\operatorname{pd}(G \oplus E) \leqslant n$. Then, by Theorem 7, we obtain $G' \in \mathscr{G}(\mathscr{X},\mathscr{Y})$.

On the other hand, using the $\mathscr{X}$-resolution of $M \oplus Q \oplus L$, we have, for all $Y \in \mathscr{Y}$ and all positive integers $i \geq 1$, $\operatorname{Ext}^i_{\mathscr{A}}(J_n, Y) \cong \operatorname{Ext}^{i+n}_{\mathscr{A}}(M \oplus Q \oplus L, Y) = 0$ by Lemma 7 because $\operatorname{id}(Y) \leqslant n$. Therefore, $0 \to J_n \to G' \to J_n \to 0$ is $\operatorname{Hom}_{\mathscr{A}}(-,\mathscr{Y})$-exact.

Now, join all the sequences $0 \to J_n \to G' \to J_n \to 0$ to obtain an exact, $\operatorname{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact, and $\operatorname{Hom}_{\mathscr{A}}(-,\mathscr{Y})$-exact sequence of the form $\cdots \to G' \to G' \to G' \to \ldots$. Then, we have $J_n \in \mathscr{G}(\mathscr{X},\mathscr{Y})$ by [8] (Theorem 4.2) so $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\operatorname{pd}(M \oplus Q \oplus L) \leqslant n$. Therefore, $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\operatorname{pd}(M) \leqslant n$ by Proposition 12. □

Corollary 5. *Assume that $\mathscr{A}$ has arbitrary direct sums and let $(\mathscr{X},\mathscr{Y})$ be left G-perfect. If $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\operatorname{PD}(\mathscr{A}) \leqslant n$ for some positive integer n, then $\mathscr{A}$ has enough projectives and*

$$\mathscr{G}(\mathscr{X},\mathscr{Y}) = \mathscr{G}(\mathcal{P}roj(\mathscr{A}),\mathscr{Y}).$$

Proof. By Theorem 9 every object of $\mathscr{A}$ has an epic $\mathscr{X}$-precover, and by the hypotheses $\mathscr{X} \subseteq \mathcal{P}roj(\mathscr{A})$, so any object of $\mathscr{A}$ is a quotient of a projective object.

Now, since $\mathscr{X} \subseteq \mathcal{P}roj(\mathscr{A})$ and every exact sequence is $\operatorname{Hom}_{\mathscr{A}}(\mathcal{P}roj(\mathscr{A}),-)$-exact, we already have the first inclusion.

Conversely, let $M \in \mathscr{G}(\mathcal{P}roj(\mathscr{A}),\mathscr{Y})$. Then, $M \in \widehat{\operatorname{cores}}(\mathscr{Y})$ and $M \in {}^{\perp}\mathscr{Y}$ by Proposition 1, and, by Theorem 9, $M \in \widehat{\operatorname{res}}(\mathscr{X})$ and $\mathscr{X} \subseteq \mathcal{P}roj(\mathscr{A})$ (hence $M \in \mathscr{X}^{\perp}$). Then, we conclude, by Proposition 1, that $M \in \mathscr{G}(\mathscr{X},\mathscr{Y})$. □

Dually, we have the following.

Theorem 10. *Assume that $\mathscr{A}$ has enough projectives and arbitrary direct products and let $(\mathscr{X},\mathscr{Y})$ be right G-perfect. Then, the following assertions are equivalent for any positive integer n:*

1. $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\operatorname{ID}(\mathscr{A}) \leqslant n$.
2. *The following statements hold:*
   (a) *$\mathscr{G}(\mathscr{X},\mathscr{Y})$ is closed under direct products.*
   (b) *$\widehat{\operatorname{cores}}(\mathscr{Y}) = \mathscr{A}$ (all object of $\mathscr{A}$ admit a monic $\mathscr{Y}$-preenvelope).*
   (c) *$\mathscr{Y} \subseteq \mathcal{I}nj(\mathscr{A})$.*
   (d) *$\operatorname{pd}(X) \leqslant n \; \forall X \in \mathscr{X}$.*
   (e) *$\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\operatorname{id}(P) \leqslant n \; \forall P \in \mathcal{P}roj(\mathscr{A})$.*

Corollary 6. *Under the assumptions of Theorem 10, if $\mathscr{G}(\mathscr{X},\mathscr{Y})\text{-}\operatorname{ID}(\mathscr{A}) \leqslant n$ for some integer n, then $\mathscr{G}(\mathscr{X},\mathscr{Y}) = \mathscr{G}(\mathscr{X},\mathcal{I}nj(\mathscr{A}))$.*

6. Relative Foxby Classes

Throughout this section, $\mathscr{C}$ and $\mathscr{D}$ will be two abelian categories with arbitrary direct products and coproducts, and we will assume that $\mathscr{C}$ has a projective generator and that $\mathscr{D}$ has an injective cogenerator. $F : \mathscr{C} \to \mathscr{D}$ and $H : \mathscr{D} \to \mathscr{C}$ will be two functors in an adjoint situation, (F, H), and their derived functors will be denoted by $\mathbf{L}_i F$ and $\mathbf{R}^i H$ $\forall i \geq 0$ respectively.

The Auslander class of $\mathscr{C}$ relative to F, denoted by $\mathcal{A}(\mathscr{C})$, consists of all objects X of $\mathscr{C}$ satisfying the following:

1. $\mathbf{L}_i F(X) = 0 \; \forall i \geq 1$.
2. $\mathbf{R}^i H(F(X)) = 0 \; \forall i \geq 1$.
3. The unit $\mu_X : X \to HF(X)$ is an isomorphism.

Similarly, the Bass class of $\mathscr{D}$ relative to H, denoted by $\mathcal{B}(\mathscr{D})$, consists of all objects Y of $\mathscr{D}$, satisfying the following:

1. $\mathbf{R}^i H(Y) = 0 \; \forall i \geq 1$.
2. $\mathbf{L}_i F(H(Y)) = 0 \; \forall i \geq 1$.

3. The counit $\nu_Y : FH(Y) \to Y$ is an isomorphism.

Definition 6 ([9] Definition 2.1)**.**

1. *The adjoint pair (F, H) is said to be right semidualizing if the class of injective objects is contained in $\mathcal{B}(\mathscr{D})$.*
2. *The adjoint pair (F, H) is said to be left semidualizing if the class of projective objects is contained in $\mathcal{A}(\mathscr{C})$.*

Theorem 11. *If (F, H) is a right semidualizing adjoint pair, then*

$$\mathcal{A}(\mathscr{C}) = \mathscr{G}(\mathcal{P}roj(\mathscr{C}), H(\mathcal{I}nj(\mathscr{D}))).$$

Proof. Given any $(\mathcal{P}roj(\mathscr{C}), H(\mathcal{I}nj(\mathscr{D})))$-Gorenstein object M, we know it has a complete $(\mathcal{P}roj(\mathscr{C}), H(\mathcal{I}nj(\mathscr{D})))$-resolution

$$X : \cdots \to P_1 \to P_0 \to H(E_0) \to H(E_1) \to \cdots,$$

so $\mathrm{Hom}_{\mathscr{C}}(X, H(E))$ is exact for any injective object E of $\mathscr{D}$. In particular, if E' is an injective cogenerator of $\mathscr{D}$, then $\mathrm{Hom}_{\mathscr{C}}(X, H(E'))$ is exact, so the complex $\mathrm{Hom}_{\mathscr{D}}(F(X), E')$ is also exact (where we denote by $F(X)$ the complex $\cdots \to F(P_1) \to F(P_0) \to FH(E_0) \to FH(E_1) \to \cdots$).

But then $F(X)$ is exact, so we can apply [9] (Theorem 2.11) to obtain that $M \in \mathcal{A}(\mathscr{C})$.

Conversely, given any $M \in \mathcal{A}(\mathscr{C})$, we know by [9] (Theorem 2.11) that there is an exact sequence

$$X : \cdots \to P_1 \to P_0 \to H(E_0) \to H(E_1) \to \cdots$$

such that the following hold:

1. $E_i \in \mathcal{I}nj(\mathscr{D})$ and $P_i \in \mathcal{P}roj(\mathscr{C})$ for all i.
2. $M = \mathrm{Ker}(H(E_0) \to H(E_1))$.
3. $F(X)$ is exact.

Thus, to obtain $M \in \mathscr{G}(\mathcal{P}roj(\mathscr{C}), H(\mathcal{I}nj(\mathscr{D}))$, we only need to prove that the sequence X is $\mathrm{Hom}_{\mathscr{C}}(-, H(\mathcal{I}nj(\mathscr{D})))$-exact. But $F(X)$ is exact, so for any $E \in \mathcal{I}nj(\mathscr{D})$, the complex $\mathrm{Hom}_{\mathscr{D}}(F(X), E)$ is exact, and therefore the complex $\mathrm{Hom}_{\mathscr{C}}(X, H(E))$ is also exact. □

Corollary 7. *Let (F, H) be both a right and a left semidualizing adjoint pair of functors. Then, the pair $(\mathcal{P}roj(\mathscr{C}), H(\mathcal{I}nj(\mathscr{D}))$ is G-perfect.*

Proof. H is right adjoint, so it preserves direct sums, so both $\mathcal{P}roj(\mathscr{C})$ and $H(\mathcal{I}nj(\mathscr{D}))$ are closed under finite direct sums. Moreover, we know that $\mathcal{P}roj(\mathscr{C}) \perp H(\mathcal{I}nj(\mathscr{D}))$, that $\mathcal{P}roj(\mathscr{C})$ is closed under kernels of epimorphisms and that both $\mathcal{P}roj(\mathscr{C})$ and $H(\mathcal{I}nj(\mathscr{D}))$ are self-orthogonal classes (see [9] Lemma 2.8).

Now, since (F, H) is both a right and a left semidualizing adjoint pair, $\mathcal{P}roj(\mathscr{C}) \subseteq \mathcal{A}(\mathscr{C})$ and $H(\mathcal{I}nj(\mathscr{D})) \subseteq \mathcal{A}(\mathscr{C})$ (see [9] Proposition 2.1).

Then,

$$\mathcal{P}roj(\mathscr{C}), H(\mathcal{I}nj(\mathscr{D})) \subseteq \mathscr{G}(\mathcal{P}roj(\mathscr{C}), H(\mathcal{I}nj(\mathscr{D}))$$

by Theorem 11. Thus, $\mathcal{P}roj(\mathscr{C}) \subseteq \widehat{\mathrm{cores}}(H(\mathcal{I}nj(\mathscr{D})))$ and $H(\mathcal{I}nj(\mathscr{D})) \subseteq \widehat{\mathrm{res}}(\mathcal{P}roj(\mathscr{C}))$ by Propostions 4 and 5.

Therefore, it only remains to prove that $H(\mathcal{I}nj(\mathscr{D}))$ is closed under cokernels of monomorphisms.

Consider the exact sequence

$$0 \to H(E) \to H(E') \to K \to 0$$

with $E, E' \in \mathcal{I}nj(\mathscr{D})$. By [9] (Proposition 2.12) we have $K \in \mathcal{A}(\mathscr{C})$. Applying F, since $E, E' \in \mathcal{B}(\mathscr{D})$, we have the exact sequence

$$\mathbf{L}_i F(K) = 0 \to E \to E' \to F(K) \to 0.$$

Since $\mathcal{I}nj(\mathscr{D})$ is closed under cokernels of monomorphisms, we have $F(K) \in \mathcal{I}nj(\mathscr{D})$. Therefore, since $K \cong H(F(K))$, $K \in H(\mathcal{I}nj(\mathscr{D}))$. $\square$

Dually, we have the following.

Theorem 12. *Let (F, H) be a left semidualizing adjoint pair. Then,*

$$\mathcal{B}(\mathscr{D}) = \mathscr{G}(F(\mathcal{P}roj(\mathscr{C})), \mathcal{I}nj(\mathscr{D})).$$

Corollary 8. *Let (F, H) be both a right and a left semidualizing adjoint pair. Then, the pair $(F(\mathcal{P}roj(\mathscr{C})), \mathcal{I}nj(\mathscr{D}))$ is G-perfect.*

7. Conclusions

As a result of the study carried out in this paper, we obtain that a common framework can be (and has been) established in the general categorical environment of an abelian category for the development of a general Gorenstein homological theory involving all the existing variants so far introduced. This framework consists in defining $(\mathscr{X}, \mathscr{Y})$-Gorenstein objects, or simply $\mathscr{G}(\mathscr{X}, \mathscr{Y})$ objects, as the 0 syzygies of complete $(\mathscr{X}, \mathscr{Y})$-resolutions.

Under certain compatibility conditions between the classes $\mathscr{X}$ and $\mathscr{Y}$ (compatibility conditions which are clear in the particular cases treated so far, such as Gorenstein projective or injective modules in all their variants), this theory becomes (as one would wish) very strong in the sense that all kernels of complete $(\mathscr{X}, \mathscr{Y})$-resolutions are $(\mathscr{X}, \mathscr{Y})$-Gorenstein objects. In other words, the category $\mathscr{G}(\mathscr{X}, \mathscr{Y})$ of all $(\mathscr{X}, \mathscr{Y})$-Gorenstein objects is stable ($\mathscr{G}(\mathscr{G}(\mathscr{X}, \mathscr{Y})) = \mathscr{G}(\mathscr{X}, \mathscr{Y})$) as proved in Theorem 3.

Of course, to give maximum value and impact to this development of general Gorenstein objects, the dimensions associated with them must be consistent. Thus, the Comparison Lemma is proved in Theorems 4 and 5.

But also, as in the classical cases, there must be a useful tool to calculate these dimensions, beyond the use of the resolution lengths indicated in the definition. And as in the classical cases, Ext functors are these tools. Thus, for the projective dimension, Theorem 7 says that whenever M is of finite $\mathscr{G}(\mathscr{X}, \mathscr{Y})$-projective dimension, one has $\mathscr{G}(\mathscr{X}, \mathscr{Y})\text{-}\operatorname{pd}(M) \leqslant n$ if and only if $\operatorname{Ext}^i_{\mathscr{A}}(M, Y) = 0$ for all $i > n$ and all $Y \in \mathscr{Y}$. In other words, if $\mathscr{G}(\mathscr{X}, \mathscr{Y})\text{-}\operatorname{pd}(M)$ is finite, then

$$\mathscr{G}(\mathscr{X}, \mathscr{Y})\text{-}\operatorname{pd}(M) = \sup\{i \in \mathbb{N} : \operatorname{Ext}^i_{\mathscr{A}}(M, Y) \neq 0 \text{ for some } Y \in \mathscr{Y}\}.$$

This, of course, has its dual statement for the injective dimension whose proof does not go beyond a mere dualization of arguments.

With the tools for computing and comparing $(\mathscr{X}, \mathscr{Y})$-Gorenstein dimensions in hand, the study of the global dimension of the category seems to be the natural question to address. More specifically, the interest lies in finding conditions that guarantee that the global $(\mathscr{X}, \mathscr{Y})$-Gorenstein dimension of the category is finite. When the category has arbitrary direct sums, these conditions are found in Theorem 9: if $(\mathscr{X}, \mathscr{Y})$ is left G-perfect, then $\mathscr{G}(\mathscr{X}, \mathscr{Y})\text{-}\operatorname{PD}(\mathscr{A}) \leqslant n$ if and only if $\mathscr{G}(\mathscr{X}, \mathscr{Y})$ is closed under direct sums, every object of $\mathscr{A}$ admits an epic $\mathscr{X}$-precover, $\mathscr{X} \subseteq \mathcal{P}roj(\mathscr{A})$, $\operatorname{id}(Y) \leqslant n\ \forall Y \in \mathscr{Y}$, and $\mathscr{G}(\mathscr{X}, \mathscr{Y})\text{-}\operatorname{pd}(I) \leqslant n\ \forall I \in \mathcal{I}nj(\mathscr{A})$. As consequence, it follows that, under these conditions for $\mathscr{A}$ and the pair $(\mathscr{X}, \mathscr{Y})$, if $\mathscr{G}(\mathscr{X}, \mathscr{Y})\text{-}\operatorname{PD}(\mathscr{A}) \leqslant n$ for some n, then $\mathscr{A}$ has enough projectives, and $\mathscr{G}(\mathscr{X}, \mathscr{Y}) = \mathscr{G}(\mathcal{P}roj(\mathscr{A}), \mathscr{Y})$.

Finally, it was of particular interest to relate the classes of Gorenstein objects to the Foxby classes. Indeed, Foxby classes are defined in module categories based on the functors Hom and $\otimes$ (and their derived functors). However, the nature of these classes can be considered on the basis of general adjoint pairs of functors. And even at this level of generality (always assuming conditions on the pair of functors, of course), it is possible to give a satisfactory relation between these two types of classes. Thus, if $F : \mathscr{C} \to \mathscr{D}$ and $H : \mathscr{D} \to \mathscr{C}$ are such that (F, H) is a right semidualizing adjoint pair of functors (see Definition 6), then $\mathcal{A}(\mathscr{C}) = \mathscr{G}(\mathcal{P}roj(\mathscr{C}), H(\mathcal{I}nj(\mathscr{D})))$ (Theorem 11), and if (F, H) is left semidualizing, then $\mathcal{B}(\mathscr{D}) = \mathscr{G}(F(\mathcal{P}roj(\mathscr{C})), \mathcal{I}nj(\mathscr{D}))$ (Theorem 12).

It would be interesting to study the possible translation of the results obtained in this paper to the context of generalizations of Gorenstein-derived categories with the perspective of applying them in categories with special interest, such as those of quasi-coherent sheaves.

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