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On Complex Pisot Numbers That Are Roots of Borwein Trinomials

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Abstract: Let $n > m$ be positive integers. Polynomials of the form $z^n \pm z^m \pm 1$ are called Borwein trinomials. Using an old result of Bohl, we derive explicit formulas for the number of roots of a Borwein trinomial inside the unit circle $|z| < 1$. Based on this, we determine all Borwein trinomials that have a complex Pisot number as a root. There are exactly 29 such trinomials.

Keywords: Borwein trinomial; complex Pisot number; unimodular number; root of unity

MSC: 11R18; 11R04; 11A63; 11D41

1. Introduction

A real algebraic integer $\alpha > 1$ is called a *Pisot number* after [1,2], if all the algebraic conjugates of α over the field of rational numbers $\mathbb{Q}$ (other than α itself) are of absolute value < 1. Pisot numbers attract a lot of attention in the study of number expansions with algebraic number bases [3,4], substitution tilings [5–7], integer sequences with particular regard to linear recurrences [8–10], distributions of the fractional parts of the powers of real numbers [11,12] and many other areas [13,14].

Recently, there has been a surge in interest in complex-base number expansions [15–18]: in the distributions of the powers of algebraic numbers [19,20]; in the complex plane $\mathbb{C}$ with respect to the Gaussian lattice $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}, i^2 = -1\}$; and in complex algebraic integers with special multiplicative properties [21–24]. In these kinds of problems, the complex analogues of the Pisot numbers in $\mathbb{C}$ play the same pivotal role as the Pisot numbers in $\mathbb{R}$. Recall that an algebraic number $\beta \in \mathbb{C} \setminus \mathbb{R}$, $|\beta| > 1$ is called *a complex Pisot number* if all of its algebraic conjugates $\beta' \notin \{\beta, \bar{\beta}\}$ satisfy $|\beta'| < 1$. Complex Pisot numbers were considered first by Kelly and Samet [25,26]. The smallest complex Pisot numbers were identified by Chamfy [27]; later, Garth [28,29] significantly expanded Chamfy's list. Nonetheless, recent research has increased the general interest in the spectra of complex Pisot numbers.

In the present paper, we are interested in complex Pisot numbers that originate from the simplest possible polynomials, namely Borwein trinomials. If the polynomial $f(z) \in \mathbb{Z}[z]$

$$f(z) = a_n z^n + \cdots + a_1 z + a_0, \qquad a_n a_0 \neq 0 \tag{1}$$

(1) has exactly three or four nonzero terms, then it is called *a trinomial* or *a quadrinomial*, respectively. The polynomials that have all their coefficients $a_j \in \{-1, 0, 1\}$ are called Borwein polynomials (in honor of the late P. Borwein, as in [30]). Thus, Borwein trinomials are polynomials of the form $z^n \pm z^m \pm 1$. For example, $z^9 - z^5 + 1$ is a Borwein trinomial. The main result of our paper is Theorem 1.

Theorem 1. *Any Borwein trinomial that has a complex Pisot number as its root is of the form $\pm f(\pm z)$, where $f(z)$ is one of the* 17 *polynomials listed in Table 1.*

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Table 1. Representative Borwein trinomials that have a complex Pisot number as their root.

z^3+z+1	z^6-z^2+1
z^3-z^2+1	z^6+z^4+1
z^4+z+1	z^6+z^5+1
z^4+z^2-1	z^7+z^5+1
z^4+z^3+1	z^7-z^6+1
z^5+z+1	z^8+z^6-1
z^5-z^2+1	z^8+z^7+1
z^5+z^3+1	$z^{10}+z^8+1$
z^5-z^4+1	

All the polynomials in Table 1 are irreducible, except for z^5+z+1, z^7+z^5+1, z^8+z^7+1, and $z^{10}+z^8+1$, which are all divisible by z^2+z+1. In comparison, all Borwein trinomials and quadrinomials that give a rise to real Pisot numbers were essentially identified in [31] (after taking into account the irreducibility theorem of Ljunggren [32]). The proof of Theorem 1 is based on the following result.

Theorem 2. *Let $n>m$ be positive integers. All Borwein trinomials $f(z)=z^n\pm z^m\pm 1$ with at most two roots inside the unit disc $|z|<1$ are given in Table 2.*

Table 2. Representative Borwein trinomials with at most 2 zeros inside the open unit disk $|z|<1$. Only one polynomial out of $\pm f(\pm z)$ is shown.

***N*(*f*) = 1**	***N*(*f*) = 2**	***N*(*f*) = 2**
z^2+z-1	z^4+z^2-1	z^3+z^2+1
z^3+z^2-1	z^4-z^2-1	z^3-z+1
z^3+z+1	z^6+z^4-1	z^4+z^3+1
z^4+z-1	z^6-z^4-1	z^4+z+1
z^5+z-1	z^6+z^2+1	z^5+z^4+1
	z^6+z^2-1	z^5-z^3+1
	z^8+z^2-1	z^5+z^2+1
	z^8-z^2-1	z^5-z+1
	$z^{10}+z^2-1$	z^6+z+1
	$z^{10}+z^2+1$	z^7+z^2+1
		z^7-z+1
		z^8+z+1

We also note that Borwein trinomials appear to have no multiple roots in $\mathbb{C}$ (see Proposition 2).

More generally, the number of zeros of a Borwein trinomial or a Borwein quadrinomial is interesting in the context of the distribution of zeros of polynomials with small coefficients [30]. For this, let us state the definition for the zero number $N(f)$ of a polynomial $f(z)$. First, recall that $f(z)$ splits over the field of complex numbers $\mathbb{C}$ into

$$f(z)=a_n(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n),$$

where the complex zeros $\alpha_1,\alpha_2,\ldots,\alpha_n\in\mathbb{C}$ of $f(z)$ are not necessarily distinct. The zero counting functions with respect to the unit circle are introduced through the formulas

$$N(f)=\#\{j,\ 1\leqslant j\leqslant n:|\alpha_j|<1\}$$

and

$$U(f)=\#\{j,\ 1\leqslant j\leqslant n:|\alpha_j|=1\},$$

where the zeros are counted with the multiplicities. The *reciprocal* polynomial $f^*(z)$ is defined by
$$f^*(z) = z^n f(1/z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n.$$

Note that
$$N(f^*) = \#\{j,\, 1 \leqslant j \leqslant n : |\alpha_j| > 1\}.$$

Hence, one always has that
$$N(f) + N(f^*) + U(f) = n.$$

A complex number with absolute value 1 is called a *unimodular number*. Note that every root of unity is a unimodular number. However, not every unimodular number is a root of unity, since $e^{ni} \neq 1$ for every positive integer n.

We derive Theorem 2 from Proposition 3, which gives explicit formulas for $N(f)$ for any Borwein trinomial $f(z)$. Finally, Proposition 3 is derived from an old result of Bohl (see Theorem 3).

Previous work on the smallest complex Pisot numbers [27–29] was based on the complicated computation of the coefficients of Taylor–Maclaurin series of bounded analytic functions (Schur functions), a method pioneered by Dufresnoy and Pisot [1,2]. Our new contribution to expand the list of known complex Pisot numbers is based on Bohl's formula [33,34] (discussed below in the next section).

The paper is organized as follows. In Section 2, we prove Proposition 3 and Theorems 1 and 2. The irreducibility of Borwein trinomials is considered in Section 3. We explicitly describe irreducible Borwein trinomials (see Corollary 4). This result has already been proven by Ljunggren (see Theorem 3 in [35]). Nevertheless, we give an alternative proof based on Proposition 3.

2. Proofs of Theorems

Let x be a real number. Recall that $\lfloor x \rfloor$ denotes the largest rational integer that is less than or equal to x. Similarly, $\lceil x \rceil$ denotes the smallest rational integer that is greater than or equal to x. We will need the following basic properties of $\lfloor x \rfloor$ and $\lceil x \rceil$, which follow directly from the definitions of these functions.

Proposition 1. *The following statements are true.*

(i) *For any real number x, $\lfloor x \rfloor \leqslant x \leqslant \lceil x \rceil$.*
(ii) *For any real number x, the equalities $\lceil x \rceil = -\lfloor -x \rfloor$ and $\lfloor x \rfloor = -\lceil -x \rceil$ hold.*
(iii) *For any real numbers a and b, $a < b$, the interval $(a,\, b)$ contains exactly $\lceil b \rceil - \lfloor a \rfloor - 1$ rational integers.*

The main tool in the proof of Theorem 2 is the following result due to Bohl: for a modern formulation, see the expository note [34] (also formulated as Theorem 3.2 in [36]).

Theorem 3 (Bohl's theorem, [33,36])**.** *Let $f(z) = z^n + pz^m + q$ be a trinomial, where $p, q \in \mathbb{C}$ and m and n are coprime positive integers such that $n > m$. Assume that, for a real number $v > 0$, there exists a triangle with edge lengths v^n, $|p|v^m$, and $|q|$. Let $\alpha = \angle(|p|v^m,\, |q|)$ and $\beta = \angle(v^n,\, |q|)$. Then, the number of roots of $f(z)$ that lie in the open disc $|z| < v$ is given by the number of integers located in the open interval $(C_f - \delta_f, C_f + \delta_f)$, where*
$$C_f = \frac{n(\pi + \arg(p) - \arg(q)) - m(\pi - \arg(q))}{2\pi}$$

and
$$\delta_f = \frac{n\alpha + m\beta}{2\pi}.$$

Note that if $f(z)$ is a polynomial such that $f(0) \neq 0$ and ℓ is a positive integer, then $N(f(z^\ell)) = \ell N(f(z))$.

Proposition 2. *Let $n > m$ be positive integers and $a, b \in \{\pm 1\}$. Then, the polynomial $z^n + az^m + b$ has no multiple roots in $\mathbb{C}$.*

Proof of Proposition 2. For a contradiction, assume that $z_0 \in \mathbb{C}$ is a multiple root of $f(z) = z^n + az^m + b$. Then, $f(z_0) = f'(z_0) = 0$. Since $f'(z) = nz^{n-1} + amz^{m-1} = z^{m-1}(nz^{n-m} + am)$, we have that $z_0^{n-m} = -am/n$ and $|z_0|^{n-m} = m/n < 1$. Hence, $|z_0| < 1$. On the other hand, $f'(z_0) = 0$ implies that $z_0^n = -am/nz_0^m$. Substituting this into $f(z_0) = z_0^n + az_0^m + b = 0$ yields $-am/nz_0^m + az_0^m + b = 0$. Hence, $z_0^m = -bn/(a(n-m))$ and $|z_0|^m = n/(n-m) > 1$. Therefore, $|z_0| > 1$, which contradicts the previously obtained inequality $|z_0| < 1$. □

Proposition 3. *Let $n > m$ be coprime positive integers. Then,*

$$\begin{aligned}
N(z^n + z^m + 1) &= n - m - 1 - 2\left\lfloor \frac{n-2m}{3} \right\rfloor, \\
N(z^n + z^m - 1) &= 2\left\lceil \frac{n+m}{6} \right\rceil - 1, \\
N(z^n - z^m + 1) &= 2\left\lceil \frac{n+4m}{6} \right\rceil - m - 1, \\
N(z^n - z^m - 1) &= n - 2\left\lfloor \frac{2n-m}{6} \right\rfloor - 1.
\end{aligned}$$

Proof of Proposition 3. We will apply Theorem 3 to the polynomial $f(z) = z^n + pz^m + q$, where $p = \pm 1$, $q = \pm 1$, and $v = 1$. Note that, in Theorem 3, the triangle with edge lengths $v^n = 1$, $|p|v^m = 1$, and $|q| = 1$ is an equilateral triangle. Hence, $\alpha = \beta = \pi/3$ and $\delta_f = (n\alpha + m\beta)/(2\pi) = (n+m)/6$. By Theorem 3, $N(f)$ equals the number of integers located in the open interval $(C_f - \delta_f, C_f + \delta_f)$, where

$$C_f = \frac{n(\pi + \arg(p) - \arg(q)) - m(\pi - \arg(q))}{2\pi}.$$

We will consider only the case $f(z) = z^n + z^m + 1$. The remaining formulas for $N(f)$ in Proposition 3 can be obtained completely analogously.

Let $f(z) = z^n + z^m + 1$. Then, $p = q = 1$ and $\arg(p) = \arg(q) = 0$. Hence, $C_f = (n-m)/2$. By Theorem 3, $N(f)$ equals the number of integers located in the open interval $(C_f - \delta_f, C_f + \delta_f) = ((n-2m)/3,\ (2n-m)/3)$. Hence, in view of Proposition 1 (iii),

$$N(f) = \left\lceil \frac{2n-m}{3} \right\rceil - \left\lfloor \frac{n-2m}{3} \right\rfloor - 1.$$

Note that $(2n-m)/3 = n - m - (n-2m)/3$. Then, in view of Proposition 1 (ii),

$$\begin{aligned}
N(f) &= \left\lceil n - m - \frac{n-2m}{3} \right\rceil - \left\lfloor \frac{n-2m}{3} \right\rfloor - 1 \\
&= n - m - 1 + \left\lceil -\frac{n-2m}{3} \right\rceil - \left\lfloor \frac{n-2m}{3} \right\rfloor = n - m - 1 - 2\left\lfloor \frac{n-2m}{3} \right\rfloor.
\end{aligned}$$

□

The reciprocal polynomial $(z^n \pm z^m \pm 1)^*$ is of the form $\pm z^n \pm z^{n-m} + 1$. Therefore, Proposition 3 implies the following corollary.

Corollary 1. *Let $n > m$ be coprime positive integers. Then,*

$$N((z^n+z^m+1)^*) = 2\left\lceil \frac{n+m}{3} \right\rceil - m - 1,$$
$$N((z^n+z^m-1)^*) = n - 2\left\lfloor \frac{n+m}{6} \right\rfloor - 1,$$
$$N((z^n-z^m+1)^*) = 2\left\lceil \frac{5n-4m}{6} \right\rceil - n + m - 1,$$
$$N((z^n-z^m-1)^*) = 2\left\lceil \frac{2n-m}{6} \right\rceil - 1.$$

Note that for $f(z) = z^n \pm z^m \pm 1$, $N(f^*)$ equals the number of roots of $f(z)$ that lie strictly outside the unit circle $|z| = 1$. Now, in view of Proposition 3, Corollary 1, and the formula

$$N(f) + N(f^*) + U(f) = n$$

we can determine the number of unimodular roots of $f(z)$.

Corollary 2. *Let $n > m$ be coprime positive integers. Then,*

$$U(z^n+z^m+1) = 2\left(1 - \left(\left\lceil \frac{n+m}{3} \right\rceil - \left\lfloor \frac{n+m}{3} \right\rfloor\right)\right),$$
$$U(z^n+z^m-1) = 2\left(1 - \left(\left\lceil \frac{n+m}{6} \right\rceil - \left\lfloor \frac{n+m}{6} \right\rfloor\right)\right),$$
$$U(z^n-z^m+1) = 2\left(1 - \left(\left\lceil \frac{n+4m}{6} \right\rceil - \left\lfloor \frac{n+4m}{6} \right\rfloor\right)\right),$$
$$U(z^n-z^m-1) = 2\left(1 - \left(\left\lceil \frac{2n-m}{6} \right\rceil - \left\lfloor \frac{2n-m}{6} \right\rfloor\right)\right).$$

Proof of Corollary 2. We will consider only the case $f(z) = z^n + z^m + 1$. The remaining formulas for $U(f)$ can be obtained completely analogously.

Let $f(z) = z^n + z^m + 1$. By Proposition 3 and Corollary 1,

$$N(f) = n - m - 1 - 2\left\lfloor \frac{n-2m}{3} \right\rfloor \quad \text{and} \quad N(f^*) = 2\left\lceil \frac{n+m}{3} \right\rceil - m - 1.$$

Hence, the formula $N(f) + N(f^*) + U(f) = n$ implies

$$\begin{aligned} U(f) = n - N(f) - N(f^*) &= n - \left(n - m - 1 - 2\left\lfloor \frac{n-2m}{3} \right\rfloor\right) \\ &\quad - \left(2\left\lceil \frac{n+m}{3} \right\rceil - m - 1\right) = 2m + 2 + 2\left\lfloor \frac{n-2m}{3} \right\rfloor - 2\left\lceil \frac{n+m}{3} \right\rceil \\ &= 2m + 2 + 2\left\lfloor \frac{n+m}{3} - m \right\rfloor - 2\left\lceil \frac{n+m}{3} \right\rceil \\ &= 2 + 2\left\lfloor \frac{n+m}{3} \right\rfloor - 2\left\lceil \frac{n+m}{3} \right\rceil \\ &= 2\left(1 - \left(\left\lceil \frac{n+m}{3} \right\rceil - \left\lfloor \frac{n+m}{3} \right\rfloor\right)\right). \end{aligned}$$

□

The following corollary has already been proven by Ljunggren (see Theorem 3 in [35]). Nevertheless, we give an alternative proof of this result.

Corollary 3. *Let $n > m$ be coprime positive integers.*

1. *The polynomial $f(z) = z^n + z^m + 1$ has a unimodular root if and only if $n + m$ is divisible by 3. Furthermore, if $n + m$ is divisible by 3, then $f(z) = (z^2 + z + 1)g(z)$, where the polynomial $g(z)$ has no unimodular roots.*
2. *The polynomial $f(z) = z^n + z^m - 1$ has a unimodular root if and only if $n + m$ is divisible by 6. Furthermore, if $n + m$ is divisible by 6, then $f(z) = (z^2 - z + 1)g(z)$, where the polynomial $g(z)$ has no unimodular roots.*
3. *The polynomial $f(z) = z^n - z^m + 1$ has a unimodular root if and only if $n + 4m$ is divisible by 6. Furthermore, if $n + 4m$ is divisible by 6, then $f(z) = (z^2 - z + 1)g(z)$, where the polynomial $g(z)$ has no unimodular roots.*
4. *The polynomial $f(z) = z^n - z^m - 1$ has a unimodular root if and only if $2n - m$ is divisible by 6. Furthermore, if $2n - m$ is divisible by 6, then $f(z) = (z^2 - z + 1)g(z)$, where the polynomial $g(z)$ has no unimodular roots.*

Note that the polynomial $g(z)$ in this corollary is irreducible (see Theorem 4).

Proof of Corollary 3. The first part of every proposition follows directly from Corollary 2.

1. Assume that $n + m$ is divisible by 3. According to Corollary 2, the trinomial $f(z) = z^n + z^m + 1$ has precisely two unimodular roots. It suffices to show that $\zeta = e^{\frac{2\pi i}{3}}$ is a root of $f(z)$ (indeed, if $f(\zeta) = 0$, then $f(\bar{\zeta}) = \overline{f(\zeta)} = 0$, so that ζ and $\bar{\zeta} = e^{-\frac{2\pi i}{3}}$ are the only unimodular roots of $f(z)$ and $z^2 + z + 1 = (z - \zeta)(z - \bar{\zeta})$ divides $f(z)$). We have that $n + m = 3t$ for some positive integer t. Moreover, ζ is a primitive third root of unity, whose minimal polynomial is $z^2 + z + 1 = (z - \zeta)(z - \bar{\zeta})$. Since $\zeta^3 = 1$, we have that

$$f(\zeta) = \zeta^n + \zeta^m + 1 = \zeta^{3t-m} + \zeta^m + 1 = \zeta^{-m} + \zeta^m + 1 = \zeta^{-m}\left(\zeta^{2m} + \zeta^m + 1\right).$$

Note that m is not divisible by 3 since m and n are coprime and $n + m = 3t$. Hence, ζ^m is also a primitive third root of unity, and thus a root of $z^2 + z + 1$. Therefore, $f(\zeta) = \zeta^{-m}(\zeta^{2m} + \zeta^m + 1) = 0$.

2. Assume that $n + m$ is divisible by 6. According to Corollary 2, the trinomial $f(z) = z^n + z^m - 1$ has precisely two unimodular roots. As in the proof of the first proposition, it suffices to show that $\zeta = e^{\frac{\pi i}{3}}$ is a root of $f(z)$. We have that $n + m = 6t$ for some positive integer t. Moreover, ζ is a primitive sixth root of unity, whose minimal polynomial is $z^2 - z + 1 = (z - \zeta)(z - \bar{\zeta})$. Since $\zeta^6 = 1$, we have that

$$f(\zeta) = \zeta^n + \zeta^m - 1 = \zeta^{6t-m} + \zeta^m - 1 = \zeta^{-m} + \zeta^m - 1 = \zeta^{-m}\left(\zeta^{2m} - \zeta^m + 1\right).$$

Note that m is coprime to 6 since m and n are coprime and $n + m = 6t$. Hence, ζ^m is also a primitive sixth root of unity, and thus a root of $z^2 - z + 1$. Therefore, $f(\zeta) = \zeta^{-m}(\zeta^{2m} - \zeta^m + 1) = 0$.

3. Assume that $n + 4m$ is divisible by 6. According to Corollary 2, the trinomial $f(z) = z^n - z^m + 1$ has precisely two unimodular roots. As in the proof of the first proposition, it suffices to show that $\zeta = e^{\frac{\pi i}{3}}$ is a root of $f(z)$. We have that $n + 4m = 6t$ for some positive integer t. Moreover, ζ is a primitive sixth root of unity, whose minimal polynomial is $z^2 - z + 1 = (z - \zeta)(z - \bar{\zeta})$. Since $\zeta^6 = 1$, we have that

$$f(\zeta) = \zeta^n - \zeta^m + 1 = \zeta^{6t-4m} - \zeta^m + 1 = \zeta^{-4m} - \zeta^m + 1 = \zeta^{2m} - \zeta^m + 1.$$

Note that m is coprime to 6 since m and n are coprime and $n + 4m = 6t$. Hence, ζ^m is also a primitive sixth root of unity, and thus a root of $z^2 - z + 1$. Therefore, $f(\zeta) = \zeta^{2m} - \zeta^m + 1 = 0$.

4. This proposition follows from the second proposition by considering the reciprocal polynomial $-(z^n - z^m - 1)^* = z^n + z^{n-m} - 1$. $\square$

Proof of Theorem 2. Let $n > m$ be positive integers. Suppose that $f(z) = z^n \pm z^m \pm 1$ is a Borwein trinomial such that $N(f) \leqslant 2$. Consider two possible cases: $\gcd(n, m) = 1$ and $\gcd(n, m) > 1$.

Case 1. We have that $\gcd(n, m) = 1$. We will apply Theorem 3 to the polynomial $f(z) = z^n + pz^m + q$, where $p = \pm 1$, $q = \pm 1$ and $v = 1$. Note that in Theorem 3, the triangle with edge lengths $v^n = 1$, $|p|v^m = 1$, and $|q| = 1$ is an equilateral triangle. Hence, $\alpha = \beta = \pi/3$ and $\delta_f = (n\alpha + m\beta)/(2\pi) = (n + m)/6$. By Theorem 3, $N(f)$ equals the number of integers located in the open interval $(C_f - \delta_f, C_f + \delta_f)$, where

$$C_f = \frac{n(\pi + \arg(p) - \arg(q)) - m(\pi - \arg(q))}{2\pi}.$$

Hence, by (iii) and (i) of Proposition 1, we have

$$\begin{aligned} N(f) = \left\lceil C_f + \delta_f \right\rceil - \left\lfloor C_f - \delta_f \right\rfloor - 1 &\geqslant (C_f + \delta_f) - \left\lfloor C_f - \delta_f \right\rfloor - 1 \\ &\geqslant (C_f + \delta_f) - (C_f - \delta_f) - 1 = 2\delta_f - 1 = \frac{n + m - 3}{3}. \end{aligned}$$

Recall that $N(f) \leqslant 2$. Thus, $(n + m - 3)/3 \leqslant 2$, which is equivalent to $n + m \leqslant 9$. Thus, we are left to compute $N(f)$ for every polynomial $f(z) = z^n \pm z^m \pm 1$, where $n > m$, $\gcd(n, m) = 1$, and $n + m \leqslant 9$. In total, there are 13 pairs (n, m) satisfying these conditions, namely

$$\begin{aligned} &(8,1),\ (7,2),\ (7,1),\ (6,1),\ (5,4),\ (5,3),\ (5,2),\ (5,1), \\ &\qquad\qquad\qquad\qquad (4,3),\ (4,1),\ (3,2),\ (3,1),\ (2,1). \end{aligned}$$

Hence, there are exactly $4 \times 13 = 52$ polynomials $f(z) = z^n \pm z^m \pm 1$ to be considered. Applying Proposition 3 (one can use any mathematics software, e.g., SageMath [37]), we obtain all such polynomials with $N(f) = 1$ and $N(f) = 2$, which are given in the first and third columns of Table 2, respectively.

Case 2. We have that $\gcd(n, m) > 1$. Denote $a = \gcd(n, m)$. Then $n = an_1$ and $m = am_1$ for some coprime positive integers $n_1 > m_1$. Furthermore, $f(z) = z^n \pm z^m \pm 1 = g(z^a)$, where $g(z) = z^{n_1} \pm z^{m_1} \pm 1$ is a Borwein trinomial. One has that $N(f(z)) = N(g(z^a)) = aN(g(z))$. This, in view of $a > 1$ and $N(f) \leqslant 2$, implies $a = 2$, $N(f) = 2$, and $N(g(z)) = 1$. We have already determined all Borwein trinomials $g(z)$ with $N(g) = 1$ in Case 1 (see the first column in Table 2). Hence, $f(z) = g(z^2)$ for any polynomial $g(z)$ from the first column of Table 2. All such trinomials $f(z) = g(z^2)$ with $N(f) = 2$ are given in the second column of Table 2. □

Proof of Theorem 1. Let $f(z) = z^n \pm z^m \pm 1$ be a Borwein trinomial such that one of its roots, say β, is a complex Pisot number. Denote by $p(z)$ the minimal polynomial of β. Then, $p(z)$ is irreducible and divides $f(z)$. By Theorem 4, every root (if any) of the quotient $f(z)/p(z)$ is a unimodular number (if $f(z)$ is irreducible, then $f(z) = p(z)$ and $f(z)/p(z) = 1$). Hence, both polynomials $f(z)$ and $p(z)$ have the same number of roots outside the unit circle $|z| > 1$, and this number equals 2 since $p(z)$ is the minimal polynomial of a complex Pisot number. Therefore, $N(f^*) = N(p^*) = 2$. Now, we have that the Borwein trinomial $f^*(z)$ has exactly two roots inside the unit circle $|z| < 1$, namely β^{-1} and $\bar{\beta}^{-1}$. Recall that $\beta \in \mathbb{C} \setminus \mathbb{R}$. Thus, both roots of $f^*(z)$ inside the unit circle $|z| < 1$ are non-real numbers. On the other hand, Table 2 lists all Borwein trinomials $g(z)$ with $N(g) = 2$ in the second and third columns (see Theorem 2). One can easily check that all of these polynomials have two non-real roots inside the unit circle $|z| < 1$, except for polynomials $z^4 + z^2 - 1$, $z^6 + z^4 - 1$, $z^6 + z^2 - 1$, $z^8 + z^2 - 1$, and $z^{10} + z^2 - 1$, which all have two real roots inside the unit circle $|z| < 1$. Hence, all Borwein trinomials $f(z) = z^n \pm z^m \pm 1$, which have a complex Pisot number as a root, are given in Table 1. □

3. Irreducibility of Borwein Trinomials

Selmer [38] studied the irreducibility of trinomials $z^n \pm z \pm 1$. In particular, he proved that the trinomial $z^n - z - 1$ is irreducible for every positive integer $n > 1$. Tverberg [39] proved that a trinomial $z^n \pm z^m \pm 1$ is reducible if and only if it has a unimodular root. Ljunggren [35] extended this result to any quadrinomial $z^n \pm z^m \pm z^p \pm 1$.

Theorem 4 ([39] and Theorem 3 in [35])**.** *Let $n > m$ be positive integers. The trinomial $f(z) = z^n \pm z^m \pm 1$ is reducible over the rationals if and only if it has a unimodular root. If $f(z)$ has unimodular roots, these roots can be collected to give a rational factor of $f(z)$. The other factor of $f(z)$ is then irreducible.*

Note that, for any polynomial $f(z)$ and any positive integer a, one has that $U(f(z^a)) = aU(f(z))$. Thus, $f(z^a)$ has a unimodular root if and only if $f(z)$ has a unimodular root. Combining this and Theorem 4, we obtain that, for any positive integer a, the trinomial $z^n \pm z^m \pm 1$ is irreducible if and only if the trinomial $z^{na} \pm z^{ma} \pm 1$ is irreducible. Hence, considering the irreducibility of a trinomial $z^n \pm z^m \pm 1$, one can always assume that m and n are coprime.

The following corollary has already been proven by Ljunggren (see Theorem 3 in [35]). Nevertheless, we give an alternative proof of this result.

Corollary 4. *Let $n > m$ be positive integers and $a = \gcd(n, m)$.*

1. *The polynomial $f(z) = z^n + z^m + 1$ is reducible if and only if $(n + m)/a$ is divisible by 3. Furthermore, if $(n + m)/a$ is divisible by 3, then $f(z)$ has exactly $2a$ unimodular roots, which are the roots of $z^{2a} + z^a + 1$, and the quotient $f(z)/(z^{2a} + z^a + 1)$ is an irreducible polynomial.*
2. *The polynomial $f(z) = z^n + z^m - 1$ is reducible if and only if $(n + m)/a$ is divisible by 6. Furthermore, if $(n + m)/a$ is divisible by 6, then $f(z)$ has exactly $2a$ unimodular roots, which are the roots of $z^{2a} - z^a + 1$, and the quotient $f(z)/(z^{2a} - z^a + 1)$ is an irreducible polynomial.*
3. *The polynomial $f(z) = z^n - z^m + 1$ is reducible if and only if $(n + 4m)/a$ is divisible by 6. Furthermore, if $(n + 4m)/a$ is divisible by 6, then $f(z)$ has exactly $2a$ unimodular roots, which are the roots of $z^{2a} - z^a + 1$, and the quotient $f(z)/(z^{2a} - z^a + 1)$ is an irreducible polynomial.*
4. *The polynomial $f(z) = z^n - z^m - 1$ is reducible if and only if $(2n - m)/a$ is divisible by 6. Furthermore, if $(2n - m)/a$ is divisible by 6, then $f(z)$ has exactly $2a$ unimodular roots, which are the roots of $z^{2a} - z^a + 1$, and the quotient $f(z)/(z^{2a} - z^a + 1)$ is an irreducible polynomial.*

Proof of Corollary 4. We will consider only the case $f(z) = z^n + z^m + 1$. The remaining three propositions can be proven completely analogously.

Let $f(z) = z^n + z^m + 1$ and $\tilde{f}(z) = z^{n/a} + z^{m/a} + 1$. Note that n/a and m/a are coprime and $f(z) = \tilde{f}(z^a)$. Furthermore, $U(f(z)) = U(\tilde{f}(z^a)) = aU(\tilde{f}(z))$. Hence, by Theorem 4, the trinomial $f(z)$ is reducible if and only if the trinomial $\tilde{f}(z)$ has a unimodular root. By Corollary 3, the trinomial $\tilde{f}(z)$ has a unimodular root if and only if $n/a + m/a = (n + m)/a$ is divisible by 3. This proves the first part of the proposition.

Assume that $(n + m)/a$ is divisible by 3. Then, by Corollary 3, the trinomial $\tilde{f}(z)$ can be factored as $\tilde{f}(z) = (z^2 + z + 1)g(z)$, where $g(z)$ is a polynomial that has no unimodular roots. Hence, $f(z) = \tilde{f}(z^a) = (z^{2a} + z^a + 1)g(z^a)$. Note that every root of $z^{2a} + z^a + 1$ is a root of unity since $z^{3a} - 1 = (z^a - 1)(z^{2a} + z^a + 1)$. Finally, Theorem 4 implies that the quotient $f(z)/(z^{2a} + z^a + 1) = g(z^a)$ is an irreducible polynomial. □

A real algebraic integer $\alpha > 1$ is called a *Salem number* after [40–42], if all other algebraic conjugates of α lie in the unit disc $|z| \leqslant 1$ with at least one conjugate on the unit circle $|z| = 1$. In particular, the minimal polynomial of every Salem number is of even degree and self-reciprocal: $f^*(z) = f(z)$. Note that none of the algebraic conjugates of a Salem number is a root of unity. Therefore, by Corollary 4, no Salem number is the root of a Borwein trinomial.

4. Conclusions

In the present paper, we identify all complex Pisot numbers $\beta \in \mathbb{C}$ that arise from polynomials of the simplest possible shape, namely $\{-1, 0, 1\}$ – trinomials. We hope that the Pisot numbers listed in Table 1 for Theorem 1 will find application in new complex number systems [15,17,18], quasi-crystals [16], and digital filter designs [43,44]. In the future, it would be interesting to extend these results to the quadrinomial case.

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