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Double Fuzzy Ideal Multifunctions

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Abstract: In this paper, we introduce the notion of (p, q)-fuzzy local function and DF-ideal topological space. Also, we introduce the concepts DFU-ð-continuous and DFL-ð-continuous, almost ð-continuous, weakly ð-continuous and almost weakly ð-continuous multifunctions. Several properties and characterizations of the introduced multifunctions and types of continuity are established. Some examples are given to explain the correct implications between these notions.

Keywords: DF-ideal topological space; (p, q)-fuzzy local function; DF-multifunction; DFU-(DFL-) ð-continuous; almost ð-continuous; weakly ð-continuous; almost weakly ð-continuous

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1. Introduction and Preliminaries

Zadeh in [1] initiated the concept of fuzzy sets which has been applied in many branches. Chang in [2] initiated the notion of a topology of fuzzy sets. Sostak in [3] and Kubiak in [4] redefined the topological structure in the fuzzy case as a mapping with grades of openness and not as a family of fuzzy sets as introduced by Chang [2]. In [5], Sostak introduced a wide research study on the fuzzy topological structures. Atanassov in [6] initiated the concept of intuitionistic fuzzy sets. Coker in [7] constructed a topology in case of intuitionistic fuzzy sets. Following the constructed topology notion as presented by Sostak, Montal and Samanta [8] introduced the concept of the intuitionistic gradation of openness. Garcia and Rodabaugh in [9] proved their idea that the more suitable notation in the intuitionistic fuzzy case is the notation "double". That is, "intuitionistic fuzzy" will be replaced with "double fuzzy", and it will shortened to "DF". Thus, through this paper, intuitionistic fuzzy topological spaces will be denoted by DFTS and any type of intuitionistic fuzzy continuity will be denoted by DF-continuity. In [10], the author considered the normality of DFTS. In [11], the authors defined the DF-semi continuity. A fuzzy multifunction is a fuzzy set valued function ([12,13]). Fuzzy multifunctions are used and applied in many directions like economics, artificial intelligence, decision theory, uncertainty, etc. Each fuzzy multifunction has a lower inverse and an upper inverse. These lower inverses and upper inverses are numerous and imply several types of continuity as described in [14–17].

The motivation and the highlights of this paper are as follows: To introduce DF-local multifunctions related with DF-ideals and study its properties. Also, to submit new types of DF-continuity based on a DF-ideal and study the common properties of continuity and discuss the implications between these new types of continuity. Some examples are submitted to explain that these implications may be not reversed. The use of DF-ideals in defining these new types of continuity extended the usual corresponding definitions of fuzzy continuity, and so the introduced types of DF-continuity are extensions of the corresponding usual ones. We called these types of DF-ideal continuous multifunctions almost, weak and almost weak.

The paper is divided to five sections following this introduction as follows. Section 2 introduces the main definition of DF-local functions joined to a DF-ideal. Section 3 investigates the notion of DFU-almost and DFL-almost $\eth$-continuity, and introduces many characteristic properties of these defined multifunctions. Section 4 investigates the notion of DFU-weakly and DFL-weakly $\eth$-continuity, and discusses its properties, as well as studies the implications with the previous definition of DFU-almost and DFL-almost $\eth$-continuity. Section 5 investigates the notion of DFU-almost weakly and DFL-almost weakly $\eth$-continuity, and discusses its properties, as well as studying the implications with the previous definitions of DFU-almost and DFL-almost $\eth$-continuity and DFU-weakly and DFL-weakly $\eth$-continuity. Section 6 outlines the conclusion.

Let X be a universal set, $I = [0,1]$, $I_0 = (0,1]$, and $I_1 = [0,1)$.

I^X refers to the set of all fuzzy sets in X. $\underline{0}$ and $\underline{1}$ refer to the empty and the whole fuzzy sets, respectively, on X. By the family Ω, we refer to the set of all compared fuzzy sets, that is, $\lambda, \mu \in \Omega$ iff $\lambda \le \mu$ or $\mu \le \lambda$. The complement $\underline{1} - \lambda$ of a fuzzy set $\lambda \in I^X$ is defined by $\underline{1} - \lambda(x) = 1 - \lambda(x)$. A fuzzy point x_t in X is a fuzzy set, so that $x_t(z) = 0\ \forall z \neq x$ and $x_t(x) = t$. $x_t \in \lambda$ iff $t \le \lambda(x), x \in X$. The difference ([18]) $\mu \barwedge \lambda$, is defined by $\mu \barwedge \lambda = \underline{0}$ if $\mu \le \lambda$, and $\mu \barwedge \lambda = \mu \wedge \lambda^c$ otherwise.

Recall that a DF-ideal on X ([19]), $(\eth, \eth^*) : I^X \to I$ satisfies the following conditions:

(1) $\eth(\lambda) + \eth^*(\lambda) \le 1$.
(2) $\lambda_1 \le \lambda_2$ implies $\eth(\lambda_1) \ge \eth(\lambda_2)$ and $\eth^*(\lambda_1) \le \eth^*(\lambda_2)$.
(3) $\eth(\lambda_1 \vee \lambda_2) \ge \eth(\lambda_1) \wedge \eth(\lambda_2)$ and $\eth^*(\lambda_1 \vee \lambda_2) \le \eth^*(\lambda_1) \vee \eth^*(\lambda_2)$.
(4) $\eth(\underline{0}) = 1, \eth(\underline{1}) = 0, \eth^*(\underline{0}) = 0, \eth^*(\underline{1}) = 1$.

The special DF-ideals $(\eth^0, \eth^{*0})$, $(\eth^1, \eth^{*1})$ are defined by: $\eth^0(\underline{0}) = \eth^{*1}(\underline{1}) = 1, \eth^{*0}(\underline{0}) = \eth^1(\underline{1}) = 0$, otherwise we have $\eth^0(\nu) = \eth^{*1}(\nu) = 0$ and $\eth^1(\nu) = \eth^{*0}(\nu) = 1$.

Let $(\eth_1, \eth_1^*)$ and $(\eth_2, \eth_2^*)$ be DF-ideals on X. Then, $(\eth_1, \eth_1^*) \le (\eth_2, \eth_2^*)$ iff $\eth_2(\nu) \le \eth_1(\nu)$ and $\eth_2^*(\nu) \ge \eth_1^*(\nu)$ for each $\nu \in I^X$. If $\nu_1 \le \nu_2$, then $\eth(\nu_1) \ge \eth(\nu_2)$ and $\eth^*(\nu_1) \le \eth^*(\nu_2)$.

A map $\Phi : X \multimap Y$ is called a fuzzy multifunction ([20]) iff $\Phi(x) \in I^Y$ for each $x \in X$. The membership value of y to $\Phi(x)$ is $\Phi(x)(y) = G_\Phi(x,y)$ for all $(x,y) \in X \times Y$. The domain of Φ, denoted by $dom(\Phi)$ and the range of Φ, denoted by $rng(\Phi)$ for any $x \in X$ and $y \in Y$ are defined by: $dom(\Phi)(x) = \bigvee_{y \in Y} G_\Phi(x,y)$ and $rng(\Phi)(x) = \bigvee_{x \in X} G_\Phi(x,y)$. Φ is called normalized fuzzy multifunction iff for each $x \in X$, there exists $y_0 \in Y$ such that $G_\Phi(x, y_0) = \underline{1}$. Φ is called crisp fuzzy multifunction iff $G_\Phi(x, y_0) = \underline{1}$ for each $x \in X$ and $y \in Y$. The image $\Phi(\lambda)$ of $\lambda \in I^X$, the lower inverse $\Phi^l(\lambda)$ and the upper inverse $\Phi^u(\lambda)$ are defined, respectively, as follows: $\Phi(\lambda)(y) = \bigvee_{x \in X} (G_\Phi(x,y) \wedge \lambda(x))$, $\Phi^l(\lambda)(x) = \bigvee_{y \in Y} (G_\Phi(x,y) \wedge \lambda(y))$, $\Phi^u(\lambda)(x) = \bigwedge_{y \in Y} (G^c_\Phi(x,y) \vee \lambda(y))$.

Let $\Phi : X \multimap Y$ and $\Xi : Y \multimap Z$ be two fuzzy multifunctions. Then, the composition $\Xi \circ \Phi : X \multimap Z$ is defined by $((\Xi \circ \Phi)(x))(z) = \bigvee_{y \in Y} (G_\Phi(x,y) \wedge G_\Xi(y,z))$.

If $\Phi : (X, \tau, \tau^\circ) \multimap (Y, \sigma, \sigma^\circ)$ is a DF-multifunction, $p \in I_0$ and $q \in I_1$, then, Φ is called ([21]):

(1) DFU-semi-continuous at a fuzzy point $x_t \in dom(\Phi)$ if $x_t \in \Phi^u(\mu)$ for each $\mu \in I^Y$ and $\sigma(\mu) \ge p, \sigma^\circ(\mu) \le q$, there exists $\lambda \in I^X$, $\tau(\lambda) \ge p$, $\tau^\circ(\lambda) \le q$ and $x_t \in \lambda$ such that $\lambda \wedge dom(\Phi) \le \Phi^u(\mu)$.
(2) DFL-semi-continuous at a fuzzy point $x_t \in dom(\Phi)$ if $x_t \in \Phi^l(\mu)$ for each $\mu \in I^Y$ and $\sigma^\circ(\mu) \ge p, \sigma(\mu) \le q$, there exists $\lambda \in I^X$, $\tau(\lambda) \ge p$, $\tau^\circ(\lambda) \le q$ and $x_t \in \lambda$ such that $\lambda \le \Phi^l(\mu)$.
(3) DFU- (DFL-) semi-continuous if it is DFU- (DFL-) semi-continuous at every each point $x_t \in dom(\Phi)$.

All definitions and properties of the image, the lower inverse, the upper inverse and the composition of fuzzy multifunctions could be found in [7,8,20].

2. DF-Local Functions

In this section, we introduce the notion of (p,q)-fuzzy local functions related with a DF-ideal.

Definition 1. *Let $(X,\tau,\tau^\circ,(\eth,\eth^*))$ be a DF-$\eth$-topological space, $\lambda \in I^X$, $p \in I_0$ and $q \in I_1$. Then, the (p,q)-fuzzy local function $\Psi(\lambda,p,q)$ of λ is defined as follows:*

$$\Psi(\lambda,p,q) = \bigwedge\{\mu \in I^X : \eth(\lambda \bar{\wedge} \mu) \geq p, \eth^*(\lambda \bar{\wedge} \mu) \leq q, \tau(\mu^c) \geq p, \tau^\circ(\mu^c) \leq q\}.$$

Remark 1. (1) *If $\eth = \eth^0$ and $\eth^* = \eth^{*0}$ for each $\lambda \in I^X$, we obtain that $\Psi(\lambda,p,q) = \bigwedge\{\mu \in I^X : \lambda \leq \mu, \tau(\mu^c) \geq p, \tau^\circ(\mu^c) \leq q\} = C_{\tau,\tau^\circ}(\lambda,p,q)$.*

(2) *If $\eth = \eth^1$ and $\eth^* = \eth^{*1}$, $\eth(\lambda) \geq p$, $\eth^*(\lambda) \leq q$) for each $\lambda \in I^X$, we obtain that $\Psi(\lambda,p,q) = \underline{0}$.*

Note that: $\Psi(\lambda,p,q)$ or $\Psi(\lambda,\eth,\eth^*,p,q)$ stands for the same local function Ψ.

Theorem 1. *Let $(X,\tau,\tau^\circ,(\eth,\eth^*))$ be a DF-ideal topological space and $(\eth_1,\eth_1^*)$, $(\eth_2,\eth_2^*)$ be DF-ideals on X. Then, for any sets $\lambda,\nu \in I^X$, $p \in I_0$ and $q \in I_1$:*

(1) $\Psi(\underline{0},p,q) = \underline{0}$.
(2) *If $\lambda \leq \nu$, then $\Psi(\lambda,p,q) \leq \Psi(\nu,p,q)$.*
(3) *If $(\eth_1,\eth_1^*) \leq (\eth_2,\eth_2^*)$, then $\Psi(\lambda,\eth_1,\eth_1^*,p,q) \leq \Psi(\lambda,\eth_2,\eth_2^*,p,q)$.*
(4) $\Psi(\lambda,p,q) = C_{\tau,\tau^\circ}(\Psi(\lambda,p,q),p,q) \leq C_{\tau,\tau^\circ}(\lambda,p,q)$.
(5) $\Psi(\Psi(\lambda,p,q),p,q) \leq \Psi(\lambda,p,q)$ *and* $(\Psi(\lambda,p,q))^c \neq \Psi(\lambda^c,p,q)$.
(6) $\Psi(\lambda \vee \nu,p,q) \geq \Psi(\lambda,p,q) \vee \Psi(\nu,p,q)$ *and* $\Psi(\lambda \wedge \nu,p,q) \leq \Psi(\lambda,p,q) \wedge \Psi(\nu,p,q)$.
(7) *If $\eth(\nu) \geq p$ and $\eth^*(\nu) \leq q$, then $\Psi(\lambda \vee \nu,p,q) \geq \Psi(\lambda,p,q)$.*

Proof. (1) From Definition 1, we have $\Psi(\underline{0},p,q) = \underline{0}$.

(2) Assume that $\Psi(\lambda,p,q) \not\leq \Psi(\nu,p,q)$. If $\lambda \leq \nu$, then using the definition of $\Psi(\nu,p,q)$, there exists $\mu \in I^X$ with $\Psi(\nu,p,q) \leq \mu$, $\eth(\nu \bar{\wedge} \mu) \geq p$, $\eth^*(\nu \bar{\wedge} \mu) \leq q$, $\tau(\mu^c) \geq p$, $\tau^\circ(\mu^c) \leq q$ such that $\Psi(\lambda,p,q) \not\leq \mu$. Since $\lambda \leq \nu$ implies that $\lambda \bar{\wedge} \mu \leq \nu \bar{\wedge} \mu$, $\eth(\lambda \bar{\wedge} \mu) \geq \eth(\nu \bar{\wedge} \mu) \geq p$ and $\eth^*(\lambda \bar{\wedge} \mu) \leq \eth^*(\nu \bar{\wedge} \mu) \leq q$. Hence, $\Psi(\lambda,p,q) \leq \mu$, and this is a contradiction. Thus, $\Psi(\lambda,p,q) \leq \Psi(\nu,p,q)$.

(3) Suppose that $\Psi(\lambda,\eth_1,\eth_1^*,p,q) \not\leq \Psi(\lambda,\eth_2,\eth_2^*,p,q)$ and $(\eth_1,\eth_1^*) \leq (\eth_2,\eth_2^*)$. From the definition of $\Psi(\lambda,\eth_2,\eth_2^*,p,q)$, there exists $\mu \in I^X$ with $\Psi(\lambda,\eth_2,\eth_2^*,p,q) \leq \mu$, $\eth_2(\lambda \bar{\wedge} \mu) \geq p$, $\eth_2^*(\lambda \bar{\wedge} \mu) \leq q$, $\tau(\mu^c) \geq p$, $\tau^\circ(\mu^c) \leq q$ such that $\Psi(\lambda,\eth_1,\eth_1^*,p,q) \not\leq \mu$. Since $(\eth_1,\eth_1^*) \leq (\eth_2,\eth_2^*)$ implies that $\eth_1(\lambda \bar{\wedge} \mu) \geq \eth_2(\lambda \bar{\wedge} \mu) \geq p$ and $\eth_1^*(\lambda \bar{\wedge} \mu) \leq \eth_2^*(\lambda \bar{\wedge} \mu) \leq q$. Hence, $\Psi(\lambda,\eth_1,\eth_1^*,p,q) \leq \mu$, and this is a contradiction. Thus, $\Psi(\lambda,\eth_1,\eth_1^*,p,q) \leq \Psi(\lambda,\eth_2,\eth_2^*,p,q)$.

(4) Definition 1 implies that $\Psi(\lambda,p,q) = C_{\tau,\tau^\circ}(\Psi(\lambda,p,q),p,q)$. Since $(\eth^0,\eth^{*0}) \leq (\eth,\eth^*)$ for any DF-ideal $(\eth,\eth^*)$, $\Psi(\lambda,\eth,\eth^*,p,q) \leq \Psi(\lambda,\eth^0,\eth^{*0},p,q) = C_{\tau,\tau^\circ}(\lambda,p,q)$. Then, $\Psi(\lambda,p,q) = C_{\tau,\tau^\circ}(\Psi(\lambda,p,q),p,q) \leq C_{\tau,\tau^\circ}(\lambda,p,q)$.

(5) From (4), we have $\Psi(\Psi(\lambda,p,q),p,q) = C_{\tau,\tau^\circ}(\Psi(\Psi(\lambda,p,q),p,q),p,q) \leq C_{\tau,\tau^\circ}(\Psi(\lambda,p,q),p,q) = \Psi(\lambda,p,q)$. (In general the converse is not true as shown by Example 1).

(6) Since $\lambda \leq \lambda \vee \nu$ and $\nu \leq \lambda \vee \nu$ imply that $\Psi(\lambda,p,q) \leq \Psi(\lambda \vee \nu,p,q)$ and $\Psi(\nu,p,q) \leq \Psi(\lambda \vee \nu,p,q)$. Thus, $\Psi(\lambda,p,q) \vee \Psi(\nu,p,q) \leq \Psi(\lambda \vee \nu,p,q)$. Also, since $\lambda \wedge \nu \leq \lambda$ and $\lambda \wedge \nu \leq \nu$ imply that $\Psi(\lambda \wedge \nu,p,q) \leq \Psi(\lambda,p,q)$ and $\Psi(\lambda \wedge \nu,p,q) \leq \Psi(\nu,p,q)$. Hence, $\Psi(\lambda \wedge \nu,p,q) \leq \Psi(\lambda,p,q) \wedge \Psi(\nu,p,q)$.

(7) Since $\eth(\nu) \geq p$ and $\eth^*(\nu) \leq q$ imply that $\Psi(\nu,p,q) = \underline{0}$. Thus, $\Psi(\lambda \vee \nu,p,q) \geq \Psi(\lambda,p,q) \vee \Psi(\nu,p,q) \geq \Psi(\lambda,p,q)$. □

Lemma 1. *Let $(\tau,\tau^\circ) : \Omega \to I$ be a DF-topology on X and $(\eth,\eth^*) : \Omega \to I$ be a DF-ideal on X. Then, for each $\lambda,\mu \in \Omega$, $p \in I_0$ and $q \in I_1$,*

(1) $\Psi(\lambda \vee \mu,p,q) = \Psi(\lambda,p,q) \vee \Psi(\mu,p,q)$.
(2) *If $\eth(\mu) \geq p$ and $\eth^*(\mu) \leq q$, then $\Psi(\lambda \vee \mu,p,q) = \Psi(\lambda,p,q)$.*

Proof. (1) Suppose $\Psi(\lambda \vee \mu,p,q) \not\leq \Psi(\lambda,p,q) \vee \Psi(\mu,p,q)$, then there exist $\nu_1,\nu_2 \in \Omega$, $\eth(\lambda \bar{\wedge} \nu_1) \geq p$, $\eth^*(\lambda \bar{\wedge} \nu_1) \leq q$ with $\tau(\nu_1^c) \geq p$, $\tau^\circ(\nu_1^c) \leq q$ and $\eth(\mu \bar{\wedge} \nu_2) \geq p$, $\eth^*(\mu \bar{\wedge}$

$\nu_2) \leq q$ with $\tau(\nu_2^c) \geq p$, $\tau^\circ(\nu_2^c) \leq q$ such that $\Psi(\lambda, p, q) \vee \Psi(\mu, p, q) \leq \nu_1 \vee \nu_2 \leq \Psi(\lambda \vee \mu, p, q)$. However, $(\lambda \vee \mu) \overline{\wedge} (\nu_1 \vee \nu_2) \leq (\lambda \overline{\wedge} \nu_1) \vee (\mu \overline{\wedge} \nu_2)$, then $\eth[(\lambda \vee \mu) \overline{\wedge} (\nu_1 \vee \nu_2)] \geq p$, $\eth^*[(\lambda \vee \mu) \overline{\wedge} (\nu_1 \vee \nu_2)] \leq q$ and $\tau((\nu_1 \vee \nu_2)^c) \geq p$, $\tau^\circ((\nu_1 \vee \nu_2)^c) \leq q$. So, $\Psi(\lambda \vee \mu, p, q) \leq \nu_1 \vee \nu_2$, and this is a contradiction. Thus, $\Psi(\lambda \vee \mu, p, q) \leq \Psi(\lambda, p, q) \vee \Psi(\mu, p, q)$.

(2) Since $\eth(\mu) \geq p$ and $\eth^*(\mu) \leq q$ imply that $\Psi(\mu, p, q) = \underline{0}$. Then, $\Psi(\lambda \vee \mu, p, q) = \Psi(\lambda, p, q) \vee \Psi(\mu, p, q) = \Psi(\lambda, p, q)$. □

The following example shows that generally, $\Psi(\Psi(\lambda, p, q), p, q) \neq \Psi(\lambda, p, q)$ and $(\Psi(\lambda, p, q))^c \neq \Psi(\lambda^c, p, q)$ for any $\lambda \in I^X$, $p \in I_0$ and $q \in I_1$.

Example 1. *Define $\tau, \tau^\circ, \eth, \eth^* : I^X \to I$ as follows:*

$$\tau(\lambda) = \begin{cases} 1, & \lambda \in \{\underline{0}, \underline{1}\} \\ \frac{1}{2}, & \lambda = \underline{0.5} \\ \frac{1}{3}, & \lambda = \underline{0.9} \\ 0, & o.w, \end{cases} \qquad \tau^\circ(\lambda) = \begin{cases} 0, & \lambda \in \{\underline{0}, \underline{1}\} \\ \frac{1}{2}, & \lambda = \underline{0.5} \\ \frac{2}{3}, & \lambda = \underline{0.9} \\ 1, & o.w, \end{cases}$$

$$\eth(\nu) = \begin{cases} 1, & \nu = \underline{0} \\ \frac{3}{4}, & \underline{0} < \nu < \underline{0.5} \\ \frac{1}{2}, & \nu = \underline{0.5} \\ 0, & o.w, \end{cases} \qquad \eth^*(v) = \begin{cases} 0, & \nu = \underline{0} \\ \frac{1}{4}, & \underline{0} < \nu < \underline{0.5} \\ \frac{1}{2}, & \nu = \underline{0.5} \\ 1, & o.w. \end{cases}$$

Then,

$$\underline{0} = \Psi(\Psi(\underline{0.7}, \frac{1}{3}, \frac{2}{3}), \frac{1}{3}, \frac{2}{3}) \neq \Psi(\underline{0.7}, \frac{1}{3}, \frac{2}{3}) = \underline{0.5},$$

and

$$\underline{1} = \left(\Psi(\underline{0.5}, \frac{1}{3}, \frac{2}{3})\right)^c \neq \Psi(\underline{(0.5)^c}, \frac{1}{3}, \frac{2}{3}) = \underline{0}.$$

Lemma 2. *Let $(X, \tau, \tau^\circ, (\eth, \eth^*))$ be a DF-ideal topological space and $\{\lambda_j,\ j \in J\} \subset I^X$. Then, $\bigvee_{j \in J} \Psi(\lambda_j, p, q) \leq \Psi(\bigvee_{j \in J} \lambda_j, p, q)$ (resp. $\Psi(\bigwedge_{j \in J} \lambda_j, p, q) \leq \bigwedge_{j \in J} \Psi(\lambda_j, p, q)$).*

Proof. Since $\lambda_j \leq \bigvee_{j \in J} \lambda_j$ for each $j \in J$ and from Theorem 1 (2), we obtain that $\Psi(\lambda_j, p, q) \leq \Psi(\bigvee_{j \in J} \lambda_j, p, q)$ for each $j \in J$. This implies that

$$\bigvee_{j \in J} \Psi(\lambda_j, p, q) \leq \Psi(\bigvee_{j \in J} \lambda_j, p, q).$$

The other case is similarly proven. □

Definition 2. *Let $(X, \tau, \tau^\circ, (\eth, \eth^*))$ be a DF-ideal topological space. Then, for each $\lambda \in I^X$, $p \in I_0$ and $q \in I_1$, we define an operator $cl^* : I^X \times I_0 \times I_1 \to I^X$ as follows:*

$$cl^*(\lambda, p, q) = \lambda \vee \Psi(\lambda, p, q).$$

If $(\eth, \eth^*) = (\eth^0, \eth^{*0})$, then $cl^*(\lambda, p, q) = \lambda \vee \Psi(\lambda, p, q) = \lambda \vee C_{\tau, \tau^\circ}(\lambda, p, q) = C_{\tau, \tau^\circ}(\lambda, p, q)$ for each $\lambda \in I^X$. Again, if $(\eth, \eth^*) = (\eth^1, \eth^{*1})$ (resp. $\eth(\nu) \geq p$ and $\eth^*(\nu) \leq q$), then $cl^*(\lambda, p, q) = \lambda$ for each $\lambda \in I^X$.

Theorem 2. *Let $(X, \tau, \tau^\circ, (\eth, \eth^*))$ be a DF-ideal topological space. Then, for any $\lambda, \nu \in I^X$, $p \in I_0$, and $q \in I_1$, the operator $cl^* : I^X \times I_0 \times I_1 \to I^X$ satisfies the following properties:*

(1) $cl^*(\underline{0}, p, q) = \underline{0}$.
(2) $\lambda \leq cl^*(\lambda, p, q) \leq C_{\tau,\tau^\circ}(\lambda, p, q)$.
(3) *If* $\lambda \leq \nu$, *then* $cl^*(\lambda, p, q) \leq cl^*(\nu, p, q)$.
(4) $cl^*(\lambda \vee \nu, p, q) \geq cl^*(\lambda, p, q) \vee cl^*(\nu, p, q)$.
(5) $cl^*(\lambda \wedge \nu, p, q) \leq cl^*(\lambda, p, q) \wedge cl^*(\nu, p, q)$.

Proof. (1) Since $cl^*(\underline{0}, p, q) = \underline{0} \vee \Psi(\underline{0}, p, q)$ and $\Psi(\underline{0}, p, q) = \underline{0}$, then $cl^*(\underline{0}, p, q) = \underline{0}$.

(2) $cl^*(\lambda, p, q) = \lambda \vee \Psi(\lambda, p, q)$ implies that $\lambda \leq cl^*(\lambda, p, q)$. From $\lambda \leq C_{\tau,\tau^\circ}(\lambda, p, q)$, we have $\Psi(\lambda, p, q) \leq C_{\tau,\tau^\circ}(\lambda, p, q)$ implies that $cl^*(\lambda, p, q) \leq C_{\tau,\tau^\circ}(\lambda, p, q)$. Thus, $\lambda \leq cl^*(\lambda, p, q) \leq C_{\tau,\tau^\circ}(\lambda, p, q)$.

(3) From $\lambda \leq \nu$ and Theorem 1 (2), we have $\lambda \vee \Psi(\lambda, p, q) \leq \nu \vee \Psi(\nu, p, q)$, that is, $cl^*(\lambda, p, q) \leq cl^*(\nu, p, q)$.

(4) Since $\lambda \leq \lambda \vee \nu$ and $\nu \leq \lambda \vee \nu$ imply that $cl^*(\lambda, p, q) \leq cl^*(\lambda \vee \nu, p, q)$ and $cl^*(\nu, p, q) \leq cl^*(\lambda \vee \nu, p, q)$. Thus, $cl^*(\lambda, p, q) \vee cl^*(\nu, p, q) \leq cl^*(\lambda \vee \nu, p, q)$.

(5) $\lambda \wedge \nu \leq \lambda$ and $\lambda \wedge \nu \leq \nu$ imply that $cl^*(\lambda \wedge \nu, p, q) \leq cl^*(\lambda, p, q)$ and $cl^*(\lambda \wedge \nu, p, q) \leq cl^*(\nu, p, q)$. Thus, $cl^*(\lambda \wedge \nu, p, q) \leq cl^*(\lambda, p, q) \wedge cl^*(\nu, p, q)$. □

Lemma 3. *Let $(\tau, \tau^\circ) : \Omega \to I$ be a DF-topology on X and $(\eth, \eth^*) : \Omega \to I$ be a DF-ideal on X. Then, for each $\lambda, \mu \in \Omega$, $p \in I_0$ and $q \in I_1$, the operator $cl^* : \Omega \times I_0 \times I_1 \to \Omega$ satisfies the following:*

(1) $cl^*(\lambda \vee \mu, p, q) = cl^*(\lambda, p, q) \vee cl^*(\mu, p, q)$.
(2) $cl^*(cl^*(\lambda, p, q), p, q) = cl^*(\lambda, p, q)$.

Proof. (1) By the definition of cl^*, we have

$$\begin{aligned} cl^*(\lambda \vee \mu, p, q) &= (\lambda \vee \mu) \vee \Psi(\lambda \vee \mu, p, q) \\ &= (\lambda \vee \mu) \vee (\Psi(\lambda, p, q) \vee \Psi(\mu, p, q)) \\ &= (\lambda \vee \Psi(\lambda, p, q)) \vee (\mu \vee \Psi(\mu, p, q)) \\ &= cl^*(\lambda, p, q) \vee cl^*(\mu, p, q). \end{aligned}$$

(2) $cl^*(\lambda, p, q) \leq cl^*(cl^*(\lambda, p, q), p, q)$. Now, it will be shown that $cl^*(\lambda, p, q) \geq cl^*(cl^*(\lambda, p, q), p, q)$, where

$$\begin{aligned} cl^*(cl^*(\lambda, p, q), p, q) &= cl^*(\lambda \vee \Psi(\lambda, p, q), p, q) \\ &= cl^*(\lambda, p, q) \vee cl^*(\Psi(\lambda, p, q), p, q) \\ &= cl^*(\lambda, p, q) \vee \Psi(\lambda, p, q) \vee \Psi(\Psi(\lambda, p, q), p, q) \\ &= cl^*(\lambda, p, q) \vee \Psi(\Psi(\lambda, p, q), p, q) \\ &\leq cl^*(\lambda, p, q) \vee \Psi(\lambda, p, q) = cl^*(\lambda, p, q), \end{aligned}$$

and thus the proof is completed. □

From Lemma 3, $\tau(\lambda) = \bigvee\{p : cl^*(\lambda^c, p, q) = \lambda^c\}$, $\tau^\circ(\lambda) = \bigwedge\{q : cl^*(\lambda^c, p, q) = \lambda^c\}$, and then (τ, τ°) is a DF-topology generated by cl^*.

Theorem 3. *Let $(X, \tau, \tau^\circ, (\eth, \eth^*))$ be a DF-ideal topological space. Then, for each $\lambda \in I^X$, $p \in I_0$ and $q \in I_1$, we define an operator $int^* : I^X \times I_0 \times I_1 \to I^X$ as follows:*

$$int^*(\lambda, p, q) = \lambda \wedge (\Psi(\lambda^c, p, q))^c.$$

For $\lambda, \nu \in I^X$, the operator int^* fulfills the following:

(1) $int^*(\underline{1}, p, q) = \underline{1}$.
(2) $I_{\tau,\tau^\circ}(\lambda, p, q) \leq int^*(\lambda, p, q) \leq \lambda$.

(3) *If* $\lambda \leq \nu$, *then* $int^*(\lambda, p, q) \leq int^*(\nu, p, q)$.
(4) $int^*(\lambda \wedge \nu, p, q) \leq int^*(\lambda, p, q) \wedge int^*(\nu, p, q)$.
(5) $int^*(\lambda, p, q) = I_{\tau,\tau^\circ}(\lambda, p, q)$ *if* $(\eth, \eth^*) = (\eth^0, \eth^{*0})$.
(6) $int^*(\lambda^c, p, q) = (cl^*(\lambda, p, q))^c$.

Proof. Direct. □

Lemma 4. *Let* $(\tau, \tau^\circ) : \Omega \to I$ *be a DF-topology on X and* $(\eth, \eth^*) : \Omega \to I$ *be a DF-ideal on X. Then, for each* $\lambda, \mu \in \Omega$, $p \in I_0$ *and* $q \in I_1$, *the operator* $int^* : \Omega \times I_0 \times I_1 \to \Omega$ *satisfies the following:*

(1) $int^*(int^*(\lambda, p, q), p, q) = int^*(\lambda, p, q)$.
(2) $int^*(\lambda \wedge \nu, p, q) = int^*(\lambda, p, q) \wedge int^*(\nu, p, q)$.

Proof. It is similarly proven as that proof of Lemma 3. □

Definition 3. *Let* $\Phi : (X, \tau, \tau^\circ, (\eth, \eth^*)) \multimap (Y, \sigma, \sigma^\circ)$ *be a DF-multifunction,* $p \in I_0$ *and* $q \in I_1$. *Then,* Φ *is called:*

(1) *DFU-*$\eth$*-continuous at a fuzzy point* $x_t \in dom(\Phi)$ *iff* $x_t \in \Phi^u(\mu)$ *for each* $\mu \in I^Y$, $\sigma(\mu) \geq p$ *and* $\sigma^\circ(\mu) \leq q$, *there exists* $\lambda \in I^X, \tau(\lambda) \geq p$ *and* $\tau^\circ(\lambda) \leq q$ *and* $x_t \in \lambda$ *such that* $\lambda \wedge dom(\Phi) \leq \Psi(\Phi^u(\mu), p, q)$.
(2) *DFL-*$\eth$*-continuous at a fuzzy point* $x_t \in dom(\Phi)$ *iff* $x_t \in \Phi^l(\mu)$ *for each* $\mu \in I^Y$, $\sigma(\mu) \geq p$ *and* $\sigma^\circ(\mu) \leq q$, *there exists* $\lambda \in I^X, \tau(\lambda) \geq p$ *and* $\tau^\circ(\lambda) \leq q$ *and* $x_t \in \lambda$ *such that* $\lambda \leq \Psi(\Phi^l(\mu), p, q)$.
(3) *DFU- (resp. DFL-)* $\eth$*-continuous iff it is DFU- (resp. DFL-)* $\eth$*-continuous at every fuzzy point* $x_t \in dom(\Phi)$.

Remark 2. *If* Φ *is a normalized multifunction, then* Φ *is DFU-* $\eth$*-continuous at* $x_t \in dom(\Phi)$ *iff* $x_t \in \Phi^u(\mu)$ *for each* $\mu \in I^Y$, $\sigma(\mu) \geq p$ *and* $\sigma^\circ(\mu) \leq q$, *there exists* $\lambda \in I^X$, $\tau(\lambda) \geq p$, $\tau^\circ(\lambda) \leq q$ *and* $x_t \in \lambda$ *such that* $\lambda \leq \Psi(\Phi^u(\mu), p, q)$.

Theorem 4. *Let* $\Phi : (X, \tau, \tau^\circ, (\eth, \eth^*)) \multimap (Y, \sigma, \sigma^\circ)$ *be a DF- (resp. normalized) multifunction, Then,* Φ *is DFL- (DFU-)* $\eth$*-continuous iff* $\Phi^l(\mu) \leq I_{\tau,\tau^\circ}\Big(\Psi(\Phi^l(\mu), p, q), p, q\Big)$ *(resp.* $\Phi^u(\mu) \leq I_{\tau,\tau^\circ}(\Psi(\Phi^u(\mu), p, q), p, q)$*) for each* $\mu \in I^Y$, $\sigma(\mu) \geq p$, $\sigma^\circ(\mu) \leq q$, $p \in I_0$ *and* $q \in I_1$.

Proof. ($\Rightarrow$) Let $x_t \in dom(\Phi)$, $\mu \in I^Y$, $\sigma(\mu) \geq p$, $\sigma^\circ(\mu) \leq q$ and $x_t \in \Phi^l(\mu)$. Then, there exists $\lambda \in I^X, \tau(\lambda) \geq p, \tau^\circ(\lambda) \leq q$ and $x_t \in \lambda$ such that $\lambda \leq \Psi(\Phi^l(\mu), p, q)$. Thus, $x_t \in \lambda \leq I_{\sigma,\sigma^\circ}(\Psi(\Phi^l(\mu), p, q), p, q)$, and hence $\Phi^l(\mu) \leq I_{\sigma,\sigma^\circ}(\Psi(\Phi^l(\mu), p, q), p, q)$.

($\Leftarrow$) Let $x_t \in dom(\Phi)$, $\mu \in I^Y$, $\sigma(\mu) \geq p$, $\sigma^\circ(\mu) \leq q$ and $x_t \in \Phi^l(\mu)$. Then, $\Phi^l(\mu) \leq I_{\sigma,\sigma^\circ}(\Psi(\Phi^l(\mu), p, q), p, q)$, and hence $x_t \in I_{\sigma,\sigma^\circ}(\Psi(\Phi^l(\mu), p, q), p, q) \leq \Psi(\Phi^l(\mu), p, q)$. Thus, Φ is DFL-$\eth$-continuous.

The other case is similarly proven. □

Remark 3. (1) *DFU- (resp. DFL-)* $\eth^0$*-continuity* $\Leftrightarrow$ *DFU- (resp. DFL-) semi-continuity.*

(2) *DFU- (resp. DFL-)* $\eth$*-continuity and DFU- (resp. DFL-) semi-continuity are independent notions as shown by Example 2.*

Example 2. *Let* $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$ *and* $\Phi : X \multimap Y$ *be a DF-multifunction defined by* $G_\Phi(x_1, y_1) = 0.3$, $G_\Phi(x_1, y_2) = 0.1$, $G_\Phi(x_1, y_3) = 1$, $G_\Phi(x_2, y_1) = 1$, $G_\Phi(x_2, y_2) = 0.1$, $G_\Phi(x_2, y_3) = 0.5$. *Define the DF-topologies* $(\tau_1, \tau_1^*), (\tau_2, \tau_2^*) : I^X \to I$, $(\sigma, \sigma^\circ) : I^Y \to I$, *and the DF-ideals* $(\eth_1, \eth_1^*)$, $(\eth_2, \eth_2^*) : I^X \to I$ *as follows:*

$$\tau_1(\lambda) = \begin{cases} 1, & \lambda \in \{\underline{0}, \underline{1}\} \\ 0.5, & \lambda = \underline{0.3} \\ 0.7, & \lambda = \underline{0.4} \\ 0, & o.w, \end{cases} \qquad \tau_1^*(\lambda) = \begin{cases} 0, & \lambda \in \{\underline{0}, \underline{1}\} \\ 0.5, & \lambda = \underline{0.3} \\ 0.3, & \lambda = \underline{0.4} \\ 1, & o.w, \end{cases}$$

$$\tau_2(\lambda) = \begin{cases} 1, & \lambda \in \{\underline{0}, \underline{1}\} \\ 0.5, & \lambda = \underline{0.4} \\ 0, & o.w, \end{cases} \qquad \tau_2^*(\lambda) = \begin{cases} 0, & \lambda \in \{\underline{0}, \underline{1}\} \\ 0.5, & \lambda = \underline{0.4} \\ 1, & o.w, \end{cases}$$

$$\eth_1(\nu) = \begin{cases} 1, & \nu = \underline{0} \\ 0.6, & \underline{0} < \nu < \underline{0.4} \\ 0, & o.w, \end{cases} \qquad \eth_1^*(\nu) = \begin{cases} 0, & \nu = \underline{0} \\ 0.4, & \underline{0} < \nu < \underline{0.4} \\ 1, & o.w, \end{cases}$$

$$\eth_2(\nu) = \begin{cases} 1, & \nu = \underline{0} \\ 0.5, & \underline{0} < \nu < \underline{0.3} \\ 0, & o.w, \end{cases} \qquad \eth_2^*(\nu) = \begin{cases} 0, & \nu = \underline{0} \\ 0.5, & \underline{0} < \nu < \underline{0.3} \\ 1, & o.w, \end{cases}$$

$$\sigma(\mu) = \begin{cases} 1, & \mu \in \{\underline{0}, \underline{1}\} \\ 0.5, & \mu = \underline{0.3} \\ 0, & o.w, \end{cases} \qquad \sigma^\circ(\mu) = \begin{cases} 0, & \mu \in \{\underline{0}, \underline{1}\} \\ 0.5, & \mu = \underline{0.3} \\ 1, & o.w. \end{cases}$$

Let $p = 0.5$ and $q = 0.5$. Then, (1) $\Phi : (X, \tau_1, \tau_1^*, (\eth_1, \eth_1^*)) \multimap (Y, \sigma, \sigma^\circ)$ *is DFU- (resp. DFL-) semi-continuous but it is not DFU- (resp. DFL-) $\eth$-continuous because*

$$\Phi^u(\underline{0.3}) = \underline{0.3} \text{ (resp. } \Phi^l(\underline{0.3}) = \underline{0.3}) \text{ and } \underline{0.3} \leq I_{\tau,\tau^*}(\underline{0.3}, 0.5, 0.5) = \underline{0.3}$$

and $\underline{0.3} \not\leq I_{\tau_1,\tau_1^*}((\Psi(\underline{0.3}, 0.5, 0.5), 0.5, 0.5) = \underline{0}$.

(2) $\Phi : (X, \tau_2, \tau_2^*, (\eth_2, \eth_2^*)) \multimap (Y, \sigma, \sigma^\circ)$ *is DFU- (resp. DFL-) $\eth$-continuous but it is not DFU- (resp. DFL-) semi-continuous because* $\Phi^u(\underline{0.3}) = \underline{0.3}$ (resp. $\Phi^l(\underline{0.3}) = \underline{0.3}$), $\underline{0.3} \leq I_{\tau_2,\tau_2^*}(\Psi(\underline{0.3}, 0.5, 0.5), 0.5, 0.5) = \underline{0.4}$ *and* $\underline{0.3} \not\leq I_{\tau,\tau^\circ}(\underline{0.3}, 0.5, 0.5) = \underline{0}$.

Corollary 1. *Let $\Phi : (X, \tau, \tau^\circ, (\eth, \eth^*)) \multimap (Y, \sigma, \sigma^\circ)$ and $\Xi : (Y, \sigma, \sigma^\circ) \multimap (Z, \eta, \eta^\circ)$ be two DF- (resp. normalized) multifunctions. Then, $\Xi \circ \Phi$ is DFL- (resp. DFU-) $\eth$-continuous multifunction if Φ is DFL- (resp. DFU-) $\eth$-continuous multifunction and Ξ is a DFL- (resp. DFU-) semi-continuous multifunction.*

3. DFU-Almost and DFL-Almost $\eth$-Continuity

This section investigates the notions of DFU-almost and DFL- almost $\eth$-continuity, and introduces many characteristic properties of the defined multifunctions.

Definition 4. *Let $\Phi : (X, \tau, \tau^\circ) \multimap (Y, \sigma, \sigma^\circ, \eth, \eth^*)$ be a DF-multifunction, $p \in I_0$ and $q \in I_1$. Then, Φ is called:*

(1) *DFU-almost $\eth$-continuous at $x_t \in dom(\Phi)$ iff $x_t \in \Phi^u(\mu)$ for each $\mu \in I^Y$, $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$, there exists $\lambda \in I^X, \tau(\lambda) \geq p$ and $\tau^\circ(\lambda) \leq q$ and $x_t \in \lambda$ such that $\lambda \wedge dom(\Phi) \leq \Phi^u(I_{\sigma,\sigma^\circ}(cl^*(\mu, p, q), p, q))$.*

(2) *DFL-almost $\eth$-continuous at $x_t \in dom(\Phi)$ iff $x_t \in \Phi^l(\mu)$ for each $\mu \in I^Y$, $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$, there exists $\lambda \in I^X, \tau(\lambda) \geq p$ and $\tau^\circ(\lambda) \leq q$ and $x_t \in \lambda$ such that $\lambda \leq \Phi^l(I_{\sigma,\sigma^\circ}(cl^*(\mu, p, q), p, q))$.*

(3) *DFU- (resp. DFL-) almost $\eth$-continuous iff it is DFU- (resp. DFL-) almost $\eth$-continuous at every fuzzy point $x_t \in dom(\Phi)$.*

If we take $cl^* = C_{\tau,\tau^\circ}$, then we have the definition of DF-almost continuous multifunctions.

Remark 4. (1) *If Φ is a normalized DF-multifunction, then Φ is DFU-almost $\eth$-continuous at $x_t \in dom(\Phi)$ iff $x_t \in \Phi^u(\mu)$ for each $\mu \in I^Y$, $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$, there exists $\lambda \in I^X$, $\tau(\lambda) \geq p$, $\tau^\circ(\lambda) \leq q$ and $x_t \in \lambda$ such that $\lambda \leq \Phi^u(I_{\sigma,\sigma^\circ}(cl^*(\mu, p, q), p, q))$.*

(2) *DFU- (resp. DFL-) semi-continuity $\Rightarrow$ DFU- (resp. DFL-) almost $\eth$-continuity $\Rightarrow$ DFU- (resp. DFL-) almost continuity.*

(3) *DFU- (resp. DFL-) almost $\eth^0$-continuity $\Leftrightarrow$ DFU- (resp. DFL-) almost continuity.*

Theorem 5. *For a DF-multifunction Φ: $(X, \tau, \tau^\circ) \multimap (Y, \sigma, \sigma^\circ, (\eth, \eth^*))$, $\mu \in I^Y$, $p \in I_0$ and $q \in I_1$, these statements are equivalent:*

(1) *Φ is DFL-almost $\eth$-continuous.*

(2) *$\Phi^l(\mu) \leq I_{\tau,\tau^\circ}\left(\Phi^l(I_{\sigma,\sigma^\circ}(cl^*(\mu, p, q), p, q)\right), p, q)$ if $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$.*

(3) *$C_{\tau,\tau^\circ}(\Phi^u(C_{\sigma,\sigma^\circ}(int^*(\mu, p, q), p, q)), p, q) \leq \Phi^u(\mu)$ if $\sigma(\mu^c) \geq p$ and $\sigma^\circ(\mu^c) \leq q$.*

Proof. (1) $\Longrightarrow$ (2) Let $x_t \in dom(\Phi)$, $\mu \in I^Y$, $\sigma(\mu) \geq p$, $\sigma^\circ(\mu) \leq q$ and $x_t \in \Phi^l(\mu)$. Then, there exists $\lambda \in I^X, \tau(\lambda) \geq p, \tau^\circ(\lambda) \leq q$ and $x_t \in \lambda$ such that $\lambda \leq \Phi^l(I_{\sigma,\sigma^\circ}(cl^*(\mu,p,q),p,q))$. Thus, $x_t \in \lambda \leq I_{\tau,\tau^\circ}\phi^l(I_{\sigma,\sigma^\circ}(cl^*(\mu,p,q),p,q)),p,q)$, and hence $\Phi^l(\mu) \leq I_{\tau,\tau^\circ}\left(\Phi^l(I_{\sigma,\sigma^\circ}(cl^*(\mu,p,q),p,q)),p,q\right)$.

(2) $\Longrightarrow$ (3) Let $\mu \in I^Y$ with $\sigma(\mu^c) \geq p$ and $\sigma^\circ(\mu^c) \leq q$. Then,

$$\begin{aligned}[\Phi^u(\mu)]^c &= \Phi^l(\mu^c) \leq I_{\tau,\tau^\circ}\left(\Phi^l(I_{\sigma,\sigma^\circ}(cl^*(\mu^c, p, q), p, q)), p, q\right) \\ &= [C_{\tau,\tau^\circ}(\Phi^u(C_{\sigma,\sigma^\circ}(int^*(\mu, p, q), p, q)), p, q)]^c.\end{aligned}$$

Thus, $C_{\tau,\tau^\circ}(\Phi^u(C_{\sigma,\sigma^\circ}(int^*(\mu, p, q), p, q)), p, q) \leq \Phi^u(\mu)$.

(3) $\Longrightarrow$ (1) Let $x_t \in dom(\Phi)$, $\mu \in I^Y$, $\sigma(\mu) \geq p$, $\sigma^\circ(\mu) \leq q$ and $x_t \in \Phi^l(\mu)$. Then,

$$\begin{aligned}&\left[I_{\tau,\tau^\circ}\left(\Phi^l(I_{\sigma,\sigma^\circ}(cl^*(\mu, p, q), p, q)\right), p, q)\right]^c \\ =\ & C_{\tau,\tau^\circ}(\Phi^u(C_{\sigma,\sigma^\circ}(int^*(\mu^c, p, q), p, q)), p, q) \leq \Phi^u(\mu^c) = \left[\Phi^l(\mu)\right]^c,\end{aligned}$$

and hence $\Phi^l(\mu) \leq I_{\tau,\tau^\circ}\left(\Phi^l(I_{\sigma,\sigma^\circ}(cl^*(\mu, p, q), p, q)\right), p, q)$. Therefore, $x_t \in I_{\tau,\tau^\circ}\left(\Phi^l(I_{\sigma,\sigma^\circ}(cl^*(\mu, p, q), p, q)\right), p, q) \leq \Phi^l(I_{\sigma,\sigma^\circ}(cl^*(\mu, p, q), p, q))$. Thus, Φ is DFL-almost $\eth$-continuous. □

Theorem 6. *For a normalized DF-multifunction $\Phi : (X, \tau, \tau^\circ) \multimap (Y, \sigma, \sigma^\circ, (\eth, \eth^*))$, $\mu \in I^Y$, $p \in I_0$ and $q \in I_1$, these statements are equivalent:*

(1) *Φ is DFU-almost $\eth$-continuous.*

(2) *$\Phi^u(\mu) \leq I_{\tau,\tau^\circ}(\Phi^u(I_{\sigma,\sigma^\circ}(cl^*(\mu, p, q), p, q)), p, q)$ if $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$.*

(3) *$C_{\tau,\tau^\circ}\left(\Phi^l(C_{\sigma,\sigma^\circ}(int^*(\mu, p, q), p, q)), p, q\right) \leq \Phi^l(\mu)$ if $\sigma(\mu^c) \geq p$ and $\sigma^\circ(\mu^c) \leq q$.*

The following example shows that generally the implications of Remark 4 (2) are not reversed.

Example 3. *Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$ and $\Phi : X \multimap Y$ be a DF-multifunction defined by $G_\Phi(x_1, y_1) = 0.2$, $G_\Phi(x_1, y_2) = 1$, $G_\Phi(x_1, y_3) = 0$, $G_\Phi(x_2, y_1) = 0.4$, $G_\Phi(x_2, y_2) = 0.2$, $G_\Phi(x_2, y_3) = 1$. Define the DF-topologies $(\tau_1, \tau_1^*), (\tau_2, \tau_2^*) : I^X \to I$, (σ_1, σ_1^*), $(\sigma_2, \sigma_2^*) : I^Y \to I$, and the DF-ideals $(\eth_1, \eth_1^*)$, $(\eth_2, \eth_2^*) : I^Y \to I$ as follow:*

$$\tau_1(\lambda) = \begin{cases} 1, & \lambda \in \{\underline{0}, \underline{1}\} \\ 0.3, & \lambda = \underline{0.4} \\ 0, & o.w, \end{cases} \qquad \tau_1^*(\lambda) = \begin{cases} 0, & \lambda \in \{\underline{0}, \underline{1}\} \\ 0.7, & \lambda = \underline{0.4} \\ 1, & o.w, \end{cases}$$

$$\tau_2(\lambda) = \begin{cases} 1, & \lambda \in \{\underline{0}, \underline{1}\} \\ 0.5, & \lambda = \underline{0.5} \\ 0, & o.w, \end{cases} \qquad \tau_2^*(\lambda) = \begin{cases} 0, & \lambda \in \{\underline{0}, \underline{1}\} \\ 0.5, & \lambda = \underline{0.5} \\ 1, & o.w, \end{cases}$$

$$\sigma_1(\mu) = \begin{cases} 1, & \mu \in \{\underline{0}, \underline{1}\} \\ 0.4, & \mu = \underline{0.3} \\ 0.6, & \mu = \underline{0.4} \\ 0, & o.w, \end{cases} \qquad \sigma_1^*(\mu) = \begin{cases} 0, & \mu \in \{\underline{0}, \underline{1}\} \\ 0.6, & \mu = \underline{0.3} \\ 0.4, & \mu = \underline{0.4} \\ 1, & o.w, \end{cases}$$

$$\sigma_2(\mu) = \begin{cases} 1, & \mu \in \{\underline{0}, \underline{1}\} \\ 0.4, & \mu = \underline{0.5} \\ 0.7, & \mu = \underline{0.4} \\ 0, & o.w, \end{cases} \qquad \sigma_2^*(\mu) = \begin{cases} 0, & \mu \in \{\underline{0}, \underline{1}\} \\ 0.6, & \mu = \underline{0.5} \\ 0.3, & \mu = \underline{0.4} \\ 1, & o.w, \end{cases}$$

$$ð_1(\nu) = \begin{cases} 1, & \nu = \underline{0} \\ 0.7, & \underline{0} < \nu < \underline{0.3} \\ 0, & o.w, \end{cases} \qquad ð_1^*(\nu) = \begin{cases} 0, & \nu = \underline{0} \\ 0.3, & \underline{0} < \nu < \underline{0.3} \\ 1, & o.w, \end{cases}$$

$$ð_2(\nu) = \begin{cases} 1, & \nu = \underline{0} \\ 0.7, & \underline{0} < \nu \leq \underline{0.4} \\ 0, & o.w, \end{cases} \qquad ð_2^*(\nu) = \begin{cases} 0, & \nu = \underline{0} \\ 0.3, & \underline{0} < \nu \leq \underline{0.4} \\ 1, & o.w. \end{cases}$$

Then, (1) $\Phi : (X, \tau_1, \tau_1^*) \multimap (Y, \sigma_1, \sigma_1^*, (ð_1, ð_1^*))$ *is a DFU- (resp. DFL-) almost ð-continuous and it is not DFU- (resp. DFL-) semi-continuous because*

$$\underline{0.3} = \Phi^u(\underline{0.3}) \leq I_{\tau_1,\tau_1^*}\Big(\Phi^u(I_{\sigma_1,\sigma_1^*}(cl^*(\underline{0.3}, 0.3, 0.7), 0.3, 0.7)\Big), 0.3, 0.7) = \underline{0.4},$$

$$\underline{0.4} = \Phi^u(\underline{0.4}) \leq I_{\tau_1,\tau_1^*}\Big(\Phi^u(I_{\sigma_1,\sigma_1^*}(cl^*(\underline{0.4}, 0.3, 0.7), 0.3, 0.7)\Big), 0.3, 0.7) = \underline{0.4},$$

and $\underline{0.3} \nleq I_{\tau_1,\tau_1^*}(\underline{0.3}, 0.3, 0.7) = \underline{0}$.

(2) $\Phi : (X, \tau_2, \tau_2^*) \multimap (Y, \sigma_2, \sigma_2^*, (ð_2, ð_2^*))$ *is DFU- (resp. DFL-) almost continuous and it is not DFU- (resp. DFL-) almost ð-continuous because*

$$\underline{0.4} = \Phi^u(\underline{0.4}) \leq I_{\tau_2,\tau_2^*}\Big(\Phi^u(I_{\sigma_2,\sigma_2^*}(C_{\sigma_2,\sigma_2^*}(\underline{0.4}, 0.3, 0.7), 0.3, 0.7)\Big), 0.3, 0.7) = \underline{0.5},$$

$$\underline{0.5} = \Phi^u(\underline{0.5}) \leq I_{\tau_2,\tau_2^*}\Big(\Phi^u(I_{\sigma_2,\sigma_2^*}(C_{\sigma_2,\sigma_2^*}(\underline{0.5}, 0.3, 0.7), 0.3, 0.7)\Big), 0.3, 0.7) = \underline{0.5},$$

and $\underline{0.4} = \Phi^u(\underline{0.4}) \nleq I_{\tau_2,\tau_2^*}\Big(\Phi^u(I_{\sigma_2,\sigma_2^*}(cl^*(\underline{0.4}, 0.3, 0.7), 0.3, 0.7)\Big), 0.3, 0.7) = \underline{0}$.

Theorem 7. *For a DF-multifunction* $\Phi : (X, \tau, \tau^\circ) \multimap (Y, \sigma, \sigma^\circ, (ð, ð^*))$, $\mu \in I^Y$, $p \in I_0$ *and* $q \in I_1$, *these statements are equivalent:*

(1) Φ *is DFL-almost ð-continuous.*

(2) $\tau\Big(\Phi^l(\mu)\Big) \geq p$ *and* $\tau^\circ\Big(\Phi^l(\mu)\Big) \leq q$ *if* $\mu = I_{\sigma,\sigma^\circ}(cl^*(\mu, p, q), p, q)$.

(3) $\tau\Big(\Phi^l(I_{\sigma,\sigma^\circ}(cl^*(\mu, p, q), p, q))\Big) \geq p$, *and* $\tau^\circ\Big(\Phi^l(I_{\sigma,\sigma^\circ}(cl^*(\mu, p, q), p, q)\Big)) \leq q$ *if* $\sigma(\mu) \geq p$ *and* $\sigma^\circ(\mu) \leq q$.

Proof. (1) $\Longrightarrow$ (2) If $\mu = I_{\sigma,\sigma^\circ}(cl^*(\mu,p,q),p,q)$, then $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$. From Theorem 5 (2), $\Phi^l(\mu) \leq I_{\tau,\tau^\circ}\Big(\Phi^l(I_{\sigma,\sigma^\circ}(cl^*(\mu,p,q),p,q)\Big),p,q) = I_{\tau,\tau^\circ}\Big(\Phi^l(\mu),p,q\Big)$. Thus, $\tau\Big(\Phi^l(\mu)\Big) \geq p$ and $\tau^\circ\Big(\Phi^l(\mu)\Big) \leq q$.

(2) $\Leftrightarrow$ (3) Obvious.

(3) $\Longrightarrow$ (1) Let $x_t \in dom(\Phi)$, $\mu \in I^Y$, $\sigma(\mu) \geq p$, $\sigma^\circ(\mu) \leq q$ and $x_t \in \Phi^l(\mu)$. Then, from $\mu \leq I_{\sigma,\sigma^\circ}(cl^*(\mu,p,q),p,q)$, we obtain that $\tau\Big(\Phi^l(I_{\sigma,\sigma^\circ}(cl^*(\mu,p,q),p,q))\Big) \geq p$, $\tau^\circ\Big(\Phi^l(I_{\sigma,\sigma^\circ}(cl^*(\mu,p,q),p,q)\Big)) \leq q$ and $x_t \in \Phi^l(\mu) \leq \Phi^l(I_{\sigma,\sigma^\circ}(cl^*(\mu,p,q),p,q))$. Thus, Φ is DFL-almost ð-continuous. □

Theorem 8. *For a DF-multifunction $\Phi : (X,\tau,\tau^\circ) \multimap (Y,\sigma,\sigma^\circ,(ð,ð^*))$, $\mu \in I^Y$, $p \in I_0$ and $q \in I_1$, these statements are equivalent:*

(1) *Φ is DFL-almost ð-continuous.*
(2) *$\tau\big([\Phi^u(\mu)]^c\big) \geq p$ and $\tau^\circ\big([\Phi^u(\mu)]^c\big) \leq q$, if $\mu = C_{\sigma,\sigma^\circ}(int^*(\mu,p,q),p,q)$.*
(3) *$\tau\big([\Phi^u(C_{\sigma,\sigma^\circ}(int^*(\mu,p,q),p,q))]^c\big) \geq p$*
and $\tau^\circ\big([\Phi^u(C_{\sigma,\sigma^\circ}(int^*(\mu,p,q),p,q))]^c\big) \leq q$ if $\sigma(\mu^c) \geq r$ and $\sigma^\circ(\mu^c) \leq q$.

Theorem 9. *For a normalized DF-multifunction $\Phi : (X,\tau,\tau^\circ) \multimap (Y,\sigma,\sigma^\circ,(ð,ð^*))$, $\mu \in I^Y$, $p \in I_0$ and $q \in I_1$, these statements are equivalent:*

(1) *Φ is DFU-almost ð-continuous.*
(2) *$\tau(\Phi^u(\mu)) \geq p$ and $\tau^\circ(\Phi^u(\mu)) \leq q$ if $\mu = I_{\sigma,\sigma^\circ}(cl^*(\mu,p,q),p,q)$.*
(3) *$\tau(\Phi^u(I_{\sigma,\sigma^\circ}(cl^*(\mu,p,q),p,q))) \geq p$ and $\tau^\circ(\Phi^u(I_{\sigma,\sigma^\circ}(cl^*(\mu,p,q),p,q))) \leq q$ if $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$.*

Theorem 10. *For a normalized DF-multifunction $\Phi : (X,\tau,\tau^\circ) \multimap (Y,\sigma,\sigma^\circ,(ð,ð^*))$, $\mu \in I^Y$, $p \in I_0$ and $q \in I_1$, these statements are equivalent:*

(1) *Φ is DFU-almost ð-continuous.*
(2) *$\tau\Big(\Big[\Phi^l(\mu)\Big]^c\Big) \geq p$ and $\tau^\circ\Big(\Big[\Phi^l(\mu)\Big]^c\Big) \leq q$ if $\mu = C_{\sigma,\sigma^\circ}(int^*(\mu,p,q),p,q)$.*
(3) *$\tau\Big(\Big[\Phi^l(C_{\sigma,\sigma^\circ}(int^*(\mu,p,q),p,q))\Big]^c\Big) \geq p$ and $\tau^\circ\Big[\Phi^l(C_{\sigma,\sigma^\circ}(int^*(\mu,p,q),p,q))\Big]^c \leq q$ if $\sigma(\mu^c) \geq p$ and $\sigma^\circ(\mu^c) \leq q$.*

Theorem 11. *Let $\Phi : (X,\tau,\tau^\circ) \multimap (Y,\sigma,\sigma^\circ,(ð,ð^*))$ be a DF-multifunction. Then, Φ is DFL-almost ð-continuous iff $C_{\tau,\tau^\circ}(\Phi^u(\mu),p,q) \leq \Phi^u(C_{\sigma,\sigma^\circ}(\mu,p,q))$ for any $\mu \in I^Y$ with $\mu \leq C_{\sigma,\sigma^\circ}(int^*(\mu,p,q),p,q)$, $p \in I_0$ and $q \in I_1$.*

Proof. $\Rightarrow$ Let Φ be a DFL-almost ð-continuous. Then, for any $\mu \in I^Y$ with $\mu \leq C_{\sigma,\sigma^\circ}(int^*(\mu,p,q),p,q) = \nu$ (say), where $\nu = C_{\sigma,\sigma^\circ}(int^*(\nu,p,q),p,q)$. By Theorem 8, then $\tau\big([\Phi^u(\nu)]^c\big) \geq p$ and $\tau^\circ\big([\Phi^u(\nu)]^c\big) \leq q$, and thus

$$\begin{aligned} C_{\tau,\tau^\circ}(\Phi^u(\mu),p,q) &\leq C_{\tau,\tau^\circ}(\Phi^u(\nu),p,q) \\ &= \Phi^u(C_{\sigma,\sigma^\circ}(int^*(\nu,p,q),p,q)) \leq \Phi^u(C_{\sigma,\sigma^\circ}(\mu,p,q)). \end{aligned}$$

($\Leftarrow$) Let $\mu \in I^Y$ with $\mu = C_{\sigma,\sigma^\circ}(int^*(\mu,p,q),p,q)$. Then, $\mu \leq C_{\sigma,\sigma^\circ}(int^*(\mu,p,q),p,q)$ and $C_{\tau,\tau^\circ}(\Phi^u(\mu),p,q) \leq \Phi^u(C_{\sigma,\sigma^\circ}(\mu,p,q)) = \Phi^u(\mu)$.

Therefore, $\tau\big([\Phi^u(\mu)]^c\big) \geq p$ and $\tau^\circ\big([\Phi^u(\mu)]^c\big) \leq q$. Hence, Φ is DFL-almost ð-continuous. □

Theorem 12. *Let $\Phi : (X,\tau,\tau^\circ) \multimap (Y,\sigma,\sigma^\circ,(ð,ð^*))$ be a normalized DF-multifunction. Then, Φ is DFU-almost ð-continuous iff $C_{\tau,\tau^\circ}\Big(\Phi^l(\mu),p,q\Big) \leq \Phi^l(C_{\sigma,\sigma^\circ}(\mu,p,q))$ for any $\mu \in I^Y$ with $\mu \leq C_{\sigma,\sigma^\circ}(int^*(\mu,p,q),p,q)$, $p \in I_0$ $q \in I_1$.*

Theorem 13. *Let* $\{\Phi_i : (X, \tau, \tau^\circ) \multimap (Y, \sigma, \sigma^\circ, (\eth, \eth^*)), i \in \Gamma\}$ *be a family of DFL-almost* $\eth$*-continuous multifunctions. Then,* $\bigcup_{i\in\Gamma} \Phi_i$ *is a DFL-almost* $\eth$*-continuous multifunction.*

Proof. Let $\mu \in I^Y$. Then, $\left(\bigcup_{i\in\Gamma} \Phi_i\right)^l(\mu) = \bigvee_{i\in\Gamma}\left(\Phi_i^l(\mu)\right)$. Since $\{\Phi_i, i \in \Gamma\}$ is a family of DFL-almost $\eth$-continuous multifunctions, $\tau\left(\Phi_i^l(\mu)\right) \geq p$ and $\tau^\circ\left(\Phi_i^l(\mu)\right) \leq q$, for any $\mu = I_{\sigma,\sigma^\circ}(cl^*(\mu, p, q), p, q)$ and $i \in \Gamma$. Then,

$$\tau\left(\left(\bigcup_{i\in\Gamma} \Phi_i\right)^l(\mu)\right) = \tau\left(\bigvee_{i\in\Gamma}\left(\Phi_i^l(\mu)\right)\right) \geq \bigwedge_{i\in\Gamma} \tau\left(\Phi_i^l(\mu)\right) \geq p$$

and $\tau^\circ\left(\left(\bigcup_{i\in\Gamma} \Phi_i\right)^l(\mu)\right) = \tau^\circ\left(\bigvee_{i\in\Gamma}\left(\Phi_i^l(\mu)\right)\right) \leq \bigvee_{i\in\Gamma} \tau^\circ\left(\Phi_i^l(\mu)\right) \leq q$. Hence, $\bigcup_{i\in\Gamma} \Phi_i$ is DFL-almost $\eth$-continuous. $\square$

Theorem 14. *Let* $\Phi_1, \Phi_2 : (X, \tau, \tau^\circ) \multimap (Y, \sigma, \sigma^\circ, (\eth, \eth^*))$ *be two normalized DFU-almost* $\eth$*-continuous multifunctions. Then,* $\Phi_1 \cup \Phi_2$ *is a DFU-almost* $\eth$*-continuous multifunction.*

Proof. Let $\mu \in I^Y$. Then, $(\Phi_1 \cup \Phi_2)^u(\mu) = \Phi_1^u(\mu) \wedge \Phi_2^u(\mu)$. Since Φ_1 and Φ_2 are two normalized DFU-almost $\eth$-continuous, $\tau\left(\Phi_i^u(\mu)\right) \geq p$ and $\tau^\circ\left(\Phi_i^u(\mu)\right) \leq q$ for any $\mu = I_{\sigma,\sigma^\circ}(cl^*(\mu, p, q), p, q)$ and $i \in \{1, 2\}$. Then,

$$\tau\left((\Phi_1 \cup \Phi_2)^u(\mu)\right) = \tau(\Phi_1^u(\mu) \wedge \Phi_2^u(\mu)) \geq \tau(\Phi_1^u(\mu)) \wedge \tau(\Phi_2^u(\mu)) \geq p,$$

and $\tau^\circ\left((\Phi_1 \cup \Phi_2)^u(\mu)\right) = \tau^\circ\left(\Phi_1^u(\mu) \wedge \Phi_2^u(\mu)\right) \leq \tau^\circ\left(\Phi_1^u(\mu)\right) \vee \tau^\circ(\Phi_2^u(\mu)) \leq q$. Hence, $\Phi_1 \cup \Phi_2$ is a DFU-almost $\eth$-continuous multifunction. $\square$

Corollary 2. *Let* $\Phi : (X, \tau, \tau^\circ) \multimap (Y, \sigma, \sigma^\circ)$ *and* $\Xi : (Y, \sigma, \sigma^\circ) \multimap (Z, \eta, \eta^\circ, (\eth, \eth^*))$ *be two DF- (resp. normalized) multifunctions. Then,* $\Xi \circ \Phi$ *is a DFL- (resp. DFU-) almost* $\eth$*-continuous if* Ξ *is DFL- (resp. DFU-) almost* $\eth$*-continuous and* Φ *is DFL- (resp. DFU-) semi-continuous.*

4. DFU-Weakly and DFL-Weakly $\eth$-Continuity

In this section, we introduce the notion of DFU-weakly and DFL-weakly $\eth$-continuity, and discuss its relations with the previous definitions of DFU-almost and DFL-almost $\eth$-continuity.

Definition 5. *Let* $\Phi : (X, \tau, \tau^\circ) \multimap (Y, \sigma, \sigma^\circ, (\eth, \eth^*))$ *be a DF-multifunction,* $p \in I_0$ *and* $q \in I_1$*. Then,* Φ *is called:*

(1) *DFU-weakly* $\eth$*-continuous at* $x_t \in dom(\Phi)$ *iff* $x_t \in \Phi^u(\mu)$ *for each* $\mu \in I^Y$, $\sigma(\mu) \geq p$ *and* $\sigma^\circ(\mu) \leq q$*, there exists* $\lambda \in I^X$, $\tau(\lambda) \geq p$ *and* $\tau^\circ(\lambda) \leq q$ *and* $x_t \in \lambda$ *such that* $\lambda \wedge dom(\Phi) \leq \Phi^u(cl^*(\mu, p, q))$.

(2) *DFL-weakly* $\eth$*-continuous at* $x_t \in dom(\Phi)$ *iff* $x_t \in \Phi^l(\mu)$ *for each* $\mu \in I^Y$, $\sigma(\mu) \geq p$ *and* $\sigma^\circ(\mu) \leq q$*, there exists* $\lambda \in I^X$, $\tau(\lambda) \geq p$ *and* $\tau^\circ(\lambda) \leq q$ *and* $x_t \in \lambda$ *such that* $\lambda \leq \Phi^l(cl^*(\mu, p, q))$.

(3) *DFU- (resp. DFL-) weakly* $\eth$*-continuous iff it is DFU- (resp. DFL-) weakly* $\eth$*-continuous at every* $x_t \in dom(\Phi)$.

If $cl^* = C_{\tau,\tau^\circ}$, then we have the definition of DF-weakly continuous multifunctions.

Remark 5. (1) *If* Φ *is normalized DF-multifunction, then* Φ *is DFU-weakly* $\eth$*-continuous at* $x_t \in dom(\Phi)$ *iff* $x_t \in \Phi^u(\mu)$ *for each* $\mu \in I^Y$, $\sigma(\mu) \geq p$ *and* $\sigma^\circ(\mu) \leq q$ *there exists* $\lambda \in I^X$, $\tau(\lambda) \geq p$, $\tau^\circ(\lambda) \leq q$ *and* $x_t \in \lambda$ *such that* $\lambda \leq \Phi^u(cl^*(\mu, p, q))$.

(2) *DFU- (resp. DFL-) almost* $\eth$*-continuity* $\Rightarrow$ *DFU- (resp. DFL-) weakly* $\eth$*-continuity* $\Rightarrow$ *DFU- (resp. DFL-) weakly continuity.*

(3) *DFU- (resp. DFL-) weakly* $\eth^0$*-continuity* $\Leftrightarrow$ *DFU- (resp. DFL-) weakly continuity.*

Theorem 15. *A DF-multifunction* $\Phi : (X, \tau, \tau^\circ) \multimap (Y, \sigma, \sigma^\circ, (\eth, \eth^*))$ *is DFL-weakly* $\eth$*-continuous iff* $\Phi^l(\mu) \leq I_{\tau,\tau^\circ}(\Phi^l(cl^*(\mu, p, q)), p, q)$ *for each* $\mu \in I^Y$ *with* $\sigma(\mu) \geq p$, $\sigma^\circ(\mu) \leq q$, $p \in I_0$ *and* $q \in I_1$.

Proof. ($\Rightarrow$) Let $x_t \in dom(\Phi)$, $\mu \in I^Y$ with $\sigma(\mu) \geq p$, $\sigma^\circ(\mu) \leq q$ and $x_t \in \Phi^l(\mu)$. Then, there exists $\lambda \in I^X$, $\tau(\lambda) \geq p$, $\tau^\circ(\lambda) \leq q$ and $x_t \in \lambda$ such that $\lambda \leq \Phi^l(cl^*(\mu, p, q))$. Thus, $x_t \in \lambda \leq I_{\tau,\tau^\circ}(\Phi^l(cl^*(\mu, p, q)), p, q)$, and hence $\Phi^l(\mu) \leq I_{\tau,\tau^\circ}(\Phi^l(cl^*(\mu, p, q)), p, q)$.

($\Leftarrow$) Let $x_t \in dom(\Phi)$, $\mu \in I^Y$ with $\sigma(\mu) \geq p$, $\sigma^\circ(\mu) \leq q$ and $x_t \in \Phi^l(\mu)$. Then, $x_t \in \Phi^l(\mu) \leq I_{\tau,\tau^\circ}(\Phi^l(cl^*(\mu, p, q)), p, q)$. Thus, $x_t \in \lambda \leq I_{\tau,\tau^\circ}(\Phi^l(cl^*(\mu, p, q)), p, q) \leq \Phi^l(cl^*(\mu, p, q))$. Hence, Φ is DFU-weakly $\eth$-continuous. □

Theorem 16. *A normalized DF-multifunction* $\Phi : (X, \tau, \tau^\circ) \multimap (Y, \sigma, \sigma^\circ, (\eth, \eth^*))$ *is DFU-weakly* $\eth$*-continuous iff* $\Phi^u(\mu) \leq I_{\tau,\tau^\circ}(\Phi^u(cl^*(\mu, p, q)), p, q)$ *for each* $\mu \in I^Y$ *with* $\sigma(\mu) \geq p$, $\sigma^\circ(\mu) \leq q$, $p \in I_0$ *and* $q \in I_1$.

The following example shows that generally, the implications in Remark 5 (2) are not reversed.

Example 4. *Let* $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$ *and* $\Phi : X \multimap Y$ *be a DF-multifunction defined by* $G_\Phi(x_1, y_1) = 0.7$, $G_\Phi(x_1, y_2) = 0.3$, $G_\Phi(x_1, y_3) = 0.3$, $G_\Phi(x_2, y_1) = 0.1$, $G_\Phi(x_2, y_2) = 1$, $G_\Phi(x_2, y_3) = 0.1$, $G_\Phi(x_3, y_1) = 0.2$, $G_\Phi(x_3, y_2) = 0.1$, $G_\Phi(x_3, y_3) = 1$. *Define* $\mu_1 \in I^X$ *and* $\mu_2 \in I^Y$ *as follows:* $\mu_1 = \{0.3, 0.1, 0.2\}$ *and* $\mu_2 = \{0.3, 0.1, 0.2\}$. *Define the DF-topologies* $(\tau, \tau^\circ) : I^X \to I$, $(\sigma, \sigma^*) : I^Y \to I$, *and the DF-ideals* $(\eth_1, \eth_1^*), (\eth_2, \eth_2^*) : I^Y \to I$ *as follows:*

$$\tau(\lambda) = \begin{cases} 1, & \lambda \in \{\underline{0}, \underline{1}\} \\ 0.3, & \lambda = \underline{0.5} \\ 0, & o.w, \end{cases} \qquad \tau^\circ(\lambda) = \begin{cases} 0, & \lambda \in \{\underline{0}, \underline{1}\} \\ 0.7, & \lambda = \underline{0.5} \\ 1, & o.w, \end{cases}$$

$$\sigma(\mu) = \begin{cases} 1, & \mu \in \{\underline{0}, \underline{1}\} \\ 0.25, & \mu = \mu_2 \\ 0, & o.w, \end{cases} \qquad \sigma^*(\mu) = \begin{cases} 0, & \mu \in \{\underline{0}, \underline{1}\} \\ 0.75, & \mu = \mu_2 \\ 1, & o.w, \end{cases}$$

$$\eth_1(\nu) = \begin{cases} 1, & \nu = \underline{0} \\ 0.4, & \underline{0} < \nu < \underline{0.3} \\ 0, & o.w, \end{cases} \qquad \eth_1^*(\nu) = \begin{cases} 0, & \nu = \underline{0} \\ 0.6, & \underline{0} < \nu < \underline{0.3} \\ 1, & o.w, \end{cases}$$

$$\eth_2(\nu) = \begin{cases} 1, & \nu = \underline{0} \\ 0.5, & \underline{0} < \nu \leq \underline{0.2} \\ 0, & o.w, \end{cases} \qquad \eth_2^*(\nu) = \begin{cases} 0, & \nu = \underline{0} \\ 0.5, & \underline{0} < \nu \leq \underline{0.2} \\ 1, & o.w. \end{cases}$$

Then, (1) $\Phi : (X, \tau, \tau^\circ) \multimap (Y, \sigma, \sigma^*, (\eth_1, \eth_1^*))$ *is DFU- (resp. DFL-) weakly continuous and it is not DFU- (resp. DFL-) weakly* $\eth$*-continuous because*

$$\mu_1 = \Phi^u(\mu_2) \leq I_{\tau,\tau^\circ}(\Phi^u(C_{\sigma,\sigma^*}(\mu_2, 0.25, 0.75)), 0.25, 0.75) = \underline{0.5},$$
$$\mu_1 = \Phi^l(\mu_2) \leq I_{\tau,\tau^\circ}(\Phi^l(C_{\sigma,\sigma^*}(\mu_2, 0.25, 0.75)), 0.25, 0.75) = \underline{0.5},$$

and

$$\mu_1 = \Phi^u(\mu_2) \nleq I_{\tau,\tau^\circ}(\Phi^u(cl^*(\mu_2, 0.25, 0.75)), 0.25, 0.75) = \underline{0},$$
$$\mu_1 = \Phi^l(\mu_2) \nleq I_{\tau,\tau^\circ}(\Phi^l(cl^*(\mu_2, 0.25, 0.75)), 0.25, 0.75) = \underline{0}.$$

(2) $\Phi : (X, \tau, \tau^{\circ}) \multimap (Y, \sigma, \sigma^*, (\eth_2, \eth_2^*))$ *is DFU- (resp. DFL-) weakly* $\eth$*-continuous and it is not DFU- (resp. DFL-) almost* $\eth$*-continuous because*

$$\mu_1 = \Phi^u(\mu_2) \leq I_{\tau,\tau^{\circ}}(\Phi^u(cl^*(\mu_2, 0.25, 0.75)), 0.25, 0.75) = \underline{0.5},$$

$$\mu_1 = \Phi^l(\mu_2) \leq I_{\tau,\tau^{\circ}}(\Phi^l(cl^*(\mu_2, 0.25, 0.75)), 0.25, 0.75) = \underline{0.5},$$

and

$$\mu_1 = \Phi^u(\mu_2) \nleq I_{\tau,\tau^{\circ}}(\Phi^u(I_{\sigma,\sigma^*}(cl^*(\mu_2, 0.25, 0.75), 0.25, 0.75)), 0.25, 0.75) = \underline{0},$$

$$\mu_1 = \Phi^l(\mu_2) \nleq I_{\tau,\tau^{\circ}}(\Phi^l(I_{\sigma,\sigma^*}(cl^*(\mu_2, 0.25, 0.75), 0.25, 0.75)), 0.25, 0.75) = \underline{0}.$$

Theorem 17. *A DF-multifunction* $\Phi : (X, \tau, \tau^{\circ}) \multimap (Y, \sigma, \sigma^{\circ}, (\eth, \eth^*))$ *is DFL-weakly* $\eth$*-continuous iff* $C_{\tau,\tau^{\circ}}(\Phi^u(int^*(\mu, p, q)), p, q) \leq \Phi^u(\mu)$ *for each* $\mu \in I^Y$ *with* $\sigma(\mu^c) \geq p$, $\sigma^{\circ}(\mu^c) \leq q$, $p \in I_0$ *and* $q \in I_1$.

Proof. ($\Rightarrow$) Let $\mu \in I^Y$ with $\sigma(\mu^c) \geq p$, $\sigma^{\circ}(\mu^c) \leq q$. Then, by Theorem 15,

$$\begin{aligned} [\Phi^u(\mu)]^c &= \Phi^l(\mu^c) \leq I_{\tau,\tau^{\circ}}(\Phi^l(cl^*(\mu^c, p, q)), p, q) \\ &= [C_{\tau,\tau^{\circ}}(\Phi^u(int^*(\mu, p, q)), p, q)]^c. \end{aligned}$$

Thus, $C_{\tau,\tau^{\circ}}(\Phi^u(int^*(\mu, p, q)), p, q) \leq \Phi^u(\mu)$.

($\Leftarrow$) Let $x_t \in dom(\Phi)$, $\mu \in I^Y$with $\sigma(\mu) \geq p$, $\sigma^{\circ}(\mu) \leq q$ and $x_t \in \Phi^l(\mu)$. Then,

$$\begin{aligned} \Big[I_{\tau,\tau^{\circ}}(\Phi^l(cl^*(\mu, p, q)), p, q)\Big]^c &= C_{\tau,\tau^{\circ}}(\Phi^u(int^*(\mu, p, q)), p, q) \\ &\leq \Phi^u(\mu^c) = \Big[\Phi^l(\mu)\Big]^c, \end{aligned}$$

and hence $\Phi^l(\mu) \leq I_{\tau,\tau^{\circ}}(\Phi^l(cl^*(\mu, p, q)), p, q)$. Thus, Φ is DFL-weakly $\eth$-continuous. □

Theorem 18. *A normalized DF-multifunction* $\Phi : (X, \tau, \tau^{\circ}) \multimap (Y, \sigma, \sigma^{\circ}, (\eth, \eth^*))$ *is DFU-weakly* $\eth$*-continuous iff* $C_{\tau,\tau^{\circ}}(\Phi^l(int^*(\mu, p, q)), p, q) \leq \Phi^l(\mu)$ *for each* $\mu \in I^Y$ *with* $\sigma(\mu^c) \geq p$, $\sigma^{\circ}(\mu^c) \leq q$, $p \in I_0$ *and* $q \in I_1$.

Theorem 19. *If* $\Phi : (X, \tau, \tau^{\circ}) \multimap (Y, \sigma, \sigma^{\circ}, (\eth, \eth^*))$ *is a normalized DFU-weakly* $\eth$*-continuous multifunction and* $\Phi(\lambda) \leq I_{\sigma,\sigma^{\circ}}(cl^*(\Phi(\lambda), p, q), p, q)$ *for each* $\lambda \in I^X$*, and then* Φ *is DFU-almost* $\eth$*-continuous.*

Proof. Let $x_t \in dom(\Phi)$, $\mu \in I^Y$, $\sigma(\mu) \geq p$, $\sigma^{\circ}(\mu) \leq q$ and $x_t \in \Phi^u(\mu)$. Then, there exists $\lambda \in I^X$ with $\tau(\lambda) \geq p, \tau^{\circ}(\lambda) \leq q$ and $x_t \in \lambda$ such that $\lambda \leq \Phi^u(cl^*(\mu, p, q))$, $\Phi(\lambda) \leq \Phi(\Phi^u(cl^*(\mu, p, q))) \leq cl^*(\mu, p, q)$. Since $\Phi(\lambda) \leq I_{\sigma,\sigma^{\circ}}(cl^*(\Phi(\lambda), p, q), p, q) \leq I_{\sigma,\sigma^{\circ}}(cl^*(\mu, p, q), p, q)$ and $\lambda \leq \Phi^u(\Phi(\lambda)) \leq \Phi^u(I_{\sigma,\sigma^{\circ}}(cl^*(\mu, p, q), p, q))$. Thus, Φ is DFU-almost $\eth$-continuous. □

Corollary 3. *Let* $\Phi : (X, \tau, \tau^{\circ}) \multimap (Y, \sigma, \sigma^{\circ})$ *and* $\Xi : (Y, \sigma, \sigma^{\circ}) \multimap (Z, \eta, \eta^{\circ}, (\eth, \eth^*))$ *be two DF- (resp. normalized) multifunctions. Then,* $\Xi \circ \Phi$ *is a DFL- (resp. DFU-) weakly* $\eth$*-continuous if* Ξ *is DFL- (resp. DFU-) weakly* $\eth$*-continuous and* Φ *is DFL- (resp. DFU-) semi-continuous.*

Theorem 20. *Let* $\Phi : (X, \tau, \tau^{\circ}) \multimap (Y, \sigma, \sigma^{\circ}, (\eth, \eth^*))$ *be a DFL-weakly* $\eth$*-continuous multifunction. Then,* $\Phi^l(\mu) \leq I_{\tau,\tau^{\circ}}(\Phi^l(cl^*(\mu, p, q)), p, q)$ *for any* $\mu \in I^Y$ *with* $\mu \leq I_{\sigma,\sigma^{\circ}}(cl^*(\mu, p, q), p, q)$, $p \in I_0$ *and* $q \in I_1$.

Proof. Let Φ be a DFL-weakly $\eth$-continuous and $\mu \in I^Y$ with $\mu \leq I_{\sigma,\sigma^{\circ}}(cl^*(\mu, p, q), p, q)$. Then, $x_t \in \Phi^l(\mu) \leq \Phi^l(I_{\sigma,\sigma^{\circ}}(cl^*(\mu, p, q), p, q))$ and there exists $\lambda \in I^X$, $\tau(\lambda) \geq p$, $\tau^{\circ}(\lambda) \leq q$ and $x_t \in \lambda$ such that

$$\lambda \leq (\Phi^l(cl^*(I_{\sigma,\sigma^\circ}(cl^*(\mu,p,q),p,q),p,q)) \leq \Phi^l(cl^*(\mu,p,q)).$$

Thus, $\lambda \leq I_{\tau,\tau^\circ}(\Phi^l(cl^*(\mu,p,q)),p,q)$ and $\Phi^l(\mu) \leq I_{\tau,\tau^\circ}(\Phi^l(cl^*(\mu,p,q)),p,q)$. □

Theorem 21. *Let* $\Phi : (X,\tau,\tau^\circ) \multimap (Y,\sigma,\sigma^\circ,(\eth,\eth^*))$ *be a normalized DFU-weakly* $\eth$*-continuous multifunction. Then,* $\Phi^u(\mu) \leq I_{\tau,\tau^\circ}(\Phi^u(cl^*(\mu,p,q)),p,q)$ *for any* $\mu \in I^Y$ *with* $\mu \leq I_{\sigma,\sigma^\circ}(cl^*(\mu,p,q),p,q)$, $p \in I_0$ *and* $q \in I_1$.

Proof. Clear. □

5. DFU-Almost Weakly and DFL-Almost Weakly $\eth$-Continuity

This section introduces the notions of DFU-almost weakly and DFL-almost weakly $\eth$-continuity, and discusses its relations with the previous definitions of DFU-$\eth$-continuity and DFL-$\eth$-continuity (weakly or almost).

Definition 6. *Let* $\Phi : (X,\tau,\tau^\circ) \multimap (Y,\sigma,\sigma^\circ,(\eth,\eth^*))$ *be a DF-multifunction,* $p \in I_0$ *and* $q \in I_1$. *Then,* Φ *is called:*

(1) *DFU-almost weakly* $\eth$*-continuous at* $x_t \in dom(\Phi)$ *iff* $x_t \in \Phi^u(\mu)$ *for each* $\mu \in I^Y$, $\sigma(\mu) \geq p$ *and* $\sigma^\circ(\mu) \leq q$, *there exists* $\lambda \in I^X, \tau(\lambda) \geq p$ *and* $\tau^\circ(\lambda) \leq q$ *and* $x_t \in \lambda$ *such that* $\lambda \wedge dom(\Phi) \leq C_{\tau,\tau^\circ}(\Phi^u(cl^*(\mu,p,q)),p,q)$.

(2) *DFL-almost weakly* $\eth$*-continuous at* $x_t \in dom(\Phi)$ *iff* $x_t \in \Phi^l(\mu)$ *for each* $\mu \in I^Y$, $\sigma(\mu) \geq p$ *and* $\sigma^\circ(\mu) \leq q$, *there exists* $\lambda \in I^X, \tau(\lambda) \geq p$ *and* $\tau^\circ(\lambda) \leq q$ *and* $x_t \in \lambda$ *such that* $\lambda \leq C_{\tau,\tau^\circ}(\Phi^l(cl^*(\mu,p,q)),p,q)$.

(3) *DFU- (resp. DFL-) almost weakly* $\eth$*-continuous iff it is DFU- (resp. DFL-) almost weakly* $\eth$*-continuous at every* $x_t \in dom(\Phi)$.

If $cl^* = C_{\tau,\tau^\circ}$, then we have the definition of DF-almost weakly continuous multifunctions.

Remark 6. (1) *If* Φ *is a normalized DF-multifunction, then* Φ *is DFU-almost weakly* $\eth$*-continuous at* $x_t \in dom(\Phi)$ *iff* $x_t \in \Phi^u(\mu)$ *for each* $\mu \in I^Y$, $\sigma(\mu) \geq p$ *and* $\sigma^\circ(\mu) \leq q$, *there exists* $\lambda \in I^X, \tau(\lambda) \geq p$, $\tau^\circ(\lambda) \leq q$ *and* $x_t \in \lambda$ *such that* $\lambda \leq C_{\tau,\tau^\circ}(\Phi^u(cl^*(\mu,p,q)),p,q)$.

(2) *DFU- (resp. DFL-) weakly* $\eth$*-continuity* $\Rightarrow$ *DFU-(resp. DFL-) almost weakly* $\eth$*-continuity* $\Rightarrow$ *DFU- (resp. DFL-) almost weakly continuity.*

(3) *DFU- (resp. DFL-) almost weakly* $\eth^0$*-continuity* $\Leftrightarrow$ *DFU- (resp. DFL-) almost weakly continuity.*

Theorem 22. *For a DF-multifunction* $\Phi : (X,\tau,\tau^\circ) \multimap (Y,\sigma,\sigma^\circ,(\eth,\eth^*))$, $\mu \in I^Y$, $p \in I_0$ *and* $q \in I_1$, *these statements are equivalent:*

(1) Φ *is DFL-almost weakly* $\eth$*-continuous.*

(2) $\Phi^l(\mu) \leq I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ}(\Phi^l(cl^*(\mu,p,q)),p,q),p,q)$ *if* $\sigma(\mu) \geq p$ *and* $\sigma^\circ(\mu) \leq q$.

(3) $C_{\tau,\tau^\circ}(I_{\tau,\tau^\circ}(\Phi^u(int^*(\mu,p,q)),p,q),p,q) \leq \Phi^u(\mu)$ *if* $\sigma(\mu^c) \geq p$ *and* $\sigma^\circ(\mu^c) \leq q$.

Proof. (1) $\Longrightarrow$ (2) Let $x_t \in dom(\Phi)$, $\mu \in I^Y$, $\sigma(\mu) \geq p$, $\sigma^\circ(\mu) \leq q$ and $x_t \in \Phi^l(\mu)$. Then, there exists $\lambda \in I^X, \tau(\lambda) \geq p, \tau^\circ(\lambda) \leq q$ and $x_t \in \lambda$ such that $\lambda \leq C_{\tau,\tau^\circ}(\Phi^l(cl^*(\mu,p,q)),p,q)$. Thus, $x_t \in \lambda \leq I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ}(\Phi^l(cl^*(\mu,p,q)),p,q),p,q)$, and hence $\Phi^l(\mu) \leq I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ}(\Phi^l(cl^*(\mu,p,q)),p,q),p,q)$.

(2) $\Longrightarrow$ (3) Let $\mu \in I^Y$ with $\sigma(\mu^c) \geq p$ and $\sigma^\circ(\mu^c) \leq q$. Then,

$$\begin{aligned}
[\Phi^u(\mu)]^c &= \Phi^l(\mu^c) \leq I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ}(\Phi^l(cl^*(\mu^c,p,q)),p,q),p,q) \\
&= [C_{\tau,\tau^\circ}(I_{\tau,\tau^\circ}(\Phi^u(int^*(\mu,p,q)),p,q),p,q]^c.
\end{aligned}$$

Thus, $C_{\tau,\tau^\circ}(I_{\tau,\tau^\circ}(\Phi^u(int^*(\mu,p,q)),p,q),p,q) \leq \Phi^u(\mu)$.

(3) $\Longrightarrow$ (1) Let $x_t \in dom(\Phi)$, $\mu \in I^Y$, $\sigma(\mu) \geq p$, $\sigma^\circ(\mu) \leq q$ and $x_t \in \Phi^l(\mu)$. Then,

$$\begin{aligned} & \left[I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ}(\Phi^l(cl^*(\mu,p,q)),p,q),p,q)\right]^c \\ = \;& C_{\tau,\tau^\circ}(I_{\tau,\tau^\circ}(\Phi^u(int^*(\mu^c,p,q)),p,q),p,q) \leq \Phi^u(\mu^c) = \left[\Phi^l(\mu)\right]^c, \end{aligned}$$

and hence $\Phi^l(\mu) \leq I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ}(\Phi^l(cl^*(\mu,p,q)),p,q),p,q)$. Therefore, $x_t \in I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ}(\Phi^l(cl^*(\mu,p,q)),p,q),p,q) \leq C_{\tau,\tau^\circ}(\Phi^l(cl^*(\mu,p,q)),p,q)$. Thus, Φ is a DFL-almost weakly ð-continuous multifunction. □

Theorem 23. *For a normalized DF-multifunction $\Phi : (X,\tau,\tau^\circ) \multimap (Y,\sigma,\sigma^\circ,(ð,ð^*))$, $\mu \in I^Y$, $p \in I_0$ and $q \in I_1$, these statements are equivalent:*

(1) *Φ is DFU-almost weakly ð-continuous.*
(2) $\Phi^u(\mu) \leq I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ}(\Phi^u(cl^*(\mu,p,q)),p,q),p,q)$ *if $\sigma(\mu) \geq p$ and $\sigma^\circ(\mu) \leq q$.*
(3) $C_{\tau,\tau^\circ}\left(I_{\tau,\tau^\circ}(\Phi^l(int^*(\mu,p,q)),p,q),p,q\right) \leq \Phi^l(\mu)$ *if $\sigma(\mu^c) \geq p$ and $\sigma^\circ(\mu^c) \leq q$.*

The following example shows that generally, the implications in Remark 6 (2) are not reversed.

Example 5. *Let$X = \{x_1,x_2,x_3\}$, $Y = \{y_1,y_2,y_3\}$ and $\Phi : X \multimap Y$ be a DF-multifunction defined by $G_\Phi(x_1,y_1) = 0.7$, $G_\Phi(x_1,y_2) = 0.3$, $G_\Phi(x_1,y_3) = 0.3$, $G_\Phi(x_2,y_1) = 0.1$, $G_\Phi(x_2,y_2) = 1$, $G_\Phi(x_2,y_3) = 0.1$, $G_\Phi(x_3,y_1) = 0.2$, $G_\Phi(x_3,y_2) = 0.1$, $G_\Phi(x_3,y_3) = 1$. Define $\mu_1 \in I^X$ and $\mu_2 \in I^Y$ as follows: $\mu_1 = \{0.3, 0.1, 0.2\}$ and $\mu_2 = \{0.3, 0.1, 0.2\}$. Define the DF-topologies $(\tau_1,\tau_1^*), (\tau_2,\tau_2^*) : I^X \to I$, $(\sigma,\sigma^\circ) : I^Y \to I$, and a DF-ideal $(ð,ð^*) : I^Y \to I$ as follows:*

$$\tau_1(\lambda) = \begin{cases} 1, & \lambda \in \{\underline{0},\underline{1}\} \\ 0.3, & \lambda = \underline{0.5} \\ 0, & o.w, \end{cases} \qquad \tau_1^*(\lambda) = \begin{cases} 0, & \lambda \in \{\underline{0},\underline{1}\} \\ 0.7, & \lambda = \underline{0.5} \\ 1, & o.w, \end{cases}$$

$$\tau_2(\lambda) = \begin{cases} 1, & \lambda \in \{\underline{0},\underline{1}\} \\ 0.25, & \lambda = \underline{0.2} \\ 0, & o.w, \end{cases} \qquad \tau_2^*(\lambda) = \begin{cases} 0, & \lambda \in \{\underline{0},\underline{1}\} \\ 0.75, & \lambda = \underline{0.2} \\ 1, & o.w, \end{cases}$$

$$\sigma(\mu) = \begin{cases} 1, & \mu \in \{\underline{0},\underline{1}\} \\ 0.25, & \mu = \mu_2 \\ 0, & o.w, \end{cases} \qquad \sigma^\circ(\mu) = \begin{cases} 0, & \mu \in \{\underline{0},\underline{1}\} \\ 0.75, & \mu = \mu_2 \\ 1, & o.w, \end{cases}$$

$$ð(\nu) = \begin{cases} 1, & \nu = \underline{0} \\ 0.4, & \underline{0} < \nu \leq \underline{0.3} \\ 0, & o.w, \end{cases} \qquad ð^*(\nu) = \begin{cases} 0, & \nu = \underline{0} \\ 0.3, & \underline{0} < \nu \leq \underline{0.3} \\ 1, & o.w. \end{cases}$$

Then, (1) $\Phi : (X,\tau_2,\tau_2^*) \multimap (Y,\sigma,\sigma^\circ,(ð,ð^*))$ is DFU- (resp. DFL-) almost weakly continuous and it is not DFU- (resp. DFL-) almost weakly ð-continuous because

$$\mu_1 = \Phi^u(\mu_2) \leq I_{\tau_2,\tau_2^*}(C_{\tau_2,\tau_2^*}(\Phi^u(C_{\sigma,\sigma^\circ}(\mu_2,0.25,0.75)),0.25,0.75),0.25,0.75) = \underline{1},$$

$$\mu_1 = \Phi^l(\mu_2) \leq I_{\tau_2,\tau_2^*}(C_{\tau_2,\tau_2^*}(\Phi^l(C_{\sigma,\sigma^\circ}(\mu_2,0.25,0.75)),0.25,0.75),0.25,0.75) = \underline{1},$$

$$\mu_1 = \Phi^u(\mu_2) \nleq I_{\tau_2,\tau_2^*}(C_{\tau_2,\tau_2^*}(\Phi^u(cl^*(\mu_2,0.25,0.75)),0.25,0.75),0.25,0.75) = \underline{0.2},$$

$$\mu_1 = \Phi^l(\mu_2) \nleq I_{\tau_2,\tau_2^*}(C_{\tau_2,\tau_2^*}(\Phi^l(cl^*(\mu_2,0.25,0.75)),0.25,0.75),0.25,0.75) = \underline{0.2}.$$

(2) $\Phi : (X, \tau_1, \tau_1^*) \multimap (Y, \sigma, \sigma^\circ, (\eth, \eth^*))$ *is DFU- (resp. DFL-) almost weakly* $\eth$*-continuous and it is not DFU- (resp. DFL-) weakly* $\eth$*-continuous because*

$$\mu_1 = \Phi^u(\mu_2) \le I_{\tau_1,\tau_1^*}(C_{\tau_1,\tau_1^*}(\Phi^u(cl^*(\mu_2, 0.25, 0.75)), 0.25, 0.75), 0.25, 0.75) = \underline{0.5},$$

$$\mu_1 = \Phi^l(\mu_2) \le I_{\tau_1,\tau_1^*}(C_{\tau_1,\tau_1^*}(\Phi^l(cl^*(\mu_2, 0.25, 0.75)), 0.25, 0.75), 0.25, 0.75) = \underline{0.5},$$

$$\mu_1 = \Phi^u(\mu_2) \not\le I_{\tau_1,\tau_1^*}(\Phi^u(cl^*(\mu_2, 0.25, 0.75)), 0.25, 0.75) = \underline{0},$$

$$\mu_1 = \Phi^l(\mu_2) \not\le I_{\tau_1,\tau_1^*}(\Phi^l(cl^*(\mu_2, 0.25, 0.75)), 0.25, 0.75) = \underline{0}.$$

Theorem 24. *Let* $\Phi : (X, \tau, \tau^\circ) \multimap (Y, \sigma, \sigma^\circ, (\eth, \eth^*))$ *be a normalized DF-multifunction, and* Φ *is DFU-almost weakly* $\eth$*-continuous and DFL-almost* $\eth$*-continuous. Then,* Φ *is DFU-weakly* $\eth$*-continuous.*

Proof. Let $\mu \in I^Y$ with $\sigma(\mu) \ge p$, $\sigma^\circ(\mu) \le q$ and Φ be DFU-almost weakly $\eth$-continuous. Then, by Theorem 23 (2),

$$\Phi^u(\mu) \le I_{\tau,\tau^\circ}(C_{\tau,\tau^\circ}(\Phi^u(cl^*(\mu, p, q)), p, q), p, q).$$

Since $C_{\sigma,\sigma^\circ}(\mu, p, q) = C_{\sigma,\sigma^\circ}(int^*(C_{\sigma,\sigma^\circ}(\mu, p, q), p, q), p, q)$, it follows that
$\tau\big([\Phi^u(C_{\sigma,\sigma^\circ}(\mu, p, q))]^c\big) \ge p$ and $\tau^\circ\big([\Phi^u(C_{\sigma,\sigma^\circ}(\mu, p, q))]^c\big) \le q$, and then
$\tau\big([\Phi^u(cl^*(\mu, p, q))]^c\big) \ge p,\ \tau^\circ\big([\Phi^u(cl^*(\mu, p, q))]^c\big) \le q$ and
$\Phi^u(\mu) \le I_{\tau,\tau^\circ}(\Phi^u(cl^*(\mu, p, q)), p, q)$. Thus, by Theorem 16, Φ is DFU-weakly $\eth$-continuous. □

Theorem 25. *Let* $\Phi : (X, \tau, \tau^\circ) \multimap (Y, \sigma, \sigma^\circ, (\eth, \eth^*))$ *be a normalized DF-multifunction,* Φ *be DFL-almost weakly* $\eth$*-continuous and DFU-almost* $\eth$*-continuous. Then,* Φ *is DFL-weakly* $\eth$*-continuous.*

Corollary 4. *Let* $\Phi : (X, \tau, \tau^\circ) \multimap (Y, \sigma, \sigma^\circ)$ *and* $\Xi : (Y, \sigma, \sigma^\circ) \multimap (Z, \eta, \eta^\circ, (\eth, \eth^*))$ *be two DF- (resp. normalized) multifunctions. Then,* $\Xi \circ \Phi$ *is a DFL- (resp. DFU-) almost weakly* $\eth$*-continuous multifunction if* Ξ *is DFL- (resp. DFU-) almost weakly* $\eth$*-continuous and* Φ *is DFL- (resp. DFU-) semi-continuous.*

6. Conclusions

This paper submitted the notions of DFU-almost, DFU-weakly, DFU-almost weakly $\eth$-continuous multifunctions and also submitted the notions of DFL-almost, DFL-weakly, DFL-almost weakly $\eth$-continuous multifunctions depending on a DF-ideal $\eth$. Some characterizations of these types of DF-continuity are proven, and many examples are submitted to explain the allowed implications between these types of DF-continuity. That is, the variety of continuity of DF-multifunctions based on DF-ideals and the implications in between are meaningful and have been discussed in detail. In future work, we will generalize these notions to wider forms of DF-semi continuity. Also, we will try to study the variety of DF-continuity in the fuzzy soft set theory using special operators.

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