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General Mean-Field BDSDEs with Stochastic Linear Growth and Discontinuous Generator

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Abstract: In this paper, we consider the general mean-field backward doubly stochastic differential equations (mean-field BDSDEs) whose generator f can be discontinuous in y. We prove the existence theorem of solutions under stochastic linear growth conditions and also obtain the related comparison theorem. Naturally, we present those results under the linear growth condition, which is a special case of the stochastic condition. Finally, a financial claim sale problem is discussed, which demonstrates the application of the general mean-field BDSDEs in finance.

Keywords: backward doubly stochastic differential equations; mean-field; Wasserstein metric; discontinuous; stochastic linear growth

MSC: 60H10

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1. Introduction

It is well known that backward stochastic differential equations (BSDEs) can be regarded as a class of stochastic differential equations (SDEs) with a given terminal condition (not an initial condition). In 1990, Pardoux and Peng [1] published a famous article and studied nonlinear BSDEs for the first time,

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s. \tag{1}$$

In the past 30 years, research on nonlinear BSDEs has developed rapidly. Many scholars have discovered that this theory has important applications in many fields, such as mathematical finance, stochastic control, partial differential equations (PDEs), and so on. Afterward, Pardoux and Peng [2] proposed backward doubly stochastic differential equations (BDSDEs),

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\overleftarrow{B}_s - \int_t^T Z_s dW_s, \tag{2}$$

which contain two random integrals in opposite directions, leading to two opposite information flows, and thus have more complex measurability. Then. Shi, Gu, and Liu [3] proved the comparison theorem of BDSDEs. Recently, Owo [4–6] generalized these results under a series of stochastic conditions, including the existence and uniqueness theorem of solution for BDSDEs with stochastic Lipschitz generator, the existence theorem of solutions under stochastic linear growth and continuous or discontinuous conditions, and he also proved the associated comparison theorems. Inspired by this literature, in this paper, we study a new class of BDSDEs called general mean-field BDSDEs to obtain the corresponding results, and the equation's form is as follows:

$$Y_t = \xi + \int_t^T f(s, P_{Y_s}, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\overleftarrow{B}_s - \int_t^T Z_s dW_s, \tag{3}$$

where the coefficients of BDSDEs depend not only on the solution processes but also on the law of the solution processes, which acts as the mean-field term.

Mean-field theory is also a hot research topic that has infiltrated various fields, such as statistical mechanics, physics, economics, finance, and so on. In 2007, Lasry and Lions [7] formally proposed the concept of mean-field games, which studied the problem of stochastic differential games with N particles and the limit behavior of random moving particles when N goes to infinity. Inspired by this idea, Buckdahn, Djehiche, Li, Peng [8] and Buckdahn, Li, Peng [9] used purely random methods to investigate a special class of mean-field problems, and proposed a new type of BSDEs, called mean-field BSDEs. Since then, more and more scholars have devoted their energies to the study of mean-field problems (see [10,11], etc.). Li, Liang and Zhang [12] studied mean-field BSDEs under continuous conditions and proposed a technical lemma by which the existence of solutions was obtained. Wang, Zhao and Shi [13] extended this result to discontinuous conditions. In recent years, Li and Xing [14] combined the results of BDSDEs and mean-field theory, and investigated the existence of a solution for general mean-field BDSDEs with continuous coefficients. Furthermore, Shi, Wang and Zhao [15] obtained the related results of the general mean-field BDSDEs under stochastic linear growth and continuous conditions.

It is worth emphasizing that the theory of mean-field is new, and there are still many conclusions to explore. On the one hand, the ordinary continuous condition or linear growth condition cannot be satisfied in many applications, which the example in Section 4 can reflect: Consider a financial claim with a contingent ξ and there is an investor who has additional information not detected in the financial market and wants to sell the claim. Moreover, suppose that the interest rate is applied only to portfolios whose value remains above a nominal value at any time. This problem is equivalent to solving the following mean-field BDSDE:

$$Y_t = \xi + \int_t^T \left(\theta(s)e^{-\frac{\beta A(s)}{2}} E[Y_s] + \mu(s)Y_s I_{\{Y_s>1\}} + \gamma(s)Z_s\right)\mathrm{d}s + \int_t^T c(s)Z_s \mathrm{d}\overleftarrow{B}_s - \int_t^T Z_s \mathrm{d}W_s. \tag{4}$$

Since $f(t,p,y,z) = \theta(t)e^{-\frac{\beta A(t)}{2}} E[y] + r(t)yI_{\{y>1\}} + \gamma(t)z$ is not continuous in y, we cannot apply the existence result in [15]. Therefore, we relax the restriction on the generator $f(t,p,y,z)$ that f is discontinuous in y, continuous in p and z, and we solve the above problem, shown in Section 4. On the other hand, mean field theory is a useful tool when we study problems related to large numbers of particles. Because when the number of particles N tends to infinity, it is impractical to deal with the behavior of each particle, but through the mean-field term, we only need to pay attention to the limited behavior of randomly moving particles when N tends to infinity. In conclusion, it is meaningful to study the general mean-field BDSDE (3) with discontinuous and stochastic linear growth coefficients, which can solve some problems in physics, finance and so on.

Our paper is organized as follows: In Section 2, we give some preliminary results of general mean-field BDSDEs which are needed in what follows, and we also list some existing results related to our paper. Section 3 is devoted to giving the main results, including the existence theorem of solutions and the related comparison theorem under stochastic linear growth and discontinuous conditions. Then, we naturally introduce the existence theorem of solutions under linear growth conditions, which is a special case of stochastic conditions, and we also propose the associated comparison theorem. In Section 4, we study the application of the general mean-field BDSDEs to the financial claim sales problem. Finally, we conclude in Section 5.

2. Preliminaries

Now, we begin with introducing some necessary notations and concepts.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, that is, all subsets of zero probability sets belong to $\mathcal{F}$, and let $T > 0$ be an arbitrarily fixed time horizon throughout this paper. Let $\{W_t; 0 \le t \le T\}$ and $\{B_t; 0 \le t \le T\}$ be two mutually independent standard Brownian Motions with values respectively in $\mathbb{R}^d$ and $\mathbb{R}^\ell$, defined on $(\Omega, \mathcal{F}, P)$. Let $\mathcal{N}$ denote the class of P-null sets of $\mathcal{F}$, and $\mathcal{P}_2(\mathbb{R}^k)$ denotes the set of the probability measures p over $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ with a finite second moment, that is, $\int_{\mathbb{R}^k} |x|^2 \, p(dx) < \infty$. Here, $\mathcal{B}(\mathbb{R}^k)$ denotes the Borel σ-field over $\mathbb{R}^k$, and the probability space $(\Omega, \mathcal{F}, P)$ needs to be rich, so we assume that there is a sub-σ-field $\mathcal{F}^0$, $\mathcal{N} \subset \mathcal{F}^0 \subset \mathcal{F}$, such that

(i) Brownian motion (B, W) is independent of $\mathcal{F}^0$;
(ii) $\mathcal{F}^0$ is 'rich enough', that is, for every $p \in \mathcal{P}_2(\mathbb{R}^k)$ there is a random variable $\xi \in L^2(\Omega, \mathcal{F}^0, P; \mathbb{R}^k)$ such that $P_\xi = p$.

Besides, $\{a(t)\}_{t\in[0,T]}$ is a jointly measurable process with positive values and square-integrable in $[0, T]$, and we define an increasing process $\{A(t)\}_{t\in[0,T]}$ by setting $A(t) = \int_0^t a^2(s)ds$. Every β that appears throughout this paper must satisfy $\beta > 0$ and be big enough. Here, are the following spaces:

- $L^{2,\beta}(\Omega, \mathcal{F}_t, P; \mathbb{R}^k) := \Big\{ \mathbb{R}^k\text{-value } \mathcal{F}_t\text{-measurable random variables}$
  $\xi : \|\xi\|^2_{L^{2,\beta}} := E[e^{\beta A(t)}|\xi|^2] < +\infty \Big\}$;
- $\mathcal{H}^{2,\beta}(0, T; \mathbb{R}^k) := \Big\{ \mathbb{R}^k\text{-value processes } \zeta : \text{ for any } t \in [0, T], \zeta(t) \text{ is } \mathcal{F}_t\text{-measurable}$
  $\text{with } \|\zeta\|^2_{\mathcal{H}^{2,\beta}} := E[\int_0^T e^{\beta A(t)}|\zeta(t)|^2 dt] < +\infty \Big\}$;
- $\mathcal{H}^{2,\beta,a}(0, T; \mathbb{R}^k) := \Big\{ \mathbb{R}^k\text{-value processes } \zeta : \text{ for any } t \in [0, T], \zeta(t) \text{ is } \mathcal{F}_t\text{-measurable}$
  $\text{with } \|a\zeta\|^2_{\mathcal{H}^{2,\beta}} := E[\int_0^T e^{\beta A(t)}a^2(t)|\zeta(t)|^2 dt] < +\infty \Big\}$;
- $\mathcal{S}^{2,\beta}(0, T; \mathbb{R}^k) := \Big\{ \mathbb{R}^k\text{-value continuous processes } \zeta : \text{ for any } t \in [0, T], \zeta(t) \text{ is}$
  $\mathcal{F}_t\text{-measurable with } \|\zeta\|^2_{\mathcal{S}^{2,\beta}} := E[\sup_{0\le t\le T} e^{\beta A(t)}|\zeta(t)|^2] < +\infty \Big\}$.

Note, that the space $\mathcal{H}^{2,\beta}(0, T; \mathbb{R}^k)$ with the norm $\| \cdot \|_{\mathcal{H}^{2,\beta}}$ is a Banach space, so is the space

$$\mathcal{M}^{2,\beta}(0, T) := \Big(\mathcal{H}^{2,\beta,a}(0, T; \mathbb{R}^k) \cap \mathcal{S}^{2,\beta}(0, T; \mathbb{R}^k) \Big) \times \mathcal{H}^{2,\beta}(0, T; \mathbb{R}^{k\times d}),$$

with the norm $\|(Y, Z)\|^2_{\mathcal{M}^{2,\beta}} = \|aY\|^2_{\mathcal{H}^{2,\beta}} + \|Y\|_{\mathcal{S}^{2,\beta}} + \|Z\|^2_{\mathcal{H}^{2,\beta}}$.

Now, let us consider the following general mean-field BDSDEs: for all $t \in [0, T]$, given $\xi \in L^{2,\beta}(\Omega, \mathcal{F}_T, P; \mathbb{R}^k)$,

$$Y_t = \xi + \int_t^T f(s, P_{Y_s}, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\overleftarrow{B}_s - \int_t^T Z_s dW_s. \tag{5}$$

Without loss of generality, in this paper, we consider the case of $k = 1$. Before discussing the main results of this paper, we will introduce some previous results of general mean-field BDSDEs under some stochastic conditions. Let, coefficients $f : [0, T] \times \Omega \times \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $g : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^\ell$ be jointly measurable and satisfy the following assumptions:

(A1) g is stochastic Lipschitz in $(y, z) \in \mathbb{R} \times \mathbb{R}^d$: There exists a non-negative $\mathcal{F}_t^W$-measurable process $\{\nu(t)\}_{t\in[0,T]}$ and a constant $\alpha, 0 < \alpha < 1$, such that for all $y_1, y_2 \in \mathbb{R}, z_1, z_2 \in \mathbb{R}^d$,

$$|g(t, y_1, z_1) - g(t, y_2, z_2)|^2 \le \nu(t)|y_1 - y_2|^2 + \alpha|z_1 - z_2|^2;$$

(A2) f is stochastic Lipschitz in $(p, y, z) \in \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \times \mathbb{R}^d$: There exist non-negative $\mathcal{F}_t^W$-measurable processes $\{\theta(t)\}_{t\in[0,T]}$, $\{\mu(t)\}_{t\in[0,T]}$ and $\{\gamma(t)\}_{t\in[0,T]}$ such that for all $p_1, p_2 \in \mathcal{P}_2(\mathbb{R}), y_1, y_2 \in \mathbb{R}, z_1, z_2 \in \mathbb{R}^d$,

$$|f(t, p_1, y_1, z_1) - f(t, p_2, y_2, z_2)| \le \theta(t)e^{-\frac{\beta A(t)}{2}} W_2(p_1, p_2) + \mu(t)|y_1 - y_2| + \gamma(t)|z_1 - z_2|;$$

(A3) For all $t \in [0,T]$, there exists a positive process $\{a(t)\}_{t\in[0,T]}$, which satisfies $a(t)^2 = \theta(t)^2 + \mu(t) + \gamma(t)^2 + \nu(t) \geq \iota > 0$ and the definitions of $\theta(t), \mu(t), \gamma(t)$ and $\nu(t)$ are same as those in assumptions (A1) and (A2). $A(t) = \int_0^t a(s)^2 ds < \infty$;

(A4) For any $(t,\mu,y,z) \in [0,T] \times \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \times \mathbb{R}^d$, there is

$$E[\int_0^T e^{\beta A(t)} \frac{|f(t,\delta_0,0,0)|^2}{a(t)^2} dt + \int_0^T e^{\beta A(t)} |g(t,0,0)|^2 dt] < +\infty,$$

where δ_0 denotes throughout the paper, the Dirac measure with mass at $0 \in \mathbb{R}$;

(A5) For any $\theta_1, \theta_2 \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R})$ and $(t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$, there exists a non-negative $\mathcal{F}_t^W$ -measurable process $\{L(t)\}_{t\in[0,T]}$ such that

$$f(t, P_{\theta_1}, y, z) - f(t, P_{\theta_2}, y, z) \leq L(t) e^{-\frac{\beta A(t)}{2}} (E[|(\theta_1 - \theta_2)^+|^2])^{\frac{1}{2}},$$

where $L(t)^2 \leq a(t)^2$, for all $t \in [0,T]$;

(A6) For almost every $(t,\omega) \in [0,T] \times \Omega$, $f(t,\omega,\cdot,\cdot,\cdot)$ is continuous, especially, with a continuity modulus $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ with respect to p, for all $p_1, p_2 \in \mathcal{P}_2(\mathbb{R})$,$(y,z) \in \mathbb{R} \times \mathbb{R}^d$,

$$|f(t,\omega,p_1,y,z) - f(t,\omega,p_2,y,z)| \leq \rho(W_2(p_1,p_2));$$

(A7) There exist non-negative $\mathcal{F}_t^W$-measurable processes $\{\theta(t)\}_{t\in[0,T]}$, $\{\mu(t)\}_{t\in[0,T]}$, $\{\gamma(t)\}_{t\in[0,T]}$ and a non-negative $\mathcal{F}_t$-measurable process $\{\phi(t)\}_{t\in[0,T]}$ satisfying $E[\int_t^T e^{\beta A(s)} \phi(s)^2 ds] < \infty$, such that for all $p \in \mathcal{P}_2(\mathbb{R}), y \in \mathbb{R}, z \in \mathbb{R}^d$,

$$|f(t,p,y,z)| \leq \phi(t) + \theta(t) e^{-\frac{\beta A(t)}{2}} W_2(p,\delta_0) + \mu(t)|y| + \gamma(t)|z|;$$

(A8) Monotonicity in p : For all $\theta_1, \theta_2 \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R})$, and all $(y,z) \in \mathbb{R} \times \mathbb{R}^d$, when $\theta_1 \leq \theta_2$, P-a.s., we have

$$f(t, P_{\theta_1}, y, z) \leq f(t, P_{\theta_2}, y, z), \mathrm{dtd}P\text{-a.e.}$$

Lemma 1 (Existence and Uniqueness)**.** *Under the assumptions (A1)–(A4), the general mean-field BDSDE (5) has a unique solution* $(Y,Z) \in \mathcal{M}^{2,\beta}(0,T)$.

Lemma 2 (Comparison theorem)**.** *Let* $g = g(t,\omega,y,z)$ *satisfy the assumptions (A1) and (A4), and* $f^{(i)} = f^{(i)}(t,\omega,p,y,z), i = 1,2$ *be two generators satisfying (A4). Moreover, we assume that*
(i) One of $f^{(i)}$ *satisfies the assumption (A2);*
(ii) One of $f^{(i)}$ *satisfies the assumption (A5).*

Denote by $(Y^{(1)}, Z^{(1)})$ *and* $(Y^{(2)}, Z^{(2)})$ *the solutions of the general mean-field BDSDE (5) with data* $(\xi^{(1)}, f^{(1)}, g)$ *and* $(\xi^{(2)}, f^{(2)}, g)$*, respectively. Then, if for all* $(p,y,z) \in \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \times \mathbb{R}^d$, $\xi^{(1)} \leq \xi^{(2)}$*, P-a.s.,* $f^{(1)}(t,p,y,z) \leq f^{(2)}(t,p,y,z)$*, dtdP-a.e., it holds that also* $Y_t^{(1)} \leq Y_t^{(2)}$*, for all* $t \in [0,T]$*, P-a.s.*

Lemma 3 (Existence)**.** *Under the assumptions (A1), (A3), (A4) and (A6)–(A8), the general mean-field BDSDE (5) at least has one solution* $(Y,Z) \in \mathcal{M}^{2,\beta}(0,T)$*. Moreover, there is a minimal solution* $(\underline{Y}, \underline{Z}) \in \mathcal{M}^{2,\beta}(0,T)$ *of the general mean-field BDSDE (5).*

Lemma 4 (Comparison theorem)**.** *Let* $g = g(t,\omega,y,z)$ *satisfy the assumptions (A1) and (A4),* $f^{(1)} = f^{(1)}(t,\omega,p,y,z)$ *satisfy the assumptions (A3), (A4) and (A6)-(A8), and* $f^{(2)} = f^{(2)}(t,\omega,p,y,z)$ *satisfy the assumption (A4). Denote by* $(Y^{(1)}, Z^{(1)})$ *and* $(Y^{(2)}, Z^{(2)})$ *the solutions of general mean-field BDSDE (5) with data* $(\xi^{(1)}, f^{(1)}, g)$ *and* $(\xi^{(2)}, f^{(2)}, g)$*, respectively. Then, if for all* $(p,y,z) \in \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \times \mathbb{R}^d$*, there are* $\xi^{(1)} \leq \xi^{(2)}$ *P-a.s. and* $f^{(1)}(t,p,y,z) \leq f^{(2)}(t,p,y,z)$*, dtdP-a.e., it holds that also* $Y_t^{(1)} \leq Y_t^{(2)}$*, for all* $t \in [0,T]$*, P-a.s.*

For the detailed proofs of Lemmas 1–4, readers can refer to [15].

3. General Mean-Field BDSDEs with Stochastic Linear Growth and Discontinuous Generator

In this section, we focus on general mean-field BDSDEs (5) with stochastic linear growth and a discontinuous generator. We need to add some assumptions for the generator f as follows:

(B1) For a.e. $(t,\omega) \in [0,T]$, $f(t,p,y,z)$ is left-continuous in y, continuous in p and z, especially, with a continuity modulus $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ for p: for all $p_1, p_2 \in \mathcal{P}_2(\mathbb{R}), (t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$, there is

$$|f(t,p_1,y,z) - f(t,p_2,y,z)| \leq \rho(W_2(p_1,p_2)).$$

Here, ρ is supposed to be non-decreasing, such that $\rho(0_+) = 0$;

(B2) There exists a continuous function $K(t,p,y,z)$ defined on $[0,T] \times \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \times \mathbb{R}^d$, which is non-decreasing with respect to p, and there exist three non-negative $\mathcal{F}_t^W$-measurable processes $\{\theta(t)\}_{t\in[0,T]}$, $\{\mu(t)\}_{t\in[0,T]}$, $\{\gamma(t)\}_{t\in[0,T]}$ such that for all $t \in [0,T], p \in \mathcal{P}_2(\mathbb{R}), y \in \mathbb{R}, z \in \mathbb{R}^d$, there is

$$|K(t,p,y,z)| \leq \theta(t)e^{-\frac{\beta A(t)}{2}} W_2(p,\delta_0) + \mu(t)|y| + \gamma(t)|z|,$$

and for all $y_1 \geq y_2 \in \mathbb{R}, p_1, p_2 \in \mathcal{P}_2(\mathbb{R}), z_1, z_2 \in \mathbb{R}^d$, we have

$$f(t,p_1,y_1,z_1) - f(t,p_2,y_2,z_2) \geq K(t,\Delta(p_1,p_2), y_1 - y_2, z_1 - z_2),$$

where $\Delta(p_1,p_2) \in \mathcal{P}_2(\mathbb{R})$ and satisfies $W_2\big(\Delta(p_1,p_2),\delta_0\big) = W_2(p_1,p_2)$.

3.1. The Existence of Solutions

Next, we will give the first important conclusion of this paper, the existence of solutions for the general mean-field BDSDEs (5) under discontinuous and stochastic linear growth conditions. In order to facilitate readers to understand the logic of proof, we refer to the idea of proof in [16]. We first introduce the following technical lemma:

Lemma 5. *for $n \geq 1$, let us define*

$$\begin{aligned} K_n(t,p,y,z) = \inf_{(\nu,r,h)\in\mathcal{P}_2(\mathbb{R})\times\mathbb{R}\times\mathbb{R}^d} \Big\{ & K(t,\nu,r,h) + n\big(\theta(t)e^{-\frac{\beta A(t)}{2}} W_{2,+}(\mu,\nu) \\ & + \mu(t)|y-r| + \gamma(t)|z-h|\big)\Big\}, \end{aligned} \tag{6}$$

where $K(t,\nu,r,h)$ is similar to that in assumption (B2). From Lemma 3.1 in [12], we know these equations are well-defined and satisfy the following properties:

(i) Linear growth: for all $(t,p,y,z) \in [0,T] \times \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \times \mathbb{R}^d$, P-a.s., we have

$$|K_n(t,p,y,z)| \leq \theta(t)e^{-\frac{\beta A(t)}{2}} W_2(p,\delta_0) + \mu(t)|y| + \gamma(t)|z|;$$

(ii) Monotonicity in p: for all $\theta_1, \theta_2 \in L^{2,\beta}(\Omega, \mathcal{F}_T, P; \mathbb{R})$ with $\theta_1 \leq \theta_2$, P-a.s., we have

$$K_n\big(t, P_{\theta_1}, y, z\big) \leq K_n\big(t, P_{\theta_2}, y, z\big), dtdP\text{-a.s.};$$

(iii) Monotonicity in n: for all $(t,p,y,z) \in [0,T] \times \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \times \mathbb{R}^d$, P-a.s., $n < m$, we have

$$K_n(t,p,y,z) \leq K_m(t,p,y,z), dtdP\text{-a.s.};$$

(iv) Stochastic Lipschitz continuous: for all $(t, p_1, y_1, z_1), (t, p_2, y_2, z_2) \in [0, T] \times \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \times \mathbb{R}^d$, P-a.s., we have

$$
\begin{aligned}
&|K_n(t, p_1, y_1, z_1) - K_n(t, p_2, y_2, z_2)| \\
\le& n\big(\theta(t)e^{-\frac{\beta A(t)}{2}} W_2(p_1, p_2) + \mu(t)|y_1 - y_2| + \gamma(t)|z_1 - z_2|\big);
\end{aligned}
$$

(v) Strong convergence: if $(p_n, y_n, z_n) \to (p, y, z)$ as $n \to \infty$ in $\mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \times \mathbb{R}^d$, then

$$\lim_{n\to\infty} K_n(t, p_n, y_n, z_n) = K(t, p, y, z).$$

Proposition 1. *Let $\xi \in L^{2,\beta}(\Omega, \mathcal{F}_T, P; \mathbb{R})$, $\varphi_t \in \mathcal{H}^{2,\beta}(0, T; \mathbb{R})$, $K(t, p, u, v)$ and $G(t, u, v)$ satisfy the assumptions (A1), (A3), (A4) and (B2), and $G(t, 0, 0) = 0$. We consider the following general mean-field BDSDE: for all $t \in [0, T]$,*

$$U_t = \xi + \int_t^T (K(s, P_{U_s}, U_s, V_s) + \varphi_s)\, \mathrm{d}s + \int_t^T G(s, U_s, V_s)\, \mathrm{d}\overleftarrow{B}_s - \int_t^T V_s\, \mathrm{d}W_s, \tag{7}$$

then, we have:

(i) The Equation (7) has at least one solution $(U, V) \in \mathcal{M}^{2,\beta}(0, T)$;

(ii) For any solution (U, V) of (7), if $\varphi_t \ge 0, \xi \ge 0$, we can obtain $U_t \ge 0$, P-a.s., $t \in [0, T]$.

Proof. Since K is continuous and $|K(t, p, y, z)| \le \theta(t)e^{-\frac{\beta A(t)}{2}} W_2(p, \delta_0) + \mu(t)|y| + \gamma(t)|z|$, we note that (7) has at least one solution. For each n, because of Lemma 5, the following general mean-field BDSDEs are as follows: for all $t \in [0, T]$,

$$U_t^n = \xi + \int_t^T \big(K_n\big(s, P_{U_s^n}, U_s^n, V_s^n\big) + \varphi_s\big)\, \mathrm{d}s + \int_t^T G(s, U_s^n, V_s^n)\, \mathrm{d}\overleftarrow{B}_s - \int_t^T V_s^n\, \mathrm{d}W_s, \quad n \ge 1 \tag{8}$$

has a unique adapted solution, and the solution $\{(U^n, V^n)\}_{n=1}^\infty$ of Equation (8) converge to the minimal solution $(\underline{U}, \underline{V})$ of Equation (7).

Next, we consider the following general mean-field BDSDEs: for all $t \in [0, T]$,

$$\tilde{U}_t^n = \int_t^T K_n\Big(s, P_{\tilde{U}_s^n}, \tilde{U}_s^n, \tilde{V}_s^n\Big)\, \mathrm{d}s + \int_t^T G(s, \tilde{U}_s^n, \tilde{V}_s^n)\, \mathrm{d}\overleftarrow{B}_s - \int_t^T \tilde{V}_s^n\, \mathrm{d}W_s, \quad n \ge 1. \tag{9}$$

For each n, there exists a unique solution to Equation (9). Since $K_n(s, \delta_0, 0, 0) = 0$, $G(s, 0, 0) = 0$, then $(\tilde{U}^n, \tilde{V}^n) = (0, 0)$ is the unique solution to Equation (9). From the Lemma 2 and $\varphi_t \ge 0$, it follows that $U_t^n \ge \tilde{U}_t^n = 0$. Therefore, $\underline{U}_t \ge 0$. □

Before proving the existence of solutions of (5), we first construct a sequence of general mean-field BDSDEs as follows: given $\xi \in L^{2,\beta}(\Omega, \mathcal{F}_T, P; \mathbb{R})$, $t \in [0, T]$, $n \ge 1$,

$$
\begin{aligned}
\underline{Y}_t^0 =& \xi + \int_t^T \Big[-\theta(s)e^{-\frac{\beta A(s)}{2}} W_2(P_{\underline{Y}_s^0}, \delta_0) - \mu(s)|\underline{Y}_s^0| - \gamma(s)|\underline{Z}_s^0| - \phi(s)\Big] \mathrm{d}s \\
&+ \int_t^T g(s, \underline{Y}_s^0, \underline{Z}_s^0) \mathrm{d}\overleftarrow{B}_s - \int_t^T \underline{Z}_s^0 \mathrm{d}W_s, \qquad (10)\\
\underline{Y}_t^n =& \xi + \int_t^T \Big[f\big(s, P_{\underline{Y}_s^{n-1}}, \underline{Y}_s^{n-1}, \underline{Z}_s^{n-1}\big) + K\big(s, \Delta(P_{\underline{Y}_s^n}, P_{\underline{Y}_s^{n-1}}), \underline{Y}_s^n - \underline{Y}_s^{n-1}, \underline{Z}_s^n - \underline{Z}_s^{n-1}\big)\Big] \mathrm{d}s \\
&+ \int_t^T g(s, \underline{Y}_s^n, \underline{Z}_s^n) \mathrm{d}\overleftarrow{B}_s - \int_t^T \underline{Z}_s^n \mathrm{d}W_s, \qquad (11)\\
\overline{Y}_t^0 =& \xi + \int_t^T \Big[\theta(s)e^{-\frac{\beta A(s)}{2}} W_{2,+}(P_{\overline{Y}_s^0}, \delta_0) + \mu(s)|\overline{Y}_s^0| + \gamma(s)|\overline{Z}_s^0| + \phi(s)\Big]\, \mathrm{d}s \\
&+ \int_t^T g(s, \overline{Y}_s^0, \overline{Z}_s^0)\, \mathrm{d}\overleftarrow{B}_s - \int_t^T \overline{Z}_s^0\, \mathrm{d}W_s. \qquad (12)
\end{aligned}
$$

By Lemma 3, there exists at least one solution of (11). Here, we only consider the minimal solution, denoted as $(\underline{Y}^n, \underline{Z}^n)$. By Lemma 1 we know that (10) and (12) have unique solutions and are denoted as $(\underline{Y}^0, \underline{Z}^0)$ and $(\overline{Y}^0, \overline{Z}^0)$, respectively.

Proposition 2. *Under assumptions (A1), (A3), (A4), (A7), (B1) and (B2), then:*
(i) For any $n \geq 0$, $\underline{Y}_t^{n+1} \geq \underline{Y}_t^n$, $t \leq T$, P-a.s.
(ii) For any $n \geq 0$, $\overline{Y}_t^0 \geq \underline{Y}_t^n$, $t \leq T$, P-a.s.

Proof. Before the proof of this proposition, we first define

$$\begin{aligned} & G(s, Y_s^{n+1} - Y_s^n, Z_s^{n+1} - Z_s^n) \\ & := g(s, Y_s^{n+1} - Y_s^n + Y_s^n, Z_s^{n+1} - Z_s^n + Z_s^n) - g(s, Y_s^n, Z_s^n) \\ & \ = g(s, Y_s^{n+1}, Z_s^{n+1}) - g(s, Y_s^n, Z_s^n), \end{aligned} \tag{13}$$

thus, we know $G(s, 0, 0) = 0$.

(i) The conclusion can be proved by the induction method. First, prove that $\underline{Y}_t^1 \geq \underline{Y}_t^0$. From the Equations (10), (11) and (13) we have

$$\begin{aligned} \underline{Y}_t^1 - \underline{Y}_t^0 = & \int_t^T \Big[K\Big(s, \Delta(P_{\underline{Y}_s^1}, P_{\underline{Y}_s^0}), \underline{Y}_s^1 - \underline{Y}_s^0, \underline{Z}_s^1 - \underline{Z}_s^0\Big) + \psi_s^0 \Big] \mathrm{d}s \\ & + \int_t^T G(s, \underline{Y}_s^1 - \underline{Y}_s^0, \underline{Z}_s^1 - \underline{Z}_s^0)\, \mathrm{d}\overleftarrow{B}_s \int_t^T \Big(\underline{Z}_s^1 - \underline{Z}_s^0\Big) \mathrm{d}W_s, \end{aligned}$$

where $\psi_s^0 = f(s, P_{\underline{Y}_s^0}, \underline{Y}_s^0, \underline{Z}_s^0) + \theta(s) e^{-\frac{\beta A(s)}{2}} W_2(P_{\underline{Y}_s^0}, \delta_0) + \mu(s)|\underline{Y}_s^0| + \gamma(s)|\underline{Z}_s^0| + \phi(s)$. From (A7), we know $\psi_s^0 \geq 0$. Because $\Big(\underline{Y}_t^0, \underline{Z}_t^0\Big)$ is the solution of Equation (10), so $\psi_s^0 \in \mathcal{H}^{2,\beta}(0, T, \mathbb{R})$. Therefore, from Proposition 1, we can obtain $\underline{Y}_t^1 - \underline{Y}_t^0 \geq 0$, i.e., $\underline{Y}_t^1 \geq \underline{Y}_t^0$, for all $t \in [0, T]$, P-a.s.

Next, suppose $\underline{Y}_t^n \geq \underline{Y}_t^{n-1}$, then we prove $\underline{Y}_t^{n+1} \geq \underline{Y}_t^n$.

From Equations (11) and (13), we can obtain

$$\begin{aligned} \underline{Y}_t^{n+1} - \underline{Y}_t^n = & \int_t^T \Big[K\Big(s, \Delta(P_{\underline{Y}_s^{n+1}}, P_{\underline{Y}_s^n}), \underline{Y}_s^{n+1} - \underline{Y}_s^n, \underline{Z}_s^{n+1} - \underline{Z}_s^n\Big) + \psi_s^n \Big] \mathrm{d}s \\ & + \int_t^T G(s, \underline{Y}_s^{n+1} - \underline{Y}_s^n, \underline{Z}_s^{n+1} - \underline{Z}_s^n)\, \mathrm{d}\overleftarrow{B}_s - \int_t^T \Big(\underline{Z}_s^{n+1} - \underline{Z}_s^n\Big) \mathrm{d}W_s, \end{aligned}$$

where $\psi_s^n = f(s, P_{\underline{Y}_s^n}, \underline{Y}_s^n, \underline{Z}_s^n) - f(s, P_{\underline{Y}_s^{n-1}}, \underline{Y}_s^{n-1}, \underline{Z}_s^{n-1}) - K\big(\Delta(P_{\underline{Y}_s^n}, P_{\underline{Y}_s^{n-1}}), \underline{Y}_s^n - \underline{Y}_s^{n-1}, \underline{Z}_s^n - \underline{Z}_s^{n-1}\big)$. From (B2), we know $\psi_s^n \geq 0$. Similarly, we can also obtain that $\underline{Y}_t^{n+1} \geq \underline{Y}_t^n$, for all $t \in [0, T]$, P-a.s.

(ii) We still use the induction method to prove $\overline{Y}_t^0 \geq \underline{Y}_t^n, n \geq 0$.

Before proving $\overline{Y}_t^0 \geq \underline{Y}_t^0$, we first consider the following general mean-field BDSDEs:

$$\begin{aligned} I'_t = & \int_t^T \theta(s) e^{-\frac{\beta A(s)}{2}} W_2\Big(P_{I'_s}, \delta_0\Big) \mathrm{d}s + \int_t^T G(s, I'_s, J'_s)\, \mathrm{d}\overleftarrow{B}_s - \int_t^T J'_s \mathrm{d}W_s, \\ I_t = & \int_t^T \Big(-\theta(s) e^{-\frac{\beta A(s)}{2}} W_2(P_{I_s}, \delta_0) - \mu(s) I_s - \gamma(s) J(s)\Big) \mathrm{d}s \\ & + \int_t^T G(s, I_s, J_s) \mathrm{d}\overleftarrow{B}_s - \int_t^T J_s \mathrm{d}W_s, \end{aligned}$$

where $G(s, 0, 0) = 0$. Under assumptions (A1), (A3) and (A4), each of these two equations has a unique solution $(I', J') = (0, 0)$ and $(I, J) = (0, 0)$.

From (10) and (12), we can obtain

$$\overline{Y}_t^0 - \underline{Y}_t^0 = \int_t^T \left(\theta(s)e^{-\frac{\beta A(s)}{2}} W_2\left(P_{\overline{Y}_s^0}, P_{\underline{Y}_t^0}\right) + \Phi_s^0\right)\mathrm{d}s + \int_t^T G(s, \overline{Y}_s^0 - \underline{Y}_s^0, \overline{Z}_s^0 - \underline{Z}_s^0)\,\mathrm{d}\overleftarrow{B}_s - \int_t^T \left(\overline{Z}_s^0 - \underline{Z}_s^0\right)\mathrm{d}W_s,$$

where

$$\begin{aligned}\Phi_s^0 =& \theta(s)e^{-\frac{\beta A(s)}{2}}(W_{2,+}(P_{\overline{Y}_s^0}, \delta_0) + W_2(P_{\underline{Y}_s^0}, \delta_0) - W_2(P_{\overline{Y}_s^0}, P_{\underline{Y}_s^0})) + \mu(s)(|\overline{Y}_s^0| + |\underline{Y}_s^0|) \\ &+ \gamma(s)(|\overline{Z}_s^0| + |\underline{Z}_s^0|) + 2\phi(s) \geq 0.\end{aligned}$$

From Lemma 2, it follows

$$\overline{Y}_t^0 - \underline{Y}_t^0 \geq I_t' = 0.$$

Therefore, $\overline{Y}_t^0 \geq \underline{Y}_t^0$, for all $t \in [0, T]$, P-a.s.

Next, for $n = 1$, we have

$$\overline{Y}_t^0 - \underline{Y}_t^1 = \int_t^T \left(-\theta(s)e^{-\frac{\beta A(s)}{2}} W_2(P_{\overline{Y}_s^0}, P_{\underline{Y}_s^1}) - \mu(s)|\overline{Y}_s^0 - \underline{Y}_s^1| - \gamma(s)|\overline{Z}_s^0 - \underline{Z}_s^1| + \Phi_s^1\right)\mathrm{d}s + \int_t^T G(s, \overline{Y}_s^0 - \underline{Y}_s^1, \overline{Z}_s^0 - \underline{Z}_s^1)\,\mathrm{d}\overleftarrow{B}_s - \int_t^T (\overline{Z}_s^0 - \underline{Z}_s^1)\mathrm{d}W_s,$$

where

$$\begin{aligned}\Phi_s^1 =& \theta(s)e^{-\frac{\beta A(s)}{2}} W_2(P_{\overline{Y}_s^0}, P_{\underline{Y}_s^1}) + \mu(s)|\overline{Y}_s^0 - \underline{Y}_s^1| + \gamma(s)|\overline{Z}_s^0 - \underline{Z}_s^1| + \theta(s)e^{-\frac{\beta A(s)}{2}} W_{2,+}(P_{\overline{Y}_s^0}, \delta_0) \\ &+ \mu(s)|\overline{Y}_s^0| + \gamma(s)|\overline{Z}_s^0| + \phi(s) - f\left(s, P_{\underline{Y}_s^0}, \underline{Y}_s^0, \underline{Z}_s^0\right) - K\left(s, \Delta(P_{\underline{Y}_s^1}, P_{\underline{Y}_s^0}), \underline{Y}_s^1 - \underline{Y}_s^0, \underline{Z}_s^1 - \underline{Z}_s^0\right) \\ \geq& f\left(s, P_{\underline{Y}_s^1}, \underline{Y}_s^1, \underline{Z}_s^1\right) - f\left(s, P_{\underline{Y}_s^0}, \underline{Y}_s^0, \underline{Z}_s^0\right) - K\left(s, \Delta(P_{\underline{Y}_s^1}, P_{\underline{Y}_s^0}), \underline{Y}_s^1 - \underline{Y}_s^0, \underline{Z}_s^1 - \underline{Z}_s^0\right) \\ \geq& 0.\end{aligned}$$

From Lemma 2, it follows

$$\overline{Y}_t^0 - \underline{Y}_t^1 \geq I_t = 0.$$

Therefore, $\overline{Y}_t^0 \geq \underline{Y}_t^1$, for all $t \in [0, T]$, P-a.s.

Next, suppose $\overline{Y}_t^0 \geq \underline{Y}_t^{n-1}$, then we prove $\overline{Y}_t^0 \geq \underline{Y}_t^n$.

$$\overline{Y}_t^0 - \underline{Y}_t^n = \int_t^T \left(-\theta(s)e^{-\frac{\beta A(s)}{2}} W_2(P_{\overline{Y}_s^0}, P_{\underline{Y}_s^n}) - \mu(s)|\overline{Y}_s^0 - \underline{Y}_s^n| - \gamma(s)|\overline{Z}_s^0 - \underline{Z}_s^n| + \Phi_s^n\right)\mathrm{d}s + \int_t^T G(s, \overline{Y}_s^0 - \underline{Y}_s^n, \overline{Z}_s^0 - \underline{Z}_s^n)\,\mathrm{d}\overleftarrow{B}_s - \int_t^T \left(\overline{Z}_s^0 - \underline{Z}_s^n\right)\mathrm{d}W_s,$$

where

$$\begin{aligned}\Phi_s^n =& \theta(s)e^{-\frac{\beta A(s)}{2}}(W_2(P_{\overline{Y}_s^0}, P_{\underline{Y}_s^n}) + W_{2,+}(P_{\overline{Y}_s^0}, \delta_0)) + \mu(s)\left(|\overline{Y}_s^0 - \underline{Y}_s^n| + |\overline{Y}_s^0|\right) + \gamma(s)\left(|\overline{Z}_s^0 - \underline{Z}_s^n|\right. \\ &\left.+ |\overline{Z}_s^0|\right) + \phi(s) - f\left(s, P_{\underline{Y}_s^{n-1}}, \underline{Y}_s^{n-1}, \underline{Z}_s^{n-1}\right) - K\left(s, \Delta(P_{\underline{Y}_s^n}, P_{\underline{Y}_s^{n-1}}), \underline{Y}_s^n - \underline{Y}_s^{n-1}, \underline{Z}_s^n - \underline{Z}_s^{n-1}\right) \\ \geq& f\left(s, P_{\underline{Y}_s^n}, \underline{Y}_s^n, \underline{Z}_s^n\right) - f\left(s, P_{\underline{Y}_s^n}, \underline{Y}_s^n, \underline{Z}_s^n\right) - K\left(s, \Delta(P_{\underline{Y}_s^n}, P_{\underline{Y}_s^{n-1}}), \underline{Y}_s^n - \underline{Y}_s^{n-1}, \underline{Z}_s^n - \underline{Z}_s^{n-1}\right) \\ \geq& 0.\end{aligned}$$

Similarly, $\overline{Y}_t^0 \geq \underline{Y}_t^n$, for all $t \in [0, T]$, P-a.s. □

Remark 1. *Proposition 2 implies that the minimal solution sequence of general mean-field BDSDEs (11) is increasing and has an upper bound, i.e.,*

$$\overline{Y}_t^0 \geq \underline{Y}_t^{n+1} \geq \underline{Y}_t^n \geq \underline{Y}_t^0,\ t \leq T,\ P\text{-a.s.},\ n \geq 1. \tag{14}$$

Furthermore, we obtain our main theorem:

Theorem 1. *Under the assumptions (A1), (A3), (A4), (A7), (B1) and (B2), the sequence of solutions of the family of Equation (11) $\{(Y^n, Z^n)\}_{n=1}^{\infty} \in \mathcal{M}^{2,\beta}(0,T)$ converges to $(\underline{Y}, \underline{Z})$, which is the minimal solution of Equation (5).*

Proof. From Equation (14), we know that $(\underline{Y}^n)_{n=1}^{\infty}$ is increasing and bounded in $\mathcal{H}^{2,\beta}(0,T;\mathbb{R})$. By the monotone convergence theorem, we can deduce that $(\underline{Y}^n)_{n=1}^{\infty}$ converges in $\mathcal{H}^{2,\beta}(0,T;\mathbb{R})$, and denote $\underline{Y}$ as the limit of $\{\underline{Y}^n\}_{n=1}^{\infty}$. Notice that

$$\sup_n E\big[\sup_{0\leq t\leq T} e^{\beta A(t)}|\underline{Y}_t^n|^2\big] \leq E\big[\sup_{0\leq t\leq T} e^{\beta A(t)}|\underline{Y}_t^0|^2\big] + E\big[\sup_{0\leq t\leq T} e^{\beta A(t)}|\overline{Y}_t^0|^2\big] < \infty.$$

Using the Itô formula to $e^{\beta A(t)}\left|\underline{Y}_t^{n+1}\right|^2$, then we have

$$\begin{aligned}
&\left|\underline{Y}_0^{n+1}\right|^2 + \beta\int_0^T e^{\beta A(t)}a^2(t)\left|\underline{Y}_t^{n+1}\right|^2 \mathrm{d}t + \int_0^T e^{\beta A(t)}\left|\underline{Z}_t^{n+1}\right|^2 \mathrm{d}t\\
=&e^{\beta A(T)}|\xi|^2 + 2\int_0^T e^{\beta A(t)}\underline{Y}_t^{n+1}\Big(f\Big(t, P_{\underline{Y}_t^n}, \underline{Y}_t^n, \underline{Z}_t^n\Big) + K(t, \Delta(P_{\underline{Y}_t^{n+1}}, P_{\underline{Y}_t^n}), \underline{Y}_t^{n+1} - \underline{Y}_t^n,\\
&\underline{Z}_t^{n+1} - \underline{Z}_t^n)\Big)\mathrm{d}t + \int_0^T e^{\beta A(t)}|g(t, \underline{Y}_t^{n+1}, \underline{Z}_t^{n+1})|^2 \mathrm{d}t + 2\int_0^T e^{\beta A(t)}\underline{Y}_t^{n+1}g(t, \underline{Y}_t^{n+1}, \underline{Z}_t^{n+1})\mathrm{d}\overleftarrow{B}_t\\
&-2\int_0^T e^{\beta A(t)}\underline{Y}_t^{n+1}\underline{Z}_t^{n+1}\mathrm{d}W_t.
\end{aligned}$$

Noticing that $\int_0^T e^{\beta A(t)}\underline{Y}_t^{n+1}\underline{Z}_t^{n+1}\mathrm{d}W_t$ and $\int_0^T e^{\beta A(t)}\underline{Y}_t^{n+1}g(t, \underline{Y}_t^{n+1}, \underline{Z}_t^{n+1})\mathrm{d}\overleftarrow{B}_t$ are martingales. Next, we take the expectation of the above equation. From $2ab \leq \frac{1}{\delta}a^2 + \delta b^2$, $\delta > 0$, we know that there exist constants $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned}
&E[|\underline{Y}_0^{n+1}|^2] + \beta E[\int_0^T e^{\beta A(t)}a^2(t)|\underline{Y}_t^{n+1}|^2 \mathrm{d}t] + E[\int_0^T e^{\beta A(t)}|\underline{Z}_t^{n+1}|^2 \mathrm{d}t]\\
=&E[e^{\beta A(T)}|\xi|^2] + 2E[\int_0^T e^{\beta A(t)}\underline{Y}_t^{n+1}(f(t, P_{\underline{Y}_t^n}, \underline{Y}_t^n, \underline{Z}_t^n) + K(t, \Delta(P_{\underline{Y}_t^{n+1}}, P_{\underline{Y}_t^n}), \underline{Y}_t^{n+1} - \underline{Y}_t^n,\\
&\underline{Z}_t^{n+1} - \underline{Z}_t^n))\mathrm{d}t] + E[\int_0^T e^{\beta A(t)}|g(t, \underline{Y}_t^{n+1}, \underline{Z}_t^{n+1})|^2 \mathrm{d}t]\\
\leq&E[e^{\beta A(T)}|\xi|^2] + 2E[\int_0^T e^{\beta A(t)}\underline{Y}_t^{n+1}f(t, P_{\underline{Y}_t^{n+1}}, \underline{Y}_t^{n+1}, \underline{Z}_t^{n+1})\mathrm{d}t] + E[\int_0^T e^{\beta A(t)}|g(t, \underline{Y}_t^{n+1}, \underline{Z}_t^{n+1})|^2 \mathrm{d}t]\\
\leq&E[e^{\beta A(T)}|\xi|^2] + 2E[\int_0^T e^{\beta A(t)}|\underline{Y}_t^{n+1}|(\phi(t) + \theta(t)e^{-\frac{\beta A(t)}{2}}W_2(P_{\underline{Y}_t^{n+1}}, \delta_0) + \mu(t)|\underline{Y}_t^{n+1}| + \gamma(t)\\
&|\underline{Z}_t^{n+1}|)\mathrm{d}t] + E[\int_0^T e^{\beta A(t)}((1+\delta_1)|g(t, \underline{Y}_t^{n+1}, \underline{Z}_t^{n+1}) - g(t,0,0)|^2 + (1 + \frac{1}{\delta_1})|g(t,0,0)|^2)\mathrm{d}t]\\
\leq&E[e^{\beta A(T)}|\xi|^2] + E[\int_0^T e^{\beta A(t)}|\underline{Y}_t^{n+1}|^2 \mathrm{d}t] + E[\int_0^T e^{\beta A(t)}\phi(t)^2 \mathrm{d}t] + E[\int_0^T e^{\beta A(t)}\theta(t)^2|\underline{Y}_t^{n+1}|^2 \mathrm{d}t]\\
&+ E[\int_0^T E[|\underline{Y}_t^{n+1}|^2]\mathrm{d}t] + 2E[\int_0^T e^{\beta A(t)}\mu(t)|\underline{Y}_t^{n+1}|^2 \mathrm{d}t] + \delta_2 E[\int_0^T e^{\beta A(t)}\gamma(t)^2|\underline{Y}_t^{n+1}|^2 \mathrm{d}t]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\delta_2}E[\int_0^T e^{\beta A(t)}|\underline{Z}_t^{n+1}|^2\mathrm{d}t]+(1+\delta_1)E[\int_t^T e^{\beta A(t)}\nu(t)|\underline{Y}^{n+1}(t)|^2\mathrm{d}t]\\
&+(1+\delta_1)\alpha E[\int_0^T e^{\beta A(t)}|\underline{Z}_s^{n+1}|^2\mathrm{d}t]+(1+\frac{1}{\delta_1})E[\int_0^T e^{\beta A(t)}|g(t,0,0)|^2\mathrm{d}t]\\
\leq& E[e^{\beta A(T)}|\xi|^2]+2E[\int_0^T e^{\beta A(t)}|\underline{Y}_t^{n+1}|^2\mathrm{d}t]+E[\int_0^T e^{\beta A(t)}\big(\theta(t)^2+2\mu(t)+\delta_2\gamma(t)^2+(1+\delta_1)\nu(t)\big)\\
&|\underline{Y}_t^{n+1}|^2\mathrm{d}t]+(\frac{1}{\delta_2}+(1+\delta_1)\alpha)E[\int_0^T e^{\beta A(t)}|\underline{Z}_t^{n+1}|^2\mathrm{d}t]+E[\int_0^T e^{\beta A(t)}\big(\phi(t)^2+(1+\frac{1}{\delta_1})|g(t,0,0)|^2\big)\mathrm{d}t].
\end{aligned}
$$

As we mentioned earlier, β is big enough, so let $\beta > 4+\delta_1+\delta_2$. Taking $\delta_1 = \frac{1-\alpha}{4\alpha}$ and $\delta_2 = \frac{4}{1-\alpha}$, then the above inequality can be simplified as follows

$$
\begin{aligned}
&E[|\underline{Y}_0^{n+1}|^2]+(\beta-4-\frac{1-\alpha}{4\alpha}-\frac{4}{1-\alpha})E[\int_0^T e^{\beta A(t)}a^2(t)|\underline{Y}_t^{n+1}|^2\mathrm{d}t]+\frac{1-\alpha}{2}E[\int_0^T e^{\beta A(t)}|\underline{Z}_t^{n+1}|^2\mathrm{d}t]\\
\leq& E[e^{\beta A(T)}|\xi|^2]+2TE\big[\sup_{0\leq t\leq T}e^{\beta A(t)}\big|\underline{Y}_t^{n+1}\big|^2\big]+E[\int_0^T e^{\beta A(t)}\big(\phi(t)^2+\frac{1+3\alpha}{1-\alpha}|g(t,0,0)|^2\big)\mathrm{d}t]<\infty,
\end{aligned}
$$

which implies that $\sup_n E[\int_0^T e^{\beta A(t)}\left|\underline{Z}_t^{n+1}\right|^2\mathrm{d}t] < \infty$, and $\eta_t^{n+1} = f\big(t, P_{\underline{Y}_t^n}, \underline{Y}_t^n, \underline{Z}_t^n\big) + K\big(t, \Delta(P_{\underline{Y}_t^{n+1}}, P_{\underline{Y}_t^n}), \underline{Y}_t^{n+1}-\underline{Y}_t^n, \underline{Z}_t^{n+1}-\underline{Z}_t^n\big)$ is uniformly bounded in $\mathcal{H}^{2,\beta}(0,T;\mathbb{R})$. Let $C_0 = \sup_n E[\int_0^T e^{\beta A(t)}|\eta_t^n|^2\mathrm{d}t]$. For any $m, n > 0$, using Itô formula to $e^{\beta A(t)}|\underline{Y}_t^m - \underline{Y}_t^n|^2$, we can obtain

$$
\begin{aligned}
&E[\int_0^T e^{\beta A(t)}|\underline{Z}_t^m-\underline{Z}_t^n|^2\mathrm{d}t]\\
\leq& 2E[\int_0^T e^{\beta A(t)}(\underline{Y}_t^m-\underline{Y}_t^n)(\eta_t^m-\eta_t^n)\mathrm{d}t]+E[\int_0^T e^{\beta A(t)}|g(t,\underline{Y}_t^m,\underline{Z}_t^m)-g(t,\underline{Y}_t^n,\underline{Z}_t^n)|^2\mathrm{d}t]\\
\leq& 2\Big(E[\int_0^T e^{\beta A(t)}|\underline{Y}_t^m-\underline{Y}_t^n|^2\mathrm{d}t]\Big)^{\frac{1}{2}}\Big(E[\int_t^T e^{\beta A(t)}\big(|\eta_t^m|+|\eta_t^n|\big)^2\mathrm{d}t]\Big)^{\frac{1}{2}}\\
&+E[\int_0^T e^{\beta A(t)}\nu(t)|\underline{Y}_t^m-\underline{Y}_t^n|^2\mathrm{d}t]+E[\int_0^T e^{\beta A(t)}\alpha|\underline{Z}_t^m-\underline{Z}_t^n|^2\mathrm{d}t]\\
\leq& \frac{4\sqrt{C_0}}{1-\alpha}\Big(E[\int_0^T e^{\beta A(t)}|\underline{Y}_t^m-\underline{Y}_t^n|^2\mathrm{d}t]\Big)^{\frac{1}{2}}+\frac{1}{1-\alpha}E[\int_0^T e^{\beta A(t)}\nu(t)|\underline{Y}_t^m-\underline{Y}_t^n|^2\mathrm{d}t].
\end{aligned}
$$

Because $\{\underline{Y}^n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{H}^{2,\beta}(0,T;\mathbb{R})$, $\{\underline{Z}^n\}_{n=1}^\infty$ is also a Cauchy sequence in $\mathcal{H}^{2,\beta}(0,T;\mathbb{R})$. Therefore, $\{\underline{Z}^n\}_{n=1}^\infty$ converges in $\mathcal{H}^{2,\beta}(0,T;\mathbb{R})$, we denote it by $\underline{Z}$. Now, we pass to the limit, as $n\to\infty$ on both sides of (11), it follows that

$$
\underline{Y}_t=\xi+\int_t^T f(s,P_{\underline{Y}_s},\underline{Y}_s,\underline{Z}_s)\mathrm{d}s+\int_t^T g(s,\underline{Y}_s,\underline{Z}_s)\mathrm{d}\overleftarrow{B}_s-\int_t^T \underline{Z}_s\mathrm{d}W_s.
$$

Obviously, $(\underline{Y},\underline{Z})$ solves the general mean-field BDSDEs (5).

Let, $(Y,Z)\in\mathcal{M}^{2,\beta}(0,T)$ be any solution of (5) and consider (11) with its minimal solution $(\underline{Y}^n,\underline{Z}^n)\in\mathcal{M}^{2,\beta}(0,T)$ for every $n\geq 0$.

For $n=0$, we first denote $\underline{f}_0(s,P_{Y_s},Y_s,Z_s) := -\theta(s)e^{-\frac{\beta A(s)}{2}}W_2(P_{Y_s},\delta_0)-\mu(s)|Y_s|-\gamma(s)|Z_s|-\phi(s)$. From assumption (A7), there is

$$
f(s,P_{Y_s},Y_s,Z_s)\geq\underline{f}_0(s,P_{Y_s},Y_s,Z_s),\ \text{for all } s\in[0,T].
$$

Since $\underline{f}_0$ satisfies assumptions (A2), (A4) and (A5), we obtain from Lemma 2 that $\underline{Y}_s^0\leq Y_s$, P-a.s., for all $s\in[0,T]$.

Now, suppose that there exists $n > 1$ such that $\underline{Y}_s^n \le Y_s$ and prove that $\underline{Y}_s^{n+1} \le Y_s$, P-a.s., for all $s \in [0, T]$. Denote $\underline{f}_{n+1}(s, P_{Y_s}, Y_s, Z_s) := f(s, P_{\underline{Y}_s^n}, \underline{Y}_s^n, \underline{Z}_s^n) + K(s, \Delta(P_{Y_s}, P_{\underline{Y}_s^n}), Y_s - \underline{Y}_s^n, Z_s - \underline{Z}_s^n)$, from assumption (B2) and $Y_s \ge \underline{Y}_s^n$, it follows that

$$f(s, P_{Y_s}, Y_s, Z_s) \ge \underline{f}_{n+1}(s, P_{Y_s}, Y_s, Z_s), \text{ for all } s \in [0, T].$$

Since, $\underline{f}_{n+1}$ satisfies assumptions (A2), (A4) and (A5), we obtain from Lemma 2 that, $\underline{Y}_s^{n+1} \le Y_s$, a.s., for all $s \in [0, T]$. Consequently, for all $n \ge 0$, we have $\underline{Y}_s^n \le Y_s$, P-a.s., for all $s \in [0, T]$.

Since $(\underline{Y}^n, \underline{Z}^n)$ converges to $(\underline{Y}, \underline{Z})$, we can obtain $\underline{Y}_s \le Y_s$, P-a.s., for all $s \in [0, T]$, which proves that $(\underline{Y}, \underline{Z})$ is the minimal solution of (5). □

Remark 2. *Similar to the proof of Theorem 1, we can obtain another existence result that Equation (5) has a maximal solution. Replace (B1) with (B1)′:*

(B1)′: For a.e. $(s, \omega) \in [0, T]$, $f(s, p, y, z)$ is right-continuous in y, and continuous in p and z, especially with a continuity modulus $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ for p: for all $p_1, p_2 \in \mathcal{P}_2(\mathbb{R}), (s, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$, there is $|f(s, p_1, y, z) - f(s, p_2, y, z)| \le \rho(W_2(p_1, p_2))$. Here ρ is supposed to be non-decreasing and such that $\rho(0_+) = 0$.

Consider the Equation (12) and the following equation: Given $\xi \in L^{2,\beta}(\Omega, \mathcal{F}_T, P; \mathbb{R}), t \in [0, T]$,

$$\begin{aligned}\overline{Y}_t^n =& \xi + \int_t^T \Big[f(s, P_{\overline{Y}_s^{n-1}}, \overline{Y}_s^{n-1}, \overline{Z}_s^{n-1}) + K(s, \Delta(P_{\overline{Y}_s^{n-1}}, P_{\overline{Y}_s^n}), \overline{Y}_s^{n-1} - \overline{Y}_s^n, \overline{Z}_s^{n-1} - \overline{Z}_s^n)\Big]\mathrm{d}s \\ &+ \int_t^T g(s, \overline{Y}_s^n, \overline{Z}_s^n)\mathrm{d}\overleftarrow{B}_s - \int_t^T \overline{Z}_s^n \mathrm{d}W_s, \ n \ge 1.\end{aligned} \tag{15}$$

For all $n \ge 1$, there exists at least one solution to the general mean-field BDSDEs (15), and here we give the sequence of maximal solutions denoted by $\{(\overline{Y}^n, \overline{Z}^n)\}_{n=1}^\infty$, which will limit to the maximal solution of Equation (5).

Working similarly to Lemma 5 and Theorem 1, we conclude the following:

Corollary 1. *Under the assumptions (A1), (A3), (A4), (A7), (B1)′ and (B2), if $\{(\overline{Y}^n, \overline{Z}^n)\}_{n=1}^\infty$ is the maximal solution of the Equation (15), then*

(i) For $n \ge 0$, $\overline{Y}_t^0 \ge \overline{Y}_t^n \ge \overline{Y}_t^{n+1} \ge \underline{Y}_t^0$, $t \in [0, T]$, P-a.s.;

(ii) $\{(\overline{Y}^n, \overline{Z}^n)\}_{n=1}^\infty \in \mathcal{M}^{2,\beta}(0, T)$ converges to $(\overline{Y}, \overline{Z})$, which is the maximal solution of Equation (5).

3.2. Comparison Theorem

The comparison theorem is also an important result in the theory of general mean-field BDSDEs; therefore, we will prove the comparison theorem to the case where the generator f is discontinuous.

Theorem 2. *Assume that $\xi^{(1)}, \xi^{(2)} \in L^{2,\beta}(\Omega, \mathcal{F}_T, P; \mathbb{R})$, g and $f^{(i)}$, $i = 1, 2$ satisfy the assumptions (A1), (A3), (A4), (A7), (B1) and (B2). Let $(\underline{Y}^{(1)}, \underline{Z}^{(1)})$ be the minimal solution of the general mean-field BDSDEs (5) with the data $\left(\xi^{(1)}, f^{(1)}, g\right)$; $(Y^{(2)}, Z^{(2)})$ be a solution of the general mean-field BDSDEs (5) with the data $\left(\xi^{(2)}, f^{(2)}, g\right)$. Then, if $\xi^{(1)} \le \xi^{(2)}$, P-a.s. and $f^{(1)}(t, p, y, z) \le f^{(2)}(t, p, y, z)$, dtdP-a.e., it holds that $\underline{Y}_t^{(1)} \le Y_t^{(2)}$, for all $t \in [0, T]$, P-a.s..*

Proof. Let $(\underline{Y}_t^n, \underline{Z}_t^n)_{t \in [0,T]} (n = 0, 1, \cdots)$ be the minimal solutions of the following general mean-field BDSDEs:

$$
\begin{aligned}
\underline{Y}_t^0 = \xi^{(1)} &+ \int_t^T \Big[-\theta(s)e^{-\frac{\beta A(s)}{2}} W_2(P_{\underline{Y}_s^0}, \delta_0) - \mu(s)|\underline{Y}_s^0| - \gamma(s)|\underline{Z}_s^0| - \phi(s) \Big] \mathrm{d}s \\
&+ \int_t^T g(s, \underline{Y}_s^0, \underline{Z}_s^0) \mathrm{d}\overleftarrow{B}_s - \int_t^T \underline{Z}_s^0 \mathrm{d}W_s, \qquad (16)\\
\underline{Y}_t^n = \xi^{(1)} &+ \int_t^T \Big[f^{(1)}(s, P_{\underline{Y}_s^{n-1}}, \underline{Y}_s^{n-1}, \underline{Z}_s^{n-1}) + K\big(s, \Delta(P_{\underline{Y}_s^n}, P_{\underline{Y}_s^{n-1}}), \underline{Y}_s^n - \underline{Y}_s^{n-1}, \underline{Z}_s^n - \underline{Z}_s^{n-1}\big) \Big] \mathrm{d}s \\
&+ \int_t^T g(s, \underline{Y}_s^n, \underline{Z}_s^n) \mathrm{d}\overleftarrow{B}_s - \int_t^T \underline{Z}_s^n \mathrm{d}W_s. \qquad (17)
\end{aligned}
$$

For any $(s, p, y, z) \in [0, T] \times \mathcal{P}(\mathbb{R}) \times \mathbb{R} \times \mathbb{R}^d$, we denote

$$
\begin{aligned}
F_0^{(1)}(s, p, y, z) &= -\theta(s)e^{-\frac{\beta A(s)}{2}} W_2(p, \delta_0) - \mu(s)|y| - \gamma(s)|z| - \phi(s), \\
F_n^{(1)}(s, p, y, z) &= f^{(1)}(s, P_{\underline{Y}_s^{n-1}}, \underline{Y}_s^{n-1}, \underline{Z}_s^{n-1}) + K(s, \Delta(p, P_{\underline{Y}_s^{n-1}}), y - \underline{Y}_s^{n-1}, z - \underline{Z}_t^{n-1}), \quad n \geq 1.
\end{aligned}
$$

First, we prove $Y_s^{(2)} \geq \underline{Y}_s^0$. From the assumption (A7), we have

$$
\begin{aligned}
& f^{(2)}(s, P_{Y_s^{(2)}}, Y_s^{(2)}, Z_s^{(2)}) \\
& \geq -\theta(s)e^{-\frac{\beta A(s)}{2}} W_2(P_{Y_s^{(2)}}, \delta_0) - \mu(s)|Y_s^{(2)}| - \gamma(s)|Z_s^{(2)}| - \phi(s) \\
& = F_0^{(1)}(s, P_{Y_s^{(2)}}, Y_s^{(2)}, Z_s^{(2)}).
\end{aligned}
$$

Since $F_0^{(1)}$ satisfies the assumptions (A6)–(A8), we obtain from Lemma 4 that $Y_s^{(2)} \geq \underline{Y}_s^0$, for all $s \in [0, T]$, P-a.s.

Next, we prove $Y_s^{(2)} \geq \underline{Y}_s^1$. From the assumption (B2), it follows that

$$
\begin{aligned}
& f^{(2)}(s, P_{Y_s^{(2)}}, Y_s^{(2)}, Z_s^{(2)}) \\
& \geq f^{(2)}(s, P_{\underline{Y}_s^0}, \underline{Y}_s^0, \underline{Z}_s^0) + K(s, \Delta(Y_s^{(2)}, \underline{Y}_s^0), Y_s^{(2)} - \underline{Y}_s^0, Z_s^{(2)} - \underline{Z}_s^0) \\
& \geq f^{(1)}(s, P_{\underline{Y}_s^0}, \underline{Y}_s^0, \underline{Z}_s^0) + K(s, \Delta(Y_s^{(2)}, \underline{Y}_s^0), Y_s^{(2)} - \underline{Y}_s^0, Z_s^{(2)} - \underline{Z}_s^0) \\
& = F_1^{(1)}(s, P_{Y_s^2}, Y_s^2, Z_s^2).
\end{aligned} \qquad (18)
$$

Since $F_1^{(1)}$ satisfies the assumptions (A6)–(A8), we obtain from Lemma 4 that $Y_s^{(2)} \geq \underline{Y}_s^1$, for all $s \in [0, T]$, P-a.s.

Then, we assume that there exists $n \geq 1$ such that $Y_s^{(2)} \geq \underline{Y}_s^n$, P-a.s., following the same procedure as (18), we can prove that $Y_s^{(2)} \geq \underline{Y}_s^{n+1}$, for all $s \in [0, T]$, P-a.s.

Finally, from Theorem 1 we know $\{(\underline{Y}^n, \underline{Z}^n)\}_{n \geq 0}$ converges in $\mathcal{M}^{2,\beta}(0, T)$ to the minimal solution $(\underline{Y}^{(1)}, \underline{Z}^{(1)})$ of general mean-field BDSDEs $(\xi^{(1)}, f^{(1)}, g)$, so there is $\underline{Y}_s^{(1)} \leq Y_s^{(2)}$, for all $s \in [0, T]$, P-a.s. □

3.3. A Special Case: General Mean-Field BDSDEs with Linear Growth and Discontinuous Generator

Next, we will discuss the general mean-field BDSDEs under non-stochastic conditions, which is a special case of that under the above stochastic conditions. Let $\beta = 0$, and for all $t \in [0, T]$, let processes $\theta(t), \mu(t), \gamma(t), \phi(t), \nu(t)$ equal to the constant A, then the results under stochastic conditions will degenerate into some classical results, which are shown in Theorems 3 and 4.

At first, when $\beta = 0, \theta(t), \mu(t), \gamma(t), \phi(t), \nu(t) \equiv A$, for all $t \in [0, T]$, the corresponding assumptions will be modified as follows:

(C1) $g(t,\omega,0,0) \in \mathcal{H}^{2,0}(0,T,\mathbb{R}^\ell)$;

(C2) g is Lipschitz in (y,z) : There are constants $\alpha_1 > 0$ and $0 < \alpha_2 < 1$, such that for all $y_1, y_2 \in \mathbb{R}, z_1, z_2 \in \mathbb{R}^d$,

$$|g(t,y_1,z_1) - g(t,y_2,z_2)|^2 \leq \alpha_1 |y_1 - y_2|^2 + \alpha_2 |z_1 - z_2|^2;$$

(C3) Linear growth: There exists $A \geq 0$, such that for all $(p,y,z) \in \mathcal{P}(\mathbb{R}) \times \mathbb{R} \times \mathbb{R}^d$, there is

$$|f(t,p,y,z)| \leq A(1 + W_2(p,\delta_0) + |y| + |z|), \text{ d}t\text{d}P\text{-a.e.};$$

(C4) Monotonicity in p : For all $\theta_1, \theta_2 \in L^2(\Omega, \mathcal{F}, P; \mathbb{R})$, and all $(y,z) \in \mathbb{R} \times \mathbb{R}^d$, when $\theta_1 \leq \theta_2$, P-a.s., we have

$$f(t, P_{\theta_1}, y, z) \leq f(t, P_{\theta_2}, y, z), \text{d}s\text{d}P\text{-a.e.};$$

(C5) There exists a continuous function $K(t,p,y,z)$ defined on $[0,T] \times \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \times \mathbb{R}^d$, which is non-decreasing with respect to p and for all $A \geq 0$ satisfying

$$|K(t,p,y,z)| \leq A(W_2(p,\delta_0) + |y| + |z|),$$

such that for all $y_1 \geq y_2 \in \mathbb{R}, p_1, p_2 \in \mathcal{P}_2(\mathbb{R}), z_1, z_2 \in \mathbb{R}^d$, we have

$$f(t,p_1,y_1,z_1) - f(t,p_2,y_2,z_2) \geq K(t, \Delta(p_1,p_2), y_1 - y_2, z_1 - z_2),$$

where $\Delta(p_1,p_2) \in \mathcal{P}_2(\mathbb{R})$ and satisfies $W_2\big(\Delta(p_1,p_2),\delta_0\big) = W_2(p_1,p_2)$.

Theorem 3. *Under the assumptions (C1)-(C5) and (B1), the general mean-field BDSDE (5) at least has one solution $(Y,Z) \in \mathcal{M}^{2,0}(0,T)$. Moreover, there is a minimal solution $(\underline{Y},\underline{Z}) \in \mathcal{M}^{2,0}(0,T)$ of the general mean-field BDSDE (5).*

Here, we give an example to show the rationality of those mentioned assumptions.

Example 1. *Let $\xi \in L^{2,0}(\Omega, \mathcal{F}_T, P; \mathbb{R})$, consider the following mean-field BDSDE: for $t \in [0,T]$,*

$$Y_t = \xi + \int_t^T \Big(1 + E[Y_s] + Y_s I_{\{Y_s > 1\}} + Z_s\Big)\text{d}s + \int_t^T \left(\frac{1}{2}Y_s + \frac{1}{2}Z_s\right)\text{d}\overleftarrow{B}_s - \int_t^T Z_s \text{d}W_s. \quad (19)$$

Taking $A = 1$, $\alpha_1 = \frac{1}{2}$ and $\alpha_2 = \frac{1}{2}$, the above equation satisfies assumptions (C1)–(C4) and (B1). Given the continuous function $K(t,p,y,z) = z$, the assumptions (C5) will also be satisfied. Therefore, from Theorem 3 we know that the Equation (19) at least has one solution $(Y,Z) \in \mathcal{M}^{2,0}(0,T)$.

Theorem 4 (Comparison theorem). *Assume that $\xi^{(1)}, \xi^{(2)} \in L^{2,0}(\Omega, \mathcal{F}_T, P; \mathbb{R})$, g and $f^{(i)}$, $i = 1,2$ satisfy the assumptions (C1)–(C5) and (B1). Let $(\underline{Y}^{(1)}, \underline{Z}^{(1)})$ be the minimal solution of the general mean-field BDSDEs (5) with the data $(\xi^{(1)}, f^{(1)}, g)$; $(Y^{(2)}, Z^{(2)}) \in \mathcal{M}^{2,0}(0,T)$ be a solution of the general mean-field BDSDEs (5) with the data $(\xi^{(2)}, f^{(2)}, g)$. If $\xi^{(1)} \leq \xi^{(2)}$, P-a.s. and $f^{(1)}(t,p,y,z) \leq f^{(2)}(t,p,y,z)$, dtdP-a.e., it holds that $\underline{Y}_t^{(1)} \leq Y_t^{(2)}$, for all $t \in [0,T]$, P-a.s.*

Proof. The proof of Theorems 3 and 4 is similar to that of Theorems 1 and 2, so it is omitted here. □

4. Application in Finance: Selling a Financial Claim

Considering a financial claim with a contingent ξ and there an investor who wants to sell the claim and hedge it. Suppose that the investor has additional information not detected in the financial market, and his decision is also affected by the distribution of all investors' decisions in the market. Moreover, suppose that the interest rate is applied only to portfolios whose value remains above a nominal value at any time. This problem is equivalent to solving the following mean-field BDSDE: $\xi \in L^{2,\beta}(\Omega, \mathcal{F}_T, P; \mathbb{R})$, for $t \in [0,T]$,

$$Y_t = \xi + \int_t^T \left(\theta(s)e^{-\frac{\beta A(s)}{2}} E[Y_s] + \mu(s)Y_s I_{\{Y_s>1\}} + \gamma(s)Z_s\right)\mathrm{d}s + \int_t^T c(s)Z_s \mathrm{d}\overleftarrow{B}_s - \int_t^T Z_s \mathrm{d}W_s, \tag{20}$$

where the mean-field term $E[y]$ reflects that the investor relies on the distribution of all investors' decisions in the market to make a decision, $\mu(t)$ is the interest rate, $\gamma(t)$ is the risk premium vector and $c(t)$ is the volatility caused by the systemic risks.

We have $f(t,p,y,z) = \theta(t)e^{-\frac{\beta A(t)}{2}} E[y] + \mu(t)yI_{\{y>1\}} + \gamma(t)z$ and $g(t,y,z) = c(t)z$. Obviously, it follows that the assumptions (A7), (B1) and (B2) are satisfied with $\phi(t) \equiv 0$ and $K(t,p,y,z) = \gamma(t)z$. If we let $c(t) = \alpha 1_{\{B_T - B_t > 0\}}$ and $\nu(t) \equiv 0$, $0 < \alpha < 1$, then assumption (A1) is also satisfied. For any $t \in [0,T]$, let $X_t = \sqrt{2t}\,(X \wedge 1)$, where $X \sim \mathcal{N}(0,1)$. Now, for any $\varepsilon > 0$, we put

$$\theta(t) = \sqrt{\frac{1}{2}X_t^2 1_{\{W_t \geq 0\}}}, \quad \mu(t) = \frac{1}{2}X_t^2 1_{\{W_t \geq 0\}}, \quad \text{and} \quad \gamma(t) = \sqrt{X_t^2 1_{\{W_t<0\}} + \varepsilon}.$$

Then, $\theta(t), \mu(t), \gamma(t)$ and $\nu(t)$ are positive $\mathcal{F}_t^W$-adapted processes. Indeed, for any $t \in [0,T]$, we have

$$a^2(t) = \theta(t)^2 + \mu(t) + \gamma^2(t) + \nu(t) = X_t^2 + \varepsilon > 0, \quad \text{and}$$
$$A(t) = \int_0^t \left(X_s^2 + \varepsilon\right) ds \leq (X \wedge 1)^2 T^2 + \varepsilon T < +\infty,$$

thus, the assumptions (A3) and (A4) are satisfied. Therefore, from Theorem 1 we know, the Equation (20) at least has one solution $(Y,Z) \in \mathcal{M}^{2,\beta}(0,T)$, that is, the investor can sell the financial claim at a certain price Y.

5. Conclusions

This paper studies a class of general mean-field BDSDEs whose generator f depends not only on the solution processes but also on their distribution.

We present the main result in Section 3, that is, the existence of the solutions for the general mean-field BDSDEs and the comparison theorem under discontinuous and stochastic linear growth conditions.

It is worth emphasizing that the general mean-field BDSDEs with discontinuous generators can help to deal with some financial problems, for example, we discuss a financial claim sale problem in Section 4, which can be solved by a class of general mean-field BDSDE.

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References

1. Pardoux, E.; Peng, S. Adapted solution of a backward stochastic differential equation. *Syst. Control Lett.* **1990**, 14, 55–61. [CrossRef]
2. Pardoux, E.; Peng, S. Backward doubly stochastic differential equations and systems of quasilinear SPDEs. *Probab. Theory Relat. Fields* **1994**, *98*, 209–227. [CrossRef]

3. Shi, Y.; Gu, Y.; Liu, K. Comparison theorems of backward doubly stochastic differential equations and applications. *Stoch. Anal. Appl.* **2005**, *23*, 97–110. [CrossRef]
4. Owo, J.M. Backward doubly stochastic differential equations with stochastic Lipschitz condition. *Stat. Probab. Lett.* **2015**, *96*, 75–84. [CrossRef]
5. Owo, J.M. Backward doubly SDEs with continuous and stochastic linear growth coefficients. *Random Oper. Stoch. Equ.* **2018**, *26*, 175–184. [CrossRef]
6. Owo, J.M. Backward doubly stochastic differential equations with discontinuous and stochastic linear growth generator. *Stoch. Anal. Appl.* **2022**, *41*, 958–973. [CrossRef]
7. Lasry, J.; Lions, P. Mean field games. *Jpn. J. Math.* **2007**, *2*, 229–260. [CrossRef]
8. Buckdahn, R.; Djehiche, B.; Li, J.; Peng, S. Mean-field backward stochastic differential equation: A limit approach. *Ann. Probab.* **2009**, *37*, 1524–1565. [CrossRef]
9. Buckdahn, R.; Li, J.; Peng, S. Mean-field backward stochastic differential equations and related partial differential equations. *Stoch. Process. Their Appl.* **2009**, *119*, 3133–3154. [CrossRef]
10. Moon, J.; Kim, W. Explicit characterization of feedback nash equilibria for indefinite, linear-quadratic, mean-field-type stochastic zero-sum differential games with jump-diffusion Models. *Mathematics* **2020**, *8*, 1669. [CrossRef]
11. Zhang, P.; Mohamed, N.A.; Ibrahim, A.I.N. Mean-field and anticipated BSDEs with time-felayed generator. *Mathematics* **2023**, *11*, 888. [CrossRef]
12. Li, J.; Liang, H.; Zhang, X. General mean-field BSDEs with continuous coefficients. *J. Math. Anal. Appl.* **2018**, *466*, 264–280. [CrossRef]
13. Wang, J.; Zhao, N.; Shi, Y. General mean-field backward stochastic differential equations with discontinuous coefficients. In Proceedings of the 41th Chinese Control Conference, Hefei, China, 25–27 July 2022; pp. 1287–1292.
14. Li, J.; Xing, C. General mean-field BDSDEs with continuous coefficients. *J. Math. Anal. Appl.* **2022**, *506*, 125699. [CrossRef]
15. Shi, Y.; Wang, J.; Zhao, N. General mean-field BDSDEs with continuous and stochastic linear growth coefficients. *J. Syst. Sci. Complex.* **2024**. *in press*
16. Bibi Ruhomally, Y.; Zaid Dauhoo, M.; Dumas, L. Stochastic modelling of marijuana use in Washington: Pre-and post-Initiative-502 (I-502). *IMA J. Appl. Math.* **2022**, *87*, 1121–1150. [CrossRef]

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