mathematics

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Some Properties of the Functions Representable as Fractional Power Series

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Abstract: The α-fractional power moduli series are introduced as a generalization of α-fractional power series and the structural properties of these series are investigated. Using the fractional Taylor's formula, sufficient conditions for a function to be represented as an α-fractional power moduli series are established. Beyond theoretical formulations, a practical method to represent solutions to boundary value problems for fractional differential equations as α-fractional power series is discussed. Finally, α-analytic functions on an open interval I are defined, and it is shown that a non-constant function is α-analytic on I if and only if $1/\alpha$ is a positive integer and the function is real analytic on I.

Keywords: Caputo fractional derivative operator; fractional power series; fractional analytic function

MSC: 26A33; 26E05; 34A08

Citation: Groza, G.; Jianu, M.; Mierluş-Mazilu, I. Some Properties of the Functions Representable as Fractional Power Series. *Mathematics* **2024**, *12*, 961. https://doi.org/10.3390/math12070961

Academic Editor: Leonid Piterbarg

Received: 6 February 2024
Revised: 19 March 2024
Accepted: 22 March 2024
Published: 24 March 2024

1. Introduction

The power series method is a classical tool to approximate solutions to initial value problems for ordinary differential equations. Since fractional calculus became an useful instrument for modeling various phenomena in science and engineering, a lot of classical notions were extended to the fractional case (see [1–5]). For example, the classical power series were generalized to α-fractional power series (with α a positive number) and some classical methods in calculus were extended to the fractional case (see [6]).

The α-fractional power series are used to approximate solutions to fractional ordinary differential equations (FODE). For instance, in [7,8], the solutions to the Bagley–Torvik equation and the fractional Laguerre-type logistic equation are approximated by using α-fractional power series. Numerical approximations of solutions to fractional ordinary differential equations using α-fractional power series can be found in [9–11] and references therein. Results on the solutions to systems of fractional ordinary differential equations are presented in [12]. A generalization of the α-fractional power series is studied in [13], and is applied to obtain solutions to linear fractional order differential equations. The theoretical background and the applications of the fractional-calculus operators which are based upon the general Fox–Wright function and its special forms as Mittag–Leffler-type functions are presented in [14,15].

The research in the field of fractional differential equations has focused mostly on initial value problems, but there are also some papers dealing with boundary value problems (see [16–19]). For instance, in [17,18], the existence and uniqueness of a solution to the boundary value problems for fractional order differential equations and nonlocal boundary condition are studied. In [19], the authors use the fractional central formula, based on the generalized Taylor theorem [20], for approximating the fractional derivatives of order α and 2α, respectively.

In this paper, we introduce the α-fractional power moduli series as a generalization of the α-fractional power series. We study the properties of these series in Section 2, using sequential fractional derivatives (Theorems 1 and 2). Using the generalized Taylor's formula, sufficient conditions for a function to be represented as an α-fractional power moduli series are established in Corollary 2.

A practical method to approximate solutions to boundary value problems for FODE using the partial sums of α-fractional power series is presented in Section 3 and is applied in some illustrative examples. The α-fractional analytic functions (on an open interval I) are studied in Section 4. The real analytic functions (obtained for $\alpha = 1$) seem to be a particular case of α-fractional analytic functions, but it is proved (see [6]) that a function representable as an α-fractional power series at a point x_0, that is, $f(x) = \sum_{n\geq 0} a_n(x-x_0)^{n\alpha}$ for all $x \in [x_0, x_0 + r)$, must be a real analytic function on the open interval $(x_0, x_0 + r)$. As a consequence, non-constant α-analytic functions exist only for $\alpha = \frac{1}{m}$ with m a positive integer and they are exactly the real analytic functions on the interval I.

2. Fractional Power Series

A series of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^{n\alpha}, x \geq x_0, \tag{1}$$

with $a_n \in \mathbb{R}$ and $\alpha \in (0,1]$ is called an *α-fractional power series* about x_0. We note that any series of the form $\sum\limits_{n=0}^{\infty} a'_n(x-x_0)^{n\alpha'}$ with $\alpha' > 1$ is also a series of the form (1) with $\alpha = \frac{\alpha'}{\lceil \alpha' \rceil}$, where $\lceil x \rceil = \min\{z \in \mathbb{Z} : z \geq x\}$ denotes the ceiling function.

Similarly, a series of the form

$$\sum_{n=0}^{\infty} a_n|x-x_0|^{n\alpha}, \tag{2}$$

with $a_n \in \mathbb{R}$ and $\alpha \in (0,1]$ is called an *α-fractional power moduli series* about x_0.

Fractional power series can be studied using the fractional differential and fractional integral operators. We shortly present the most important definitions and results in fractional calculus (see [1–5]). Moreover, starting from these classical results, we introduce a general frame which is needed in the case of fractional power moduli series.

Definition 1. *Let I be a real interval, $I = (x_0, b]$. A function $f : I \to \mathbb{R}$ is said to be of class $C_{\rho,x_0+}(I)$ if $f(x) = (x-x_0)^\rho g(x)$, where $g : [x_0, b] \to \mathbb{R}$ is a continuous function. If there exists $f^{(n)}(x)$, for every $x \in I$ and $f^{(n)} \in C_{\rho,x_0+}(I)$, then the function f is said to be of class $C^{(n)}_{\rho,x_0+}(I)$.*

Similarly, if $I = [a, x_0)$, then $f : I \to \mathbb{R}$ is said to be a function of class $C_{\rho,x_0-}(I)$ if $f(x) = (x_0-x)^\rho g(x)$, where $g : [a, x_0] \to \mathbb{R}$ is a continuous function. If there exists $f^{(n)}(x)$, for every $x \in I$ and $f^{(n)} \in C_{\rho,x_0-}(I)$, then the function f is said to be of class $C^{(n)}_{\rho,x_0-}(I)$.

If $I = [a, b]$ is a real interval and $x_0 \in (a, b)$, then $f : I \to \mathbb{R}$ is said to be a function of class $C_{\rho,x_0\pm}(I)$ if $f(x) = |x-x_0|^\rho g(x)$, where $g : [a, b] \to \mathbb{R}$ is a continuous function. If there exists $f^{(n)}(x)$, for every $x \in I$ and $f^{(n)} \in C_{\rho,x_0\pm}(I)$, then the function f is said to be of class $C^{(n)}_{\rho,x_0\pm}(I)$.

Definition 2. *Let I be a real interval, $I = (x_0, b]$ and $f : I \to \mathbb{R}$ be a function of class $C_{\rho,x_0+}(I)$, with $\rho > -1$. Then, for any $x \in I$, the left-sided Riemann–Liouville fractional integral of order $\alpha > 0$ of f is defined as*

$$(J^{\alpha}_{x_0+}f)(x) = \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} \frac{f(t)dt}{(x-t)^{1-\alpha}}.$$

If $I = [a, x_0)$ is a real interval and $f : I \to \mathbb{R}$ is a function of class $C_{\rho,x_0-}(I)$, with $\rho > -1$, then the right-sided Riemann–Liouville fractional integral of order $\alpha > 0$ of f is defined as

$$(J^{\alpha}_{x_0-}f)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{x_0} \frac{f(t)dt}{(t-x)^{1-\alpha}}.$$

Lemma 1. *Let $I = [a,b]$ be a real interval, $x_0 \in (a,b)$, $\alpha > 0$, and $f : I \to \mathbb{R}$ be a function of class $C_{\rho,x_0\pm}(I)$, where $\rho > -1$ and $\rho \geq -\alpha$. Then, there exist the following limits, and they are finite and equal: $(J^{\alpha}_{x_0+}f)(x_0) := \lim_{x \to x_0} (J^{\alpha}_{x_0+}f)(x)$, $(J^{\alpha}_{x_0-}f)(x_0) := \lim_{x \to x_0} (J^{\alpha}_{x_0+}f)(x)$. Thus, the Riemann–Liouville fractional integral of order α can be defined on both sides of x_0 by the following continuous function:*

$$(J^{\alpha}_{x_0\pm}f)(x) := \begin{cases} (J^{\alpha}_{x_0+}f)(x), & \text{if } x \geq x_0 \\ (J^{\alpha}_{x_0-}f)(x), & \text{if } x < x_0. \end{cases}$$

Proof. Since $f(x) = |x - x_0|^{\rho} g(x)$, where $g(x)$ is a continuous function, we can write:

$$(J^{\alpha}_{x_0+}f)(x) = \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} \frac{(t-x_0)^{\rho} g(t)dt}{(x-t)^{1-\alpha}} = \frac{(x-x_0)^{\rho+\alpha}}{\Gamma(\alpha)} \int_0^1 \tau^{\rho}(1-\tau)^{\alpha-1} g(x_0 + \tau(x-x_0))d\tau.$$

From the Mean Value Theorem, it follows that there exists $\xi_x \in (x_0, x)$ such that

$$(J^{\alpha}_{x_0+}f)(x) = \frac{(x-x_0)^{\rho+\alpha}}{\Gamma(\alpha)} g(\xi_x) \int_0^1 \tau^{\rho}(1-\tau)^{\alpha-1} d\tau = (x-x_0)^{\rho+\alpha} g(\xi_x) \frac{\Gamma(\rho+1)}{\Gamma(\alpha+\rho+1)}$$

and we obtain

$$(J^{\alpha}_{x_0+}f)(x_0) = \lim_{x \to x_0} (J^{\alpha}_{x_0+}f)(x) = \begin{cases} 0 & \text{if } \rho > -\alpha, \\ g(x_0)\Gamma(1-\alpha) & \text{if } \rho = -\alpha. \end{cases}$$

In a similar way, it can be proved that the limit $(J^{\alpha}_{x_0-}f)(x_0) = \lim_{x \to x_0} (J^{\alpha}_{x_0-}f)(x)$ exists and is equal to $(J^{\alpha}_{x_0+}f)(x_0)$. □

Introduced by M. Caputo in 1967 (see [21]), the fractional derivative operator expressed by Definition 3 can definitely share some similarities with the fractional derivatives considered by J. Liouville in 1832 (see [22], p. 10, formula (B)). That is why recent studies refer to the Caputo fractional derivative as the *Liouville–Caputo fractional derivative* (see [14–17]). We thank the reviewer who brought this issue to our attention.

Definition 3. *Let $\alpha > 0$ and $m = \lceil \alpha \rceil$. Consider the interval $I = (x_0, b]$, and $f : I \to \mathbb{R}$ a function of class $C^{(m)}_{\rho,x_0+}(I)$ with $\rho > -1$. For any $x \in I$, the left-sided Liouville–Caputo fractional derivative of order α of f is defined as*

$$(D^{\alpha}_{x_0+}f)(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_{x_0}^{x} \frac{f^{(m)}(t)dt}{(x-t)^{\alpha-m+1}}, & \text{if } \alpha \notin \mathbb{N} \\ f^{(\alpha)}(x), & \text{if } \alpha \in \mathbb{N}. \end{cases}$$

If $I = [a, x_0)$ is an interval, and $f : I \to \mathbb{R}$ is a function of class $C^{(m)}_{\rho,x_0-}(I)$ with $\rho > -1$, then, for any $x \in I$, the right-sided Liouville–Caputo fractional derivative of order α of f is defined as

$$(D^{\alpha}_{x_0-}f)(x) = \begin{cases} \frac{(-1)^m}{\Gamma(m-\alpha)} \int_{x}^{x_0} \frac{f^{(m)}(t)dt}{(t-x)^{\alpha-m+1}}, & \text{if } \alpha \notin \mathbb{N} \\ (-1)^{\alpha} f^{(\alpha)}(x), & \text{if } \alpha \in \mathbb{N} \end{cases}.$$

If x_0 is an interior point of an interval I and f is a function of class $C^{(m)}_{\alpha-m,x_0\pm}(I)$, then there exist $(D^\alpha_{x_0+}f)(x_0) := \lim_{x\to x_0}(D^\alpha_{x_0+}f)(x)$, $(D^\alpha_{x_0-}f)(x_0) := \lim_{x\to x_0}(D^\alpha_{x_0-}f)(x)$ and they are finite and equal. The Liouville–Caputo derivative of f is defined on both sides of x_0 by the continuous function:

$$(D^\alpha_{x_0\pm}f)(x) := \begin{cases} (D^\alpha_{x_0+}f)(x), & \text{if } x \ge x_0 \\ (D^\alpha_{x_0-}f)(x), & \text{if } x < x_0. \end{cases}$$

A remarkable property of the Riemann–Liouville fractional integral operators is the "semigroup property" ([1], Theorem 2.4): if $J^\alpha_{x_0}$ is any one of the operators $J^\alpha_{x_0+}$, $J^\alpha_{x_0-}$, and $J^\alpha_{x_0\pm}$, then, for any $\alpha, \beta > 0$ and for any suitable function f, we have

$$J^\alpha_{x_0}J^\beta_{x_0}f = J^\beta_{x_0}J^\alpha_{x_0}f = J^{\alpha+\beta}_{x_0}f.$$

It follows that, for any $\alpha > 0$ and $n \in \mathbb{N}$, one can write $\underbrace{J^\alpha_{x_0}J^\alpha_{x_0}\dots J^\alpha_{x_0}}_{n-times}f = J^{n\alpha}_{x_0}f$.

The equality above does not hold in the case of Liouville–Caputo fractional differential operators. Let us take, for instance, the function $f(x) = \sqrt{x}$:

$$D^{\frac{1}{2}}_{0+}D^{\frac{1}{2}}_{0+}\sqrt{x} = D^{\frac{1}{2}}_{0+}\left(D^{\frac{1}{2}}_{0+}\sqrt{x}\right) = D^{\frac{1}{2}}_{0+}\left(\sqrt{\pi}/2\right) = 0,$$
$$D^{\frac{1}{2}+\frac{1}{2}}_{0+}\sqrt{x} = D^1_{0+}\sqrt{x} = \left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}.$$

Let $D^\alpha_{x_0}$ be one of the Liouville–Caputo fractional differential operators $D^\alpha_{x_0+}$, $D^\alpha_{x_0-}$, and $D^\alpha_{x_0\pm}$, and n be a positive integer. We denote by $\hat{D}^{n\alpha}_{x_0}f$ the *sequential fractional derivative of order n* of the function f:

$$\hat{D}^{n\alpha}_{x_0}f := \underbrace{D^\alpha_{x_0}D^\alpha_{x_0}\dots D^\alpha_{x_0}}_{n-times}f.$$

As noted above, $\hat{D}^{n\alpha}_{x_0}f \neq D^{n\alpha}_{x_0}f$ for $n > 1$.

For any positive integer n and $\alpha \in (0,1)$, we denote by $\hat{C}^{n,\alpha}_{x_0\pm}(I)$ (resp. $\hat{C}^{n,\alpha}_{x_0+}(I)$) the set of all the functions possessing sequential fractional derivatives of order k, $\hat{D}^{k\alpha}_{x_0\pm}f$ (resp. $\hat{D}^{k\alpha}_{x_0+}f$), which are continuous on $\bar{I}$, for every $k \le n$.

Lemma 2. *Let $f : I = (x_0 - r, x_0 + r) \to \mathbb{R}$ be a function continuous on $I\backslash\{x_0\}$ such that*

$$f(x_0 - x) = f(x_0 + x), \text{ for all } x \in (0,r), \tag{3}$$

that is, the graph of f is symmetric with respect to the straight line $x = x_0$. Then, for any $\rho > -1$ and $\alpha \in (0,1)$ we have:

(i) $f \in C_{\rho,x_0+}(x_0, x_0 + r)$ if and only if $f \in C_{\rho,x_0-}(x_0 - r, x_0)$, and

$$(J^\alpha_{x_0-}f)(x_0 - x) = (J^\alpha_{x_0+}f)(x_0 + x), \text{ for all } x \in (0,r);$$

(ii) $f \in C^{(1)}_{\rho,x_0+}(x_0, x_0 + r)$ if and only if $f \in C^{(1)}_{\rho,x_0-}(x_0 - r, x_0)$, and

$$(D^\alpha_{x_0-}f)(x_0 - x) = (D^\alpha_{x_0+}f)(x_0 + x), \text{ for all } x \in (0,r).$$

Proof. (i) First of all, we notice that the function f satisfies (3) if and only if there is a continuous function $h : (0,r) \to \mathbb{R}$ such that

$$f(x) = h(|x - x_0|), \text{ for all } x \in (x_0 - r, x_0 + r). \tag{4}$$

Obviously, we have

$$f \in C_{\rho,x_0+}(x_0, x_0+r) \Leftrightarrow h \in C_{\rho,0+}(0,r) \Leftrightarrow f \in C_{\rho,x_0-}(x_0-r, x_0)$$

and

$$(J^{\alpha}_{x_0+}f)(x_0+x) = (J^{\alpha}_{0+}h)(x) = (J^{\alpha}_{x_0-}f)(x_0-x), \text{ for all } x \in (0,r).$$

(ii) We notice that

$$f'(x) = \begin{cases} h'(x-x_0) & \text{if } x > x_0 \\ -h'(x_0-x) & \text{if } x < x_0, \end{cases}$$

and

$$f \in C^{(1)}_{\rho,x_0+}(x_0, x_0+r) \Leftrightarrow h \in C^{(1)}_{\rho,0+}(0,r) \Leftrightarrow f \in C^{(1)}_{\rho,x_0-}(x_0-r, x_0).$$

We prove that

$$(D^{\alpha}_{x_0-}f)(x_0-x) = (D^{\alpha}_{x_0+}f)(x_0+x) = (D^{\alpha}_{0+}h)(x), \tag{5}$$

for any $x \in (0,r)$. We can write

$$\begin{aligned}(D^{\alpha}_{x_0-}f)(x_0-x) &= \frac{-1}{\Gamma(1-\alpha)} \int_{x_0-x}^{x_0} \frac{f'(t)dt}{(t-x_0+x)^{\alpha}} \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{-f'(x_0-\tau)d\tau}{(x-\tau)^{\alpha}} = (D^{\alpha}_{0+}h)(x).\end{aligned}$$

In a similar way, it can be proved that $(D^{\alpha}_{x_0+}f)(x_0+x) = (D^{\alpha}_{0+}h)(x)$ for all $x \in (0,r)$ and so the lemma is proved. □

The generalized Taylor's formula using sequential fractional derivatives was introduced in [20]. The next result shows how this formula can be extended on both sides of x_0 for functions satisfying (3).

Theorem 1. *Consider $\alpha \in (0,1)$, $I = (x_0-r, x_0+r)$, and $f \in \hat{C}^{n+1,\alpha}_{x_0}(\bar{I})$ a function satisfying (3). Then, for all $x \in I$, there exists $\xi \in I$ such that*

$$f(x) = T_n(x,x_0) + R_n(x,x_0),$$

where

$$T_n(x,x_0) = \sum_{k=0}^{n} \frac{(\hat{D}^{k\alpha}_{x_0\pm}f)(x_0)}{\Gamma(k\alpha+1)} |x-x_0|^{k\alpha} \tag{6}$$

and

$$R_n(x,x_0) = \frac{(\hat{D}^{(n+1)\alpha}_{x_0\pm}f)(\xi)}{\Gamma((n+1)\alpha+1)} |x-x_0|^{(n+1)\alpha}.$$

Proof. For $x \geq x_0$, the theorem is proved in [20]. Thus, there exists $\xi \in [x_0, x_0+r)$ such that

$$f(x) = \sum_{k=0}^{n} \frac{(\hat{D}^{k\alpha}_{x_0+}f)(x_0)}{\Gamma(k\alpha+1)} (x-x_0)^{k\alpha} + \frac{(\hat{D}^{(n+1)\alpha}_{x_0+}f)(\xi)}{\Gamma((n+1)\alpha+1)} (x-x_0)^{(n+1)\alpha}.$$

For $x < x_0$, the theorem follows by Lemma 2. □

By Lemma 2, it follows that

$$D^{\alpha}_{x_0\pm}|x-x_0|^{n\alpha} = \begin{cases} \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} |x-x_0|^{(n-1)\alpha}, & \text{if } n \geq 1 \\ 0, & \text{if } n = 0. \end{cases} \tag{7}$$

The following theorem establishes the basic properties of the α-fractional power moduli series and extends the results from [6,23] regarding fractional power series.

Theorem 2. *Let* (2) *be an α-fractional power moduli series, with $\alpha \in (0,1]$, and $R = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}}$.*

Then, $r = \begin{cases} R^{\frac{1}{\alpha}}, \text{ if } R < \infty \\ \infty, \text{ if } R = \infty \end{cases}$ *is called the radius of convergence of the series* (2)*. If $r > 0$, then*

(i) For any $b \in (0,r)$, the series (2) *converges absolutely and uniformly on $[x_0 - b, x_0 + b]$, and there exists a positive integer $n(b)$ such that $|a_n| \leq b^{-n\alpha}$, for all $n \geq n(b)$. If*

$$f(x) = \sum_{n=0}^{\infty} a_n |x - x_0|^{n\alpha}, \; x \in (x_0 - r, x_0 + r), \tag{8}$$

then f is a continuous function and the equality (3) *holds for all $x \in [0, r)$.*

(ii) There exists the fractional derivative $D^{\alpha}_{x_0\pm} f : (x_0 - r, x_0 + r) \to \mathbb{R}$. Moreover, the series of the fractional derivatives $\sum\limits_{n=0}^{\infty} a_n D^{\alpha}_{x_0\pm} |x - x_0|^{n\alpha} = \sum\limits_{n=1}^{\infty} a_n \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} |x - x_0|^{(n-1)\alpha}$ converges absolutely and uniformly on $[x_0 - b, x_0 + b]$, for any $b \in (0, r)$, and

$$(D^{\alpha}_{x_0\pm} f)(x) = \sum_{n=0}^{\infty} a_n D^{\alpha}_{x_0\pm} |x - x_0|^{n\alpha} = \sum_{n=1}^{\infty} a_n \frac{\Gamma(n\alpha + 1)}{\Gamma((n-1)\alpha + 1)} |x - x_0|^{(n-1)\alpha}, \tag{9}$$

for all $x \in (x_0 - r, x_0 + r)$.

Proof. (i) Let us consider the power series $v(t) = \sum\limits_{n=0}^{\infty} a_n t^n$, where $t = |x - x_0|^{\alpha}$. Then, the statement follows by the well-known properties of the power series and the definition of f.

(ii) Let $h : [0, r) \to \mathbb{R}$ be the sum of the fractional power series

$$h(x) = \sum_{n=0}^{\infty} a_n x^{n\alpha}.$$

Then, h is continuous and there exists the fractional Liouville–Caputo derivative $D^{\alpha}_{0+} h : [0, r) \to \mathbb{R}$ (see [23], Theorem 1). Moreover, the series of the fractional derivatives $\sum\limits_{n=0}^{\infty} a_n D^{\alpha}_{0+} x^{n\alpha}$ is absolutely and uniformly convergent on $[0, b]$, for any $b \in (0, r)$ and

$$(D^{\alpha}_{0+} h)(x) = \sum_{n=0}^{\infty} a_n D^{\alpha}_{0+} x^{n\alpha} = \sum_{n=1}^{\infty} a_n \frac{\Gamma(n\alpha + 1)}{\Gamma((n-1)\alpha + 1)} x^{(n-1)\alpha}, \text{ for all } x \in [0, r).$$

Since $f(x) = h(|x - x_0|)$, for all $x \in (x_0 - r, x_0 + r)$, the theorem follows by (5). □

We note that the operator $\mathcal{D}^{\alpha}$, defined for power series $v(t) = \sum\limits_{n=0}^{\infty} a_n t^n$ as

$$\mathcal{D}^{\alpha} v(t) = \sum_{n=1}^{\infty} a_n \frac{\Gamma(n\alpha + 1)}{\Gamma((n-1)\alpha + 1)} t^{n-1}$$

is known as the Gelfond–Leontiev operator with respect to the Mittag–Leffler function $E_{\alpha}(t) = \sum\limits_{n=0}^{\infty} \frac{t^n}{\Gamma(n\alpha+1)}$ [1,24]. Using this operator, the Liouville–Caputo fractional derivative of the function (8) can be written as $(D^{\alpha}_{x_0\pm} f)(x) = \mathcal{D}^{\alpha} v(|x - x_0|^{\alpha})$.

Corollary 1. *Assume that the series* (2) *has a positive radius of convergence r. Then, for every non-negative integer k, there exists the sequential fractional derivative of order k $(\hat{D}^{k\alpha}_{x_0\pm} f)$, which is a continuous function on $I = (x_0 - r, x_0 + r)$ and*

$$(\hat{D}^{k\alpha}_{x_0\pm}f)(x) = \sum_{n=k}^{\infty} a_n \frac{\Gamma(n\alpha+1)}{\Gamma((n-k)\alpha+1)}|x-x_0|^{(n-k)\alpha}. \tag{10}$$

Moreover, $a_n = \frac{(\hat{D}^{n\alpha}_{x_0\pm}f)(x_0)}{\Gamma(n\alpha+1)}$, *for every* $n \geq 0$.

Proof. The relation (10) follows by applying Theorem 2 k times. By taking $x = x_0$ in (10), we get the last statement. □

The following corollary provides a sufficient condition for a function satisfying (3) to be represented as a fractional power moduli series.

Corollary 2. *Assume that* $\alpha \in (0,1)$, $I = [x_0 - r, x_0 + r]$ *is a real interval, and* $f \in \hat{C}^{n,\alpha}_{x_0\pm}(I)$, *for every* n, *is a function satisfying* (3). *If* $M_n = o\left(\frac{\Gamma(n\alpha+1)}{r^{n\alpha}}\right)$, *where* $M_n = \sup_{x\in I}|(\hat{D}^{n\alpha}_{x_0\pm}f)(x)|$, *then* f *is represented as an* α*-fractional power moduli series on* I*:*

$$f(x) = \sum_{n=0}^{\infty} \frac{(\hat{D}^{n\alpha}_{x_0\pm}f)(x_0)}{\Gamma(n\alpha+1)}|x-x_0|^{n\alpha}, \text{ for all } x \in I.$$

Proof. By Theorem 1, we get

$$\left|f(x) - \sum_{k=0}^{n} \frac{(\hat{D}^{k\alpha}_{x_0\pm}f)(x_0)}{\Gamma(k\alpha+1)}|x-x_0|^{k\alpha}\right| = \left|\frac{(\hat{D}^{(n+1)\alpha}_{x_0\pm}f)(\xi)}{\Gamma((n+1)\alpha+1)}|x-x_0|^{(n+1)\alpha}\right|,$$

which implies the corollary. □

Remark 1. *If* $\alpha \in (0,1]$, $I = [x_0, x_0 + r)$ *(resp.* $I = (x_0 - r, x_0 + r)$*) and* $f \in \hat{C}^{n,\alpha}_{x_0\pm}(\bar{I})$, *for every* n, *is a function represented as an* α*-fractional power series* (1) *(resp. an* α*-fractional power moduli series* (2)*) at* x_0 *on* I, *then, by Theorem* 2 *and Corollary* 2, *it follows that the coefficients* a_n *are uniquely determined. Moreover, if* f *can be also represented as a* β*-fractional power series with* $\beta \in (0,1]$, *then* $\frac{\alpha}{\beta}$ *must be a rational number.*

3. Boundary Value Problems for Fractional Differential Equations

In this section, we present a method to study the existence and the uniqueness of solutions to boundary value problems for fractional linear differential equations, solutions which are representable as α-fractional power series. This is based on the result below.

Let us consider $\alpha \in (0,1]$ and the fractional linear differential equation

$$(\hat{D}^{n\alpha}_{0+}y)(x) + a_1(x)(\hat{D}^{(n-1)\alpha}_{0+}y)(x) + \ldots + a_n(x)y(x) = f(x),\ x \in [0,b]. \tag{11}$$

Theorem 3 (see [23], Theorem 3)**.** *Suppose that* $b' > b$ *and* $f, a_j,\ j = 1,2,\ldots,n$, *are representable as* α*-fractional power series at* 0 *on* $[0,b')$. *If* $y_i^{(0)}$, $i = 0,1,\ldots,n-1$, *are arbitrary real numbers, then there exists* $y = y(x)$ *representable as an* α*-fractional power series at* 0 *on* $[0,b]$, *which is a solution to Equation* (11), *uniquely determined such that*

$$(\hat{D}^{i\alpha}_{0}y)(0) = y_i^{(0)},\ i = 0,1,...,n-1.$$

Example 1. *Let us consider the boundary value problem for the inhomogeneous fractional Airy Equation (see ([25], 10.4) and ([2], Example 7.10))*

$$(\hat{D}^{2\alpha}_{0+}y)(x) - x^\alpha y(x) = 1 - x^\alpha + x^{2\alpha},\ x \in [0,b],\ \alpha \in (0,1], \tag{12}$$

$$y(0) = \delta_1,\ y(b) = \delta_2, \tag{13}$$

where δ_1 *and* δ_2 *are arbitrary real numbers.*

In order to prove that the boundary value problem (12), (13) has a (unique) solution representable as an α-fractional power series at 0 on the interval $[0, b]$, we firstly solve an initial value problem for the same equation (see, for example, [26], p. 88).

Le us assume that $y = y(x)$ is represented as a fractional power series at $x_0 = 0$ on an interval $I = [0, b')$, with $b' > b$. Thus, by Theorem 2, we can write for all $x \in I$:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n\alpha}, \tag{14}$$

$$(\hat{D}^{\alpha}_{0+} y)(x) = \sum_{n=0}^{\infty} a_{n+1} \frac{\Gamma((n+1)\alpha + 1)}{\Gamma(n\alpha + 1)} x^{n\alpha},$$

$$(\hat{D}^{2\alpha}_{0+} y)(x) = \sum_{n=0}^{\infty} a_{n+2} \frac{\Gamma((n+2)\alpha + 1)}{\Gamma(n\alpha + 1)} x^{n\alpha}. \tag{15}$$

If the function y given by (14) is a solution to the fractional differential Equation (12), then

$$a_{n+2} = (a_{n-1} + c_n) \frac{\Gamma(n\alpha + 1)}{\Gamma((n+2)\alpha + 1)}, \text{ for all } n \geq 0, \tag{16}$$

where $a_{-1} = 0$, $c_0 = c_2 = 1$, $c_1 = -1$, and $c_n = 0$, for all $n \geq 3$.

Let us consider the initial value problem for Equation (13) with the initial conditions

$$y(0) = \delta_1, \ (\hat{D}^{\alpha}_{0+} y)(0) = s, \tag{17}$$

where s is a real parameter.

By Theorem 3, it follows that, for any fixed s, the initial value problem (12), (17) has a unique solution $\tilde{y} = \tilde{y}(x, s)$ which can be represented as an α-fractional power series at 0 on $[0, b')$:

$$\tilde{y}(x, s) = \sum_{n=0}^{\infty} \tilde{a}_n(s) x^{n\alpha}, x \in [0, b'). \tag{18}$$

As shown above, the coefficients $\tilde{a}_n(s)$ must verify (16), and, from the initial conditions (17), we have $\tilde{a}_0(s) = \delta_1$ and $\tilde{a}_1(s) = \frac{s}{\Gamma(\alpha+1)}$. Thus, by (16), we find

$$\tilde{a}_2(s) = \frac{1}{\Gamma(2\alpha + 1)}, \ \tilde{a}_3(s) = (\delta_1 - 1) \frac{\Gamma(\alpha + 1)}{\Gamma(3\alpha + 1)}, \ \tilde{a}_4(s) = \frac{\Gamma(2\alpha + 1)}{\Gamma(4\alpha + 1)} \left(\frac{s}{\Gamma(\alpha + 1)} + 1 \right),$$

and

$$\tilde{a}_n(s) = \tilde{a}_{n-3}(s) \frac{\Gamma((n-2)\alpha + 1)}{\Gamma(n\alpha + 1)}, \text{ for all } n \geq 5. \tag{19}$$

Hence, for every $k \geq 1$, it follows that

$$\tilde{a}_{3k}(s) = (\delta_1 - 1) \prod_{j=0}^{k-1} \frac{\Gamma((3j+1)\alpha + 1)}{\Gamma((3j+3)\alpha + 1)},$$

$$\tilde{a}_{3k+1}(s) = \left(\frac{s}{\Gamma(\alpha + 1)} + 1 \right) \prod_{j=0}^{k-1} \frac{\Gamma((3j+2)\alpha + 1)}{\Gamma((3j+4)\alpha + 1)},$$

and

$$\tilde{a}_{3k+2}(s) = \frac{1}{\Gamma(2\alpha + 1)} \prod_{j=0}^{k-1} \frac{\Gamma((3j+3)\alpha + 1)}{\Gamma((3j+5)\alpha + 1)}.$$

We notice that the coefficients $\tilde{a}_{3k}(s)$ and $\tilde{a}_{3k+2}(s)$ do not depend on s. For every $n \geq 3$, we denote

$$d_n = \begin{cases} \prod_{j=0}^{k-1} \frac{\Gamma((3j+2)\alpha+1)}{\Gamma((3j+4)\alpha+1)} & \text{if } n = 3k+1 \\ \tilde{a}_n(s) & \text{otherwise,} \end{cases}$$

and $d_1 = 1$, $d_2 = \tilde{a}_2(s) = \frac{1}{\Gamma(2\alpha+1)}$, $d_3 = \tilde{a}_3(s) = (\delta_1 - 1)\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}$. *By* (19), *we get*

$$d_n = d_{n-3}\frac{\Gamma((n-2)\alpha+1)}{\Gamma(n\alpha+1)}, \text{ for all } n \geq 4.$$

By replacing in (18), *we obtain*

$$\tilde{y}(x,s) = \delta_1 + \sum_{n=2}^{\infty} d_n x^{n\alpha} + \frac{s}{\Gamma(\alpha+1)} \sum_{k=0}^{\infty} d_{3k+1} x^{(3k+1)\alpha}, x \in [0, b'). \tag{20}$$

It can be easily proved that the fractional power series in the formula (20) *have the radius of convergence* $r = \infty$ *(hence, they are uniformly convergent on* $[0, b]$*). We denote the sum of the series by* $g(x) = \sum\limits_{n=2}^{\infty} d_n x^{n\alpha}$ *and* $h(x) = \sum\limits_{k=0}^{\infty} d_{3k+1} x^{(3k+1)\alpha}$*. To obtain a solution to the boundary value problem* (12), (13), *we need to find the value of* s *for which* $\tilde{y}(b,s) = \delta_2$, *that is, to solve the equation*

$$\delta_1 + g(b) + \frac{s}{\Gamma(\alpha+1)}h(b) = \delta_2.$$

Hence, for

$$s = \frac{\Gamma(\alpha+1)(\delta_2 - \delta_1 - g(b))}{h(b)}, \tag{21}$$

the series (18) *is a solution to the boundary value problem* (12), (13).

Let us suppose that $y_1(x)$ *is another solution to the boundary value problem* (12), (13) *which can be represented as an* α*-fractional power series at 0 on* $[0, b]$ *and denote* $s_1 = (\hat{D}^{\alpha}_{0+}\tilde{y})(0)$. *Then, the function* $y_1(x)$, *is the solution to the initial value problem*

$$\tilde{y}(0) = \delta_1, \; (\hat{D}^{\alpha}_{0+}\tilde{y})(0) = s_1,$$

for Equation (12). *Since* $y_1(x)$ *satisfies the boundary condition* (13) *and* s_0 *is uniquely defined by* (21), *it follows that* $s_1 = s_0$. *Hence* $y_1(x) = \tilde{y}(x, s_0)$, *which implies the uniqueness of the solution to the boundary value problem* (12), (13).

As a numerical example, if we take $b = 2$, $\delta_1 = -1$, *and* $\delta_2 = 5$, *the solutions to the BVP* (12), (13), *for* $\alpha = 0.2$, $\alpha = 0.5$, *and* $\alpha = 1$ *are presented in Figure 1 by using red, blue, and black lines, respectively.*

Example 2. *Let us consider the inhomogeneous fractional differential equation*

$$(\hat{D}^{3\alpha}_{0+}y)(x) - x^{\alpha}(\hat{D}^{\alpha}_{0+}y)(x) - y(x) = 1 - 2x^{\alpha}; x \in [0, b], \; \alpha \in (0, 1]. \tag{22}$$

and the boundary value problem

$$y(0) = \delta_1, (\hat{D}^{\alpha}_{0+}y)(0) = \delta_2, y(b) = \delta_3, \tag{23}$$

where δ_1, δ_2, *and* δ_3 *are arbitrary real numbers.*

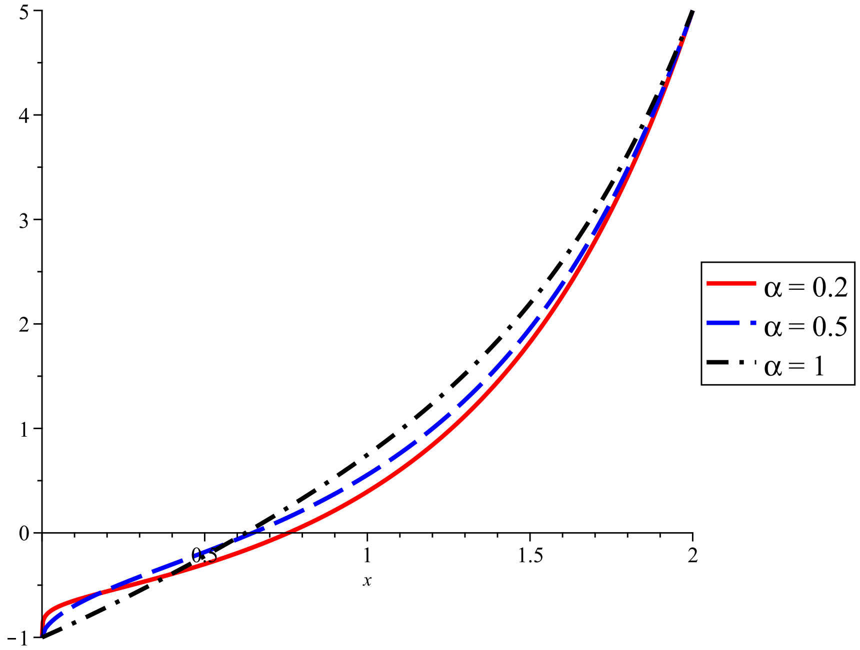

Figure 1. Solutions to BVP in Example 1 for different values of α.

As in Example 1, let us assume that the function $y = y(x)$ of the form (14) is a solution to the fractional differential Equation (22). Then, for $n \geq 1$, we find

$$a_{n+3} = \frac{\Gamma(n\alpha+1)}{\Gamma((n+3)\alpha+1)}\left(a_n\left(\frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)}+1\right)+c_n\right) \tag{24}$$

and $a_3 = \frac{a_0+c_0}{\Gamma(3\alpha+1)}$, where $c_0 = 1, c_1 = -2$, and $c_n = 0$, for all $n \geq 2$.

Let $\tilde{y}$ be a solution to the fractional differential Equation (22) satisfying the initial conditions

$$\tilde{y}(0) = \delta_1, (\hat{D}^{\alpha}_{0+}\tilde{y})(0) = \delta_2, (\hat{D}^{2\alpha}_{0+}\tilde{y})(0) = s, \tag{25}$$

where s is a real parameter. Then, by Theorem 3, the initial value problem (22), (25) has a unique solution $\tilde{y}(x,s)$, given by (18), where the coefficients $\tilde{a}_n$ satisfy (24) and $\tilde{a}_0 = \delta_1$, $\tilde{a}_1 = \frac{\delta_2}{\Gamma(\alpha+1)}$, $\tilde{a}_2 = \frac{s}{\Gamma(2\alpha+1)}$, and $\tilde{a}_3 = \frac{\delta_1+c_0}{\Gamma(3\alpha+1)}$. Hence, for $k \geq 1$, we find

$$\tilde{a}_{3k} = \frac{\delta_1+c_0}{\Gamma(3k\alpha+1)}\prod_{j=1}^{k-1}\left(\frac{\Gamma(3j\alpha+1)}{\Gamma((3j-1)\alpha+1)}+1\right),$$

$$\tilde{a}_{3k+1} = \frac{\Gamma(\alpha+1)(\delta_2(1+\frac{1}{\Gamma(\alpha+1)})+c_1)}{\Gamma((3k+1)\alpha+1)}\prod_{j=1}^{k-1}\left(\frac{\Gamma((3j+1)\alpha+1)}{\Gamma(3j\alpha+1)}+1\right)$$

and

$$\tilde{a}_{3k+2} = \frac{s}{\Gamma((3k+2)\alpha+1)}\prod_{j=0}^{k-1}\left(\frac{\Gamma((3j+2)\alpha+1)}{\Gamma((3j+1)\alpha+1)}+1\right). \tag{26}$$

Thus, we notice that only the coefficients of the form $\tilde{a}_{3k+2}$ depend on s. We denote $\tilde{a}_{3k+2} = sd_{3k+2}$ and $\tilde{a}_{3k} = d_{3k}$, $\tilde{a}_{3k+1} = d_{3k+1}$, for every $k = 0, 1, \ldots$ and

$$g(x) = \sum_{k=0}^{\infty} d_{3k}x^{3k\alpha} + \sum_{k=0}^{\infty} d_{3k+1}x^{(3k+1)\alpha}, \quad h(x) = \sum_{k=0}^{\infty} d_{3k+2}x^{(3k+2)\alpha}.$$

The series above have the radius of convergence ∞, so they are absolutely convergent $[0,b]$ and we have

$$\tilde{y}(x,s) = g(x) + sh(x). \tag{27}$$

To obtain a solution to the boundary value problem (22), (23) we have to find s such that

$$\tilde{y}(b,s) = g(b) + sh(b) = \delta_3,$$

so

$$s = \frac{\delta_3 - f(b)}{g(b)}.$$

The uniqueness of the solution follows as in Example 1.

As a numerical example, we take $b = 2$, $\delta_1 = 1$, $\delta_2 = 2$, and $\delta_3 = 3$. The solutions to the BVP (22), (23), for $\alpha = 0.2$, $\alpha = 0.5$, and $\alpha = 1$ are presented in Figure 2, by using red, blue, and black lines, respectively.

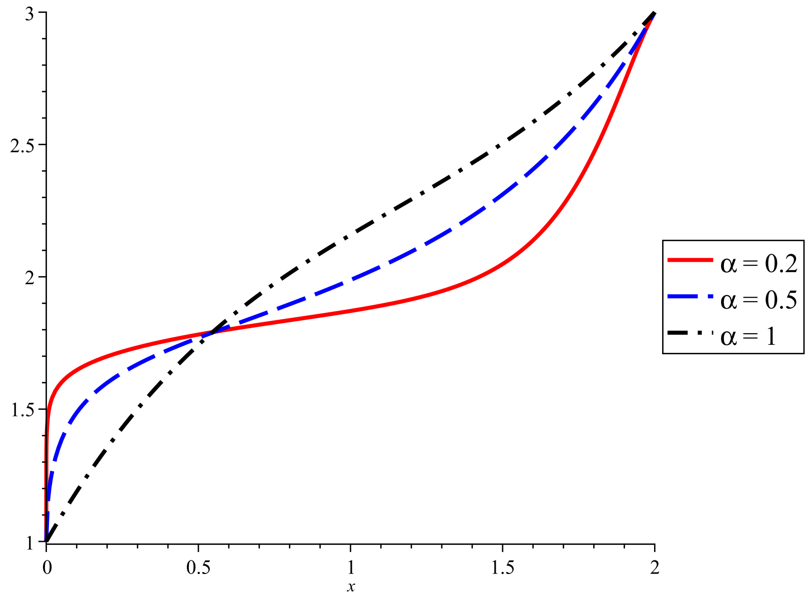

Figure 2. Solutions to BVP in Example 2 for different values of α.

4. Fractional Analytic Functions

A function is said to be representable as an α-fractional power series at x_0 if it is equal to the sum of the fractional Taylor series about x_0 on an interval $[x_0, x_0 + r)$. Some authors (see [2], Definition 7.8) call such functions *α-analytic at x_0*. In the following, we define the α-analytic functions *on an open interval*.

Definition 4. *Let I be an open interval and $\alpha \in (0,1]$. A real function f defined on I is called α-analytic on I, if, for every $x_0 \in I$, there exists $\varepsilon > 0$ such that $J_{x_0,\varepsilon} = [x_0, x_0 + \varepsilon) \subset I$ and f can be represented as an α-fractional power series at x_0, $f(x) = \sum\limits_{n=0}^{\infty} a_n (x - x_0)^{n\alpha}$, for all $x \in J_{x_0,\varepsilon}$.*

Remark 2. *By Definition 4, it follows that f is an α-fractional analytic on I, if, for every $x_0 \in I$, there exists $\varepsilon > 0$ such that $\bar{J}_{x_0,\varepsilon} = (x_0 - \varepsilon, x_0 + \varepsilon) \subset I$ and the real function $\mathcal{S}_{x_0,\varepsilon}(f)$ defined by*

$$\mathcal{S}_{x_0,\varepsilon}(f)(x) = \begin{cases} f(x) & \text{if } x \in [x_0, x_0 + \varepsilon) \\ f(2x_0 - x) & \text{if } x \in (x_0 - \varepsilon, x_0), \end{cases}$$

can be represented as an α-fractional power moduli series (2) at x_0 on $\bar{J}_{x_0,\varepsilon}$.

If $\alpha = 1$ in Definition 4, then the classical real analytic functions are obtained. A well-known result (see [27], Corollary 1.2.4) establishes that the sum of a power series, $f(x) = \sum_{n\geq 0} a_n(x-x_0)^n$, is analytic on the interval $(x_0 - r, x_0 + r)$, where r is the radius of convergence of the series. In the following, we discuss the analyticity of the functions representable as *fractional* power series.

Let $f(x) = \sum_{n\geq 0} a_n(x-x_0)^{n\alpha}$, for all $x \in (x_0, x_0 + r)$, $\alpha \in (0,1)$ be a function representable as an α-fractional power series at x_0. Then, f can be written as $f = g \circ h$, where $g : (-r^\alpha, r^\alpha) \to \mathbb{R}$, $g(y) = \sum_{n\geq 0} a_n y^n$, and $h : (x_0, x_0 + r) \to \mathbb{R}$, $h(x) = (x - x_0)^\alpha$. By the remark above, g is an analytic function. Since, for any $x_1 \in (x_0, x_0 + r)$, $h(x)$ can be written as $h(x) = \sum_{n\geq 0} (-1)^n \alpha(\alpha-1)\ldots(\alpha-n+1)(x_1-x_0)^{\alpha-n}(x-x_1)^n$ in a small neighbourhood of x_1, it follows that $h(x)$ is also analytic on $(x_0, x_0 + r)$, so the composite function $f = g \circ h$ is also an analytic function (see [27], Proposition 1.4.2).

Theorem 4 ([6])**.** *If $f(x)$ can be represented as an α-fractional power series at x_0,*

$$f(x) = \sum_{n\geq 0} a_n(x-x_0)^{n\alpha}, \text{ for all } x \in [x_0, x_0 + r), \tag{28}$$

where $\alpha \in (0,1]$ and $r > 0$, then f is a real analytic function on the interval $(x_0, x_0 + r)$.

Obviously, an (integer) power series can be also considered as an α-fractional power series, for any $\alpha = \frac{1}{m}$, $m \in \mathbb{N}^*$. Hence, any real analytic function on an open interval I is also an α-analytic function on I (with $\alpha = \frac{1}{m}$). By Theorem 4, it follows that any α-analytic function on I is real analytic on I and, from Remark 1, we obtain that α must be a rational number if f is non-constant. On the other hand, if $\alpha = \frac{k}{m}$ with $k > 1$, then, for any $x_0 \in I$, we have $f(x) = \sum_{n=0}^{\infty} \frac{f^{(kn)}(x_0)}{(kn)!}(x-x_0)^{kn}$ in a neighbourhood of x_0, so $f'(x_0) = 0$ for all $x_0 \in I$ and the next corollary follows.

Corollary 3. *Let I be an open interval and $f : I \to \mathbb{R}$ be a non-constant function. Then, f is an α-analytic function if and only if $\alpha = \frac{1}{m}$, $m \in \mathbb{N}^*$, and f is real analytic on I.*

5. Conclusions

In this paper, we study the properties of the α-fractional power moduli series as a generalization of the α-fractional power series. Using the generalized Taylor's formula with fractional derivatives, a sufficient condition for a function to be represented as an α-fractional power moduli series is established.

Moreover, we present a practical method to solve the boundary value problems for fractional differential equations, the solution being expressed as an α-fractional power series.

Finally, α-analytic functions are defined and we prove that a non-constant function is α-analytic on an open interval I if and only if $\alpha = \frac{1}{m}$ with m a positive integer, and the function is real analytic on I.

Author Contributions: Conceptualization, G.G.; methodology, G.G. and M.J.; software, G.G. and I.M.-M.; validation, G.G., M.J. and I.M.-M.; writing—original draft preparation, G.G. and M.J.; writing—review and editing, G.G., M.J. and I.M.-M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflicts of interest.

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