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Anisotropic Moser–Trudinger-Type Inequality with Logarithmic Weight

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Abstract: Our main purpose in this paper is to study the anisotropic Moser–Trudinger-type inequalities with logarithmic weight $\omega_\beta(x) = [-\ln F^o(x)]^{(n-1)\beta}$. This can be seen as a generation result of the isotropic Moser–Trudinger inequality with logarithmic weight. Furthermore, we obtain the existence of extremal function when β is small. Finally, we give Lions' concentration-compactness principle, which is the improvement of the anisotropic Moser–Trudinger-type inequality.

Keywords: anisotropic Moser–Trudinger-type inequality; logarithmic weight; existence of extremal function

MSC: 35A23; 35A01; 35B38

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1. Introduction

It is well-known that important geometric inequalities, for example, the Sobolev inequality, Moser–Trudinger inequality, etc., and the existence of extreme functions play a key role to study partial differential equations. For a bounded domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$, we have $W_0^{1,p}(\Omega) \subset L^q(\Omega)$, $1 \leq q \leq \frac{np}{n-p}$ for $1 \leq p < n$ by the calssical Sobolev embedding theorem. Particularly, for $p = n$, $W_0^{1,n}(\Omega) \subset L^q(\Omega)$, $\forall q \geq 1$. But $W_0^{1,n}(\Omega) \not\subseteq L^\infty(\Omega)$. For the borderline case $p = n$, the Moser–Trudinger inequality is the perfect replacement. In 1971, Moser [1] proved the sharpening of Trudinger's inequality as follows:

$$\sup_{u \in W_0^{1,n}(\Omega), \|\nabla u\|_n \leq 1} \int_\Omega e^{\alpha |u|^{\frac{n}{n-1}}} dx \leq C, \tag{1}$$

for $\forall \alpha \leq \alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$ and ω_{n-1} stands for area of the $(n-1)$-sphere. Moreover, α_n is sharp, which means that if $\alpha > \alpha_n$, then the inequality (1) can no longer hold. Inequality (1) is the so-called Moser–Trudinger inequality, the extremal of which is related to the existence of solutions of some semi-linear Liouville-type equations.

As far as we know, there have been many important studies related to the Moser–Trudinger inequality, for example, [2–8], etc. In the references listed above, readers can see the Moser–Trudinger inequality in $\mathbb{R}^n$ and in hyperbolic spaces, the existence of an extremal function for the Moser–Trudinger inequality, etc. These important geometric inequalities play a key role in geometry analysis, calculus of variations, and PDEs; we refer to [9–14] and references therein. And recently, the authors of [15] studied a system of Kirchhoff type driven by the Q-Laplacian in the Heisenberg group $\mathbb{H}^n$. They obtained the existence of solutions via variational methods based on a new Moser–Trudinger-type inequality for the Heisenberg group $\mathbb{H}^n$. Moreover, in [16], the authors also focus on a Kirchhoff-type problem and establish the existence of a radial solution in the subcritical growth case by the Moser–Trudinger inequality and minimax method.

Let $\varrho_\beta(x) = (-\ln|x|)^{\beta(n-1)}$, $0 \le \beta < 1$. Clalnchi and Ruf [17,18] proved the weighed Moser–Trudinger-type inequality involving the radical functions in unit ball B:

$$\sup_{u \in W^{1,n}_{0,rad}(B,\varrho_\beta), \|u\|_{\varrho_\beta \le 1}} \int_B e^{\alpha|u|^{\frac{n}{(n-1)(1-\beta)}}} dx < \infty \tag{2}$$

for any $\alpha \le \alpha_{\beta,n} = n[(1-\beta)\omega_{n-1}^{\frac{1}{n-1}}]^{\frac{1}{1-\beta}}$, $\|u\|_{\varrho_\beta} = (\int_B |\nabla u|^n \varrho_\beta(x))^{\frac{1}{n}}$. Moreover, the constant $\alpha_{\beta,n}$ is sharp, i.e., if $\alpha > \alpha_{\beta,n}$, the supremum in (2) will be infinite. We note that the authors applied Leckband's inequality [19] to prove the weighed Moser–Trudinger-type inequality (2). Note that when $\beta = 0$, by the Pólya-Szegö principle, (2) recovers the classical Moser–Trudinger inequality (1). Furthermore, Roy [20] proved the existence of an extremal function for inequality (2).

Recently, many researchers have intended to establish anisotropic Moser–Trudinger-type inequalities. Let $F \in C^2(\mathbb{R}^n \backslash \{0\})$ be a nonnegative and convex function, the polar $F^o(x)$ of which represents a Finsler metric on $\mathbb{R}^n$. By $F(x)$, a Finsler–Laplacian operator Δ_F is defined by

$$\Delta_F u := \sum_{i=1}^n \frac{\partial}{\partial x_i}(F(\nabla u)F_{\xi_i}(\nabla u)),$$

where $F_{\xi_i} = \frac{\partial F}{\partial \xi_i}$. In Euclidean modulus, Δ_F is nothing but the common Laplacian. The Finsler–Laplacian operator is closely related to the Wulff shape, which was initiated in Wulff' work [21]. More details about the properties of $F(x)$ and $F^o(x)$ can be seen in Section 2.

For a bounded smooth domain $\Omega \subset \mathbb{R}^n$, Wang and Xia [22] proved that, for $\forall \lambda \le \lambda_n = n^{\frac{n}{n-1}} \kappa_n^{\frac{1}{n-1}}$,

$$\sup_{u \in W_0^{1,n}(\Omega), \int_\Omega F^n(\nabla u)dx \le 1} \int_\Omega e^{\lambda|u|^{\frac{n}{n-1}}} dx \le C, \tag{3}$$

where

$$\kappa_n = |x \in \mathbb{R}^n | F^o(x) \le 1| \tag{4}$$

denotes the volume of a unit Wulff ball in $\mathbb{R}^n$.

In this paper, we intend to establish the anisotropic Moser–Trudinger-type inequality with logarithmic weight. We believe that these sharp inequalities will be the key tools to study the existence of solutions for some quasi-linear elliptic equations, such as the Finsler–Laplacian equation. For $\beta \in [0,1)$, we let

$$\omega_\beta(x) = [-\ln F^o(x)]^{\beta(n-1)},$$

which is the weight of logarithmic type defined on a unit Wulff ball $\mathcal{W}_1 = \{x \in \mathbb{R}^n : F^o(x) < 1\}$. And $W_0^{1,n}(\mathcal{W}_1, \omega_\beta)$ represents the functions of completion of $C_0^1(\mathcal{W}_1)$ with respect to the norm

$$\|u\|_{\omega_\beta} = (\int_{\mathcal{W}_1} F^n(\nabla u)\omega_\beta(x)dx)^{\frac{1}{n}}, \qquad u \in C_0^1(\mathcal{W}_1).$$

Let $W^{1,n}_{0,rad}(\mathcal{W}_1, \omega_\beta)$ be the subspace of $W_0^{1,n}(\mathcal{W}_1, \omega_\beta)$ of all radial functions with respect to F. In this paper, radial functions with respect to F means that $u(x) = \tilde{u}(r)$, where $r = F^o(x)$.

In the following, for convenience, we denote

$$AMT(n,\lambda,\beta) = \sup_{u \in W^{1,n}_{0,rad}(\mathcal{W}_1,\omega_\beta), \|u\|_{\omega_\beta \le 1}} \int_{\mathcal{W}_1} e^{\lambda\|u\|^{\frac{n}{(n-1)(1-\beta)}}} dx. \tag{5}$$

We now state our main results.

Theorem 1. *For any*

$$\lambda \leq \lambda_{\beta,n} = n^{1+\frac{1}{(n-1)(1-\beta)}} [\kappa_n^{\frac{1}{n-1}}(1-\beta)]^{\frac{1}{1-\beta}}, \tag{6}$$

we have $AMT(n,\lambda,\beta) < \infty$. Moreover, this constant $\lambda_{\beta,n}$ is sharp, i.e., if $\lambda > \lambda_{\beta,n}$, $AMT(n,\lambda,\beta)$ is infinite.

Next, we prove the existence of an extremal function for the anisotropic Moser–Trudinger-type inequality with logarithmic weight.

Theorem 2. *There exists $\beta_0 \in [0,1)$ such that, for $\forall \beta \in [0,\beta_0)$, $AMT(n,\lambda_{\beta,n},\beta)$ is attained.*

Finally, we establish the Lions-type concentration-compactness property, which can be seen as an improvement of the anisotropic Moser–Trudinger-type inequality in Theorem 1 for some situations.

Theorem 3. *Let $\{u_k\}$ be a sequence in $W^{1,n}_{0,rad}(\mathcal{W}_1,\omega_\beta)$ such that $\|u_k\|_{\omega_\beta} = 1$ and $u_k \rightharpoonup u_0$ in $\mathcal{W}^{1,n}_{0,rad}(\mathcal{W}_1,\omega_\beta)$. Then we have*

$$\limsup_{k\to\infty} \int_{\mathcal{W}_1} e^{p\lambda_{\beta,n}|u_k|^{\frac{n}{(n-1)(1-\beta)}}} dx < \infty, \tag{7}$$

for any $p < p(u_0) := (1 - \|u_0\|^n_{\omega_\beta})^{-\frac{1}{(n-1)(1-\beta)}}$.

2. Preliminaries

In this section, we give preliminaries involving the Finsler–Laplacian, co-area formula with respect to F and convex symmetrization $u^\sharp$ of u with respect to F.

Let $F : \mathbb{R}^n \mapsto \mathbb{R}$ be a function that is $C^2(\mathbb{R}^n \backslash \{0\})$, convex, and even. And $F(x)$ is a homogenous function, that is, for any $t \in \mathbb{R}$, $\xi \in \mathbb{R}^n$,

$$F(t\xi) = |t|F(\xi).$$

Furthermore, we assume for any $\xi \neq 0$, $F(\xi) > 0$.

By the homogeneity property of F, we can find two positive constants $0 < c_1 \leq c_2 < \infty$ such that

$$c_1|\xi| \leq F(\xi) \leq c_2|\xi|, \quad \forall \xi \in \mathbb{R}^n.$$

The operator

$$\Delta_F u := \sum_{i=1}^n \frac{\partial}{\partial x_i}(F(\nabla u)F_{\xi_i}(\nabla u))$$

is called Finsler–Laplacian, which was studied by many mathematicians. For some important works involving the Finsler-Laplacian, we refer to [22–26] and the references therein.

$F^o(x)$ is the support function of $F(x)$, which is defined as $F^o(x) := \sup_{\xi \in K}\langle x,\xi\rangle$, where $K = \{x \in \mathbb{R}^n : F(x) \leq 1\}$. Then we can check that $F^o(x)$ is also a function that is $C^2(\mathbb{R}^n \backslash \{0\})$. And $F^o(x)$ is also a convex and homogeneous function. What is more, $F^o(x)$ is dual to $F(x)$ in the sense that

$$F^o(x) = \sup_{\xi\neq 0} \frac{\langle x,\xi\rangle}{F(\xi)}, \qquad F(x) = \sup_{\xi\neq 0} \frac{\langle x,\xi\rangle}{F^o(\xi)}.$$

Denote the unit Wulff ball of center at origin as

$$\mathcal{W}_1 := \{x \in \mathbb{R}^n | F^o(x) \leq 1\}$$

and

$$\kappa_n := |\mathcal{W}_1|,$$

which is the volume of a unit Wulff ball $\mathcal{W}_1$. Also, we denote $\mathcal{W}_r$ as the Wulff ball of center at origin with radius r, i.e.,

$$\mathcal{W}_r := \{x \in \mathbb{R}^n | F^o(x) \le r\}.$$

For later use, by the assumptions of $F(x)$, we can obtain some properties of the function $F(x)$; see also [25,27,28].

Lemma 1. *We have*

(i) $|F(m) - F(n)| \le F(m+n) \le F(m) + F(n)$;
(ii) $\frac{1}{C} \le |\nabla F(m)| \le C$, *and* $\frac{1}{C} \le |\nabla F^o(m)| \le C$ *for some* $C > 0$ *and* $m \neq 0$;
(iii) $\langle m, \nabla F(m)\rangle = F(m), \langle m, \nabla F^o(m)\rangle = F^o(m)$ *for* $m \neq 0$;
(iv) $F(\nabla F^o(m)) = 1$, $F^o(\nabla F(m)) = 1$ *for* $m \neq 0$;
(v) $F^o(m) F_\xi(\nabla F^o(m)) = m$ *for* $m \neq 0$.

Now, we give the co-area formula and isoperimetric inequality with respect to F, respectively. For a domain $\Omega \subset \mathbb{R}^n$, $G \subset \Omega$, let $u \in BV(\Omega)$, which we denote as a function of bounded variation. The anisotropic bounded variation of u with respect to F is defined by

$$\int_\Omega |\nabla u|_F = \sup\{\int_\Omega u \operatorname{div}\tau dx, \tau \in C_0^1(\Omega; \mathbb{R}^n), F^o(\tau) \le 1\},$$

and the anisotropic perimeter of G with respect to F is defined by

$$H_F(G) := \int_\Omega |\nabla \mathcal{X}_G|_F dx,$$

where $\mathcal{X}_G$ is the characteristic function defined on the subset G. Then we have the co-area formula (see [26])

$$\int_\Omega |\nabla u|_F = \int_0^\infty H_F(|u| > t) dt \tag{8}$$

and the isoperimetric inequality

$$H_F(G) \ge n\kappa_n^{\frac{1}{n}} |G|^{1-\frac{1}{n}}. \tag{9}$$

Furthermore, (9) becomes an equality if and only if G is a Wulff ball.

3. Anisotropic Moser–Trudinger-Type Inequality with Logarithmic Weight

In this section, we prove Theorem 1. Firstly, we give a useful formula involving the change in functions in a unit Wulff ball $\mathcal{W}_1$. For $u \in W_{0,rad}^{1,n}(\mathcal{W}_1, \omega_\beta)$ and any $0 \le \tilde{\beta} < \beta$, we let

$$v(x) = \left(\frac{\lambda_{\beta,n}}{\lambda_{\tilde{\beta},n}}\right)^{\frac{(n-1)(1-\tilde{\beta})}{n}} u(x)|u(x)|^{\frac{\beta-\tilde{\beta}}{1-\beta}}. \tag{10}$$

Then we have the following lemma.

Lemma 2. *Let* $u \in W_{0,rad}^{1,n}(\mathcal{W}_1, \omega_\beta)$ *with* $\|u\|_{\omega_\beta} \le 1$. *Define* v *by (10); then we have* $\|v\|_{\omega_{\tilde{\beta}}} \le 1$.

Proof. By the property of $F(x)$ in Lemma 1, we have

$$\begin{aligned} F^n(\nabla v) &= (\frac{\lambda_{\beta,n}}{\lambda_{\tilde{\beta},n}})^{(n-1)(1-\tilde{\beta})}\frac{(1-\tilde{\beta})^n}{(1-\beta)^n}F^n(\nabla u)|u(x)|^{\frac{n(\beta-\tilde{\beta})}{1-\beta}} \\ &\leq \frac{(1-\tilde{\beta})}{(1-\beta)}F^n(\nabla u)\frac{\omega_\beta(x)}{\omega_{\tilde{\beta}}(x)}(\int_{\mathcal{W}_1\setminus\mathcal{W}_{F^o(x)}}F^n(\nabla u)\omega_\beta dy)^{\frac{\beta-\tilde{\beta}}{1-\beta}}. \end{aligned}$$

Hence, by the co-area Formula (8), we have

$$\begin{aligned} &\|v\|^n_{\omega_{\tilde{\beta}}} = \int_{\mathcal{W}_1}F^n(\nabla v)\omega_{\tilde{\beta}}(x)dx \\ \leq\ & \frac{(1-\tilde{\beta})}{(1-\beta)}\int_{\mathcal{W}_1}F^n(\nabla u)\omega_\beta(x)(\int_r^1 n\kappa_n|u'(s)|^n s^{n-1}\omega_\beta(s)ds)^{\frac{\beta-\tilde{\beta}}{1-\beta}}dx \\ =\ & \frac{(1-\tilde{\beta})}{(1-\beta)}(n\kappa_n)^{\frac{(1-\tilde{\beta})}{(1-\beta)}}\int_0^1|u'(r)|^n\omega_\beta(r)r^{n-1}(\int_r^1|u'(s)|^n\omega_\beta(s)s^{n-1}ds)^{\frac{\beta-\tilde{\beta}}{1-\beta}}dr \\ =\ & -(n\kappa_n)^{\frac{(1-\tilde{\beta})}{(1-\beta)}}\int_0^1\frac{d}{dr}[(\int_r^1|u'(s)|^n\omega_\beta(s)s^{n-1}ds)^{\frac{(1-\tilde{\beta})}{(1-\beta)}}]dr \\ =\ & (n\kappa_n)^{\frac{(1-\tilde{\beta})}{(1-\beta)}}(\int_0^1|u'(r)|^n\omega_\beta(r)r^{n-1}dr)^{\frac{(1-\tilde{\beta})}{(1-\beta)}} \\ =\ & (\int_{\mathcal{W}_1}F^n(\nabla u)\omega_\beta dx)^{\frac{1-\tilde{\beta}}{1-\beta}} \\ \leq\ & 1. \end{aligned}$$

□

Next, in this paper, we frequently need to change the variable in the following way. For $u\in W^{1,n}_{0,rad}(\mathcal{W}_1,\omega_\beta)$, we change the variable as follows:

$$F^o(x)=e^{-\frac{t}{n}}$$

and set

$$\psi(t)=\kappa_n^{\frac{1}{n}}n^{\frac{1+(n-1)(1-\beta)}{n}}(1-\beta)^{\frac{n-1}{n}}u(x). \tag{11}$$

Then we have $\psi'(t)=-n^{-\beta\frac{n-1}{n}}\kappa_n^{\frac{1}{n}}(1-\beta)^{\frac{n-1}{n}}\tilde{u}'(e^{-\frac{t}{n}})e^{-\frac{t}{n}}$. By Lemma 1 and co-area Formula (8), we can transform the norm as follows:

$$\begin{aligned} &\textstyle\int_{\mathcal{W}_1}F^n(\nabla u)|\log F^o(x)|^{\beta(n-1)}dx \\ &=\textstyle\int_{\mathcal{W}_1}F^n(\tilde{u}'(F^o(x))\nabla F^o(x))|\log F^o(x)|^{\beta(n-1)}dx \\ &=\textstyle\int_{\mathcal{W}_1}[\tilde{u}'(F^o(x))]^n|\log F^o(x)|^{\beta(n-1)}dx \\ &=\textstyle\int_0^1 n\kappa_n[\tilde{u}'(F^o(x))]^n|\log F^o(x)|^{\beta(n-1)}(F^o(x))^{n-1}dF^o(x) \\ &=\textstyle\int_0^{+\infty}\kappa_n[\tilde{u}'(e^{-\frac{t}{n}})]^n|\frac{t}{n}|^{\beta(n-1)}e^{-t}dt \\ &=\textstyle\int_0^{+\infty}\frac{|\psi'|^n t^{\beta(n-1)}}{(1-\beta)^{n-1}}dt. \end{aligned} \tag{12}$$

The functional changes as follows:

$$\frac{1}{\kappa_n}\int_{\mathcal{W}_1}e^{\lambda_{n,\beta}|u|^{\frac{n}{(n-1)(1-\beta)}}}dx=\int_0^{+\infty}e^{|\psi|^{\frac{n}{(n-1)(1-\beta)}}-t}dt \tag{13}$$

and

$$\frac{1}{\kappa_n}\int_{\mathcal{W}_1} e^{\lambda|u|^{\frac{n}{(n-1)(1-\beta)}}}dx = \int_0^{+\infty} e^{\bar{\lambda}|\psi|^{\frac{n}{(n-1)(1-\beta)}}-t}dt, \tag{14}$$

where $\bar{\lambda} = \frac{\lambda}{\lambda_{n,\beta}}$.

Now it is easy to prove Theorem 1 by Lemma 2.

Proof of Theorem 1. Let $u \in W^{1,n}_{0,rad}(\mathcal{W}_1, \omega_\beta)$ with $\|u\|_{\omega_\beta} \le 1$. Define v by (10). By Lemma 2, we have $\|v\|_{\omega_{\tilde{\beta}}} \le 1$ for $\forall \tilde{\beta} \le \beta$. By the definition of $AMT(n, \lambda_{\beta,n}, \beta)$, we obtain

$$\int_{\mathcal{W}_1} e^{\lambda_{\beta,n}|u|^{\frac{n}{(n-1)(1-\beta)}}}dx = \int_{\mathcal{W}_1} e^{\lambda_{\tilde{\beta},n}|v|^{\frac{n}{(n-1)(1-\tilde{\beta})}}}dx \le AMT(n, \lambda_{\tilde{\beta},n}, \tilde{\beta}). \tag{15}$$

Since (15) holds for $\forall u \in W^{1,n}_{0,rad}(\mathcal{W}_1, \omega_\beta)$ with $\|u\|_{\omega_\beta} \le 1$, then we have

$$AMT(n, \lambda_{\beta,n}, \beta) \le AMT(n, \lambda_{\tilde{\beta},n}, \tilde{\beta}),$$

for any $0 \le \tilde{\beta} \le \beta < 1$. Hence, we obtain that the function $\beta \mapsto AMT(n, \lambda_{\beta,n}, \beta)$ is decreasing on $[0, 1)$. Thus, by the anisotropic Moser–Trudinger-type inequality (3), we obtain $AMT(n, \lambda_{\beta,n}, \beta) < \infty$.

Now we prove the constant $\lambda_{\beta,n}$ is sharp. We need to show that, if $\lambda > \lambda_{n,\beta}$, $AMT(n, \lambda, \beta)$ is infinite. By (14), we only need to test

$$\int_0^{+\infty} e^{\bar{\lambda}|\psi|^{\frac{n}{(n-1)(1-\beta)}}-t}dt,$$

where $\bar{\lambda} > 1$.

Consider the family of functions of Moser's type

$$\eta_m(t) = \begin{cases} \frac{t^{1-\beta}}{m^{\frac{1-\beta}{n}}}, & t \le m, \\ m^{\frac{(1-\beta)(n-1)}{n}}, & t \ge m. \end{cases}$$

By direct computation, we have $\int_0^{+\infty} \frac{|\eta_m'|^n t^{\beta(n-1)}}{(1-\beta)^{n-1}}dt = 1$. However, as $m \to +\infty$,

$$\int_0^{+\infty} e^{\bar{\lambda}|\eta_m|^{\frac{n}{(n-1)(1-\beta)}}-t}dt \ge \int_m^{+\infty} e^{\bar{\lambda}m-t}dt \to +\infty, \text{ if } \bar{\lambda} > 1.$$

The proof of Theorem 1 is completed. □

4. Existence of the Extremal Function

In this section, we complete the proof of Theorem 2. Firstly, we give a uniform bound for $u \in W^{1,n}_{0,rad}(\mathcal{W}_1, \omega_\beta)$. For $u \in W^{1,n}_{0,rad}(\mathcal{W}_1, \omega_\beta)$, we denote by $u(r)$ the value of $u(x)$ with $r = F^o(x)$. By the Hölder inequality and co-area Formula (8), for any $0 < r < s \le 1$ and $u \in W^{1,n}_{0,rad}(\mathcal{W}_1, \omega_\beta)$, we have

$$\begin{aligned} & |u(r) - u(s)| \\ = \;& |\textstyle\int_s^r u'(t)dt| \\ \le \;& \textstyle\int_s^r |\tilde{u}'(t)| t^{\frac{n-1}{n}} |\log t|^{\frac{(n-1)\beta}{n}} t^{-\frac{n-1}{n}} |\log t|^{-\frac{(n-1)\beta}{n}} dt \\ \le \;& (n\kappa_n)^{-\frac{1}{n}} (\tfrac{1}{1-\beta})^{\frac{n-1}{n}} (\textstyle\int_{\mathcal{W}_s \setminus \mathcal{W}_r} F^n(\nabla u)\omega_\beta dx)^{\frac{1}{n}} (-\ln \tfrac{r}{s})^{\frac{(n-1)(1-\beta)}{n}} \\ = \;& (\tfrac{n}{\lambda_{\beta,n}})^{\frac{(n-1)(1-\beta)}{n}} (\textstyle\int_{\mathcal{W}_s \setminus \mathcal{W}_r} F^n(\nabla u)\omega_\beta dx)^{\frac{1}{n}} (-\ln \tfrac{r}{s})^{\frac{(n-1)(1-\beta)}{n}}. \end{aligned} \tag{16}$$

In particular, when $s=1$, for any $0<r\leq 1$ and $u\in W^{1,n}_{0,rad}(\mathcal{W}_1,\omega_\beta)$, we have

$$|u(r)|\leq (\frac{n}{\lambda_{\beta,n}})^{\frac{(n-1)(1-\beta)}{n}}(\int_{\mathcal{W}_1\setminus\mathcal{W}_r}F^n(\nabla u)\omega_\beta dx)^{\frac{1}{n}}(-\ln r)^{\frac{(n-1)(1-\beta)}{n}}. \tag{17}$$

The definition of $\psi(t)$ in (11) and (12) shows that the anisotropic norm changes as

$$\Gamma(\psi):=\int_0^{+\infty}\frac{|\psi'|^n t^{\beta(n-1)}}{(1-\beta)^{n-1}}dt=\int_{\mathcal{W}_1}F^n(\nabla u)|\log F^o(x)|^{\beta(n-1)}dx.$$

and (13) shows that the functional $I_\beta(\psi)$ and $J_\beta(u)$ changes as

$$I_\beta(\psi):=\int_0^{+\infty}e^{|\psi|^{\frac{n}{(n-1)(1-\beta)}}-t}dt=\frac{1}{\kappa_n}\int_{\mathcal{W}_1}e^{\lambda_{n,\beta}|u|^{\frac{n}{(n-1)(1-\beta)}}}dx:=J_\beta(u). \tag{18}$$

For $\delta\in[0,1)$, we define

$$\tilde{\Lambda}_\delta=\{\psi\in C^1[0,\infty)|\ \psi(0)=0,\ \Gamma(\psi)\leq\delta\}.$$

Then the existence of an extremal function in Theorem 1 reduces to find $\psi_0\in\tilde{\Lambda}_1$ such that

$$Q_\beta:=I_\beta(\psi_0)=\sup_{\psi\in\tilde{\Lambda}_1}I_\beta(\psi). \tag{19}$$

Let $\tilde{g}_k(x)$ be a maximizing sequence of (19), that is, $J_\beta(\tilde{g}_k)\to Q_\beta$. Since

$$\int_{\mathcal{W}_1}F^n(\nabla\tilde{g}_k)|\log F^o(x)|^{\beta(n-1)}dx\leq 1,$$

then there exist a subsequence (still denoted by $\tilde{g}_k$) and a function $\tilde{g}_0\in W^{1,n}_{0,rad}(\mathcal{W}_1,\omega_\beta)$ such that

$$\tilde{g}_k\rightharpoonup\tilde{g}_0,\quad \tilde{g}_k\to\tilde{g}_0\quad\text{pointwise}. \tag{20}$$

Next, we give an inequality and we will use it several times. For any $h\in C^1[0,\infty)$ and $t\geq A\geq 0$, by the Hölder inequality, we have

$$\begin{aligned} h(t) &= h(A)+\textstyle\int_A^t h'(s)ds \\ &= h(A)+\textstyle\int_A^t h'(s)s^{\frac{\beta(n-1)}{n}}s^{-\frac{(n-1)\beta}{n}}ds \\ &\leq h(A)+(\textstyle\int_A^t|h'(s)|^n s^{\beta(n-1)}ds)^{\frac{1}{n}}(\textstyle\int_A^t s^{-\beta}ds)^{\frac{n-1}{n}} \\ &= h(A)+(\textstyle\int_A^t|h'(s)|^n s^{\beta(n-1)}ds)^{\frac{1}{n}}(t^{1-\beta}-A^{1-\beta})^{\frac{n-1}{n}}. \end{aligned} \tag{21}$$

Now we give a lemma involving concentration-compactness alternative, by which we only need to prove that the maximizing sequence $\tilde{g}_k(x)$ in (20) does not concentrate at 0, and then we can pass to the limit in the functional. Firstly, we give a definition.

We say a sequence of functions $u_k\in W^{1,n}_{0,rad}(\mathcal{W}_1,\omega_\beta)$ concentrates at $x=0$, denoted by

$$F^n(\nabla u_k)\omega_\beta dx\rightharpoonup\delta_0,$$

if $\|u_k\|_{\omega_\beta}\leq 1$ and any $1>r>0$, $\int_{\mathcal{W}_1\setminus\mathcal{W}_r}F^n(\nabla u_k)\omega_\beta dx\to 0$.

Lemma 3. *[Concentration-compactness alternative] For any sequence $\tilde{v}_k$, $\tilde{v}\in W^{1,n}_{0,rad}(\mathcal{W}_1,\omega_\beta)$, such that $\tilde{v}_k\rightharpoonup\tilde{v}$ in $W^{1,n}_{0,rad}(\mathcal{W}_1,\omega_\beta)$, then up to a subsequence (still denoted by $\tilde{v}_k$), either (i) $J_\beta(\tilde{v}_k)\to J_\beta(\tilde{v})$, or (ii) $\tilde{v}_k$ concentrates at $x=0$.*

Proof. We assume that (ii) does not hold; then we only need to show that (i) holds. Since (ii) does not hold, then there exist $A > 0$ and $\delta \in (0,1)$ such that for sufficiently large k,

$$\int_{\mathcal{W}_1 \setminus \mathcal{W}_{e^{-\frac{A}{n}}}} F^n(\nabla \tilde{v}_k) |\log F^o(x)|^{\beta(n-1)} dx = \int_0^A \frac{|v_k'|^n t^{\beta(n-1)}}{(1-\beta)^{n-1}} dt \geq \delta,$$

where we use the variable of change

$$F^o(x) = e^{-\frac{t}{n}} \text{ and } \lambda_{n,\beta}^{\frac{(n-1)(1-\beta)}{n}} \tilde{v}_k(x) = v_k(t). \tag{22}$$

By (21), we have

$$|v_k(t) - v_k(A)| \leq (1-\delta)^{\frac{1}{n}} (t^{1-\beta} - A^{1-\beta})^{\frac{n-1}{n}} \leq (1-\delta)^{\frac{1}{n}} t^{\frac{(n-1)(1-\beta)}{n}}.$$

Since for any k,

$$|v_k(A)| \leq A^{\frac{(n-1)(1-\beta)}{n}},$$

we have for $t \geq T$, T sufficiently large,

$$\begin{aligned} v_k(t)^{\frac{n}{(n-1)(1-\beta)}} &\leq [A^{\frac{(n-1)(1-\beta)}{n}} + (1-\delta)^{\frac{1}{n}} t^{\frac{(n-1)(1-\beta)}{n}}]^{\frac{n}{(n-1)(1-\beta)}} \\ &\leq A + (1-\tfrac{\delta}{2})^{\frac{1}{(n-1)(1-\beta)}} t. \end{aligned} \tag{23}$$

We note that in (23), we applied the inequality if $a > b > 0$, $p > 1$. Then, for $x \in \mathbb{R}$ large enough, $(1+ax)^p \leq 1 + b^p x^p$.

We split the integral $I_\beta(v_k) = I_1(v_k) + I_2(v_k)$, where

$$I_1(v_k) = \int_0^T e^{|v_k(t)|^{\frac{(n-1)(1-\beta)}{n}} - t} dt,$$

and

$$I_2(v_k) = \int_T^\infty e^{|v_k|^{\frac{(n-1)(1-\beta)}{n}} - t} dt.$$

Since $\tilde{v}_k$ converges pointwise to $\tilde{v}$, then v_k also converges pointwise to v. Then, by $|v_k(t)| \leq t^{\frac{(n-1)(1-\beta)}{n}}$ and the dominated convergence theorem, we have that $I_1(v_k) \to I_1(v)$.

By (23), we have for any small $\epsilon > 0$ and T large enough,

$$\begin{aligned} I_2(v_k) &= \int_T^\infty e^{|v_k|^{\frac{(n-1)(1-\beta)}{n}} - t} dt \\ &\leq e^A \int_T^\infty e^{[(1-\frac{\delta}{2})^{\frac{1}{(1-\beta)(n-1)}} - 1]t} dt, \end{aligned} \tag{24}$$

which is smaller than ϵ. Then $I_\beta(v_k) \to I_\beta(v)$, that is, $J_\beta(\tilde{v}_k) \to J_\beta(\tilde{v})$. $\square$

The following lemma is proved in [29]. For δ, $a > 0$, let

$$\Lambda_\delta^a = \{\phi \in C^1[0,\infty) | \phi(0) = 0, \int_a^\infty |\phi'|^n dt \leq \delta\}.$$

Lemma 4 ([29])**.** *For each $a > 0$ and $\phi(t) \in \Lambda_\delta^a$, we have*

$$\int_a^\infty e^{\phi^{\frac{n}{n-1}}(t) - t} dt \leq \frac{e^{\phi^{\frac{n}{n-1}}(a) - a}}{1 - \delta^{\frac{1}{n-1}}} e^{\frac{c^n}{n}(\frac{n-1}{n})^{n-1}\beta_n} e^{1 + \frac{1}{2} + \cdots + \frac{1}{n-1}}, \tag{25}$$

where $\beta_n = \delta(1 - \delta^{\frac{1}{n-1}})^{-n+1}$ and $c = \frac{n}{n-1}\phi^{\frac{1}{n-1}}(a)$. The inequality tends to an equality if $c^n \beta_n \to \infty$, $a \to \infty$ and $\delta \to 0$.

Let $\tilde{f}_k(x) \in W^{1,n}_{0,rad}(\mathcal{W}_1, \omega_\beta)$ such that $\tilde{f}_k(x)$ concentrates at 0, that is, $\|\tilde{f}_k\|_{\omega_\beta} \leq 1$, $|F^n(\nabla \tilde{f}_k)|\omega_\beta \rightharpoonup \delta_0$. Define $f_k(t)$ from $\tilde{f}_k(x)$ by the same transformation as in (22). Then, since $\tilde{f}_k(x)$ concentrates at 0, we have that $\tilde{f}_k(x) \rightharpoonup 0$ in $W^{1,n}_{0,rad}(\mathcal{W}_1, \omega_\beta)$ and converges pointwise to 0.

Lemma 5. *Let $f_k(t)$ be as above. Then one of the following alternatives holds:*

(i) We can find points $a_k \in [1, \infty)$ such that

$$|f_k(a_k)|^{\frac{n}{(n-1)(1-\beta)}} - a_k = -2\log a_k; \tag{26}$$

(ii) If such a_k does not exist, then

$$\limsup_{k\to\infty} \int_0^\infty e^{|f_k(t)|^{\frac{n}{(n-1)(1-\beta)}} - t} dt = 1.$$

What is more, if the first alternative (i) holds, we can find a_k to be the first point in $[1, \infty)$ satisfying (26) and satisfying $a_k \to \infty$ as $k \to \infty$.

Proof. Since $|f_k(t)| \leq t^{\frac{(n-1)(1-\beta)}{n}}$, then if $t \in [0, 1)$, $|f_k(t)|^{\frac{n}{(n-1)(1-\beta)}} - t \leq 0$. However, if $t \in [0, 1)$, $-2\log t > 0$, then $|f_k(t)|^{\frac{n}{(n-1)(1-\beta)}} - t < -2\log t$, which implies that we cannot find a_k satisfying (26) in $[0, 1)$.

Now we assume (i) does not hold. Then we have $|f_k(t)|^{\frac{n}{(n-1)(1-\beta)}} - t < -2\log t$, $t \in [1, \infty)$. Furthermore, we have

$$e^{|f_k(t)|^{\frac{n}{(n-1)(1-\beta)}} - t} \leq t^{-2}, \text{ if } t \in [1, \infty).$$

Define the dominating function as follows:

$$h(t) = \begin{cases} 1, & t \in (0, 1), \\ \frac{1}{t^2}, & t \in [1, \infty). \end{cases}$$

Then, by the dominated convergence theorem, we obtain that $I_\beta(f_k) \to 1$.

Let (i) hold. We choose the first $a_k \geq 1$ satisfying (26). We now prove that $a_k \to \infty$ as $k \to \infty$. For any large number M, we need to prove that there exist $k_0 \in \mathbb{N}$, such that for any $k \geq k_0$, $a_k \geq M$. Firstly, we choose μ small, such that

$$\mu t < -2\log t + t, t \in [0, M).$$

Now, since $\tilde{f}_k$ concentrates, we have for $t \in [0, M)$ and any $k \geq k_0$,

$$|f_k(t)|^{\frac{n}{(n-1)(1-\beta)}} \leq \left(\int_0^M \frac{|f_k'|^n t^{\beta(n-1)}}{(1-\beta)^{n-1}} dt\right)^{\frac{1}{(n-1)(1-\beta)}} t < \mu t \leq t - 2\log t.$$

Then we obtain for any $k \geq k_0$, $a_k \geq M$. $\square$

Now we define the concentration level at 0,

$$J^\delta_{\beta,\omega_\beta}(0) = \sup_{\tilde{f}_k \in W^{1,n}_{0,rad}(\mathcal{W}_1, \omega_\beta)} \{\limsup_{k\to\infty} J_\beta(\tilde{f}_k) | \quad F^n(\nabla \tilde{f}_k)\omega_\beta \rightharpoonup \delta_0\}.$$

We can give the estimate for the concentration level.

Lemma 6. *For $\beta \in [0, 1)$, we have that*

$$J^\delta_{\beta,\omega_\beta}(0) \leq 1 + e^{1+\frac{1}{2}+\cdots+\frac{1}{n-1}}. \tag{27}$$

Proof. To prove the lemma, it is sufficient to assume the sequences $\tilde{f}_k$ satisfy the first alternative in Lemma 5, because if $\tilde{f}_k$ satisfy the second alternative, we can obtain the inequality (27) by Lemma 5.

Firstly, we show that

$$\lim_{k\to\infty}\int_0^{a_k} e^{|f_k(t)|^{\frac{n}{(n-1)(1-\beta)}}-t}=1,$$

where f_k and a_k are as in Lemma 5. Since $F^n(\nabla\tilde{f}_k)\omega_\beta\rightharpoonup 0$ and (21), we have that $f_k\to 0$ uniformly on compact subsets of $\mathbb{R}^+$. Then for any ϵ, $A>0$, we obtain $|f_k(t)|^{\frac{n}{(n-1)(1-\beta)}}\le\epsilon$ for $t\le A$ and k large enough. By the property of a_k, that is, for $t\le a_k$, $|f_k(t)|^{\frac{n}{(n-1)(1-\beta)}}\le t-2\log t$, we obtain

$$\begin{aligned}\int_0^{a_k} e^{|f_k(t)|^{\frac{n}{(n-1)(1-\beta)}}-t}dt &= \int_0^{A} e^{|f_k(t)|^{\frac{n}{(n-1)(1-\beta)}}-t}dt+\int_{a_k}^{A} e^{|f_k(t)|^{\frac{n}{(n-1)(1-\beta)}}-t}dt\\ &\le e^{\epsilon}\int_0^A e^{-t}dt+\int_A^{a_k}e^{-2\log t}dt\\ &= e^{\epsilon}(1-e^{-A})+(\frac{1}{A}-\frac{1}{a_k}).\end{aligned}$$

Therefore,

$$\limsup_{k\to\infty}\int_0^{a_k} e^{|f_k(t)|^{\frac{n}{(n-1)(1-\beta)}}-t}dt\le e^{\epsilon}(1-e^{-A})+\frac{1}{A}.$$

Now, as $\epsilon\to 0$ and $A\to\infty$, we have

$$\limsup_{k\to\infty}\int_0^{a_k} e^{|f_k(t)|^{\frac{n}{(n-1)(1-\beta)}}-t}dt\le 1.$$

On the other hand,

$$\limsup_{k\to\infty}\int_0^{a_k} e^{|f_k(t)|^{\frac{n}{(n-1)(1-\beta)}}-t}dt\ge\int_0^{a_k}e^{-t}dt=1-e^{-a_k}\to 1.$$

Next, we prove that

$$\lim_{k\to\infty}\int_{a_k}^{\infty} e^{|f_k(t)|^{\frac{n}{(n-1)(1-\beta)}}-t}\le e^{1+\frac{1}{2}+\cdots+\frac{1}{n-1}}.$$

Set $\delta_k=\int_{a_k}^{\infty}\frac{|f_k'|^n t^{\beta(n-1)}}{(1-\beta)^{n-1}}dt$. Then, by (21) with $A=0$ and $t=a_k$, we have

$$\begin{aligned}\delta_k=1-\int_0^{a_k}\frac{|f_k'|^n t^{\beta(n-1)}}{(1-\beta)^{n-1}}dt &\le 1-(\frac{|f_k(a_k)|^{\frac{n}{(n-1)(1-\beta)}}}{a_k})^{(1-\beta)(n-1)}\\ &= 1-(1-\frac{2\log a_k}{a_k})^{(1-\beta)(n-1)}.\end{aligned}\tag{28}$$

Define the function $g_k(t)=|f_k(t)|^{\frac{1}{1-\beta}}$. Then

$$\int_{a_k}^{\infty} e^{|f_k(t)|^{\frac{n}{(n-1)(1-\beta)}}-t}dt=\int_{a_k}^{\infty}e^{g_k(t)^{\frac{n}{n-1}}-t}dt.\tag{29}$$

By $|f_k^{\frac{n}{(n-1)(1-\beta)}}(t)| \le t$, we have

$$\begin{aligned} \int_{a_k}^{\infty} |g_k'|^n dt &= \frac{1}{(1-\beta)^n} \int_{a_k}^{\infty} f_k^{\frac{n\beta}{1-\beta}} |f_k'|^n dt \\ &\le \frac{1}{(1-\beta)^n} \int_{a_k}^{\infty} t^{(n-1)\beta} |f_k'|^n dt \\ &\le \frac{\delta_k}{1-\beta} := \delta_k^* \to 0. \end{aligned} \tag{30}$$

Now, applying Lemma 4 with $\delta = \delta_k^*$ and $a = a_k$, we obtain

$$\begin{aligned} &\int_{a_k}^{\infty} e^{|f_k(t)|^{\frac{n}{(n-1)(1-\beta)}} - t} dt \\ \le\ &\frac{e^{g_k(a_k)^{\frac{n}{n-1}} - a_k}}{1 - |\delta_k^*|^{\frac{1}{n-1}}} e^{\frac{c^n}{n}(\frac{n-1}{n})^{n-1}\beta_n + 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}}, \end{aligned} \tag{31}$$

where $\beta_n = \delta_k^*(1 - |\delta_k^*|^{\frac{1}{n-1}})^{-n+1}$ and $c = \frac{n}{n-1} g_k(a_k)^{\frac{1}{n-1}}$. Therefore, it is left to show that

$$\limsup_{k\to\infty} G_k := \limsup_{k\to\infty} [g_k(a_k)^{\frac{n}{n-1}} - a_k + \frac{g_k(a_k)^{\frac{n}{n-1}} \delta_k^*}{(n-1)(1 - |\delta_k^*|^{\frac{1}{n-1}})^{n-1}}] \le 0.$$

We split G_k as follows:

$$\begin{aligned} G_k &= -2\log a_k + \frac{(a_k - 2\log a_k)\delta_k}{(n-1)(1-\beta)(1-|\delta_k^*|^{\frac{1}{n-1}})^{n-1}} \\ &= -2\log a_k + \frac{a_k \delta_k}{(n-1)(1-\beta)(1-|\delta_k^*|^{\frac{1}{n-1}})^{n-1}} - \frac{2(\log a_k)\delta_k}{(n-1)(1-\beta)(1-|\delta_k^*|^{\frac{1}{n-1}})^{n-1}} \\ &= \{-2\log a_k + \frac{a_k\delta_k}{(n-1)(1-\beta)}\} + \frac{a_k \delta_k (1 - (1-|\delta_k^*|^{\frac{1}{n-1}})^{n-1})}{(n-1)(1-\beta)(1-|\delta_k^*|^{\frac{1}{n-1}})^{n-1}} \\ &\quad - \frac{2(\log a_k)\delta_k}{(n-1)(1-\beta)(1-|\delta_k^*|^{\frac{1}{n-1}})^{n-1}} \\ &=: I_1^k + I_2^k - I_3^k. \end{aligned} \tag{32}$$

We make use of the Maclaurin series expansion. Firstly,

$$\begin{aligned} \delta_k &= 1 - (-\frac{2\log a_k}{a_k} + 1)^{(1-\beta)(n-1)} \\ &= (1-\beta)(n-1)\frac{2\log a_k}{a_k} + C(\frac{2\log a_k}{a_k})^2 + o((\frac{\log a_k}{a_k})^2), \end{aligned} \tag{33}$$

for some positive constant C, which depends only on β, n. Thus, we have

$$|I_1^k| = 4C\frac{(\log a_k)^2}{a_k} + a_k o((\frac{\log a_k}{a_k})^2) \to 0, \text{ as } k \to \infty.$$

Also,

$$|I_3^k| \le C_1 \frac{(\log a_k)^2}{a_k} + C_2 \frac{(\log a_k)^3}{a_k^2} + (\log a_k) o((\frac{\log a_k}{a_k})^2) \to 0, \text{ as } k \to \infty.$$

To estimate I_2^k, we first use the binomial expansion of $(1 - |\delta_k^*|^{\frac{1}{n-1}})^{n-1}$ to obtain $|I_2^k| \le C a_k \delta_k |\delta_k^*|^{\frac{1}{n-1}}$. Now, using (30) and (33), we obtain

$$|I_2^k| \le C\frac{(\log a_k)^{\frac{n}{n-1}}}{a_k^{\frac{1}{n-1}}} \to 0, \text{ as } k \to \infty.$$

Then we have completed the proof of the Lemma. □

Proof of Theorem 2. We assume $J_\beta(\tilde{g}_k)$ does not converge to $J_\beta(\tilde{g}_0)$, where $\tilde{g}_k$, $\tilde{g}$ is as in (20). Thus, by Lemma 6, we obtain

$$M_\beta = \lim_{k\to\infty} J_\beta(\tilde{g}_k) \le 1 + e^{1+\frac{1}{2}+\cdots+\frac{1}{n-1}}.$$

If we can find some $\phi \in \tilde{\Lambda}_1$ such that

$$I_\beta(\phi) > 1 + e^{1+\frac{1}{2}+\cdots+\frac{1}{n-1}},$$

then clearly $Q_\beta > 1 + e^{1+\frac{1}{2}+\cdots+\frac{1}{n-1}}$ and thus, we obtain a contradiction.

Consider the function $h_n(t)$ as follows:

$$h_n(t) = \begin{cases} (1-\frac{1}{n})(n-1)^{-\frac{1}{n}}t, & 0 \le t \le n, \\ (t-1)^{1-\frac{1}{n}}, & n \le t \le T_n, \\ (T_n-1)^{1-\frac{1}{n}}, & t \ge T_n, \end{cases}$$

where $T_n = (n-1)e^{(\frac{n}{n-1})^n - \frac{n}{n-1}} + 1$. It has been proved in [29] that $\int_0^\infty |h_n'|^n dt \le 1$ and

$$\begin{aligned} \int_0^\infty e^{h_n(t)^{\frac{n}{n-1}}-t}dt &= 1 + e^{1+\frac{1}{2}+\cdots+\frac{1}{n-1}} + \gamma^*(n) \\ &> 1 + e^{1+\frac{1}{2}+\cdots+\frac{1}{n-1}}. \end{aligned} \tag{34}$$

Set $\phi_n^\alpha(t) = [\alpha h_n(t)]^{1-\beta}$ for $\alpha \in (0,1)$. Then

$$\begin{aligned} I_\beta(\phi_n^\alpha) &= \int_0^\infty e^{[\alpha h_n(t)]^{\frac{n}{n-1}}-t}dt \\ &= \int_0^\infty e^{(\alpha^{\frac{n}{n-1}}-1+1)h_n(t)^{\frac{n}{n-1}}-t}dt \\ &\ge e^{(\alpha^{\frac{n}{n-1}}-1)\|h_n\|_\infty}(1 + e^{1+\frac{1}{2}+\cdots+\frac{1}{n-1}} + \gamma^*(n)). \end{aligned}$$

Now we can choose $\alpha = \alpha_*$ sufficiently close to 1 such that $I_\beta(\phi_n^\alpha) > 1 + e^{1+\frac{1}{2}+\cdots+\frac{1}{n-1}}$. Let us estimate the term $\Gamma(\phi_n^{\alpha_*})$. Since $(\phi_n^{\alpha_*})' = 0$ for $t \ge T_n$, we have

$$\begin{aligned} \Gamma(\phi_n^{\alpha_*}) &= \int_0^{+\infty} \frac{|(\phi_n^{\alpha_*})'|^n t^{\beta(n-1)}}{(1-\beta)^{n-1}}dt \\ &= \int_0^{n} \frac{|(\phi_n^{\alpha_*})'|^n t^{\beta(n-1)}}{(1-\beta)^{n-1}}dt + \int_n^{T_n} \frac{|(\phi_n^{\alpha_*})'|^n t^{\beta(n-1)}}{(1-\beta)^{n-1}}dt \\ &= I_1(\beta) + I_2(\beta). \end{aligned} \tag{35}$$

Now, by direct calculation, we obtain

$$\begin{aligned} I_1(\beta) &= (1-\beta)\alpha_*^{(1-\beta)n} \int_0^n h_n^{-n\beta}|h_n'|^n t^{\beta(n-1)}dt \\ &= \alpha_*^{(1-\beta)n}(1-\beta)\int_0^n (1-\frac{1}{n})^{n(1-\beta)}(n-1)^{\beta-1}t^{-\beta} \\ &= \alpha_*^{(1-\beta)n}(1-\frac{1}{n})^{n(1-\beta)}(n-1)^{\beta-1}n^{1-\beta} \\ &= \alpha_*^{(1-\beta)n}(1-\frac{1}{n})^{(n-1)(1-\beta)} \end{aligned}$$

and

$$\begin{aligned}
I_2(\beta) &= \alpha_*^{(1-\beta)n}(1-\beta)\int_n^{T_n} h_n^{-n\beta}|h_n'|^n t^{\beta(n-1)}dt \\
&= \alpha_*^{(1-\beta)n}(1-\beta)(1-\frac{1}{n})^n\int_{n-1}^{T_n-1}\frac{1}{s}(1+\frac{1}{s})^{\beta(n-1)}ds \\
&\le \alpha_*^{(1-\beta)n}(1-\beta)(1-\frac{1}{n})^n(1+\frac{1}{n-1})^{\beta(n-1)}\int_{n-1}^{T_n-1}\frac{1}{s}ds \\
&\le \alpha_*^{(1-\beta)n}(1-\beta)(1-\frac{1}{n})^n(\frac{n}{n-1})^{\beta(n-1)}[(\frac{n}{n-1})^n-\frac{n}{n-1}] \\
&= \alpha_*^{(1-\beta)n}(1-\beta)(1-\frac{1}{n})^{n-1}(\frac{n}{n-1})^{\beta(n-1)}B_n,
\end{aligned}$$

where $B_n = (\frac{n}{n-1})^{n-1} - 1$. Note that by the above estimates, we have

$$I_1(0) + I_2(0) \le \alpha_*^n < 1.$$

Thus, we can choose $\beta = \beta_*$, depending only on n, such that $I_1(\beta_*) + I_2(\beta_*) \le 1$. Thus, we have finished the proof of the Theorem. □

5. Improvement of the Anisotropic Moser–Trudinger Inequality

In this section, we complete the proof of Theorem 3, which can be seen as an improvement of the anisotropic Moser–Trudinger inequality when $u_k \rightharpoonup u_0$.

Proof of Theorem 3. If $u_0 \equiv 0$, then we can directly obtain (7) by Theorem 1. Thus, it is left to consider the case $u_0 \not\equiv 0$. By (16), we have that $u_k \to u_0$ uniformly on $\mathcal{W}_1 \setminus \mathcal{W}_r$, $\forall r \in (0,1)$. Then, by (17) and dominated convergence theorem, we have $u_k \to u_0$ in $L^q(\mathcal{W}_1)$ for any $q < \infty$.

For any $R > 0$, $k \in \mathbb{N}$, we define the functions

$$v_{R,k} = \min\{|u_k|, L\}sign(u_k) \qquad \text{and} \qquad w_{R,k} = u_k - v_{R,k}.$$

Since $\lim\limits_{R\to\infty}\|v_{R,0}\|_{\omega_\beta}^n = \|u_0\|_{\omega_\beta}^n$, for $\forall p < p(u_0)$, then there exist R large enough such that

$$p_0 := p(1 - \|v_{R,0}\|_{\omega_\beta}^n)^{\frac{1}{(n-1)(1-\beta)}} < 1.$$

Since $v_{R,k} \to v_{R,0}$ a.e. in $\mathcal{W}_1$ as $k \to \infty$ and $v_{R,k}$ is bounded in $W_{0,rad}^{1,n}(\mathcal{W}_1, \omega_\beta)$, up to a subsequence, we can assume that $v_{R,k} \rightharpoonup v_{R,0}$ weakly in $W_{0,rad}^{1,n}(\mathcal{W}_1, \omega_\beta)$. Then we have

$$\liminf_{k\to\infty}\|v_{R,k}\|_{\omega_\beta}^n \ge \|v_{R,0}\|_{\omega_\beta}^n$$

and

$$\limsup_{k\to\infty}\|w_{R,k}\|_{\omega_\beta}^n = 1 - \liminf_{k\to\infty}\|v_{R,k}\|_{\omega_\beta}^n \le 1 - \|v_{R,0}\|_{\omega_\beta}^n.$$

Then we can find $k_0 \in \mathbb{N}$ such that for $\forall k \ge k_0$, we have

$$p\|w_{R,k}\|_{\omega_\beta}^{\frac{n}{(n-1)(1-\beta)}} \le \frac{p_0+1}{2} < 1. \tag{36}$$

Using $u_k = w_{R,k} + v_{R,k}$ and $|v_{R,k}| \le R$, we obtain

$$|u_k|^{\frac{n}{(n-1)(1-\beta)}} \le (1+\epsilon)|w_{R,k}|^{\frac{n}{(n-1)(1-\beta)}} + C(n,\beta,\epsilon)R^{\frac{n}{(n-1)(1-\beta)}},$$

where

$$C(n,\beta,\epsilon) = (1-(1+\epsilon)^{-\frac{(n-1)(1-\beta)}{n\beta+1-\beta}})^{-\frac{n\beta+1-\beta}{(n-1)(1-\beta)}}. \tag{37}$$

Now we choose $\epsilon > 0$ such that $\frac{(1+\epsilon)(1+p_0)}{2} < 1$. By (36), we have

$$\begin{aligned} & \int_{\mathcal{W}_1} e^{p\lambda_{\beta,n}|u_k|^{\frac{n}{(n-1)(1-\beta)}}} dx && (38) \\ \leq & \int_{\mathcal{W}_1} e^{p\lambda_{\beta,n}(1+\epsilon)|w_{R,k}|^{\frac{n}{(n-1)(1-\beta)}} + p\lambda_{\beta,n}C(n,\beta,\epsilon)R^{\frac{n}{(n-1)(1-\beta)}}} dx \\ \leq & \ C\int_{\mathcal{W}_1} e^{p\lambda_{\beta,n}(1+\epsilon)\|w_{R,k}\|_{\omega_\beta}^{\frac{n}{(n-1)(1-\beta)}} |\frac{w_{R,k}}{\|w_{R,k}\|_{\omega_\beta}}|^{\frac{n}{(n-1)(1-\beta)}}} dx \\ \leq & \ C\int_{\mathcal{W}_1} e^{\lambda_{\beta,n}(1+\epsilon)\frac{(1+\epsilon)(1+p_0)}{2}|\frac{w_{R,k}}{\|w_{R,k}\|_{\omega_\beta}}|^{\frac{n}{(n-1)(1-\beta)}}} dx && (39) \end{aligned}$$

for any $k \geq k_0$, where C depends only on n, β, ϵ, p and R. Combining (38) with Theorem 1 and the choice of ϵ, we obtain (7). The proof is completed. □

6. Conclusions

In this paper, we mainly study the anisotropic Moser–Trudinger-type inequality for radical Sobolev space with logarithmic weight $\omega_\beta(x) = [-\ln F^o(x)]^{\beta(n-1)}$, $\beta \in [0,1)$. Moreover, we obtain the existence of an extremal function when β is small. The extremal function is densely related to the existence of solutions of Finsler–Liouville-type equations. Finally, we obtain the Lions-type concentration-compactness principle, which is the improvement of an anisotropic Moser–Trudinger-type inequality. However, we note that the singular anisotropic Moser–Trudinger-type inequality with logarithmic weight in a unit Wulff ball $\mathcal{W}_1$ and the anisotropic Moser–Trudinger-type inequality with logarithmic weight in $\mathbb{R}^n$ are still open questions.

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