

Article

Non-Newtonian Pressure-Governed Rivulet Flows on Inclined Surface

Sergey V. Ershkov ^{1,*}  and Dmytro D. Leshchenko ² ¹ Department of Scientific Researches, Plekhanov Russian University of Economics, Scopus Number 60030998, 36 Stremyanny Lane, 117997 Moscow, Russia² Odessa State Academy of Civil Engineering and Architecture, 65029 Odessa, Ukraine; leshchenkodmytro@gmail.com

* Correspondence: sergej-ershkov@yandex.ru

Abstract: We have generalized, in the current study, the results of research presented earlier with the aim of obtaining an approximate solution for the creeping, plane-parallel flow of viscoplastic non-Newtonian fluid where the focus is on the study of rivulet fluid flows on an inclined surface. Namely, profiles of velocity of flow have been considered to be given in the same form as previously (i.e., Gaussian-like, non-stationary solutions) but with a novel type of pressure field p . The latter has been chosen for solutions correlated explicitly with the critical maximal *non-zero* level of stress τ_s in the shared plane layer of rivulet flow, when it begins to move as viscous flow (therefore, we have considered here the purely non-Newtonian case of viscoplastic flow). Correlating phenomena such as the above stem from the equations of motion of viscoplastic non-Newtonian fluid considered along with the continuity equation. We have obtained a governing sub-system of two partial differential equations of the first order for *two* functions, p and τ_s . As a result, a set of new semi-analytical solutions are presented and graphically plotted.

Keywords: rivulet flow; non-Newtonian fluid; creeping viscoplastic flow**MSC:** 35Q35; 76D17**Citation:** Ershkov, S.V.;Leshchenko, D.D. Non-Newtonian Pressure-Governed Rivulet Flows on Inclined Surface. *Mathematics* **2024**, *12*, 779. <https://doi.org/10.3390/math12050779>

Academic Editor: Vasily Novozhilov

Received: 10 February 2024

Revised: 29 February 2024

Accepted: 4 March 2024

Published: 6 March 2024



Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction, the Basic Model

During field observations and experimental studies of the motions of Newtonian and non-Newtonian fluids, flows in the form of rivulets are often recorded [1–7]. Rivulet flows can manifest themselves both as an independent (main) hydrodynamical flow and as a secondary flow due to the loss of hydrodynamic stability [7–15].

The simplest rivulet currents obviously include the classical Couette, Poiseuille, and Nusselt flows. These flows of a viscous incompressible Newtonian fluid are described by exact solutions of the Navier–Stokes equations. One review announced a generalization of the classical exact solutions of the Newtonian hydrodynamic equations from one-dimensional flows to three-dimensional fluid flows [16]. Even in this Newtonian case, finding exact or semi-analytical solutions is a difficult task.

When the non-Newtonian rheology of the liquid is taken into account, non-linear properties begin to appear, which are caused by the properties of the material. In non-Newtonian fluids, viscosity dependent in general case on strain rate is known to be changed due to the non-linear dynamical properties of the medium inside the fluid's layer, which experiences a transition phase becoming either more liquid or more solidified. In this case, the hydrodynamical equations become even more difficult to solve analytically. The complexity of integrating the equations of motion with partial derivatives can explain the rather small number of scientific papers with analytical approach describing the rivulet flows of non-Newtonian fluids.

In the current research, a particular class of semi-analytical solutions is proposed, allowing a study of the structure of hydrodynamic fields for plane-parallel rivulet flows, taking into account gravity (and additionally the dynamic Marangoni effect [17] if required in a particular engineering problem). This made it possible to identify new directions for future research. In this article, plane-parallel flows with a non-zero threshold for plasticity are investigated. The flow of a non-Newtonian fluid is realized on an inclined surface with a change in pressure field. A family of velocities is used in a form of Gaussian-type functions for each component [17] but with a much larger choice of analytical and semi-analytical solutions for the pressure field. Thus, this article provides a generalization of flow of a non-Newtonian fluid of the Poiseuille and Nusselt types for a two-dimensional velocity field.

According to [17–20], the system of Equations (1)–(3) that governs the dynamics for a particular family of viscoplastic rivulet flows (in general, a Bingham fluid if plasticity effect of flow is also taken into account) should be presented in a 2D case in the Cartesian coordinates, according to the approach suggested earlier in [17–19].

We will investigate here the novel family of solutions where non-zero plasticity effect for the flow field is also taken into account (more general solutions than those investigated previously in [17,19,20] for the case of absence of plasticity effect). This implies that we will consider the profiles of flow velocity in Equations (1)–(3), which are given by formulae suggested in [19] for solutions (here and below we have chosen $\left(\frac{8\mu}{\rho}\right) = 1$ in appropriate units of scaling in the physical sense), but with a novel type of pressure field p chosen for such solutions

$$\begin{cases} v_x = \frac{\left(\frac{8\mu}{\rho}\right)}{\sqrt{4\pi\left(\frac{2\mu}{\rho}\right)t}} \exp\left(-\frac{y^2}{4\left(\frac{2\mu}{\rho}\right)t}\right), \\ v_y = \frac{\left(\frac{8\mu}{\rho}\right)}{\sqrt{4\pi\left(\frac{2\mu}{\rho}\right)t}} \exp\left(-\frac{x^2}{4\left(\frac{2\mu}{\rho}\right)t}\right), \end{cases} \quad (1)$$

where v_x is the component of rivulet velocity in the Ox -direction; v_y is the component of rivulet velocity in the Oy -direction; μ is the coefficient of rivulet flow's dynamic viscosity (which is supposed to be locally constant); and ρ is the constant density of rivulet flow.

In addition, let us clarify that in the current study, we will consider (as previously in [17]) only the Cauchy problem in the whole space for which we suggest a Gaussian-like profile for both velocity components (1) (which are proved in [19] to satisfy the continuity equation). This is with the aim to retrieve the internal pressure field p (inside the rivulet flow) from Equations (2) and (3), as well as with the aim to retrieve from these aforementioned equations the variable *but non-zero* function for the level of flow plasticity τ_s ($\tau_s \neq 0$). This would definitely mean that we should investigate a purely non-Newtonian case of viscoplastic flow in our semi-analytical research.

Let us note, as previously in [19], that axis Ox coincides with the initial main direction of slowly moving advancing the front of flow on a surface which is inclined on angle α with respect to the horizontal plane. Internal dynamics of a rivulet is assumed to correspond to the simple case of plane-parallel flow, with absence of local intermixing of plane layers of a rivulet's flow (see [17–19]):

$$\begin{aligned} \rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right) &= -\frac{\partial p}{\partial x} + \frac{\partial s_{xx}}{\partial x} + \frac{\partial s_{xy}}{\partial y} + \rho g_x, \\ \rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} \right) &= -\frac{\partial p}{\partial y} + \frac{\partial s_{xy}}{\partial x} - \frac{\partial s_{xx}}{\partial y} + \rho g_y, \\ \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} &= 0, \quad U = \sqrt{4\left(\frac{\partial v_x}{\partial x}\right)^2 + \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)^2}, \end{aligned} \quad (2)$$

$$s_{xx} = 2\left(\mu + \frac{\tau_s}{U}\right) \frac{\partial v_x}{\partial x}, \quad s_{xy} = 2\left(\mu + \frac{\tau_s}{U}\right) \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right),$$

$$\begin{cases} U = \frac{1}{\mu} \left(\sqrt{s_{xx}^2 + \frac{s_{xy}^2}{4}} - \tau_s \right), \\ \frac{\partial v_x}{\partial x} = s_{xx} / 2\left(\mu + \frac{\tau_s}{U}\right), \end{cases} \quad (3)$$

where (according to the designations in work [17]) s_{xx} , s_{xy} are the appropriate components of the stress tensor (due to the specificity of this type of plane-parallel flow [18], $s_{yx} = s_{xy}$, $s_{yy} = -s_{xx}$); $\{\rho g_x, \rho g_y\}$ are volumetric components of the gravity field, where $\{g_x, g_y\} = \{g \sin \alpha, 0\}$; τ_s is a critical maximal level of stress in the shared plane layer of rivulet flow when it begins to move as viscous flow (we will consider here and below only the case of *non-zero* plasticity $\tau_s \neq 0$, i.e., the purely non-Newtonian case of viscoplastic flow).

2. Semi-Analytical Solving Procedure for Approximation of Creeping Flow

Let us consider the creeping but non-stationary [21–24] approximation of the hydrodynamical Equations (2) and (3) for 2D rivulet flow (1), according to the approach suggested earlier in [20]. Such essential simplification reduces convective terms in the left part of (2) to be approximately equal to zero due to the presence of negligible terms (for slowly moving rivulet flow) with respect to the right part of momentum Equation (2) (mainly, to the terms, associated with viscous forces). Indeed, we then obtain Equations (4) and (5) presented below.

It is worth noting that components of velocity (1) have been chosen, according to the results of [17], to satisfy Equations (4)–(6):

$$\begin{aligned} \rho \frac{\partial v_x}{\partial t} &= -\frac{\partial p}{\partial x} + \frac{\partial s_{xx}}{\partial x} + \frac{\partial s_{xy}}{\partial y} + \rho g \sin \alpha, \\ \rho \frac{\partial v_y}{\partial t} &= -\frac{\partial p}{\partial y} + \frac{\partial s_{xy}}{\partial x} - \frac{\partial s_{xx}}{\partial y}, \\ \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} &= 0, \end{aligned} \quad (4)$$

$$\begin{cases} s_{xx} = 2\left(\mu + \frac{\tau_s}{U}\right) \frac{\partial v_x}{\partial x} = 0, \\ s_{xy} = 2\left(\mu + \frac{\tau_s}{U}\right) \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) = 2\mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) + 2\tau_s, \\ U = \sqrt{4\left(\frac{\partial v_x}{\partial x}\right)^2 + \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)^2} = \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right), \end{cases} \quad (5)$$

$$\begin{cases} \rho \frac{\partial v_x}{\partial t} = 2\mu \frac{\partial^2 v_x}{\partial y^2}, \\ \rho \frac{\partial v_y}{\partial t} = 2\mu \frac{\partial^2 v_y}{\partial x^2}, \\ \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0, \end{cases} \Rightarrow \begin{cases} v_x = \frac{1}{\sqrt{4\pi \left(\frac{2\mu}{\rho}\right) t}} \exp\left(-\frac{y^2}{4 \left(\frac{2\mu}{\rho}\right) t}\right), \\ v_y = \frac{1}{\sqrt{4\pi \left(\frac{2\mu}{\rho}\right) t}} \exp\left(-\frac{x^2}{4 \left(\frac{2\mu}{\rho}\right) t}\right). \end{cases} \quad (6)$$

So, we can obtain the sub-system of two partial differential equations of the 1st order for *two* functions, p and τ_s , from the two first equations of system (4) (by gradually subtracting the appropriate parts of equations of system (6))

$$\begin{cases} 0 = -\frac{\partial p}{\partial x} + 2\frac{\partial \tau_s}{\partial y} + \rho g \sin \alpha, \\ 0 = -\frac{\partial p}{\partial y} + 2\frac{\partial \tau_s}{\partial x}, \end{cases} \quad (7)$$

which determines the dependence of a critical maximal level of stress in the shared plane layer of rivulet flow when it begins to move as viscous flow (*non-zero* plasticity) τ_s on pressure p or, more generally, that *one* depends on the previous *flow dynamics*.

For example, in a simple case, if p in (7) equals to hydrostatic pressure depending on height ($x \cdot \sin \alpha$) in the separate sufficiently thin layer of rivulet flow (i.e., $p = \rho g (x \sin \alpha)$ for the plane-parallel rivulet [17]), Equation (4) should be simplified to (8) below:

$$\begin{cases} \rho \frac{\partial v_x}{\partial t} = 2\mu \frac{\partial^2 v_x}{\partial y^2} + 2 \frac{\partial \tau_s}{\partial y}, \\ \rho \frac{\partial v_y}{\partial t} = 2\mu \frac{\partial^2 v_y}{\partial x^2} + 2 \frac{\partial \tau_s}{\partial x}. \end{cases} \quad (8)$$

This means, taking into account (6), that the approximation for the level of flow plasticity $\tau_s = \tau_s(t)$ ($\tau_s \neq 0$) comes into play when we choose the specific realistic dynamics representing the rivulet flow within its boundaries.

We should note that the 1st and 2nd equations of system (4) are supposed to have an appropriate solution, (1) or (6) (components of velocity v_x and v_y), with respect to which, nevertheless, the pressure field p and the level of flow plasticity τ_s should be correlated by additionally using the continuity equation as shown below

$$\left\{ \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \right\} \Rightarrow \quad (9)$$

$$0 = -\frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial y^2} + 2 \frac{\partial^2 (2\mu (\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}) + 2\tau_s)}{\partial x \partial y}, \Rightarrow \Delta p = 4 \frac{\partial^2 \tau_s}{\partial x \partial y}$$

To obtain (9), we should differentiate the 1st equation of system (4) or (7), with respect to coordinate x , and then the 2nd equation with respect to coordinate y . Then, afterwards, we should sum up these results of differentiation.

It should also be highlighted that solving the algorithm for (9) could be easier to run with the aim of obtaining the final solution for a realistic case of rivulet flow surface dynamics in engineering applications (with appropriate boundary conditions). However, untrue solutions may come through. For example, the pair of solutions $\{p = \rho g (x \sin \alpha), \tau_s = \tau_s(t)\}$ is still actual and valid in both cases (7) and (9). However, if we suggest $\tau_s = \tau_s(t)$ to be a part of solution in (9) to obtain the other part of the required solution for pressure field p , the untrue solution $p \sim \frac{1}{2\pi} \ln(\sqrt{x^2 + y^2})$ could be suggested to solve (9) (it satisfies both parts of (9) but would not satisfy both parts of (7) with a chosen function $\tau_s = \tau_s(t)$ for (7)). Hence, researchers should verify each solution of (9) to satisfy both parts of Equation (7), additionally.

Nevertheless, we can point out particular classes of solutions for function τ_s (10) which stem from (7) as well

$$0 = 2 \frac{\partial^2 \tau_s}{\partial y^2} - 2 \frac{\partial^2 \tau_s}{\partial x^2}, \quad (10)$$

where the latter equation is satisfied for the partial solutions (11) pointed out below (but is not limited to them):

$$\begin{aligned} (1) \quad \tau_s &\sim x^2 + y^2 &\Rightarrow p &\sim 4x \times y + \rho g x \sin \alpha; \\ (2) \quad \tau_s &\sim \sin x \times \sin y &\Rightarrow p &\sim -2 \cos x \times \cos y + \rho g x \sin \alpha; \\ (3) \quad \tau_s &\sim \cos x \times \cos y &\Rightarrow p &\sim -2 \sin x \times \sin y + \rho g x \sin \alpha. \end{aligned} \quad (11)$$

Let us specifically note that Equation (10) is known to have its general solution in a form (12) as follows [25]:

$$\tau_s = f(x+y) + h(x-y) \quad (12)$$

where $\{f, h\}$ are arbitrary functions.

It is worth noting that an additional class of solutions can be presented for system (7) with constant *Bernoulli*-function B (which can be associated with constant hydrodynamical head of rivulet flow) where *Bernoulli*-function B is given by the expression below:

$$B = \frac{1}{2} (\vec{v}^2) + p = \text{const} \Rightarrow p = B - \frac{1}{2} (\vec{v}^2), \quad \vec{v} = \{v_x, v_y\} \quad (13)$$

3. Discussion

Let us discuss a novel type of solution for the creeping, plane-parallel rivulet flow described by Equations (1)–(3) which generalizes the results of [17]. Namely, we have considered here profiles of velocity of flow in Equations (1)–(3), which are given below by formulae of a Gaussian type suggested in [19] for such solutions

$$\vec{v} = \{v_x, v_y\} = \left\{ \frac{\left(\frac{8\mu}{\rho}\right)}{\sqrt{4\pi\left(\frac{2\mu}{\rho}\right)t}} \exp\left(-\frac{y^2}{4\left(\frac{2\mu}{\rho}\right)t}\right), \frac{\left(\frac{8\mu}{\rho}\right)}{\sqrt{4\pi\left(\frac{2\mu}{\rho}\right)t}} \exp\left(-\frac{x^2}{4\left(\frac{2\mu}{\rho}\right)t}\right) \right\}$$

but with a novel type of pressure field p chosen for solutions correlated explicitly with the help of (7) or (9) with a critical maximal *non-zero* level of stress τ_s in the shared plane layer of rivulet flow, when it begins to move as viscous flow. Hence, we have considered here the purely non-Newtonian cases of viscoplastic flow where the critical maximal level of stress τ_s depends on the previous *flow* history in the sense of its dynamical characteristics. Namely, few partial types have been considered:

1. A simple case of hydrostatic pressure p depending on height ($x \sin \alpha$) in the separate sufficiently thin layer of rivulet flow (i.e., $p = \rho g(x \sin \alpha)$ for the plane-parallel rivulet), which corresponds via (7) to the level of flow plasticity $\tau_s = \tau_s(t)$ ($\tau_s \neq 0$);
2. Solutions (11) satisfying (7) and (10):

$$\begin{aligned} (1) \quad \tau_s &\sim x^2 + y^2 &\Rightarrow p &\sim 4x \times y + \rho g x \sin \alpha; \\ (2) \quad \tau_s &\sim \sin x \times \sin y &\Rightarrow p &\sim -2 \cos x \times \cos y + \rho g x \sin \alpha; \\ (3) \quad \tau_s &\sim \cos x \times \cos y &\Rightarrow p &\sim -2 \sin x \times \sin y + \rho g x \sin \alpha; \end{aligned}$$

3. Solutions with additional demand, still satisfying (7), but with constant *Bernoulli*-function B (which can be associated with the constant hydrodynamical head of rivulet flow) where *Bernoulli*-function B is given by the expression below:

$$p = B - \frac{1}{2}(\vec{v}^2), \quad B = \text{const}$$

$$\vec{v} = \left\{ \frac{\left(\frac{8\mu}{\rho}\right)}{\sqrt{4\pi\left(\frac{2\mu}{\rho}\right)t}} \exp\left(-\frac{y^2}{4\left(\frac{2\mu}{\rho}\right)t}\right), \frac{\left(\frac{8\mu}{\rho}\right)}{\sqrt{4\pi\left(\frac{2\mu}{\rho}\right)t}} \exp\left(-\frac{x^2}{4\left(\frac{2\mu}{\rho}\right)t}\right) \right\}$$

Finalizing the discussion, we should conclude that although a lot of exact solutions are known to be obtained in various hydrodynamical systems of equations [23,24], any new solution for resolving the aforementioned non-linear systems (4) and (5) of equations *videlicet* would be useful for practical application. As far as we know, general, analytical solutions do not exist for most cases of boundary or initial conditions for this system and the only way is to obtain some particular solutions (as, e.g., presented below in Figures 1–3 for pressure p corresponding to cases (11) pointed out above):

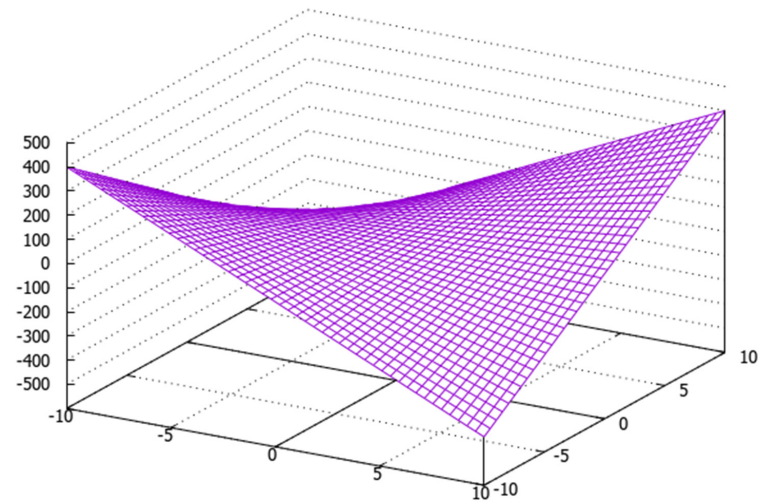


Figure 1. A schematic plot for pressure $p = 4xy + \rho g x \sin \alpha$ (meanings on vertical axis); meanings of $\{x, y\}$ are scaled for each coordinate with respect to the appropriate horizontal axis.

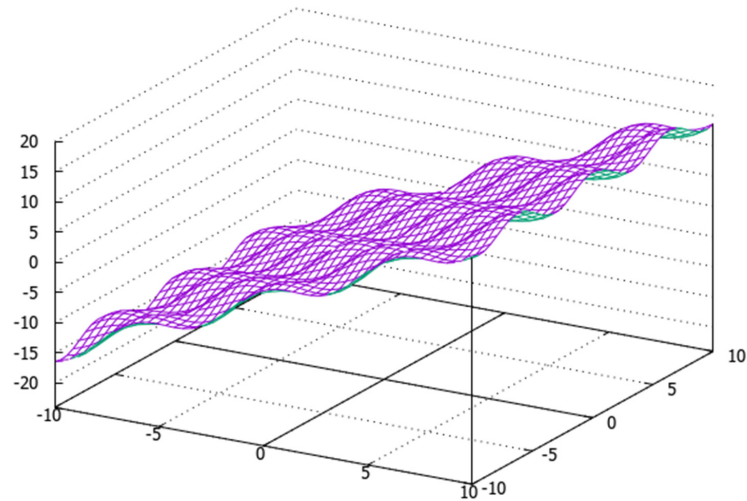


Figure 2. A schematic plot for pressure $p = -2\cos x \cos y + \rho g x \sin \alpha$ (meanings on vertical axis); meanings of $\{x, y\}$ are scaled for each coordinate with respect to the appropriate horizontal axis.

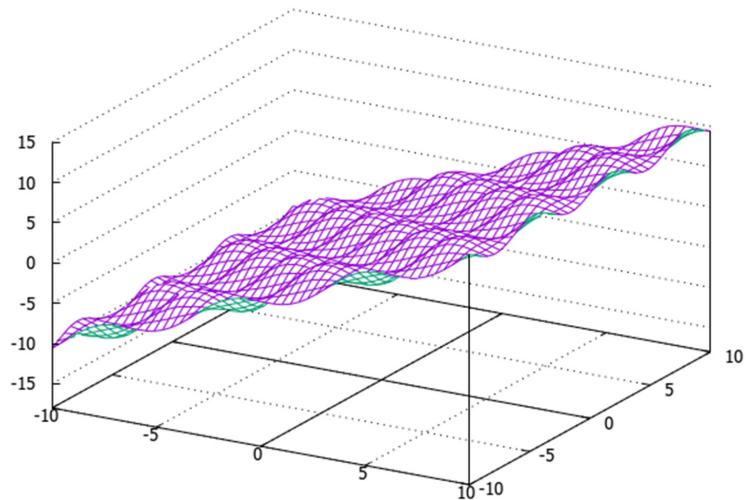


Figure 3. A schematic plot for pressure $p = -2\sin x \sin y + \rho g x \sin \alpha$ (meanings on vertical axis); meanings of $\{x, y\}$ are scaled for each coordinate with respect to the appropriate horizontal axis.

Regarding the critical maximal level of stress τ_s corresponding to cases (11), pointed out above, this is presented below in Figures 4–6:

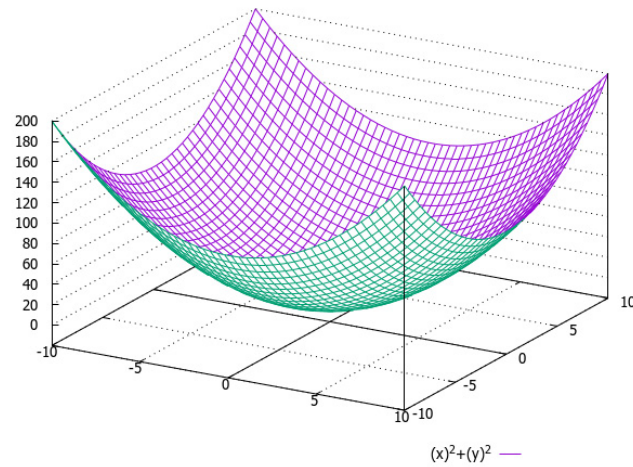


Figure 4. A schematic plot for the critical maximal level of stress $\tau_s = x^2 + y^2$ (meanings on the vertical axis); meanings of $\{x,y\}$ are scaled for each coordinate with respect to the appropriate horizontal axis.

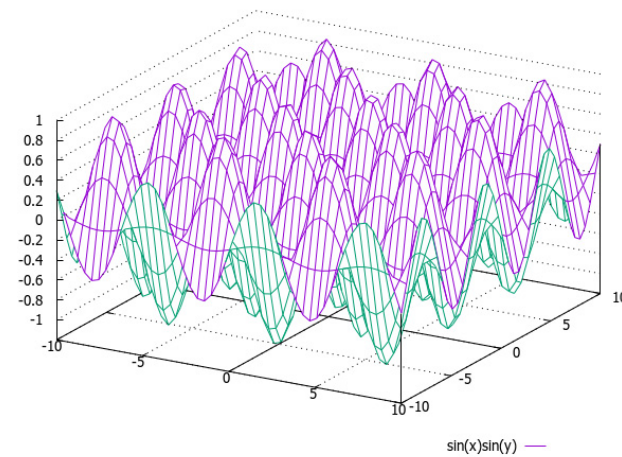


Figure 5. A schematic plot for the critical maximal level of stress $\tau_s = \sin x \sin y$ (meanings on the vertical axis); meanings of $\{x,y\}$ are scaled for each coordinate with respect to the appropriate horizontal axis.

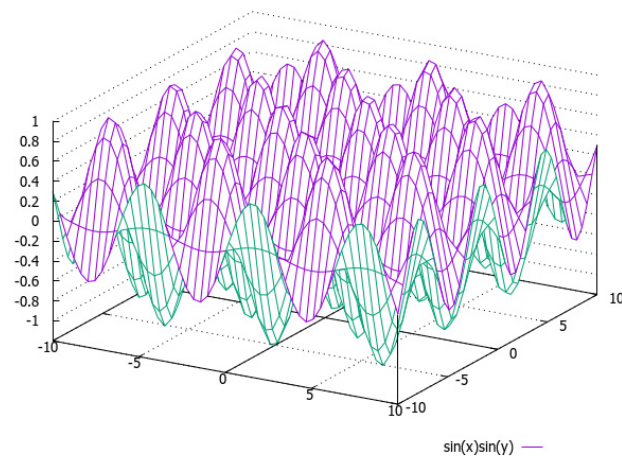


Figure 6. A schematic plot for the critical maximal level of stress $\tau_s = \cos x \cos y$ (meanings on the vertical axis); meanings of $\{x,y\}$ are scaled for each coordinate with respect to the appropriate horizontal axis.

It is worth noting that in the current study, we have generalized the results of research presented in [17]. Finally, let us outline restrictions on the range of applicability of the presented approach. Among these restricting assumptions are the constancy of the coefficient μ of rivulet flow's dynamic viscosity (which should in reality depend on heat exchange inside rivulet flow between separated layers). As for discussion of the boundary conditions at the free surface of the rivulet (a key feature of any rivulet flow), in the current research, we have considered only the Cauchy problem in the whole space. It should also be mentioned that the additional driving force of the rivulet flows is surface tension (particularly, a constant surface tension for the chosen type of solutions, as reported previously in [17]).

Also, studies [25–60] should be mentioned, where work [38] is perhaps the key paper on the rivulet flow of a viscoplastic fluid. Meanwhile, works [39–48] present a part of the contributions which stem from the large body of work on Newtonian and non-Newtonian rivulet flows carried out by Duffy and his research team in Scotland over the last couple of decades (not a complete list). In addition, handbook [25] concerns the problem under consideration, for which the theoretical basis has been described in detail, for solving a one-dimensional heat equation such as equations in (6) mentioned above. Works [61–64] describe hydrodynamical flows of incompressible couple stress or classical fluid with a constant or invariable-under-appropriate-assumptions Bernoulli function.

4. Conclusions

In the current study, we have generalized the results of research presented earlier (see [17,19,20]) but with a novel type of pressure field p chosen for solutions correlated explicitly with the critical maximal *non-zero* level of stress τ_s in the shared plane layer of rivulet flow (with help of (7) or (9)) when it begins to move as viscous flow. Therefore, we have considered here the purely non-Newtonian case of solution $\{p, \vec{v} = \{v_x, v_y\}\}$ of viscoplastic flow (1), which exactly satisfies the equations of motion (2) and (3) along with the continuity equation.

With the aim of obtaining an approximate solution for the creeping, plane-parallel flow of viscoplastic non-Newtonian fluid where the attention is focused on the study of rivulet fluid flows (4)–(5) on inclined surface, the profiles of velocity of flow have been considered to be given in the same form (1) as previously (i.e., Gaussian-like, non-stationary solutions).

These are proven to satisfy the continuity equation as well as momentum Equations (2) and (3). Profile of pressure has been chosen in a form (11) or (13) correlating to the obtained governing sub-system (7) (stemming from alma-mater system (2) and (3) or (4) and (5)) of two partial differential equations of the first order for *two* functions, p and τ_s . Thus, a novel class of semi-analytical solutions for the nonlinear system, describing the rivulet flow of viscoplastic fluid, is presented.

Let us additionally clarify that the presented profiles of velocity (1) and pressure (11) or (13) fully satisfy the equations of motion (1)–(3) or (4) and (5). These equations of motion have been formulated in book [18] earlier. Since our solutions stem from the comprehensive model presented in [18], well-known among specialists in hydrodynamics over a long period of time (one can see various types of flows presented there for different cases as well as initial or boundary conditions), they are obviously valid. To conclude, the validity is confirmed by a comprehensive model presented in [18].

5. Remarks (with Highlights)

- Semi-analytical ansatz is developed for modeling viscoplastic rivulet flows.
- The 2D creeping approximation for rivulet flow on inclined surface is considered.
- An analytical model is suggested for solving equations of momentum and continuity.
- A non-stationary solution to the system of PDEs for rivulet flow dynamics is obtained.
- Profiles of flow velocity have been considered to be Gaussian-like solutions.
- A non-zero critical maximal level of stress τ_s in the shared layer of rivulet flow is chosen.
- Pressure field p is correlated with critical maximal *non-zero* level of stress τ_s .
- Solutions satisfy gravity-driven rivulet flow, driven also by constant surface tension.

Author Contributions: In this research, S.V.E. was responsible for the general ansatz and the solving procedure, simple algebra manipulations, calculations, results of the article in Sections 1–3, and was also responsible for the search of approximate solutions. D.D.L. was responsible for theoretical investigations as well as for the deep survey of the literature on the problem under consideration. Both authors agreed with the results and conclusions of one another in Sections 1–3. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The data for this paper can be made available by contacting the corresponding author.

Acknowledgments: The authors are grateful to unknown esteemed reviewers for their valuable efforts and advice which have improved the structure of the article significantly. The authors are grateful to E.S. Baranovskii for his professional advice regarding the general solution of Equation (10). The authors appreciate the effort and time of E. Yu. Prosviryakov and his professional advice regarding the descriptions of the contributions of refs. [1–10] in the field of rivulet flow dynamics, formation, and stability.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Sinz, D.K.N.; Hanyak, M.; Zeegers, J.; Darhuber, A.A. Insoluble surfactant spreading along thin liquid films confined by chemical surface patterns. *Phys. Chem. Chem. Phys.* **2011**, *13*, 9768–9777. [CrossRef]
2. Darhuber, A.A.; Chen, J.Z.; Davis, J.M.; Troian, S.M. A Study of Mixing in Thermocapillary Flows on Micropatterned Surfaces, in Transport and Mixing at the Microscale. *Phil. Trans. R. Soc. Lond. A* **2004**, *362*, 1037–1058. [CrossRef]
3. Fraysse, N.; Homsy, G. An experimental study of rivulet instabilities in centrifugal spin coating of viscous Newtonian and non-Newtonian fluids. *Phys. Fluids* **1994**, *6*, 1491–1504. [CrossRef]
4. Kabov, O.A. Heat transfer from a small heater to a falling liquid film. *Heat Transf. Res.* **1996**, *27*, 221–226.
5. Kabov, O.A.; Chinnov, E.A. Heat transfer from a local heat source to a subcooled falling liquid film evaporating in a vapor-gas medium. *Russ. J. Engng Thermophys.* **1997**, *7*, 1–34.
6. Slade, D.; Veremieiev, S.; Lee, Y.C.; Gaskell, P.H. Gravity-driven thin film flow: The influence of topography and surface tension gradient on rivulet formation. *Chem. Eng. Process.* **2013**, *68*, 7–12. [CrossRef]
7. Scheid, B.; Oron, A.; Colinet, P.; Thiele, U.; Legros, J.C. Nonlinear evolution of nonuniformly heated falling liquid films. *Phys. Fluids* **2002**, *14*, 4130–4151. [CrossRef]
8. Cho, H.; Chou, F. Rivulet Instability with Effect of Coriolis Force. *J. Mech.* **2006**, *22*, 221–227. [CrossRef]
9. Brun, K.; Kurz, R. Analysis of Secondary Flows in Centrifugal Impellers. *Int. J. Rotating Mach.* **2005**, *2005*, 45–52. [CrossRef]
10. Kalliadasis, S.; Kiyashko, A.; Demekhin, E.A. Marangoni instability of a thin liquid film heated from below by a local heat source. *J. Fluid Mech.* **2003**, *475*, 377–408. [CrossRef]
11. Hooke, R.L. *Principles of Glacier Mechanics*, 2nd ed.; Cambridge University Press: Cambridge, UK; New York, NY, USA, 2005.
12. Paterson, W.S.B. *The Physics of Glaciers*, 3rd ed.; Pergamon Press: Oxford, UK, 1994.
13. Wu, Z.; Zhang, H.; Liu, S.; Ren, D.; Bai, X.; Xun, Z.; Ma, Z. Fluctuation analysis in the dynamic characteristics of continental glacier based on Full-Stokes model. *Sci. Rep.* **2019**, *9*, 20245. [CrossRef]
14. Slade, D.R.J. Gravity-Driven Thin Liquid Films: Rivulets and Flow Dynamics. Ph.D. Thesis, School of Mechanical Engineering, The University of Leeds, Leeds, UK, 2013. Available online: <https://core.ac.uk/download/pdf/20343955.pdf> (accessed on 10 February 2024).
15. Huppert, H.E. Flow and instability of a viscous current down a slope. *Nature* **1982**, *300*, 427–429. [CrossRef]
16. Towell, G.D.; Rothfeld, L.B. Hydrodynamics of rivulet flow. *AIChE J.* **1966**, *12*, 972–980. [CrossRef]
17. Ershkov, S.V.; Prosviryakov, E.Y.; Leshchenko, D. Marangoni-type of nonstationary rivulet-flows on inclined surface. *Int. J. Non-Linear Mech.* **2022**, *147*, 104250. [CrossRef]
18. Klimov, D.M.; Petrov, A.G.; Georgievsky, D.V. Viscous-Plastic Flows: Dynamical Chaos, Stability, and Confusion. Moscow, Science; See There Subsection 2.2.2; 2005, p. 45. Available online: <https://urait.ru/viewer/mehanika-splshnoy-sredy-vyazkoplasticheskie-tehniya-540818> (accessed on 10 February 2024). (In Russian)
19. Ershkov, S.V.; Leshchenko, D. Note on semi-analytical nonstationary solution for the rivulet flows of non-Newtonian fluids. *Math. Methods Appl. Sci.* **2022**, *45*, 7394–7403. [CrossRef]
20. Ershkov, S.V.; Leschenko, D.D. Revisiting glacier dynamics for stationary approximation of plane-parallel creeping flow. *Math. Model. Eng. Probl.* **2021**, *8*, 721–726. [CrossRef]
21. Ershkov, S.V. Non-stationary creeping flows for incompressible 3D Navier–Stokes equations. *Eur. J. Mech. B/Fluids* **2017**, *61*, 154–159. [CrossRef]
22. Ershkov, S.V.; Shamin, R.V. On a new type of solving procedure for Laplace tidal equation. *Phys. Fluids* **2018**, *30*, 127107. [CrossRef]

23. Ershkov, S.V.; Shamin, R.V. A Riccati-type solution of 3D Euler equations for incompressible flow. *J. King Saud Univ. Sci.* **2020**, *32*, 125–130. [\[CrossRef\]](#)
24. Ershkov, S.V.; Christianto, V.; Shamin, R.V.; Giniyatullin, A.R. About analytical ansatz to the solving procedure for Kelvin-Kirchhoff equations. *Eur. J. Mech. B/Fluids* **2020**, *79*, 87–91. [\[CrossRef\]](#)
25. Petrovskii, I.G. *Partial Differential Equations*; W. B. Saunders Co.: Philadelphia, PA, USA, 1967.
26. Farina, A.; Fusi, L.; Mikelić, A.; Saccomandi, G.; Sequeira, A.; Toro, E.F. *Non-Newtonian Fluid Mechanics and Complex Flows*; Farina, A., Mikelić, A., Rosso, F., Eds.; Springer International Publishing: Cham, Switzerland, 2018; 300p.
27. Baranovskii, E.S. Optimal boundary control of nonlinear-viscous fluid flows. *Sb. Math.* **2020**, *211*, 505–520. [\[CrossRef\]](#)
28. Baranovskii, E.S.; Artemov, M.A. Model for Aqueous Polymer Solutions with Damping Term: Solvability and Vanishing Relaxation Limit. *Polymers* **2022**, *14*, 3789. [\[CrossRef\]](#) [\[PubMed\]](#)
29. Astarita, G.; Marucci, G. *Principles of Non-Newtonian Fluid Hydromchanics*; McGraw Hill: Maidenhead, UK, 1974.
30. Pavlovskii, V.A. On the theoretical description of weak water solutions of polymers. *Dokl. Akad. Nauk SSSR* **1971**, *200*, 809–812.
31. Burmistrova, O.A.; Meleshko, S.V.; Pukhnachev, V.V. Exact solutions of boundary layer equations in polymer solutions. *Symmetry* **2021**, *13*, 2101. [\[CrossRef\]](#)
32. Ladyzhenskaya, O.A. On the global unique solvability of some two-dimensional problems for the water solutions of polymers. *J. Math. Sci.* **2000**, *99*, 888–897. [\[CrossRef\]](#)
33. Baranovskii, E.S. Global solutions for a model of polymeric flows with wall slip. *Math. Methods Appl. Sci.* **2017**, *40*, 5035–5043. [\[CrossRef\]](#)
34. Baranovskii, E.S.; Artemov, M.A. Optimal control for a nonlocal model of non-Newtonian fluid flows. *Mathematics* **2021**, *9*, 275. [\[CrossRef\]](#)
35. Baranovskii, E.S. Feedback optimal control problem for a network model of viscous fluid flows. *Math. Notes* **2022**, *112*, 26–39. [\[CrossRef\]](#)
36. Baranovskii, E.S. Flows of a polymer fluid in domain with impermeable boundaries. *Comput. Math. Math. Phys.* **2014**, *54*, 1589–1596. [\[CrossRef\]](#)
37. Baranovskii, E.S. An optimal boundary control problem for the motion equations of polymer solutions. *Sib. Adv. Math.* **2014**, *24*, 159–168. [\[CrossRef\]](#)
38. Wilson, S.K.; Duffy, B.R.; Ross, A.B. On the gravity-driven draining of a rivulet of viscoplastic material down a slowly varying substrate. *Phys. Fluids* **2002**, *14*, 555–571. [\[CrossRef\]](#)
39. Wilson, S.K.; Duffy, B.R.; Hunt, R. A slender rivulet of a power-law fluid driven by either gravity or a constant shear stress at the free surface. *Q. J. Mech. Appl. Math.* **2002**, *55*, 385–408. [\[CrossRef\]](#)
40. Yatim, Y.M.; Wilson, S.K.; Duffy, B.R. Unsteady gravity-driven slender rivulets of a power-law fluid. *J. Non-Newt. Fluid Mech.* **2010**, *165*, 1423–1430. [\[CrossRef\]](#)
41. Yatim, Y.M.; Duffy, B.R.; Wilson, S.K.; Hunt, R. Similarity solutions for unsteady gravity-driven slender rivulets. *Q. J. Mech. Appl. Math.* **2011**, *64*, 455–480. [\[CrossRef\]](#)
42. Yatim, Y.M.; Duffy, B.R.; Wilson, S.K. Similarity solutions for unsteady shear-stress-driven flow of Newtonian and power-law fluids: Slender rivulets and dry patches. *J. Eng. Math.* **2012**, *73*, 53–69. [\[CrossRef\]](#)
43. Paterson, C.; Wilson, S.K.; Duffy, B.R. Pinning, de-pinning and re-pinning of a slowly varying rivulet. *Eur. J. Mech. B/Fluids* **2013**, *41*, 94–108. [\[CrossRef\]](#)
44. Al Mukahal, F.H.H.; Duffy, B.R.; Wilson, S.K. A rivulet of a power-law fluid with constant contact angle draining down a slowly varying substrate. *Phys. Fluids* **2015**, *27*, 052101. [\[CrossRef\]](#)
45. Al Mukahal, F.H.H.; Wilson, S.K.; Duffy, B.R. A rivulet of a power-law fluid with constant width draining down a slowly varying substrate. *J. Non-Newt. Fluid Mech.* **2015**, *224*, 30–39. [\[CrossRef\]](#)
46. Al Mukahal, F.H.H.; Duffy, B.R.; Wilson, S.K. Advection and Taylor-Aris dispersion in rivulet flow. *Proc. Roy. Soc. A* **2017**, *473*, 20170524. [\[CrossRef\]](#) [\[PubMed\]](#)
47. Al Mukahal, F.H.H.; Duffy, B.R.; Wilson, S.K. Rivulet flow of generalized Newtonian fluids. *Phys. Rev. Fluids* **2018**, *3*, 083302. [\[CrossRef\]](#)
48. Alshaikhi, A.S.; Wilson, S.K.; Duffy, B.R. Rivulet flow over and through a permeable membrane. *Phys. Rev. Fluids* **2021**, *6*, 104003. [\[CrossRef\]](#)
49. Wang, M.; Zhao, J.; Duan, R. Rivulet formulation in the flow of film down a uniformly heated vertical substrate. *Eng. Appl. Comput. Fluid Mech.* **2019**, *13*, 396–416. [\[CrossRef\]](#)
50. Fedotkin, I.M.; Mel' nichuk, G.A.; Koval', F.F.; Klimkin, E.V. Hydrodynamics of rivulet flow on a vertical surface. *J. Eng. Phys.* **1984**, *46*, 9–14. [\[CrossRef\]](#)
51. Schmuki, P.; Laso, M. On the stability of rivulet flow. *J. Fluid Mech.* **1990**, *215*, 125–143. [\[CrossRef\]](#)
52. Bentwich, M.; Glasser, D.; Kern, J.; Williams, D. Analysis of rectilinear rivulet flow. *AIChE J.* **2004**, *22*, 772–779. [\[CrossRef\]](#)
53. Myers, T.G.; Liang, H.X.; Wetton, B. The stability and flow of a rivulet driven by interfacial shear and gravity. *Int. J. Non-Linear Mech.* **2004**, *39*, 1239–1249. [\[CrossRef\]](#)
54. Drenckhan, W.; Gatz, S.; Weaire, D. Wave patterns of a rivulet of surfactant solution in a Hele-Shaw cell. *Phys. Fluids* **2004**, *16*, 3115–3121. [\[CrossRef\]](#)

55. Weiland, R.H.; Davis, S.H. Moving contact lines and rivulet instabilities. Part 2. Long waves on flat rivulets. *J. Fluid Mech.* **2006**, *107*, 261. [[CrossRef](#)]
56. Young, G.W.; Davis, S.H. Rivulet instabilities. *J. Fluid Mech.* **2006**, *176*, 1. [[CrossRef](#)]
57. Doniec, A. Laminar flow of a liquid rivulet down a vertical solid surface. *Can. J. Chem. Eng.* **2009**, *69*, 198–202. [[CrossRef](#)]
58. Pukhnachev, V.V. Capillary/Gravity Film Flows on the Surface of a Rotating Cylinder. *J. Math. Sci.* **2005**, *130*, 4871–4883. [[CrossRef](#)]
59. Singh, R.K.; Galvin, J.E.; Sun, X. Three-dimensional simulation of rivulet and film flows over an inclined plate: Effects of solvent properties and contact angle. *Chem. Eng. Sci.* **2016**, *142*, 244–257. [[CrossRef](#)]
60. Aktershev, S.; Alekseenko, S.; Bobylev, A. Waves in a rivulet falling down an inclined cylinder. *AIChE J.* **2020**, *67*, e17002. [[CrossRef](#)]
61. Ershkov, S.V.; Prosviryakov, E.Y.; Artemov, M.A.; Leshchenko, D.D. Non-Stationary Helical Flows for Incompressible Couple Stress Fluid. *Mathematics* **2023**, *11*, 4999. [[CrossRef](#)]
62. Korobkov, M.V. Bernoulli law under minimal smoothness assumptions. *Dokl. Math.* **2011**, *83*, 107–110. [[CrossRef](#)]
63. Korobkov, M.V.; Pileckas, K.; Russo, R. Solvability in a finite pipe of steady-state Navier–Stokes equations with boundary conditions involving Bernoulli pressure. *Calc. Var.* **2020**, *59*, 32. [[CrossRef](#)]
64. Stepanyants, Y.A.; Yakubovich, E.I. The Bernoulli Integral for a Certain Class of Non-Stationary Viscous Vortical Flows of Incompressible Fluid. *Stud. Appl. Math.* **2015**, *135*, 295–309. [[CrossRef](#)]

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.