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Convergence of High-Order Derivative-Free Algorithms for the Iterative Solution of Systems of Not Necessarily Differentiable Equations

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Abstract: In this study, we extended the applicability of a derivative-free algorithm to encompass the solution of operators that may be either differentiable or non-differentiable. Conditions weaker than the ones in earlier studies are employed for the convergence analysis. The earlier results considered assumptions up to the existence of the ninth order derivative of the main operator, even though there are no derivatives in the algorithm, and the Taylor series on the finite Euclidian space restricts the applicability of the algorithm. Moreover, the previous results could not be used for non-differentiable equations, although the algorithm could converge. The new local result used only conditions on the divided difference in the algorithm to show the convergence. Moreover, the more challenging semi-local convergence that had not previously been studied was considered using majorizing sequences. The paper included results on the upper bounds of the error estimates and domains where there was only one solution for the equation. The methodology of this paper is applicable to other algorithms using inverses and in the setting of a Banach space. Numerical examples further validate our approach.

Keywords: three step eighth order algorithm; convergence; divided differences; differentiable-non-differentiable equation

MSC: 49M15; 65H10; 47H17; 65G99

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1. Introduction

Let F indicate the mapping of a subset $D \subset E$ into itself, where E is a Banach space. In a plethora of applications, researchers have reduced the problem to finding a solution $x^* \in D$ of

$$F(x) = 0. \tag{1}$$

The analytical version of the solution x^* is difficult to determine in general. Therefore, iterative algorithms have been developed that generate sequences that converge to x^* by means of some initial hypotheses [1–4].

The Newton's Scheme [1–3] defined by

$$x_0 \in D \; x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \;\; n = 0, 1, 2, \ldots, \tag{2}$$

is a popular quadratic-order algorithm. Recently, there has been a surge in the need to develop an algorithm with an order higher than two [5–9]. The Taylor series expansion provides the local order of convergence. But, there are limitations to this approach:

(C1) The convergence analysis is usually only local and $E = \mathbb{R}^j$, where j is a natural number.
(C2) The sufficient convergence hypotheses involve $F^{(d)}$, where $d = 1+$ order of convergence.
(C3) No a priori and computational error distances are available.
(C4) The isolation of the solution x^* is not discussed.
(C5) The semi-local convergence, which is considered more interesting and challenging than the local convergence, is not discussed.

Our idea addresses concerns (C1)–(C5) as follows:

(C1)′ The analysis is developed in Banach space.
(C2)′ The sufficient convergence hypotheses involve only the operators on the algorithm (see Algorithm 1), i.e., the divided differences. This is in contrast with the motivational work in [10] using hypotheses on high-order derivatives in the algorithm to show the convergence of the algorithm.
(C3)′ Error estimates become available under the concept of $\omega-$ continuity [1–3,11] in the local and majorizing sequences [3,7,12,13] in the semi-local case.
(C4)′ The isolation of the solution x^* is specified.
and
(C5)′ The semi-local convergence analysis of the algorithm is studied.

An algorithm was taken from [10] to demonstrate this idea. However, the same idea was similarly applicable in the algorithm containing the inverses of linear operators [4,11–21].

Let us redevelop the algorithm, but formatted in Banach space, as follows, $n = 0, 1, 2, \ldots$:

Algorithm 1

Step 1: Given $x_0 \in D$, solve $A_n u_1 = F(x_n)$, for u_1.
Step 2: Set $y_n = x_n - u_1$.
Step 3: Solve $A_n u_2 = F(y_n)$, for u_2.
Step 4: Solve $A_n u_3 = G_n u_2$, for u_3.
Step 5: Solve $A_n u_4 = G_n u_3$, for u_4.
Step 6: Ste $z_n = y_n - a_0 u_2 - (3 - 2a_0)u_3 - (a_0 - 2)u_4$.
Step 7: Solve $A_n u_5 = F(z_n)$, for u_5.
Step 8: Solve $A_n u_6 = Q_n u_5$, for u_6.
Step 9: Solve $A_n u_7 = Q_n u_6$, for u_7.
Step 10: Solve $A_n u_8 = Q_n u_7$, for u_8.
Step 11: Solve $A_n u_9 = Q_n u_8$, for u_9.
Step 12: Set $x_{n+1} = z_n - a_1 u_5 - a_2 u_6 - a_3 u_7 - a_4 u_8 - a_5 u_9$.
Step 13: If $x_{n+1} = x_n$, STOP. Otherwise, repeat the process with $n \longrightarrow n + 1$.

Here, a_0 and a_5 are free real parameters, $a_1 = a_5 + 4, a_2 = -4a_5 - 6, a_3 = 6a_5 + 4$ and $a_4 = -4a_5 - 1$, b_0, b_1, b_2 are fixed real numbers, $w_n = x_n + b_0 F(x_n)$, $h_n = y_n + b_1 F(y_n)$, $l_n = z_n + b_2 F(z_n)$, $A_n = [x_n, w_n; F]$, $G_n = [h_n, y_n; F]$ and $Q_n = [l_n, z_n; F]$.

Here, $[.,.; F] : D \times D \longrightarrow L(E)$ is a divided difference of an order one for the operator F [11,18,20,22], and the notation $L(E)$ is used for the set of continuous linear operators mapping E into itself. The interesting point of the algorithm is that, because of the usage of the same coefficient operator, only one LU decomposition can be performed for solving, e.g., linear systems with multiple right-hand sides. Thus, the algorithm without memory includes three steps (see Step 2, Step 6, and Step 12, where the iterations y_n, z_n, x_{n+1} are computed, respectively) and five free non-zero operator parameters. In Section 4, the parameters are further specialized. The convergence order is shown to be eight in Theorem 1 in [10]. But the existence of the ninth derivative is required for the local convergence analysis [10]. Thus, if F is not differentiable by at least that amount, the conclusions in [10] cannot assure the convergence of the algorithm to x^*. But, the algorithm can converge. Other limitations are listed in the aforementioned concerns of (C1)–(C5). As a further motivation, consider the folowing example: If $\tilde{D}$ stands for any neighborhood containing the numbers $t = 0, 1$ define $F(t) = 4t^3 \log t + 7t^5 - 7t^4$ if $t \neq 0$. Clearly, $t^* = 1 \in \tilde{D}$ solves equation $F(t) = 0$. But, the conclusions in [10] cannot assure that $\lim_{n \longrightarrow \infty} x_n = 1$, although the algorithm converges to t^*, as the function $F'''(t)$ is not continuous at $t = 0 \in \tilde{D}$. A more important semi-local analysis of the algorithm that has not yet been presented is also developed in this paper [14–17,20,22].

The rest of the paper contains the following: a local analysis of the algorithm is provided in Section 2, a semi-local analysis is provided in Section 3, numerical examples are provided in Section 4, and the paper ends with a conclusion in Section 5.

2. Local Analysis

We use the symbols $U(x, a)$ and $U[x, a]$ to denote open and closed balls in E, respectively, with center $x \in E$ and of radius $a > 0$. Let M denote the nonnegative axis, and NFC stands for a function that is nondecreasing and continuous on M or some subset of it. Then, the following hypotheses are required in the local analysis.

Assume:

(H1) Nondecreasing functions and continuous (NFC) $f_1 : M \to M$, $\varphi_0 : M \times M \to M$ exist, so that the equation

$$\varphi_0(f_1(t), t) - 1 = 0$$

admits a minimal positive solution (MPS) denoted by ρ. Let $M_0 = [0, \rho)$. It follows that for each $t \in M - \{0\}$

$$0 \leq \varphi_0(f_1(t), t) < 1 \tag{3}$$

and, consequently, the function $\lambda : M_0 \longrightarrow M$ provided by

$$\lambda(t) = \frac{1}{1 - \varphi_0(f_1(t), t)}$$

is positive.

(H2) NFC $f_2, f_3, \varphi_1 : M_0 \to M$ and $\varphi : M_0 \times M_0 \longrightarrow M$ exist, such that the equation $g_i(t) - 1 = 0$, $i = 1, 2, 3$ admits MPS denoted by ρ_i, respectively, provided $g_i : M_0 \to M$ are provided as

$$\begin{aligned} g_1(t) &= \lambda(t)\varphi(f_1(t), t), \\ g_2(t) &= [\varphi(f_1(t), g_1(t)t) + |1 - a_0|(1 + \int_0^1 \varphi_1(\theta g_1(t)t)d\theta) \\ &\quad + |3 - 2a_0|\lambda(t)(1 + \varphi_0(f_2(t), g_1(t)t)) \\ &\quad + |a_0 - 2|\lambda^2(t)(1 + \varphi_0(f_2(t), g_1(t)t))^2]\lambda(t)g_1(t), \end{aligned}$$

and for

$$\begin{aligned} \lambda_1(t) &= \varphi(f_1(t), g_2(t)t) + |1 - a_1|(1 + \int_0^1 \varphi_1(\theta g_2(t)t)d\theta) \\ &\quad + |a_2|\lambda(t)(1 + \varphi_0(f_3(t), g_2(t)t) \\ &\quad + |a_3|\lambda(t)^2(1 + \varphi_0(f_3(t), g_2(t)t))^2 \\ &\quad + |a_4|\lambda(t)^3(1 + \varphi_0(f_3(t), g_2(t)t))^3 \\ &\quad + |a_4|\lambda(t)^4(1 + \varphi_0(f_3(t), g_2(t)t))^4, \\ g_3(t) &= \lambda_1(t)\lambda(t)g_2(t). \end{aligned}$$

Define parameter

$$\rho^* = \min\{\rho_i\}, \quad i = 1, 2, 3 \ldots \tag{4}$$

and the set $M^* = [0, \rho^*)$. These definitions imply that if $t \in M^*$

$$0 \leq g_i(t) < 1. \tag{5}$$

(H3) L is an invertible operator on E, such that for each $x \in D$

$$\|w - x^*\| \leq f_1(\|x - x^*\|),$$

$$\|L^{-1}([x, w; F] - L)\| \leq \varphi_0(\|w - x^*\|, \|x - x^*\|), \ w = x + b_0 F(x).$$

Define the region $D_0 = D \cap S(x^*, \rho^*)$ with $S(x^*, \rho^*) = \{y \in D : \|y - x^*\| < \rho^*\}$.

(H4) $\|h - x^*\| \leq f_2(\|y - x^*\|)$, $\|l - x^*\| \leq f_3(\|z - x^*\|$,

$$\|L^{-1}([x, x^*; F] - L)\| \leq \varphi_1(\|x - x^*\|),$$

$$\|L^{-1}([x, w; F] - [x, x^*; F])\| \leq \varphi(\|w - x^*\|, \|x - x^*\|)$$

for $x \in D_0$, $w = x + b_0 F(x)$, $h = y + b_1 F(y)$ and $l = z + b_2 F(z)$ and y, z are provided by the last two substeps of the algorithm.
It is shown that y, z exist (see Proof of Theorem 1).
and

(H5) $S[x^*, \rho^*] \subset D$, where $S[x^*, \rho^*]$ is the closure of $S(x^*, \rho^*)$.

A local analysis of the algorithm follows.

Theorem 1. *Under conditions (H1)–(H5), the sequence $\{x_n\}$ is convergent to x^* provided that $x_0 \in U(x^*, \rho^*) - \{x^*\}$. Moreover, the following assertions hold*

$$\{x_n\} \subset S(x^*, \rho^*), \tag{6}$$

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < \rho^*, \tag{7}$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{8}$$

and

$$\|x_{n+1} - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{9}$$

where the functions g_i are as previously provided and the radius ρ^* is defined by the Formula (4).

Proof. From the hypothesis $x_0 \in S(x^*, \rho^*)$. If $F(x_0) \neq 0$ and $b_0 \neq 0$, then $w_0 \neq x_0$. It follows that the divided difference $A_0 = [x_0, w_0; F]$ is well defined. Then, by (H1), (4) and (H3), we obtain

$$\begin{aligned} \|L^{-1}(A_0 - L)\| &\leq \varphi_0(\|w_0 - x^*\|, \|x_0 - x^*\|) \\ &\leq \varphi_0(f_1(\|x_0 - x^*\|, \|x_0 - x^*\|) \\ &\leq \varphi_0(f_1(\rho^*), \rho^*) < 1. \end{aligned} \tag{10}$$

Thus, by the Banach perturbation Lemma on the linear operators with inverses [1,2,6,18] A_0^{-1} exists,

$$\|A_0^{-1} L\| \leq \frac{1}{1 - \varphi_0(f_1(\|x_0 - x^*\|, \|x_0 - x^*\|)} \tag{11}$$

and the iterate y_0 exists in the first substep of the algorithm.

Then, we can write

$$\begin{aligned} y_0 - x^* &= x_0 - x^* - A_0^{-1} F(x_0) \\ &= A_0^{-1}(A_0 - [x_0, x^*; F])(x_0 - x^*). \end{aligned} \tag{12}$$

Using (4), (5) (for $i=1$), (11) and (12)

$$\begin{aligned}\|y_0-x^*\| &\leq \|A_0^{-1}L\|\|L^{-1}(A_0-[x_0,x^*;F])\|\|x_0-x^*\| \\ &\leq \frac{\varphi(\|w_0-x^*\|,\|x_0-x^*\|)\|x_0-x^*\|}{1-\varphi_0(f_1(\|x_0-x^*\|,\|x_0-x^*\|)} \\ &\leq g_1(\|x_0-x^*\|)\|x_0-x^*\| \leq \|x_0-x^*\| < \rho^*. \end{aligned} \tag{13}$$

Thus, in the second substep of the algorithm, the following happens

$$\begin{aligned} z_0-x^* &= y_0-x^*-A_0^{-1}F(y_0) \\ &+(1-a_0)A_0^{-1}F(y_0)-(3-2a_0)A_0^{-1}G_0A_0^{-1}G_0A_0^{-1}F(y_0) \\ &-(a_0-2)A_0^{-1}G_0A_0^{-1}G_0A_0^{-1}F(y_0) \end{aligned}$$

leading to

$$\begin{aligned}\|z_0-x^*\| &\leq [\varphi(\|w_0-x^*\|,\|y_0-x^*\|)\lambda(\|x_0-x^*\|) \\ &+|1-a_0|\lambda(\|x_0-x^*\|)(1+\int_0^1\varphi_0(\theta\|y_0-x^*\|)d\theta) \\ &+|3-2a_0|\lambda^2(\|x_0-x^*\|)(1+\varphi_0(\|h_0-x^*\|,\|y_0-x^*\|)) \\ &+|a_0-2|\lambda^3(\|x_0-x^*\|)(1+\varphi_0(\|h_0-x^*\|,\|y_0-x^*\|))^2]\|y_0-x^*\| \\ &\leq g_2(\|x_0-x^*\|)\|x_0-x^*\| \leq \|x_0-x^*\|. \end{aligned} \tag{14}$$

Hence, the iterate $z_0 \in S(x^*,\rho^*)$ and the assertion (8) holds if $n=0$.

Similarly, in the last substep of the algorithm, we write

$$\begin{aligned} x_1-x^* &= z_0-x^*-A_0^{-1}F(z_0) \\ &+(1-a_1)u_5-a_2u_6-a_3u_7-a_4u_8-a_5u_9, \end{aligned}$$

leading to

$$\begin{aligned}\|x_1-x^*\| &\leq [\varphi(\|w_0-x^*\|,\|z_0-x^*\|)\lambda(\|x_0-x^*\|) \\ &+|1-a_0|\lambda(\|x_0-x^*\|)(1+\int_0^1\varphi_1(\theta\|z_0-x^*\|)d\theta) \\ &+|a_2|\lambda^2(\|x_0-x^*\|)(1+\varphi_0(\|l_0-x^*\|,\|z_0-x^*\|)) \\ &+|a_3|\lambda^3(\|x_0-x^*\|)(1+\varphi_0(\|l_0-x^*\|,\|z_0-x^*\|))^2 \\ &+|a_4|\lambda^4(\|x_0-x^*\|)(1+\varphi_0(\|l_0-x^*\|,\|z_0-x^*\|))^3 \\ &+|a_5|\lambda^5(\|x_0-x^*\|)(1+\varphi_0(\|l_0-x^*\|,\|z_0-x^*\|))^4]\|z_0-x^*\| \\ &\leq g_3(\|x_0-x^*\|)\|x_0-x^*\| \leq \|x_0-x^*\|. \end{aligned}$$

Thus, assertion (6) holds if $n=1$ and (9) if $n=0$. Repeat the preceding calculations with x_m, y_m, z_m, x_{m+1} replacing x_0, y_0, z_0, x_1 to complete the induction for assertions (7)–(9). Then, from the estimate

$$\|x_{m+1}-x^*\| \leq c\|x_m-x^*\| < \rho^*,$$

where $c=g_3(\|x_0-x^*\|)\in[0,1)$, we conclude that $\lim_{m\longrightarrow\infty}x_m=x^*$ and the iterate $x_{m+1}\in S(x^*,\rho^*)$. □

The uniqueness region can be determined.

Proposition 1. *Suppose:*

A solution of $\bar{x} \in S(x^*, \rho_4)$ exists for the equation $F(x) = 0$ for $\rho_4 > 0$; the condition (H3) holds on the ball $S(x^*, \rho_4)$ and $\rho_5 \geq \rho_4$ exists, such that

$$\int_0^1 \varphi_1(\theta\rho_5)d\theta < 1. \tag{15}$$

Define the region $D_1 = D \cap S[x^*, \rho_5]$. Then, the equation $F(x) = 0$ is uniquely solvable by x^* in the region D_1.

Proof. Let us consider the divided difference $[x^*, \bar{x}, F]$ provided that $x \neq \bar{x}$. Then, it follows by (H3) and (15) that

$$\|L^{-1}(M - L)\| \leq \int_0^1 \varphi_1(\theta\|\bar{x} - x^*\|)d\theta \leq \int_0^1 \varphi_1(\theta\rho_5)d\theta < 1.$$

Thus, the linear operator M is invertible. It follows from $\bar{x} - x^* = M^{-1}(F\bar{x}) - F(x^*)) = M^{-1}(0) = 0$ that $\bar{x} = x^*$. $\square$

Remark 1.

(i) *We can certainly choose $\rho_4 = \rho^*$ in Proposition 1.*

(ii) *Possible choice for the uncluttered functions f_i can be obtained as follows:*

$$\begin{aligned} w_0 - x^* &= x_0 - x^* + b_0F(x_0) \\ &= (I + b_0[x_0, x^*; F])(x_0 - x^*) \\ &= (I + b_0LL^{-1}([x_0, x^*; F] - L + L))(x_0 - x^*) \\ &= ((I + b_0L) + b_0LL^{-1}([x_0, x^*; F] - L))(x_0 - x^*), \\ \|w_0 - x^*\| &\leq [\|I + b_0L\| + |b_0|\|L\|\varphi_1(\|x_0 - x^*\|)]\|x_0 - x^*\|. \end{aligned}$$

Thus, we can define

$$f_1(t) = (\|I + b_0L\| + |b_0|\|L\|\varphi_1(t))t.$$

Similarly, we set

$$f_2(t) = (\|I + b_1L\| + |b_1|\|L\|\varphi_1(t))t$$

and

$$f_3(t) = (\|I + b_2L\| + |b_2|\|L\|\varphi_1(t))t.$$

(iii) *A possible choice for L in local convergence studies may be $L = F'(x^*)$, or $L = I$, or any other linear operator satisfying the conditions (H1)–(H5) (see also the Example 1 in the Section 4).*

3. Semi-Local Analysis

The role of x^*, φ_0, φ is exchanged with x_0, ψ_0 and ψ as follows:
Assume:

(E1) NFC $f_4 : M \to M, \psi_0 : M \times M \to M$ exists, such that equation

$$\psi_0(f_4(t), t) - 1 = 0$$

has MPS $\rho_6 \in M - \{0\}$. Let $M_1 = [0, \rho_6)$. Define $\psi : M_1 \longrightarrow M$ of the sequence $\{\alpha_n\}$ for $\alpha_0 = 0$, $\beta_0 \in [0, \rho_6)$ and each $n = 0, 1, 2, \ldots$ by

$$\begin{aligned}
q_n &= \frac{1}{1-\psi_0(f_4(\alpha_n),\alpha_n)},\\
p_n &= \psi(\alpha_n,\beta_n,f_4(\alpha_n))(\beta_n-\alpha_n),\\
\gamma_n &= \beta_n+|a_0|q_np_n+|3-2a_0|q_n^2(1+\psi_0(f_5(\beta_n),\beta_n))\\
&\quad +|a_0-2|q_n^3(1+\psi_0(f_5(\beta_n),\beta_n)),\\
d_n &= (1+\psi_0(\alpha_n,\gamma_n))(\gamma_n-\alpha_n)+p_n,
\end{aligned}$$

$$\begin{aligned}
\alpha_{n+1} &= \gamma_n+|a_1|q_nd_n+|a_2|q_n^2d_n(1+\psi_0(f_5(\gamma_n),\gamma_n))\\
&\quad +|a_3|q_n^3d_n(1+\psi_0(f_5(\gamma_n),\gamma_n))^2\\
&\quad +|a_4|q_n^4d_n(1+\psi_0(f_5(\gamma_n),\gamma_n))^3\\
&\quad +|a_5|q_n^5d_n(1+\psi_0(f_5(\gamma_n),\gamma_n))^4,\\
\delta_{n+1} &= \psi(\alpha_n,\alpha_{n+1},f_4(\alpha_n))(\alpha_{n+1}-\alpha_n)\\
&\quad +(1+\psi_0(f_4(\alpha_n),\alpha_n))(\alpha_{n+1}-\beta_n),
\end{aligned}$$

and

$$\beta_{n+1}=\alpha_{n+1}+\frac{\delta_{n+1}}{1-\psi_0(f_4(\alpha_{n+1}),\alpha_{n+1})}.$$

(E2) $\rho_7\in[0,\rho_6)$ exists, such that for each $n=0,1,2,\dots$

$$\psi_0(f_4(\alpha_n),\alpha_n)<1 \text{ and } \alpha_n\le\rho_7.$$

It follows that $0\le\alpha_n\le\beta_n\le\gamma_n\le\alpha_{n+1}<\rho_7$ and $\rho_8\in[0,\rho_7)$ exists, such that $\lim_{n\longrightarrow\infty}\alpha_n=\rho_8$.

(E3) An invertible linear operator of L and $x_0\in D$ exists, such that for each $x\in D$

$$\|L^{-1}([w,x;F]-L)\|\le\psi_0(\|w-x_0\|,\|x-x_0\|)$$

and

$$\|w-x_0\|\le f_4(\|x-x_0\|).$$

Notice that by condition (E1)

$$\|L^{-1}([w_0,x_0;F]-L)\|\le\psi_0(\|w_0-x_0\|,0)\le\psi_0(f_4(0),0)<1.$$

Thus, the linear operator $A_0=[w_0,x_0;F]$ is invertible and we can take

$$\|A_0^{-1}F(x_0)\|\le\beta_0.$$

(E4) Let $D_1=D\cap S(x_0,\rho_6)$

$$\|L^{-1}([y,x;F]-[w,x;F])\|\le\psi(\|x-x_0\|,\|y-x_0\|,\|w-x_0\|)$$

for each $x,y\in D_1$.
and

(E5) $S[x_0,\rho_8]\subset D$.

Then, using induction, as in the local case, we obtain the estimates

$$\begin{aligned}
\|y_0-x_0\| &= \|A_0^{-1}F(x_0)\|\le\beta_0=\beta_0-\alpha_0<\rho_8,\\
F(y_k) &= F(y_n)-F(x_n)-[w_n,x_n;F](y_n-x_n)\\
&= ([y_n,x_n;F]-[w_n,x_n;F])(y_n-x_n),\\
\|L^{-1}F(y_n)\| &\le \psi(\|x_n-x_0\|,\|y_n-x_0\|,\|w_n-x_0\|)\|y_n-x_n\|\\
&\le \psi(\alpha_n,\beta_n,f_4(\alpha_n))(\beta_n-\alpha_n)=p_n,
\end{aligned}$$

$$
\begin{aligned}
\|A_n^{-1}L\| &\leq \frac{1}{1-\psi_0(\|w_n-x_0\|,\|x_n-x_0\|)} \\
&\leq \frac{1}{1-\psi_0(f_4(\alpha_n),\alpha_n)} = q_n, \\
\|z_n-y_n\| &\leq |a_0|q_np_n + |3-2a_0|q_n^2(1+\psi_0(\|h_n-x_0\|,\|y_n-x_0\|) \\
&\quad +|a_0-2|q_n^3(1+\psi_0(\|h_n-x_0\|,\|y_0-x_0\|)^2 \\
&\leq |a_0|q_np_n + |3-2a_0|q_n^2(1+\psi_0(f_5(\beta_n),\beta_n)) \\
&\quad +|a_0-2|q_n^3 2(1+\psi_0(f_5(\beta_n),\beta_n))^2 \\
&= \gamma_n-\beta_n,
\end{aligned}
$$

$$
\begin{aligned}
\|z_n-x_0\| &\leq \|z_n-y_n\|+\|y_n-x_0\| \\
&\leq \gamma_n-\beta_n+\beta_n-\alpha_0 \\
&= \gamma_n<\rho_8, \\
\|x_{n+1}-z_n\| &\leq |a_1|q_nd_n+|a_2|q_n^2(1+\psi_0(\|l_n-x_0\|,\|z_n-x_0\|) \\
&\quad +|a_3|q_n^3d_n(1+\psi_0(f_6(\gamma_n),\gamma_n))^2 \\
&\quad +|a_4|q_n^4d_n(1+\psi_0(f_6(\gamma_n),\gamma_n))^3 \\
&\quad +|a_5|q_n^5d_n(1+\psi_0(f_6(\gamma_n),\gamma_n))^4 \\
&= \alpha_{n+1}-\gamma_n, \\
\|x_{n+1}-x_0\| &\leq \|x_{n+1}-z_n\|+\|z_n-x_0\| \\
&\leq \alpha_{n+1}-\gamma_n+\gamma_n-\alpha_0 \\
&= \alpha_{n+1}<\rho_8,
\end{aligned}
$$

$$
\begin{aligned}
F(x_{n+1}) &= F(x_{n+1})-F(x_n)-M_n(y_n-x_n) \\
&= (F(x_{n+1})-F(x_n)-M_n(x_{n+1}-x_n))+M_n(x_{n+1}-y_n) \\
&= ([x_{n+1},x_n;F]-M_n)(x_{n+1}-x_n)+M_n(x_{n+1}-y_n),
\end{aligned}
$$

$$
\begin{aligned}
\|L^{-1}F(x_{n+1})\| &\leq \psi(\|x_n-x_0\|,\|x_{n+1}-x_0\|,\|w_n-x_0\|)\|x_{n+1}-x_n\| \\
&\quad +(1+\psi_0(\|w_n-x_0\|,\|x_n-x_0\|))\|x_{n+1}-y_n\| \\
&\leq \psi(\alpha_n,\alpha_{n+1},f_4(\alpha_n))(\alpha_{n+1}-\alpha_n) \\
&\quad +(1+\psi_0(f_4(\alpha_n),\alpha_n))(\alpha_{n+1}-\beta_n) \\
&= \delta_{n+1}
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
\|y_{n+1}-x_{n+1}\| &\leq \|A_{n+1}^{-1}L\|\|L^{-1}F(x_{n+1})\| \\
&\leq \frac{\delta_{n+1}}{1-\psi_0(f_4(\|w_{n+1}-x_0\|,\|x_{n+1}-x_0\|)} \\
&\leq \frac{\delta_{n+1}}{1-\psi_0(f_4(\alpha_{n+1}),\alpha_{n+1})} \\
&= \beta_{n+1}-\alpha_{n+1},
\end{aligned}
$$

$$
\begin{aligned}
\|y_{n+1}-x_0\| &\leq \|y_{n+1}-x_{n+1}\|+\|x_{n+1}-x_0\| \\
&\leq \beta_{n+1}-\alpha_{n+1}+\alpha_{n+1}-\alpha_0 \\
&= \beta_{n+1}<\rho_8.
\end{aligned}
$$

Thus, $\{x_n\}\subset S(x_0,\rho_8)$ and is Cauchy in a Banach space E. Hence, $x^*\in S[x_0,\rho_8]$ exists, such that $\lim_{n\longrightarrow\infty} x_n = x^*$.

By letting $n\longrightarrow\infty$ in (16), we obtain $F(x^*)=0$. Notice that from the estimate,

$$
\|x_{n+j}-x_n\|\leq \alpha_{n+j}-\alpha_n. \tag{17}
$$

If $j \longrightarrow \infty$ in (17), we obtain

$$\|x^* - x_n\| \leq \alpha^* - \alpha_n.$$

Therefore, we arrive at the semi-local result for the Algorithm 1:

Theorem 2. *Suppose that conditions (E1)–(E5) hold. Then, the following assertions hold*

$$\{x_n\} \subset S(x_0, \rho_8),$$

$$\|y_n - x_n\| \leq \beta_n - \alpha_n,$$

$$\|x_{n+1} - z_n\| \leq \alpha_{n+1} - \gamma_n$$

and $x^* \in S[x_0, \rho_8]$ exists, solving the equation $F(x) = 0$.

The uniqueness of the solution for equation of $F(x) = 0$ is specified.

Proposition 2. *Suppose: A solution of $\tilde{x} \in S(x_0, \rho_9)$ exists for the equation $F(x) = 0$ for some $\rho_9 > 0$; condition (E3) holds on the ball $S(x_0, \rho_9)$ and $\rho_{10} \geq \rho_9$ exists, such that*

$$\psi_0(\rho_9, \rho_{10}) < 1. \tag{18}$$

Define the region $D_2 = D \cap S[x_0, \rho_{10}]$. Then, equation $F(x) = 0$ is uniquely solvable by $\tilde{x}$ in the region D_2.

Proof. Let $\tilde{y} \in D_2$ with $F(\tilde{y}) = 0$ and $\tilde{y} \neq \tilde{x}$. Then, the divided difference $Q = [\tilde{y}, \tilde{x}; F] = [\tilde{w}, \tilde{x}; F]$ is well defined. It then follows from (18) that

$$\begin{aligned} \|L^{-1}(Q - L)\| &\leq \psi_0(\|\tilde{y} - x_0\|, \|\tilde{x} - x_0\|) \\ &\leq \psi_0(\rho_9, \rho_{10}) < 1. \end{aligned}$$

Therefore, we deduce $\tilde{y} = \tilde{x}$. □

Remark 2.

(i) *The limit point ρ_8 can be replaced by ρ_6 in (E5) (provided in the condition (E1)).*
(ii) *Suppose that all conditions (E1)–(E5) hold in Proposition 1. Then, set $\rho_9 = \rho_8$ and $\tilde{x} = x^*$.*
(iii) *Functions f_i, $i = 4, 5, 6$ can be specified as in the local case by the following estimates:*

$$\begin{aligned} w - x_0 &= [(I + b_0 L) + b_0 L L^{-1}([x, x_0; F] - L)](x - x_0) + b_0 F(x_0), \\ \|w - x_0\| &\leq (\|I + b_0 L\| + |b_0| \|L\| \psi_1(\|x - x_0\|) \|x - x_0\| \\ &\quad + |b_0| \|F(x_0)\|. \end{aligned}$$

Hence, we can define

$$f_4(\alpha_n) = (\|I + b_0 L\| + |b_0| \|L\| \psi_1(\alpha_n)) \alpha_n + |b_0| \|F(x_0)\|.$$

Similarly, we choose

$$f_5(\beta_n) = (\|I + b_1 L\| + |b_1| \|L\| \psi_1(\beta_n)) \beta_n + |b_1| \|F(x_0)\|$$

and

$$f_6(\gamma_n) = (\|I + b_2 L\| + |b_2| \|L\| \psi_2(\gamma_n)) \gamma_n + |b_2| \|F(x_0)\|.$$

(iv) *A possible choice for L may be $L = A_0$, provided that the operator A_0 is invertible or $L = I$. Other choices are possible, as long as conditions (E1)–(E4) are validated.*

4. Numerical Examples

In this Section, we chose $a_0 = 3, a_5 = 0, b_0 = -1, b_1 = 1$, and $b_2 = -1$ for all of the examples to obtain the specialization of the algorithm, which is defined by Algorithm 2

Algorithm 2

Step1: Given $x_0 \in D$, Solve $A_n u_1 = F(x_n)$, for u_1.
Step 2: Set $y_n = x_n - u_1$.
Step 3: Solve $A_n u_2 = F(y_n)$, for u_2.
Step 4: Solve $A_n u_3 = G_n u_2$, for u_3.
Step 5: Solve $A_n u_4 = G_n u_3$, for u_4.
Step 6: Set $z_n = y_n - 3u_2 + 3u_3 - u_4$.
Step 7: Solve $A_n u_5 = F(z_n)$, for u_5.
Step 8: Solve $A_n u_6 = Q_n u_5$, for u_5.
Step 9: Solve $A_n u_7 = Q_n u_6$, for u_6.
Step 10: Solve $A_n u_8 = Q_n u_7$, for u_8.
Step 11: Set $x_{n+1} = z_n - 4u_5 + 6u_6 - 4u_7 + u_8$.
Step 12: If $x_{n+1} = x_n$, STOP. Otherwise, repeat the process with $n \longrightarrow n+1$.

Here $w_n = x_n - F(x_n)$, $h_n = y_n + F(y_n)$, $l_n = z_n - F(z_n)$, $A_n = [x_n, w_n; F]$, $G_n = [h_n, y_n; F]$, $Q_n = [l_n, z_n; F]$.

Also, we considered the choice of the divided difference $[x, y; F] = \int_0^1 F'(x + \theta(y - x))d\theta$ and $L = F'(x^*)$.

In Example 1, we provided the choice of the operator L as well as the functions φ_0, φ, and φ_1 to validate the local convergence conditions (H1)–(H5). Notice that functions f_1, f_2 and f_3 were chosen, as in Remark 1 (ii). There was no need to choose the operator L in the rest of the examples as the convergence of the aforementioned Algorithm was established (semi-local convergence). The stopping criterion is $\|x_n - x_{n-1}\| < \epsilon$, where ϵ is the desired error tolerance.

Example 1. *Let $\mathcal{M} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and $\Omega = S[x^*, 1]$. The mapping F is defined on Ω for $a = (a_1, a_2, a_3)^{tr} \in \mathbb{R}$ as*

$$F(a) = (a_1, e^{a_2} - 1, \frac{e^{-1}}{2} a_3^2 + a_3)^{tr}.$$

Then, F' is calculated to be

$$F'(a) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{a_2} & 0 \\ 0 & 0 & (e-1)a_3 + 1 \end{bmatrix}$$

Then, $x^* = (0,0,0)^{tr}$ solves equation $F(a) = 0$. Moreover, the definition of F' provides $F'(x^*) = I$. Take $L = I$.

Then, conditions (H3)–(H5) are valid if we define for f_i as provided in Remark 1.

$$\begin{aligned} \varphi_1(t) &= \frac{e-1}{2} t, \\ f_1(t) &= f_2(t) = f_3(t) = \varphi_1(t)t, \qquad (19) \\ \varphi_0(s,t) &= \frac{1}{2}(e-1)(t + f_1(t)), \\ \varphi(s,t) &= \frac{1}{2}(e-1)f_1(t). \qquad (20) \end{aligned}$$

These choices of scalar functions validate the conditions of Theorem 1. This assures the convergence of the sequence $\{x_n\}$ to solution x^*.

Then, from Formula (4), we deduce

$$\rho_1 = 0.596199595223383235547256717867 52,$$

$$\rho_2 = 0.246526977025900735005240941904 87$$

and

$$\rho_3 = \rho^* = 0.107395799514168933629261718612 38.$$

Example 2. *The solution sought for the nonlinear system*

$$\begin{aligned} 3\theta_1^2\theta_2 + \theta_2^2 - 1 + |\theta_1 - 1| &= 0 \\ \theta_1^4 + \theta_1\theta_2^3 - 1 + |\theta_2| &= 0 \end{aligned}$$

Let $F = (Q_1, Q_2)$ *for* $(\theta_1, \theta_2) \in \mathbb{R} \times \mathbb{R}$*, where*

$$\begin{aligned} Q_1 &= 3\theta_1^2\theta_2 + \theta_2^2 - 1 + |\theta_1 - 1| = 0 \quad \text{and} \\ Q_2 &= \theta_1^4 + \theta_1\theta_2^3 - 1 + |\theta_2| = 0. \end{aligned}$$

Then, the system becomes

$$F(s) = 0 \;\; for \;\; s = (\theta_1, \theta_2)^T.$$

The divided difference $L = [.,.; F]$ *belongs in the space* $M_{2\times 2}(\mathbb{R})$ *and is the standard* 2×2 *matrix in* $\mathbb{R}^2$ *[11,18]. Let us choose* $x_0 = (5,5)^T$*. It turns out that the algorithm converges to the solution of* x^**, as the initial guess of* x_0 *is close enough to it. Hence, there is no need to validate the conditions of Theorem 2, which are sufficient. Then, the application of the algorithm provides the solution* θ^* *after three iterations. The solution* $\theta^* = (\theta_1^*, \theta_2^*)^T$*, where*

$$\theta_1^* = 0.894655373334687$$

and

$$\theta_2^* = 3.27826421746298.$$

Example 3. *Consider the system of 100 equations defined by*

$$F(x) = 0,$$

where

$$F(x)(i) = \begin{cases} x_i^2 \sin(x_{i+1}) - 1, & \text{it } 1 \le i \le 99 \\ x_i^2 \sin(x_i) - 1, & \text{it } i = 100. \end{cases}$$

The results obtained for the initial point $(2, 2, \ldots, 2)$ *are provided in Table 1.*

Table 1. Iterated solutions of Example 3.

Iteration n	**Solution** x_i	$\lvert F(x)(i) \rvert$	**Time** s
1	0.524657457768460047345322189931 15	0.862114977251371	9.196941
2	1.066641788879466602224790019704 9	0.003827358620836	19.437201
3	1.068223544197249018283412719362 2	$1.110223024625157 \times 10^{-16}$	28.369249
4	1.068223544197249018283471114263 1	$1.110223024625157 \times 10^{-16}$	37.619249

Example 4. *In this example, we consider a system of five equations defined by*

$$F(x) = 0,$$

where

$$F(x)(i) = \sum_{j=1, j \neq i}^{5} x_i - e^{x_i}.$$

The results obtained for the initial points $(1, 1, 1, 1, 1)$ *are provided in Table 2.*

Table 2. Iterated solutions of Example 4.

Iteration n	**Solution** x_i	$\lvert F(x)(i) \rvert$	**Time** s
1	0.20391080591998655968666298576863	$1.081148328339054 \times 10^{-4}$	0.002347
2	0.20388835470224017654139458954887	$1.110223024625157 \times 10^{-16}$	0.002501
3	0.20388835470224017654139458954887	$1.110223024625157 \times 10^{-16}$	0.002839

We used a 4-core 64-bit Windows machine with 11th Gen Intel(R) Core (TM) i5-1135G7 CPU @ 2.40 GHz for all our computations using MATLAB R2023b.

5. Conclusions

The step eighth-order Algorithm without derivatives of the operator was studied in this paper using assumptions only on the first divided difference of the operator. Earlier studies using the Taylor series expansion algorithm made assumptions up to a ninth order derivative not on the algorithm [10].

We provided sufficient convergence conditions involving only the operators on the algorithm, computable error upper bounds on $\|x_n - x^*\|$, and presented the uniqueness of the solution results. It is worth noticing that the methodology of this study was not dependent on the convergence order of the iterative algorithm as the convergence conditions did not make use of it. Moreover, the assumption that the solution was simple was not made or implied by the convergence conditions. Thus, in case convergence conditions were satisfied, the methods also found solutions of multiplicity greater than one. The approach in this paper was applied to other algorithms with inverses to obtain the same benefits [4,20–22]. This will be the focus of our future research.

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