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Uniqueness of a Generalized Solution for a One-Dimensional Thermal Explosion Model of a Compressible Micropolar Real Gas

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Abstract: In this paper, we analyze a quasi-linear parabolic initial-boundary problem describing the thermal explosion of a compressible micropolar real gas in one spatial dimension. The model contains five variables, mass density, velocity, microrotation, temperature, and the mass fraction of unburned fuel, while the associated problem contains homogeneous boundary conditions. The aim of this work is to prove the uniqueness theorem of the generalized solution for the mentioned initial-boundary problem. The uniqueness of the solution, together with the proven existence of the solution, makes the described initial-boundary problem theoretically consistent, which provides a basis for the development of numerical methods and the engineering application of the model.

Keywords: micropolar real gas; reactive fluid; uniqueness of the solution

MSC: 35Q35

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1. Introduction

1.1. Introduction to the Topics Covered in this Paper

In this work, we study the problem of uniqueness of the solution of the system of partial differential equations describing the one-dimensional flow and thermal explosion of a micropolar compressible fluid. It is a parabolic system of quasi-linear equations that reads

$$\partial_t \rho = -\frac{1}{L}\rho^2 \partial_x v, \tag{1}$$

$$\partial_t v = -\frac{R}{L}\partial_x(\rho^p \theta) + \frac{\lambda + 2\mu}{L^2}\partial_x(\rho \partial_x v), \tag{2}$$

$$j_I \partial_t \omega = \frac{c_0 + 2c_d}{L^2}\partial_x(\rho \partial_x \omega) - 4\mu_r \frac{\omega}{\rho}, \tag{3}$$

$$c_v \partial_t \theta = \frac{\kappa}{L^2}\partial_x(\rho \partial_x \theta) - \frac{R}{L}\rho^p \theta\, \partial_x v + \frac{\lambda + 2\mu}{L^2}\rho(\partial_x v)^2 + \frac{c_0 + 2c_d}{L^2}\rho(\partial_x \omega)^2 \tag{4}$$

$$+ 4\mu_r \frac{\omega^2}{\rho} + \delta r(\rho, \theta, z), \tag{5}$$

$$\partial_t z = \frac{\sigma}{L^2}\partial_x\left(\rho^2 \partial_x z\right) - r(\rho, \theta, z), \tag{6}$$

for $(x,t) \in]0,1[\times]0,T[$. The system is coupled with the following initial conditions

$$\rho(x,0) = \rho_0(x),\ v(x,0) = v_0(x),\ \omega(x,0) = \omega_0(x),\ \theta(x,0) = \theta_0(x),\ z(x,0) = z_0(x), \tag{7}$$

for $x \in [0,1]$, and the following homogeneous boundary conditions

$$\begin{aligned} v(0,t) = v(1,t) = 0, \quad \omega(0,t) = \omega(1,t) = 0, \\ \partial_x\theta(0,t) = \partial_x\theta(1,t) = 0, \quad \partial_x z(0,t) = \partial_x z(1,t) = 0, \end{aligned} \tag{8}$$

for $t \in [0,T]$.

Here, $\rho = \rho(x,t)$, $v = v(x,t)$, $\omega = \omega(x,t)$, $\theta = \theta(x,t)$, and $z = z(x,t)$ denote mass density, velocity, microrotational velocity, absolute temperature, and mass fraction of unburned fuel, respectively. Equations (1)–(4) represent laws of conservation of mass, moment, momentum moment, and energy, while Equation (6) describes the dynamics of a chemical reaction.

Let us first give some important information about the system, which is derived from work [1]. The system is given in mass Lagrangian coordinates and the spatial coordinate x is dimensionless, whereby $L > 0$ is the dimension-bearing constant obtained during the derivation of the model. The rest of the constants that appear in the system are as follows: $j_I > 0$ is microinertia density; $c_v > 0$ is specific heat for constant volume; $\kappa > 0$ is heat conductivity coefficient; $\delta > 0$ is the reaction rate; $\sigma > 0$ is the species diffusion coefficient; λ and μ are coefficients of viscosity; and c_0, c_d, and μ_r are coefficients of microviscosity, whereby the following inequalities hold:

$$\mu \geq 0, \qquad 3\lambda + 2\mu \geq 0, \qquad c_d \geq 0, \qquad 3c_0 + 2c_d \geq 0, \qquad \mu_r > 0. \tag{9}$$

Here, we consider the real gas model which is characterized by the generalized equation of state

$$P = R\rho^p\theta, \tag{10}$$

where $P = P(x,t)$ is the pressure, while $R > 0$ and $p \geq 1$ are constants (see [2]). For $p = 1$, the model reduces to the ideal gas model.

Equation (10) resulted from the interpolation of the equations of state for ideal and barotropic fluids, taking into account the generalization of the equation of state for gases introduced in [3]. Since barotropic fluids have important applications in meteorology and astrophysics, it is reasonable to assume that the addition of a temperature component will allow the analysis of a larger number of thermodynamic problems related to this type of fluid. On the other hand, if we consider (10) as a generalization of the equation of state for an ideal gas, we can say that this equation allows the observation of gases that deviate from ideal behavior, especially under extreme conditions such as thermal explosions, as discussed in [3].

Function r represents the intensity of the chemical reaction, and Arrhenius' law (see [4] for details) is most often assumed to hold, that is

$$r(\rho,\theta,z) = \epsilon\rho^{m-1}z^m \exp\frac{\theta - 1}{\epsilon\theta}, \tag{11}$$

where $\epsilon > 0$ is the activation energy, and $m \geq 1$ is an integer representing the overall sum of the individual reaction orders of reactants. In this work, we allow r to assume a more general form

$$r(\rho,\theta,z) = z^m\tilde{r}(\rho,\theta,z), \tag{12}$$

where $m \geq 1$ is an integer, and $\tilde{r} :]0,+\infty[\times]0,+\infty[\times[0,+\infty[\mapsto [0,\infty[$ such that the following holds:

- $\tilde{r}$ is bounded on each $[a,b]\times]0,+\infty[\times[0,+\infty[$, for $0 < a < b$, ;
- $\tilde{r}$ is continuous with respect to ρ, Lipschitz continuous on bounded sets with respect to ρ, and globally Lipschitz continuous with respect to θ and z;
- $\lim\limits_{\rho\to0^+} \tilde{r}(\rho,\theta,z) = 0$ and $\lim\limits_{\theta\to0^+} \tilde{r}(\rho,\theta,z) = 0$.

These assumptions on r are inspired by the work in [2]. Let us notice that Arrhenius' law (11) satisfies the stated assumptions, which indicates that they are not too restrictive.

This problem falls into the category of quasi-linear parabolic problems (see [5] for details) that are generally written as

$$\partial_t u + Au = f, \tag{13}$$

where A is a quasi-linear elliptic spatial differential operator, where the quasi-linearity is characterized by linearity in the highest-order derivatives, more precisely, in the second-order derivatives. This form of the problem is advantageous since some general methods for its solution are known, such as the Faedo–Galerkin projection onto finite-dimensional subspaces (see [6]), which was used to prove the existence of this problem in [7].

1.2. Literature Review and Important Results

The classical continuum as a modeling tool is extremely robust in modeling a large number of materials. However, as the development of scientific models increasingly focuses on the micro- and nanolevel, it becomes apparent that neglecting certain material properties related to microlevel behavior does not yield satisfactory results [8]. In the past, there were several approaches that attempted to account for the microproperties and microbehavior of materials, but most of these models proved to be too complex for mathematical and engineering analysis [9]. An optimal model that could describe micro-behaviors was introduced by Ahmed Cemal Eringen in the second half of the last century. Eringen emphasizes the importance of microrotations, which he adds as a new model variable while neglecting microdeformations [10].

In this paper, we focus on micropolar fluid, which is increasingly used in various fields of science, for example, in chemistry [11], physics [12], metallurgy [13], biomedicine [14], thermodynamics [15], mechanical engineering [16], and many others. The mathematical analysis of micropolar fluids can be divided into two broad areas—incompressible fluid theory and compressible fluid theory, with incompressible fluids being far better studied from the point of view of both mathematical properties and applications. For an overview of the results and recent achievements in the field of incompressible fluids, we refer to the following papers [17–19]:

Classical compressible fluid is still a field with many open problems. In particular, the mathematical analysis of three-dimensional models [20], of models admitting an initial vacuum [21], and of models with time-dependent domains [22] stand out.

The mathematical analysis of the compressible micropolar fluid began with the assumption of the ideality of the fluid, i.e., Clapeyron's equation of state. The mathematical model of an ideal micropolar compressible and thermally conductive fluid with homogeneous boundary conditions was presented by N. Mujaković in 1998 [23]. The described model was also considered in three dimensions but assuming spherical [24] or cylindrical symmetry [25,26]. In all the above cases, the existence and uniqueness of the global generalized solution were proved, and the problems of regularity and stabilization of the solution were also considered [27]. The flow model of the described fluid between parallel plates has also been studied, but so far, only the uniqueness of the solution has been proved [28].

In this paper, we are concerned with a model in which Clapeyron's equation of state is generalized. In other words, it is no longer an ideal fluid but a real fluid. This type of fluid has been considered in the classical case in the context of several mathematical problems, considering in particular the problem of the existence of a solution, the problem of regularity, and the problem of stabilization of the solution [29–34]. For the micropolar case of a real fluid in one dimension, the local and global existence and the uniqueness of the generalized solution have been proved so far [35–37].

In addition to the generalization of the equation of state, we are also dealing with a more complex system of partial differential equations describing the problem of thermal explosion. This problem has been analyzed so far mainly in the case of a classical ideal fluid [4,38] and classical real fluid [2,39,40].

In this work, the problem of the thermal explosion of a real gas is extended to the micropolar case, and the corresponding model is governed by (1)–(8). For this problem, the local and global existence of a generalized solution has been proved so far [7,41], and the uniqueness of the generalized solution is proved in this work.

1.3. Research Aims and Objectives

Our primary goal in this article is to rigorously establish the uniqueness of the solution to the problem (1)–(8). A secondary goal is the development and adaptation of proof methods for the uniqueness of solutions related to the equations for real micropolar fluids, which can be applied to mathematically similar problems modeling other phenomena.

1.4. Structure of the Article

In Section 1, the problem studied is described and an overview of the relevant results is given. The Section 2 lists the specific mathematical tools needed to prove the main theorem. In Section 3, the generalized solution to the problem described is defined in detail, while the main result is described in Section 4. In Section 5, auxiliary results are given, on the basis of which the main result is proved in Section 6.

1.5. Positioning of Our Results within the Relevant Field

The analyzed problem of the thermal explosion of a micropolar real gas is at an early stage of mathematical analysis. So far, only the existence of the solution has been proven, while the uniqueness of the solution has not yet been confirmed. By proving uniqueness, the model becomes theoretically consistent and is suitable for further mathematical and technical analyses. This means that the result of this work places the observed initial-boundary problem in the realm of engineering models for which it makes sense to continue the analysis, either in the context of developing numerical methods or in the sense of research for concrete practical applications. Moreover, the proof itself shows in its technical aspect the necessary adaptations to prove uniqueness that must be made when the classical thermal explosion model is considered in the context of micropolar fluids, since the uniqueness of this problem has not yet been analyzed in the context of micropolarity. This adaptation can then be applied to similar quasi-linear parabolic problems. It is also important to point out that in similar papers dealing with the classical model, uniqueness is almost always only assumed and not rigorously analyzed and proved.

2. Preliminaries

In this section, we provide an overview of some known inequalities, namely Young's, Hölder's, Poincaré's, Ladyzhenskaya's, and Grönwall's inequality, which we use in the following. For simplicity, here and below, we use the following abbreviated notation for the L^2 and L^∞ norm:

$$\|f\| = \|f\|_{L^2(a,b)}, \qquad |f| = \|f\|_{L^\infty(a,b)}. \tag{14}$$

Proposition 1 (Young, in [42])**.** *Let $\alpha, \beta \geq 0$, $q \in]1, \infty[$ and $q' = \frac{q}{q-1}$. Then, the following inequality holds*

$$\alpha\beta \leq \frac{\alpha^q}{q} + \frac{\beta^{q'}}{q'}. \tag{15}$$

Moreover, if $\gamma > 0$, then

$$ab \leq \frac{\gamma^q a^q}{q} + \frac{b^{q'}}{q'\gamma^{q'}}. \tag{16}$$

Proposition 2 (Hölder, in [42])**.** *Let $q \in [1, \infty]$ and $q' = \frac{q}{q-1}$. If f and g are measurable functions on $[a, b]$, then*

$$\|fg\|_{L^1(a,b)} \leq \|f\|_{L^q(a,b)} \cdot \|g\|_{L^{q'}(a,b)}. \tag{17}$$

Proposition 3 (Poincaré, in [6]; Gagliardo–Ladyzhenskaya, in [5])**.** *Let $f :]a, b[\to \mathbf{R}$. If one of the following conditions holds:*

1. $f \in H_0^1(a,b)$;
2. $f \in H^1(a,b)$ *and* $\int_a^b f(x)dx = 0$;

then

$$\|f\| \leq C\|f'\|, \tag{18}$$

$$|f|^2 \leq C\|f\| \cdot \|f'\|, \tag{19}$$

$$|f| \leq C\|f'\|, \tag{20}$$

where positive constant C does not depend on function f.

Proposition 4 (Grönwall, in [43])**.** *Let ξ, f, g, h be real-valued measurable functions on $[a,b]$, with fh, gh, and ξh integrable. If f, g, h are non-negative and*

$$\xi(x) \leq f(x) + g(x)\int_a^x h(t)\xi(t)dt, \quad x \in [a,b] \tag{21}$$

then

$$\xi(x) \leq f(x) + g(x)\int_a^x f(t)h(t)\exp\left(\int_t^x g(s)h(s)ds\right)dt \tag{22}$$

holds a.e. on $[a,b]$.

3. Generalized Solution

Generalized solution [1,7] to the initial-boundary value problem (1)–(8) in $Q_T =]0,1[\times]0,T[$, for $T > 0$, is a function

$$(x,t) \mapsto (\rho, v, \omega, \theta, z)(x,t), \quad (x,t) \in Q_T, \tag{23}$$

such that

$$\rho \in L^\infty\big(0,T;H^1(0,1)\big) \cap H^1(Q_T), \quad \operatorname*{ess\,inf}_{Q_T} \rho > 0, \tag{24}$$

$$v, \omega, \theta, z \in L^\infty\big(0,T;H^1(0,1)\big) \cap H^1(Q_T) \cap L^2\big(0,T;H^2(0,1)\big). \tag{25}$$

ρ, v, ω, θ, and z satisfy Equations (1)–(6) a.e. in Q_T in the sense of weak derivatives, initial conditions (7) a.e. in $]0,1[$, and boundary conditions (8) in the sense of traces.

We call the solution to the problem generalized because the above Equations (1)–(6) are not generally satisfied in the sense of classical definition of (partial) derivatives due to functions involved not being necessarily smooth enough but in a more general sense of theory of distributions, or more specifically, Sobolev spaces and weak derivatives (see for example [6,42]). According to the definition of weak derivatives, Equations (1)–(6) translate into the following:

$$\iint_{Q_T}\left[\rho\partial_t\phi - \frac{1}{L}\rho^2\partial_x v\phi\right]dxdt = 0, \tag{26}$$

$$\iint_{Q_T}\left[v\partial_t\phi + \frac{R}{L}(\rho)^p\theta\partial_x\phi - \frac{\lambda+2\mu}{L^2}\rho\partial_x v\partial_x\phi\right]dxdt = 0, \tag{27}$$

$$\iint_{Q_T}\left[\omega\partial_t\phi - \frac{c_0+2c_d}{L^2 j_I}\rho\partial_x\omega\partial_x\phi - 4\frac{\mu_r}{j_I}\frac{\omega}{\rho}\phi\right]dxdt = 0, \tag{28}$$

$$\begin{aligned}\iint_{Q_T}\Bigg[\theta\partial_t\phi &- \frac{\kappa}{L^2 c_v}(\rho\partial_x\theta)\partial_x\phi + \frac{R}{Lc_v}(\rho)^p\theta\partial_x v\phi + \frac{\lambda+2\mu}{L^2 c_v}\rho(\partial_x v)^2\phi + 4\frac{\mu_r}{c_v}\frac{(\omega)^2}{\rho}\phi \\ &+ \frac{c_0+2c_d}{L^2 c_v}\rho(\partial_x\omega)^2\phi + \frac{\delta}{c_v}r(\rho,\theta,z)\phi\Bigg]dxdt = 0,\end{aligned} \tag{29}$$

$$\iint_{Q_T}\left[z\partial_t\phi - \frac{\sigma}{L^2}(\rho)^2\partial_x z\partial_x\phi - r(\rho,\theta,z)\phi\right]dxdt = 0, \tag{30}$$

for all test functions $\phi = \phi(x,t)$.

From the embedding theorems for function spaces (see [6,44] for details), it can be seen that for a generalized solution $(\rho, v, \omega, \theta, z)$, even stronger inclusions are valid, more precisely:

$$\rho \in C\big([0,T]; L^2(0,1)\big) \cap L^\infty\big(0,T; C([0,1])\big), \tag{31}$$

$$v, \omega, \theta, z \in L^2\big(0,T; C^1([0,1])\big) \cap C\big([0,T]; H^1(0,1)\big) \cap C\big(\overline{Q}_T\big). \tag{32}$$

If additionally $\rho_0 \in H^1(0,1)$, then

$$\rho \in C\big(\overline{Q}_T\big) \tag{33}$$

also holds.

Let us mention that in [7] it is shown that the initial-boundary value problem (1)–(8) has a generalized solution locally in time if the following holds for the initial functions:

$$\rho_0, \theta_0, z_0 \in H^1(]0,1[), \quad v_0, \omega_0 \in H_0^1(]0,1[), \tag{34}$$

$$\operatorname*{ess\,inf}_{x\in]0,1[} \rho_0(x) > 0, \quad \operatorname*{ess\,inf}_{x\in]0,1[} \theta_0(x) > 0, \quad 0 \le z_0 \le 1,\ \forall x \in [0,1], \tag{35}$$

that is, the following theorem holds.

Theorem 1 (On the local existence of the generalized solution [7])**.** *Let the functions ρ_0, v_0, ω_0, θ_0, z_0 satisfy the conditions (34)–(35). Let r be defined by (12) and m from (12) be either an odd integer or equal to* 2*. There exists $T_0 \in]0,T]$ such that the problem (1)–(8) has a generalized solution in $Q_0 := Q_{T_0}$ such that*

$$\theta > 0 \quad \text{and} \quad 0 \le z \le 1 \quad \text{in} \quad \overline{Q}_0. \tag{36}$$

For the convenience of the reader, we outline the basic ideas of the proof of Theorem 1. The proof is constructive and is divided into several main steps. First, it is necessary to construct a series of approximate solutions; the Faedo–Galekin method was used for this purpose. Subsequently, a series of a priori estimates were obtained for the constructed approximate solutions. The obtained a priori estimates allowed the choice of a sufficiently small time interval in which the approximate solutions are bounded. In the last step, the transition to the limits using the compactness theorems leads to obtaining the solution of the observed system defined on a previously chosen sufficiently small time interval.

We have also dealt with the problem of the global existence of the solution, which was proven in [41]. For the sake of completeness, we reproduce this result in the following theorem.

Theorem 2 (On global existence of the generalized solution [41])**.** *Let the functions r, ρ_0, v_0, ω_0, θ_0, and z_0 satisfy the conditions of Theorem 1. Then, for any $T > 0$, there is a generalized solution $(\rho, v, \omega, \theta, z)$ of the initial-boundary value problem (1)–(8) on $Q_T =]0,1[\times]0,T[$ with property (36).*

The proof of Theorem 2 is based on the application of the extension principle and Theorem 1, which means that the generalized solution can be extended in the time domain.

4. Main Result

In this section, we state the main result, give a few initial remarks to be used below, and describe the idea of the proof.

Let us first emphasize that, although it is known that our problem has a local and global solution (Theorems 1 and 2), the proof of the main theorem does not depend on the time interval in which the solution exists. For this reason, an arbitrary time interval is given in the statement of the theorem. We must also note that the conditions (34)–(35) that were necessary for the proof of local existence are not used in the proof of the theorem.

The following theorem expresses the main result of this work.

Theorem 3. *The generalized solution to the initial-boundary value problem (1)–(8) in* $Q_T =]0,1[\times]0,T[$ *is unique for any* $T > 0$ *for which this solution exists.*

To prove this theorem, we mainly use the techniques from the work in [28,45]. Some parts of the proof coincide with the proof in [36], where the uniqueness of the solution for a similar model is considered; so, where this is the case, we omit the detailed procedure and provide an appropriate reference.

Before we start with the proof, we write the problem in an equivalent form, where instead of mass density ρ, we use a specific volume u defined by

$$u = \frac{1}{\rho}. \tag{37}$$

We perform this because the proof is somewhat easier to write when we use this new form of system instead of the original. Transition to the new form is legitimate due to condition (24) in the definition of a generalized solution. Moreover, from (24), (31) and (37), we conclude that

$$u \in L^\infty(Q_T) \quad \text{and} \quad \operatorname*{ess\,inf}_{Q_T} u > 0. \tag{38}$$

So, the function u has the same properties as the function ρ.

To simplify the notation, we also introduce the following functions:

$$r_u(u,\theta,z) = r\left(\frac{1}{u},\theta,z\right), \quad \tilde{r}_u(u,\theta,z) = \tilde{r}\left(\frac{1}{u},\theta,z\right), \tag{39}$$

where r and $\tilde{r}$ are from (12). The obtained system reads

$$\partial_t u = \frac{1}{L}\partial_x v, \tag{40}$$

$$\partial_t v = -\frac{R}{L}\partial_x\left(\frac{\theta}{u^p}\right) + \frac{\lambda+2\mu}{L^2}\partial_x\left(\frac{\partial_x v}{u}\right), \tag{41}$$

$$j_I\partial_t\omega = \frac{c_0+2c_d}{L^2}\partial_x\left(\frac{\partial_x\omega}{u}\right) - 4\mu_r\omega u, \tag{42}$$

$$\begin{aligned} c_v\partial_t\theta = &\frac{\kappa}{L^2}\partial_x\left(\frac{\partial_x\theta}{u}\right) - \frac{R}{L}\frac{\theta\,\partial_x v}{u^p} + \frac{\lambda+2\mu}{L^2}\frac{(\partial_x v)^2}{u} + \frac{c_0+2c_d}{L^2}\frac{(\partial_x\omega)^2}{u} \\ &+ 4\mu_r\omega^2 u + \delta r_u(u,\theta,z), \end{aligned} \tag{43}$$

$$\partial_t z = \frac{\sigma}{L^2}\partial_x\left(\frac{\partial_x z}{u^2}\right) - r_u(u,\theta,z), \tag{44}$$

for $(x,t) \in]0,1[\times]0,T[$,

$$\begin{aligned} u(x,0) = \frac{1}{\rho_0(x)}, \quad v(x,0) = v_0(x), \quad \omega(x,0) = \omega_0(x), \\ \theta(x,0) = \theta_0(x), \quad z(x,0) = z_0(x), \end{aligned} \tag{45}$$

for $x \in [0,1]$,

$$\begin{aligned} v(0,t) = v(1,t) = 0, \quad \omega(0,t) = \omega(1,t) = 0, \\ \partial_x\theta(0,t) = \partial_x\theta(1,t) = 0, \quad \partial_x z(0,t) = \partial_x z(1,t) = 0, \end{aligned} \tag{46}$$

for $t \in [0,T]$.

We prove the theorem using the *reductio ad absurdum* method, i.e., we assume that the problem (40)–(46) has two solutions,

$$(u_i, v_i, \omega_i, \theta_i, z_i), \quad i = 1, 2, \tag{47}$$

in Q_T, where $T > 0$ is arbitrary but fixed. Then, we construct an auxiliary system for the difference of these two solutions, and using a series of estimates, we show that the difference between the two solutions is zero. A key tool in the proof is the integral form of the Grönwall's inequality given in Proposition 4.

Now, we construct the auxiliary system described. Let us denote differences in functions u_i, v_i, ω_i, θ_i, z_i, $i = 1, 2$, by

$$u = u_1 - u_2, \quad v = v_1 - v_2, \quad \omega = \omega_1 - \omega_2, \quad \theta = \theta_1 - \theta_2, \quad z = z_1 - z_2. \tag{48}$$

After subtracting Equations (40)–(44), which, by assumption, hold for $(u_i, v_i, \omega_i, \theta_i, z_i)$, $i = 1, 2$, we obtain

$$\partial_t u = \frac{1}{L}\partial_x v, \tag{49}$$

$$\partial_t v = -\frac{R}{L}\partial_x\left(\frac{\theta}{u_1^p} - \frac{(u_1^p - u_2^p)\,\theta_2}{u_1^p u_2^p}\right) + \frac{\lambda + 2\mu}{L^2}\partial_x\left(\frac{\partial_x v}{u_1} - \frac{u\,\partial_x v_2}{u_1 u_2}\right), \tag{50}$$

$$j_I \partial_t \omega = \frac{c_0 + 2c_d}{L^2}\partial_x\left(\frac{\partial_x \omega}{u_1} - \frac{u\,\partial_x \omega_2}{u_1 u_2}\right) - 4\mu_r(\omega\, u_1 + u\,\omega_2), \tag{51}$$

$$\begin{aligned} c_v \partial_t \theta = {} & \frac{\kappa}{L^2}\partial_x\left(\frac{\partial_x \theta}{u_1} - \frac{u\,\partial_x \theta_2}{u_1 u_2}\right) \\ & - \frac{R}{L}\left(\frac{\theta_1\,\partial_x v}{u_1^p} + \frac{\theta\,\partial_x v_2}{u_1^p} - \frac{(u_1^p - u_2^p)\,\theta_2\,\partial_x v_2}{u_1^p u_2^p}\right) \\ & + \frac{\lambda + 2\mu}{L^2}\left(\frac{(\partial_x v_1 + \partial_x v_2)\,\partial_x v}{u_1} - \frac{(\partial_x v_2)^2 u}{u_1 u_2}\right) \\ & + \frac{c_0 + 2c_d}{L^2}\left(\frac{(\partial_x \omega_1 + \partial_x \omega_2)\,\partial_x \omega}{u_1} - \frac{(\partial_x \omega_2)^2 u}{u_1 u_2}\right) \\ & + 4\mu_r\Big((\omega_1 + \omega_2)\,u_1\omega + \omega_2^2\,u\Big) \\ & + \delta(r_u(u_1, \theta_1, z_1) - r_u(u_2, \theta_2, z_2)), \end{aligned} \tag{52}$$

$$\partial_t z = \frac{\sigma}{L^2}\partial_x\left(\frac{\partial_x z}{u_1^2} - \frac{u(u_1 + u_2)\partial_x z_2}{u_1^2 u_2^2}\right) - (r_u(u_1, \theta_1, z_1) - r_u(u_2, \theta_2, z_2)), \tag{53}$$

for $(x, t) \in \Omega_T$.

Furthermore, since $(\rho_i, v_i, \omega_i, \theta_i, z_i)$ satisfy initial and boundary conditions (45)–(46), the differences $(u, v, \omega, \theta, z)$ satisfy the following homogeneous initial and boundary conditions:

$$u(x, 0) = 0, \quad v(x, 0) = 0, \quad \omega(x, 0) = 0, \quad \theta(x, 0) = 0, \quad z(x, 0) = 0, \tag{54}$$

for $x \in [0, 1]$,

$$\begin{aligned} v(0, t) = v(1, t) = 0, \quad & \omega(0, t) = \omega(1, t) = 0, \\ \partial_x \theta(0, t) = \partial_x \theta(1, t) = 0, \quad & \partial_x z(0, t) = \partial_x z(1, t) = 0, \end{aligned} \tag{55}$$

for $t \in [0, T]$.

5. Auxiliary Results

In this section, we state and prove several auxiliary estimates for functions u, v, ω, θ, and z defined by (48).

Lemma 1. *There exists $C > 0$ such that for all $t \in]0, T[$, we have*

$$\|u(t)\|^2 + \|u_1^p(t) - u_2^p(t)\|^2 + \|v(t)\|^2 + \|\omega(t)\|^2 + \int_0^t \left(\|\partial_x v(\tau)\|^2 + \|\partial_x \omega(\tau)\|^2\right) d\tau \le C \int_0^t \|\theta(\tau)\|^2 d\tau. \tag{56}$$

Proof. The proof of this lemma is analogous to the proof of Lemma 5 in [36], where estimates are obtained for u, v, and ω, which solve the nonreactive model (i.e., a model where $z = 0$). For readers' convenience, here, we outline the proof of this lemma. For more details, we refer the reader to [36].

Multiplying (49) by u and integrating over $]0, 1[$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(||u(t)||^2\right) = \frac{1}{L}\int_0^1 u\, \partial_x v\, dx. \tag{57}$$

Hölder's and Young's inequality then implies

$$\frac{d}{dt}\left(||u(t)||^2\right) \le C||u(t)|| \cdot ||\partial_x v(t)|| \le C\left(||u(t)||^2 + ||\partial_x v(t)||^2\right). \tag{58}$$

Integrating (58) over $]0, t[$ and taking into account initial conditions (54), we have

$$||u(t)||^2 \le C \int_0^t ||u(\tau)||^2 d\tau + C \int_0^t ||\partial_x v(\tau)||^2 d\tau. \tag{59}$$

Finally, Grönwall's inequality applied to (59) implies

$$||u(t)||^2 \le C \int_0^t ||\partial_x v(\tau)||^2 d\tau. \tag{60}$$

Now, let $p > 1$. Multiplying (40), for u_i and v_i, $i = 1, 2$, by u_i^{p-1}, respectively, we obtain

$$\partial_t(u_i^p) = \frac{p}{L}\partial_x v_i\, u_i^{p-1}, \quad i = 1, 2. \tag{61}$$

By subtracting Equation (61) for $i = 1$ and $i = 2$, we obtain

$$\frac{d}{dt}\left(u_1^p - u_2^p\right) = \frac{p}{L}\left(\partial_x v u_1^{p-1} + \partial_x v_2 \frac{u_1^p - u_2^p}{u_1} - \partial_x v_2 \frac{u_2^{p-1} u}{u_1}\right). \tag{62}$$

Now, similar to the proof of estimate (60), after multiplying (62) by $(u_1^p - u_2^p)$, integrating over $]0, 1[$, using property (38), Hölder's and Young's inequality, and Proposition 3, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(||u_1^p(t) - u_2^p(t)||^2\right) \\
=&\frac{p}{L}\int_0^1 \left(\partial_x v u_1^{p-1}(u_1^p - u_2^p) + \partial_x v_2 \frac{(u_1^p - u_2^p)^2}{u_1} - \partial_x v_2 \frac{u_2^{p-1} u}{u_1}(u_1^p - u_2^p)\right) dx \\
\le& C\Big[\left(||\partial_x v(t)||^2 + ||u_1^p(t) - u_2^p(t)||^2\right) + \left(1 + ||\partial_{xx} v_2(t)||^2\right)||u_1^p(t) - u_2^p(t)||^2 \\
&+ \left(1 + ||\partial_{xx} v_2(t)||^2\right)\left(||u(t)||^2 + ||u_1^p(t) - u_2^p(t)||^2\right)\Big].
\end{aligned} \tag{63}$$

Integrating (63) over $]0, t[$ and taking into account initial conditions (45), we obtain

$$||u_1^p(t) - u_2^p(t)||^2 \leq C \int_0^t \Big(1 + ||\partial_{xx} v_2(\tau)||^2\Big) ||u_1^p(\tau) - u_2^p(\tau)||^2 d\tau \\ + C \int_0^t \Big(1 + ||\partial_{xx} v_2(\tau)||^2\Big) ||u(\tau)||^2 d\tau + C \int_0^t ||\partial_x v(\tau)||^2 d\tau. \tag{64}$$

Estimates (60) and (64) imply

$$||u_1^p(t) - u_2^p(t)||^2 \leq C \int_0^t \Big(1 + ||\partial_{xx} v_2(\tau)||^2\Big) ||u_1^p(\tau) - u_2^p(\tau)||^2 d\tau \\ + C \int_0^t ||\partial_x v(\tau)||^2 d\tau, \tag{65}$$

from which Grönwall's inequality implies

$$||u_1^p(t) - u_2^p(t)||^2 \leq C \int_0^t ||\partial_x v(\tau)||^2 d\tau. \tag{66}$$

Multiplying (50)–(51) by v and ω, respectively, integrating over $]0, 1[$, and taking into account (55), we obtain

$$\frac{1}{2}\frac{d}{dt}\Big(||v(t)||^2\Big) + \frac{\lambda + 2\mu}{L^2} \int_0^1 \frac{(\partial_x v)^2}{u_1} dx = \frac{R}{L} \int_0^1 \frac{\theta \partial_x v}{u_1^p} dx \\ - \frac{R}{L} \int_0^1 \frac{(u_1^p - u_2^p)\theta_2 \partial_x v}{u_1^p u_2^p} dx + \frac{\lambda + 2\mu}{L^2} \int_0^1 \frac{u \partial_x v \, \partial_x v_2}{u_1 u_2} dx, \tag{67}$$

$$\frac{j_I}{2}\frac{d}{dt}\Big(||\omega(t)||^2\Big) + \frac{c_0 + 2c_d}{L^2} \int_0^1 \frac{(\partial_x \omega)^2}{u_1} dx \\ = \frac{c_0 + 2c_d}{L^2} \int_0^1 \frac{u \partial_x \omega \, \partial_x \omega_2}{u_1 u_2} dx - 4\mu_r \int_0^1 \Big(u_1 \omega^2 + u\, \omega_2 \omega\Big) dx. \tag{68}$$

Applying (38), we obtain estimates from below for the integrals on the left side of (67)–(68)

$$\frac{\lambda + 2\mu}{L^2} \int_0^1 \frac{(\partial_x v)^2}{u_1} dx \geq C_1 ||\partial_x v(t)||^2, \tag{69}$$

$$\frac{c_0 + 2c_d}{L^2} \int_0^1 \frac{(\partial_x \omega)^2}{u_1} dx \geq C_2 ||\partial_x \omega(t)||^2. \tag{70}$$

Applying (38), Hölder's, and Young's inequality, and Proposition 3, we estimate integrals on the right-hand side of (67)–(68) similarly as we did before. Taking that into account together with (60), (66), and (69)–(70), we obtain

$$\frac{1}{2}\frac{d}{dt}||v(t)||^2 + C_1 ||\partial_x v(t)||^2 \leq 3\alpha ||\partial_x v(t)||^2 \\ + \frac{C}{\alpha}\Big(||\theta(t)||^2 + ||u_1^p(t) - u_2^p(t)||^2 + ||u(t)||^2 \, ||\partial_{xx} v_2(t)||^2\Big), \tag{71}$$

$$\frac{j_I}{2}\frac{d}{dt}||\omega(t)||^2 + C_2 ||\partial_x \omega(t)||^2 \leq 2\alpha ||\partial_x \omega(t)||^2 + C||\omega(t)||^2 \\ + \frac{C}{\alpha}\Big(1 + ||\partial_{xx} \omega_2(t)||^2\Big) ||u(t)||^2. \tag{72}$$

where $\alpha > 0$ is to be determined in the next step.

Integrating (71)–(72) over $]0,t[$ and taking into account initial conditions (54), as well as $\alpha < \min\left\{\frac{C_1}{3}, \frac{C_2}{2}\right\}$, we obtain

$$\begin{aligned} ||v(t)||^2 + \int_0^t ||\partial_x v(\tau)||^2 d\tau \leq & C \int_0^t ||\theta(\tau)||^2 d\tau \\ & + C \int_0^t \left(1 + ||\partial_{xx} v_2(\tau)||^2\right) \int_0^\tau ||\partial_x v(s)||^2 ds\, d\tau, \end{aligned} \tag{73}$$

$$||\omega(t)||^2 + \int_0^t ||\partial_x \omega(\tau)||^2 d\tau \leq C \int_0^t ||\omega(\tau)||^2 d\tau + C \int_0^t ||\theta(\tau)||^2\, d\tau. \tag{74}$$

Applying Grönwall's inequality to (73)–(74) implies

$$||v(t)||^2 + \int_0^t ||\partial_x v(\tau)||^2 d\tau \leq C \int_0^t ||\theta(\tau)||^2 d\tau, \tag{75}$$

$$||\omega(t)||^2 + \int_0^t ||\partial_x \omega(\tau)||^2 d\tau \leq C \int_0^t ||\theta(\tau)||^2 d\tau. \tag{76}$$

Adding up (60), (66), (75), and (76), we obtain (56). □

To estimate function z, we should first determine the properties of the function r_u. From the definition of $\tilde{r}_u$ and properties of r stated in the introduction, it is easy to see that $\tilde{r}_u$ is a non-negative function defined on $]0,+\infty[\times]0,+\infty[\times[0,+\infty[$, bounded on sets of the form $[a,b]\times]0,+\infty[\times[0,+\infty[$, continuous with respect to ρ, and globally Lipschitz-continuous with respect to θ and z, with the properties

$$\lim_{\rho\to 0^+} \tilde{r}_u(\rho,\theta,z) = 0, \qquad \lim_{\theta\to 0^+} \tilde{r}_u(\rho,\theta,z) = 0. \tag{77}$$

It is not hard to show that $\tilde{r}$ is also Lipschitz-continuous on bounded sets with respect to ρ.

Let $0 < a < b$ and $u_1, u_2 \in]a,b[$. Since r is Lipschitz-continuous on bounded sets with respect to ρ, for all $(\theta, z) \in]0,+\infty[\times[0,+\infty[$, we have

$$|\tilde{r}(u_1,\theta,z) - \tilde{r}(u_2,\theta,z)| \leq L_{\rho,a,b}|u_1 - u_2|, \tag{78}$$

for some $L_{\rho,a,b} \geq 0$. From this, it follows that

$$\begin{aligned} |\tilde{r}_u(u_1,\theta,z) - \tilde{r}_u(u_2,\theta,z)| &= \left| r\left(\frac{1}{u_1},\theta,z\right) - r\left(\frac{1}{u_2},\theta,z\right)\right| \\ &\leq \frac{L_{\rho,a,b}}{|u_1u_2|}|u_1 - u_2| \leq \frac{L_{\rho,a,b}}{a^2}|u_1 - u_2|, \end{aligned} \tag{79}$$

i.e., $\tilde{r}$ is Lipschitz-continuous on $]a,b[\times]0,+\infty[\times[0,+\infty[$ with respect to ρ.

In the following lemma, we estimate the function z.

Lemma 2. *There exists $C > 0$ such that for all $t \in]0,T[$ we have*

$$\|z(t)\|^2 + \int_0^t \|\partial_x z(\tau)\|^2 d\tau \leq C \int_0^t \|\theta(\tau)\|^2 d\tau. \tag{80}$$

Proof. After multiplying Equation (53) by z and then integrating it over $]0,1[$, applying integration by parts, and substituting boundary conditions (55), we obtain

$$\frac{d}{dt}\left(\|z(t)\|^2\right) + \frac{\sigma}{L^2}\int_0^1 \frac{(\partial_x z)^2}{u_1^2}\,dx = \frac{\sigma}{L^2}\int_0^1 \frac{(u_1+u_2)u\partial_x z\,\partial_x z_2}{u_1^2u_2^2}\,dx - \int_0^1 z(r_u(u_1,\theta_1,z_1) - r_u(u_2,\theta_2,z_2))\,dx. \tag{81}$$

Using properties (38), we obtain the following estimate for the integral on the left-hand side of (81):

$$\frac{\sigma}{L^2}\int_0^1 \frac{(\partial_x z)^2}{u_1^2}\,dx \geq C_1\|\partial_x z(t)\|^2. \tag{82}$$

Notice that mapping $x \mapsto x^m$ is Lipshitz-continuous on the bounded interval

$$I = \left[\min_{i=1,2}\min_{(x,t)\in Q_T}|z_i(x,t)|, \max_{i=1,2}\max_{(x,t)\in Q_T}|z_i(x,t)|\right] \tag{83}$$

since the mean value theorem and (32) give

$$|\zeta_1^m - \zeta_2^m| \leq m\max_{\zeta\in I}|\zeta|^{m-1}|\zeta_1-\zeta_2| \leq C|\zeta_1-\zeta_2|, \tag{84}$$

for all $\zeta_1,\zeta_2 \in I$.

From (38), we have $|u_i| \leq C$, and $u_i^2 \geq C^{-1} > 0$ for $i = 1,2$. From there, we obtain the following estimate for the first integral on the right side of (81):

$$\begin{aligned} \left|\int_0^1 \frac{(u_1+u_2)u\partial_x z\,\partial_x z_2}{u_1^2u_2^2}\,dx\right| &\leq \int_0^1 \frac{(|u_1|+|u_2|)\cdot|u|\cdot|\partial_x z|\cdot|\partial_x z_2|}{u_1^2u_2^2}\,dx \\ &\leq C_2\int_0^1 |u|\cdot|\partial_x z|\cdot|\partial_x z_2|\,dx. \end{aligned} \tag{85}$$

Proposition 3 implies that $|\partial_x z_2(t)| \leq C\|\partial_{xx}z_2(t)\|$, so by using Hölder's inequality and the property (32) for the solutions, we obtain

$$\begin{aligned} \int_0^1 |u|\cdot|\partial_x z|\cdot|\partial_x z_2|\,dx &\leq C_3\|\partial_x z_2\|\int_0^1 |u|\cdot|\partial_x z|\,dx \\ &\leq C_3\|\partial_{xx}z_2(t)\|\cdot\|u(t)\|\cdot\|\partial_x z(t)\|. \end{aligned} \tag{86}$$

Young's inequality, for some $\alpha > 0$, which we determine later, implies

$$\|\partial_{xx}z_2(t)\|\cdot\|u(t)\|\cdot\|\partial_x z(t)\| \leq \frac{\alpha}{C_2C_3}\|\partial_x z(t)\|^2 + \frac{C}{\alpha}\|u(t)\|^2\,\|\partial_{xx}z_2(t)\|^2. \tag{87}$$

Combining (85)–(87), we obtain the following estimates for the first integral on the right side of (81):

$$\left|\int_0^1 \frac{(u_1+u_2)u\partial_x z\,\partial_x z_2}{u_1^2u_2^2}\,dx\right| \leq \alpha\|\partial_x z(t)\|^2 + \frac{C}{\alpha}\|u(t)\|^2\,\|\partial_{xx}z_2(t)\|^2. \tag{88}$$

Property (38) implies that u takes values on a bounded set. Taking into account that the function $\tilde{r}_u$ is bounded and globally Lipschitz-continuous with respect to θ and z, as well as Lipschitz-continuous with respect to u on all bounded sets, together with (79) and (84), we have

$$\begin{aligned}\left|\int_0^1 z(r_u(u_1,\theta_1,z_1)\ -r_u(u_2,\theta_2,z_2))\,dx\right| \leq & \left|\int_0^1 z(z_1^m - z_2^m)\tilde{r}_u(u_1,\theta_1,z_1)dx\right| \\ & + \left|\int_0^1 zz_2^m(\tilde{r}_u(u_1,\theta_1,z_1) - \tilde{r}_u(u_2,\theta_1,z_1))dx\right| \\ & + \left|\int_0^1 zz_2^m(\tilde{r}_u(u_2,\theta_1,z_1) - \tilde{r}_u(u_2,\theta_2,z_1))dx\right| \\ & + \left|\int_0^1 zz_2^m(\tilde{r}_u(u_2,\theta_2,z_1) - \tilde{r}_u(u_2,\theta_2,z_2))dx\right| \\ \leq & C\int_0^1 \Big(2|z|^2 + |z|\cdot|u| + |z|\cdot|\theta|\Big)dx.\end{aligned} \tag{89}$$

Applying Hölder's and then Young's inequality leads to

$$\begin{aligned}\int_0^1 \left(2|z|^2 + |z|\frac{|u|}{|u_1u_2|} + |z\theta|\right)dx \leq & C\Big(\|z\|^2 + \|z\|\cdot\|u\| + \|z\|\cdot\|\theta\|\Big) \\ \leq & C\Big(\|z\|^2 + \|u\|^2 + \|\theta\|^2\Big),\end{aligned} \tag{90}$$

which combined with (89) yields the following estimate for the second integral on the right side of (81)

$$\left|\int_0^1 z(r_u(u_1,\theta_1,z_1)\ -r_u(u_2,\theta_2,z_2))\,dx\right| \leq C\Big(\|z\|^2 + \|u\|^2 + \|\theta\|^2\Big). \tag{91}$$

After integrating (81) over $]0,t[$ for any $t \in [0,T]$ and then applying estimates (82) and (91) and inserting initial conditions (54) into the obtained relation, we obtain

$$\begin{aligned}\|z(t)\|^2 + C_1\int_0^t \|\partial_x z(\tau)\|^2 d\tau \leq \alpha\int_0^t \|\partial_x z(\tau)\|^2 d\tau \\ +C\int_0^t \Big(\|z(\tau)\|^2 + \|u(\tau)\|^2 + \|\theta(\tau)\|^2\Big)d\tau + \frac{C}{\alpha}\int_0^t \|\partial_{xx}z_2(\tau)\|^2\|u(\tau)\|^2 d\tau.\end{aligned} \tag{92}$$

Let $\alpha < C_1$. By applying estimate (56) in (92), we obtain

$$\begin{aligned}\|z(t)\|^2 + \int_0^t \|\partial_x z(\tau)\|^2 d\tau \leq C\int_0^t \|z(\tau)\|^2 d\tau + C\int_0^t \|\theta(\tau)\|^2 d\tau \\ +C\int_0^t \Big(1 + \|\partial_{xx}z_2(\tau)\|^2\Big)\int_0^\tau \|\theta(s)\|ds\,d\tau.\end{aligned} \tag{93}$$

Using properties (25) in (93), we obtain

$$\|z(t)\|^2 + \int_0^t \|\partial_x z(\tau)\|^2 d\tau \leq C\int_0^t \|z(\tau)\|^2 d\tau + C\int_0^t \|\theta(\tau)\|^2\,d\tau. \tag{94}$$

Finally, taking

$$\xi(t) = \|z(t)\|^2 + \int_0^t \|\partial_x z(\tau)\|^2 d\tau \tag{95}$$

in Grönwall's inequality from Proposition 4 and taking into account (94), we conclude that the assertion of the lemma holds. □

6. Proof of Main Theorem

Now, we prove our main result, i.e., Theorem 3.

First, we multiply (52) by θ and integrate the obtained equation over $]0,1[$, then after applying integration by parts and inserting boundary conditions (55), we obtain

$$\begin{aligned}
\frac{c_v}{2}\frac{d}{dt}\left(\|\theta(t)\|^2\right) + \frac{\kappa}{L^2}\int_0^1 \frac{(\partial_x\theta)^2}{u_1}\,dx &= \frac{\kappa}{L^2}\int_0^1 \frac{u\partial_x\theta\,\partial_x\theta_2}{u_1u_2}\,dx \\
&- \frac{R}{L}\int_0^1\left(\frac{\theta_1\partial_x v}{u_1^p} + \frac{\theta\partial_x v_2}{u_1^p} - \frac{(u_1^p-u_2^p)\theta_2\partial_x v_2}{u_1^pu_2^p}\right)\theta\,dx \\
&+ \frac{\lambda+2\mu}{L^2}\int_0^1\left(\frac{\partial_x v}{u_1}(\partial_x v_1+\partial_x v_2) - \frac{(\partial_x v_2)^2u}{u_1u_2}\right)\theta\,dx \\
&+ \frac{c_0+2c_d}{L^2}\int_0^1\left(\frac{\partial_x\omega}{u_1}(\partial_x\omega_1+\partial_x\omega_2) - \frac{(\partial_x\omega_2)^2u}{u_1u_2}\right)\theta\,dx \\
&+ 4\mu_r\int_0^1\left((\omega_1+\omega_2)\omega u_1 + \omega_2^2u\right)\theta\,dx \\
&+ \delta\int_0^1(r_u(u_1,\theta_1,z_1) - r_u(u_2,\theta_2,z_2))\theta\,dx.
\end{aligned} \tag{96}$$

Using properties (37) and (38), we estimate the integral on the left-hand side of (96) and obtain

$$\frac{\kappa}{L^2}\int_0^1\frac{(\partial_x\theta)^2}{u_1}\,dx \geq C_4\|\partial_x\theta(t)\|^2. \tag{97}$$

We estimate integrals on the right-hand side of (96) using properties of function $\tilde{r}_u$ defined by (32), (38), (39), Hölder's and Young's inequalities, and Proposition 3, for some $\alpha > 0$, which we specify later. Since these estimates are similar to estimates in the proof of Lemma 6 in [36] and estimate (91) from Lemma 2 in the previous section of this paper, we omit writing the details. The obtained estimates read as follows:

$$\begin{aligned}
\left|\int_0^1\frac{u\partial_x\theta\,\partial_x\theta_2}{u_1u_2}\,dx\right| &\leq \int_0^1\frac{|u|\cdot|\partial_x\theta|\cdot|\partial_x\theta_2|}{u_1u_2}\,dx \leq C\|\partial_{xx}\theta_2\|\int_0^1 u|\partial_x\theta|\,dx \\
&\leq C\|\partial_{xx}\theta_2(t)\|\cdot\|u(t)\|\cdot\|\partial_x\theta(t)\| \leq \alpha\|\partial_x\theta(t)\|^2 + \frac{C}{\alpha}\|\partial_{xx}\theta_2(t)\|^2\cdot\|u(t)\|^2,
\end{aligned} \tag{98}$$

$$\begin{aligned}
\left|\int_0^1\frac{\theta_1\theta\partial_x v}{u_1^p}\,dx\right| &\leq \int_0^1\frac{\theta_1|\theta|\cdot|\partial_x v|}{u_1^p}\,dx \leq C\|\partial_{xx}v\|\int_0^1\theta_1|\theta|\,dx \\
&\leq C\|\partial_x v(t)\|\cdot\|\theta(t)\| \leq C\|\partial_x v(t)\|^2 + C\|\theta(t)\|^2,
\end{aligned} \tag{99}$$

$$\begin{aligned}
\left|\int_0^1\frac{\theta^2\partial_x v_2}{u_1^p}\,dx\right| &\leq \int_0^1\frac{\theta^2|\partial_x v_2|}{u_1^p}\,dx \leq C\|\partial_{xx}v_2\|\int_0^1\theta^2\,dx \\
&\leq C\|\partial_{xx}v_2(t)\|\cdot\|\theta(t)\|^2 \leq C\left(1+\|\partial_{xx}v_2(t)\|^2\right)\|\theta(t)\|^2,
\end{aligned} \tag{100}$$

$$\begin{aligned}
\left|\int_0^1\frac{(u_1^p-u_2^p)\theta_2\theta\partial_x v_2}{u_1^pu_2^p}\,dx\right| &\leq C\|\partial_{xx}v_2(t)\|\cdot\|u_1^p(t)-u_2^p(t)\|\cdot\|\theta(t)\| \\
&\leq C\|\partial_{xx}v_2(t)\|^2\cdot\|u_1^p(t)-u_2^p(t)\|^2 + C\|\theta(t)\|^2,
\end{aligned} \tag{101}$$

$$\begin{aligned}
\left|\int_0^1\frac{\partial_x v}{u_1}(\partial_x v_1+\partial_x v_2)\theta\,dx\right| &\leq C(\|\partial_{xx}v_1(t)\|+\|\partial_{xx}v_2(t)\|)\|\partial_x v(t)\|\cdot\|\theta(t)\| \\
&\leq C\|\partial_x v(t)\|^2 + C\left(\|\partial_{xx}v_1(t)\|^2+\|\partial_{xx}v_2(t)\|^2\right)\|\theta(t)\|^2,
\end{aligned} \tag{102}$$

$$\begin{aligned}
\left|\int_0^1\frac{(\partial_x v_2)^2u\theta}{u_1u_2}\,dx\right| &\leq C\|\partial_{xx}v_2(t)\|^2\cdot\|u(t)\|\cdot\|\theta(t)\| \\
&\leq C\|\partial_{xx}v_2(t)\|^2\left(\|u(t)\|^2+\|\theta(t)\|^2\right),
\end{aligned} \tag{103}$$

$$\begin{aligned}\left|\int_0^1 \frac{\partial_x\omega}{u_1}(\partial_x\omega_1+\partial_x\omega_2)\theta\,dx\right| &\le \int_0^1 \frac{|\partial_x\omega|}{u_1}(|\partial_x\omega_1|+|\partial_x\omega_2|)\theta\,dx \\ &\le C\|\partial_x\omega(t)\|^2 + C\Big(\|\partial_{xx}\omega_1(t)\|^2+\|\partial_{xx}\omega_2(t)\|^2\Big)\|\theta(t)\|^2,\end{aligned} \tag{104}$$

$$\begin{aligned}\left|\int_0^1 \frac{(\partial_x\omega_2)^2 u\theta}{u_1u_2}\,dx\right| &\le \int_0^1 \frac{(\partial_x\omega_2)^2|u|\cdot|\theta|}{u_1u_2}\,dx \le C\|\partial_x\omega_2\|^2\int_0^1 |u|\cdot|\theta|\,dx \\ &\le C\|\partial_{xx}\omega_2(t)\|^2\|u(t)\|\|\theta(t)\| \le C\|\partial_{xx}\omega_2(t)\|^2\Big(\|u(t)\|^2+\|\theta(t)\|^2\Big),\end{aligned} \tag{105}$$

$$\begin{aligned}\left|\int_0^1 (\omega_1+\omega_2)\omega u_1\theta\,dx\right| &\le \int_0^1 (|\omega_1|+|\omega_2|)\cdot|\omega|\cdot u_1\cdot|\theta|\,dx \le C\int_0^1 |\omega|\cdot|\theta|\,dx \\ &\le C\|\omega(t)\|\cdot\|\theta(t)\| \le C\|\partial_x\omega(t)\|^2 + C\|u(t)\|^2,\end{aligned} \tag{106}$$

$$\begin{aligned}\left|\int_0^1 \omega_2^2 u\theta\,dx\right| &\le \int_0^1 \omega_2^2\cdot|u|\cdot|\theta|\,dx \le C\int_0^1 \cdot|u|\cdot|\theta|\,dx \\ &\le C\|\theta(t)\|\cdot\|u(t)\| \le C\|\theta(t)\|^2 + C\|u(t)\|^2,\end{aligned} \tag{107}$$

and finally,

$$\begin{aligned}\left|\int_0^1 \theta(r_u(u_1,\theta_1,z_1)-r_u(u_2,\theta_2,z_2))dx\right| &\le \left|\int_0^1 \theta(z_1^m-z_2^m)\tilde{r}_u(u_1,\theta_1,z_1)dx\right| \\ &+\left|\int_0^1 \theta z_2^m(\tilde{r}_u(u_1,\theta_1,z_1)-\tilde{r}_u(u_2,\theta_1,z_1))dx\right| \\ &+\left|\int_0^1 \theta z_2^m(\tilde{r}_u(u_2,\theta_1,z_1)-\tilde{r}_u(u_2,\theta_2,z_1))dx\right| \\ &+\left|\int_0^1 \theta z_2^m(\tilde{r}_u(u_2,\theta_2,z_1)-\tilde{r}_u(u_2,\theta_2,z_2))dx\right| \\ &\le C\int_0^1\left(|z\theta|+|\theta|\frac{|u|}{|u_1u_2|}+|\theta|^2\right)dx \\ &\le C\Big(\|z\|\cdot\|\theta\|+\|u\|\cdot\|\theta\|+\|\theta\|^2\Big) \\ &\le C\Big(\|z\|^2+\|u\|^2+\|\theta\|^2\Big).\end{aligned} \tag{108}$$

where we additionally used the Lipschitz continuity of mapping $x \mapsto x^m$.

Using estimates (97)–(108) in (96), we obtain

$$\begin{aligned}\frac{c_v}{2}\frac{d}{dt}\Big(\|\theta(t)\|^2\Big) + C_4\|\partial_x\theta(t)\|^2 &\le \alpha\|\partial_x\theta(t)\|^2 \\ &+\frac{C}{\alpha}\|\partial_{xx}\theta_2(t)\|^2\cdot\|u(t)\|^2 + C\Big(\|\partial_x v(t)\|^2+\|\partial_x\omega(t)\|^2\Big) \\ &+C\Big(\|\theta(t)\|^2+\|u(t)\|^2+\|u_1^p(t)-u_2^p(t)\|^2+\|z(\tau)\|^2\Big)W(t),\end{aligned} \tag{109}$$

where

$$W(t) = 1+\|\partial_{xx}v_1(t)\|^2+\|\partial_{xx}v_2(t)\|^2+\|\partial_{xx}\omega_1(t)\|^2+\|\partial_{xx}\omega_2(t)\|^2+\|\partial_{xx}\theta_2(t)\|^2. \tag{110}$$

Using (25), we conclude that the following holds:

$$\int_0^t W(\tau)d\tau \le C. \tag{111}$$

Let $\alpha < C_4$. After integrating (109) over $]0, t[$ for any $t \in [0, T]$ and then inserting initial conditions (54), we obtain

$$\begin{aligned} \|\theta(t)\|^2 + \int_0^t \|\partial_x \theta(\tau)\|^2 d\tau \leq C \int_0^t \Big[\|\partial_x v(\tau)\|^2 + \|\partial_x \omega(\tau)\|^2 \\ + W(\tau) \Big(\|\theta(\tau)\|^2 + \|u(\tau)\|^2 + \|u_1^p(\tau) - u_2^p(\tau)\|^2 + \|z(\tau)\|^2 \Big) \Big] d\tau, \end{aligned} \tag{112}$$

for $t \in [0, T]$. Applying estimates (56)–(80) to (112) implies

$$\begin{aligned} \|\theta(t)\|^2 + \int_0^t \|\theta(\tau)\|^2 d\tau \leq C \int_0^t \|\theta(\tau)\|^2 d\tau \\ + C \int_0^t W(\tau) \left(\|\theta(\tau)\|^2 + \int_0^\tau \|\theta(s)\|^2 ds \right) d\tau \leq C \int_0^t W(\tau) \|\theta(\tau)\|^2 d\tau. \end{aligned} \tag{113}$$

From (111), (113), and taking

$$\xi(t) = \|\theta(t)\|^2 + \int_0^t \|\theta(\tau)\|^2 d\tau \tag{114}$$

in Grönwall's inequality from Proposition 4, we conclude

$$\theta = 0 \quad \text{in} \quad Q_T, \tag{115}$$

and then, from (56)–(80), it also follows that

$$u = 0, \quad v = 0, \quad \omega = 0, \quad z = 0 \quad \text{in} \quad Q_T. \tag{116}$$

This concludes the proof of Theorem 3.

7. Conclusions

In this paper, we have analyzed the thermal explosion model of a one-dimensional micropolar real gas, which has the form of an initial-boundary value problem with a quasi-linear parabolic system of partial differential equations. Previously, only the existence of local and global solutions in time was known for this problem, which was not sufficient for the theoretical consistency of the model. In this work, we have shown the uniqueness of the solution to this problem, which makes the model theoretically consistent and suitable for further analysis and research of its mathematical properties, as well as for the development of numerical methods for its solution and finally for research on concrete practical applications. From a mathematical point of view, the research of the given model will go in several directions. Firstly, more precise mathematical properties of the solutions such as regularity and stabilization properties of parabolic problems can be investigated. In addition, it is certainly necessary to find an optimal method for the numerical solution to the problem but also to explore problems with other boundary conditions, for example, inhomogeneous boundary conditions, which can describe a larger number of phenomena. It is certainly worth highlighting that several estimation methods have been used in the work, which have been adapted to the analyzed system and, as such, can be used in the analysis of problems with a similar mathematical structure.

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