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Optimal Corrective Maintenance Policies via an Availability-Cost Hybrid Factor for Software Aging Systems

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Abstract: Availability is an important index for the evaluation of the performance of software aging systems. Although the corrective maintenance increases the system availability, the associated cost may be very high; therefore, the balancing of availability and cost during the corrective maintenance phase is a critical issue. This paper investigates optimal corrective maintenance policies via an availability-cost hybrid factor for software aging systems. The system is described by a group of coupled differential equations, where the multiplier effect of the repair rate on a system variable is bilinear term. Our aim is to drive an optimal repair rate that ensures a balance between the maximal system availability and the minimal repair cost. In a finite time interval $[0, T]$, we rigorously discuss the state space of the system and prove the existence of the optimal repair rate, and then derive the first-order necessary optimality conditions by applying a variational inequality with the adjoint variables.

Keywords: corrective maintenance; availability; cost; bilinear software systems

MSC: 93C95; 93A10

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1. Introduction

Software aging, caused by aging-related bugs and crash-related bugs, is an inevitable phenomenon that plagues many long-running software systems and manifests itself in reduced system availability [1–3]. The earliest system studied for software aging issues was the Unix system [4], followed by the Window NT system [5], Linux system [6], Android system [7], and virtual machines [8].

With the rapid growth of Internet technology and the emergence of advanced online applications, it is important to ensure the high availability of software systems. To meet such a requirement, software systems need to be maintained. When maintenance strategies are carried out, the availability of the system increases, but at the same time, the cost of maintenance also increases. Therefore, how to strike a balance between availability and cost is a worthy issue. Software rejuvenation, first proposed by Huang et al. [9], is a preventive and proactive maintenance method for performance degradation caused by software aging phenomenons. As Alonso et al. [10] pointed out, service outages that are caused by rejuvenating the software system involve an overhead because of process terminations, job drops, or data loss. Hence, it is important to determine the optimal trigger time of software rejuvenation for achieving the best trade-off between the system availability and the rejuvenation cost. Up to now, a lot of works have been performed on such optimization problems. Huang et al. [9] deduced the threshold conditions of optimal rejuvenation time for telecommunications billing applications based on the downtime and the cost caused by downtime. Dohi et al. [11] formulated the software rejuvenation models, and derived the optimal software rejuvenation schedules by considering steady-state availability and the expected cost per unit time in the steady-state. Chen and Trivedi [12] established a semi-Markov decision process of a condition-based maintenance strategy optimization for preventive maintenance problems, and proposed a method of joint optimization of maintenance rate and maintenance strategy considering the system operating cost. Eto

and Dohi [13] considered an operational software system with multiple degradations, and obtained the optimal software rejuvenation strategy with the minimum expected operating cost per unit time in steady-state through the method of dynamic programming. Meng et al. [14] investigated a periodically inspected rejuvenation policy for software systems, obtained the system availability function and cost rate function, and derived the optimal inspection time and the rejuvenation interval for maximizing system availability and minimizing cost rate. Koutras and Platis [15] developed multi-objective optimization problems to derive the rejuvenation policies for optimizing the system overall performance capability, taking into account availability and operating cost constraints. Zheng et al. [16] proposed a stochastic framework composed of a composite stochastic Petri reward net and its resulting non-Markovian availability model to capture the dynamic behavior of an operational software system in which time-based software rejuvenation and checkpointing are both aperiodically conducted. Baghi and Navimipour [17] utilized a hybrid cuckoo search (CS) and genetic algorithm (GA) to optimize the scheduling of cloud database paths, aiming to enhance the availability and reduce the cost of cloud computing.

Maintenance policies are divided into preventive maintenance and corrective maintenance [18]. Corrective maintenance, as a strategy for system maintenance, also known as post-fault maintenance, refers to taking maintenance measures after the system fails to restore the system to its normal state [18]. Since this maintenance strategy can only be implemented after the system fails, considering the maintenance costs under the control of the repair rate is important [19]. Saranga [20] proposed a general downtime cost model and used genetic algorithms to optimize the maintenance costs for complex systems. Ding et al. [21] evaluated the total expected cost and the upper and lower bounds of system availability of multi-state aging systems based on a continuously increasing failure rate function and a segmented constant approximation method, maximizing system availability, and minimizing expected costs. Özgür-Ünlüakın et al. [22] proposed a cost-effective dynamic Bayesian network modeling method, maximizing the reliability of the thermal storage air heaters systems and minimizing maintenance costs. Although there has been a rapid growth in the literature on preventive maintenance of software aging systems to combat the aging phenomenon for achieving the best balance between the software system availability and the rejuvenation cost, the research on corrective maintenance is very limited. In fact, corrective maintenance is an effective method to solve a system's sudden service failures or crashes caused by the software aging phenomenon [23]. In this paper, we study a bilinear control problem for software aging systems by taking the repair rate as corrective maintenance. Mathematically, the system is formulated as a group of coupled differential equations. Compared with the other control strategies of software aging systems in the existing literature [24,25], the significant characteristic of this system is that the repair rate takes multiplicative effect on the system variable, named a bilinear control system [26].

The rest of the paper is organized as follows: Section 2 presents the mathematical framework of the software aging system and describes the corrective rejuvenation policy as an optimal repair rate control problem. Section 3 discusses the well-posedness of the software system and the existence of the optimal repair rate within a finite time interval $[0, T]$. In Section 4, the first-order necessary condition for the optimal repair rate is derived using a variational inequality, and the Bang-Bang property of the optimal repair rate control is derived. Section 5 provides a summary of the paper.

2. System Model and Preliminary Results

Let us recall the model of the considered software aging systems [27]. The system consists of the following three states: robust state, failure-prone state, and failure state, which are represented in Figure 1 as 0, 1, and 2, respectively. The sojourn times from 0 to 1, from 1 to 2, and from 0 to 2 follows an exponential distribution with parameters α, β and γ, respectively. The repair time from 2 to 0 follows a general distribution with repair rate $\mu(x)$. In this model, the system that begins operating in state 0 can either go into state 1

and then into state 2 due to the aging-related bugs, or directly go into state 2 due to the crash-related bugs [1,2]. In state 2, the system will be repaired and return to state 0.

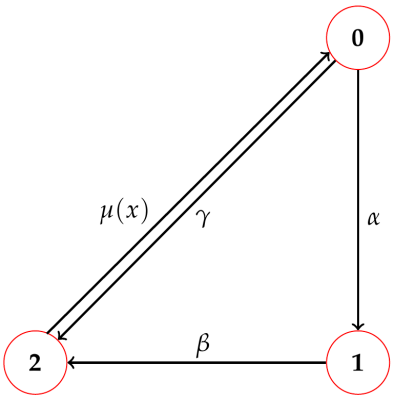


Figure 1. State transition diagram of software aging systems.

By the supplementary variable method [28], the system can be expressed as the following set of coupled differential equations:

$$\begin{cases} \dfrac{\mathrm{d}p_0(t)}{\mathrm{d}t} = -(\alpha+\gamma)p_0(t) + \displaystyle\int_0^\infty \mu(x)p_2(t,x)\mathrm{d}x \\ \dfrac{\mathrm{d}p_1(t)}{\mathrm{d}t} = \alpha p_0(t) - \beta p_1(t) \\ \dfrac{\partial p_2(t,x)}{\partial t} + \dfrac{\partial p_2(t,x)}{\partial x} = -\mu(x)p_2(t,x) \end{cases} \tag{1}$$

with the boundary condition

$$p_2(t,0) = \gamma p_0(t) + \beta p_1(t) \tag{2}$$

and the initial values

$$p_0(0) = 1,\ p_1(0) = p_2(0,x) = 0, \tag{3}$$

here $p_0(t)$ and $p_1(t)$ stand for the probability that the system is in the robust state and in the failure probable state at time t, respectively, $p_2(t,x)$ stands for the probability density that the system is in the failure state and has an elapsed repair time of x at time t.

Let T be a fixed time satisfying $0 < T < \infty$. This paper investigates the corrective maintenance policy for software aging systems (1)—(3) in the time interval $[0,T]$, with the aim of analyzing the optimal repair rate that strikes a balance between the maximal system interval availability and the minimal repair cost of the system.

For the software aging system (1)–(3), we solve $p_2(t,x)$ by using the method of characteristics [29]. Let $\bar{p}_2(t) = p_2(t,x)$, we transform the third partial differential equation of (1) into ordinary differential equation as follows:

$$\frac{\mathrm{d}\bar{p}_2(t)}{\mathrm{d}t} = \frac{\partial p_2(t,x)}{\partial t} + \frac{\partial p_2(t,x)}{\partial x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t} = -\mu(x)\bar{p}_2(t). \tag{4}$$

Let $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 1$, we obtain $x = t + \xi$ (ξ is a constant), then Equation (4) can be rewritten as follows:

$$\frac{\mathrm{d}\bar{p}_2(t)}{\mathrm{d}t} = -\mu(t+\xi)\bar{p}_2(t). \tag{5}$$

For $x < t$, we integrate (5) from $-\xi$ to t and use $\bar{p}_2(-\xi) = p_2(-\xi, 0) = p_2(t-x, 0)$ as follows:

$$\begin{aligned} p_2(t,x) = \bar{p}_2(t) =& \bar{p}_2(-\xi)\mathrm{e}^{-\int_{-\xi}^{t}\mu(s+\xi)\mathrm{d}s} \\ =& p_2(t-x,0)\mathrm{e}^{-\int_0^x \mu(s)\mathrm{d}s} \\ =& (\gamma p_0(t-x) + \beta p_1(t-x))\mathrm{e}^{-\int_0^x \mu(s)\mathrm{d}s}; \end{aligned} \tag{6}$$

for $x \geq t$, we integrate (5) from 0 to t, and use $\bar{p}_2(0) = p_2(0, \xi) = p_2(0, x-t) = 0$ and initial values (3) as follows:

$$p_2(t,x) = \bar{p}_2(t) = \bar{p}_2(0)\mathrm{e}^{-\int_0^t \mu(s+\xi)ds} = 0. \tag{7}$$

Thus, the dynamical solution $p_2(t,x)$ of the software aging system (1)–(3) can be expressed as follows:

$$p_2(t,x) = \begin{cases} (\gamma p_0(t-x) + \beta p_1(t-x))\mathrm{e}^{-\int_0^x \mu(s)\mathrm{d}s}, & x < t, \\ 0, & x \geq t, \end{cases} \tag{8}$$

so when $t \in [0, T]$, $\int_0^\infty \mu(x)p_2(t,x)\mathrm{d}x = \int_0^T \mu(x)p_2(t,x)\mathrm{d}x$, and then the system (1)–(3) is equivalent to the following:

$$\begin{cases} \dfrac{\mathrm{d}p_0(t)}{\mathrm{d}t} = -(\alpha+\gamma)p_0(t) + \displaystyle\int_0^T \mu(x)p_2(t,x)\mathrm{d}x, \\ \dfrac{\mathrm{d}p_1(t)}{\mathrm{d}t} = \alpha p_0(t) - \beta p_1(t), \\ \dfrac{\partial p_2(t,x)}{\partial t} + \dfrac{\partial p_2(t,x)}{\partial x} = -\mu(x)p_2(t,x), \\ p_2(t,0) = \gamma p_0(t) + \beta p_1(t), \\ p_0(0) = 1,\ p_1(0) = p_2(0,x) = 0. \end{cases} \tag{9}$$

To maximize the availability and minimize the repair cost of the software system in time interval $[0, T]$, we establish an optimal repair rate $\mu(x)$ as the control variable and define the set of admissible controls as follows:

$$U_{ad} = \{\mu(x) \in L^\infty(0,T) |\ 0 \leq \mu(x) \leq \bar{\mu} < \infty,\ a.e.\ x \in [0,T]\}, \tag{10}$$

in which $\bar{\mu} = \sup_{x\in(0,T)} \mu(x)$. Clearly, U_{ad} is a convex and closed set in $L^\infty(0,T)$. At this point, the system (9) within the admissible control domain is referred to as the controlled system (9).

Let $m(x) \in L^1(0,T)$ be the repair cost of the system (9) in state 2. For a given terminal time $0 < T < \infty$, the current work aims at deriving the optimal repair rate $\mu(x) \in U_{ad}$ such that the following objective functional is minimized:

$$J(\mu) = a\int_0^T \int_0^T p_2(t,x)\mathrm{d}x\mathrm{d}t + b\int_0^T \mu(x)m(x)\mathrm{d}x \tag{11}$$

where $a > 0$, $b > 0$ are the control weight parameters, $\int_0^T \int_0^T p_2(t,x)\mathrm{d}x\mathrm{d}t$ means the probability that the system (9) is in state 2 within the time interval $[0,T]$, $\int_0^T \mu(x)m(x)\mathrm{d}x$ is the total repair cost of the system (9) within the time interval $[0,T]$. Note that our control problem is a bilinear control due to the product term $\mu(x)p_2(t,x)$ of the controlled system (9).

3. Existence of an Optimal Repair Rate

In this section, we investigate the existence of the optimal repair rate $\mu^*(x) \in U_{ad}$ for the controlled system (9). To this end, we introduce the following definitions: 1 and 2, as well as Theorems 1 and 2.

Definition 1 (Weak convergence, weak * convergence [30]). *Let X be a linear normed space, and X^* be its dual space. A sequence $\{x_n\} \subset X$ is said to weakly converge to x, denoted as $x_n \rightharpoonup x$, if for all $x^* \in X^*$, $\langle x^*, x_n - x\rangle \to 0$; A sequence $\{x_n^*\} \subset X^*$ is said to weak * converge to x^*, denoted as $x_n^* \rightharpoonup^* x^*$, if for all $x \in X$, $\langle x_n^* - x^*, x\rangle \to 0$.*

Theorem 1 (Banach-Alaoglu [30]). *Let X^* be the dual space of a separable linear normed space X. Suppose $\{x_n^* \mid n = 1, 2, \cdots\} \subset X^*$ is a sequence that is bounded in norm: $M = \sup \|x_n^*\| < \infty$. Then it must have a subsequence that weakly * converges.*

Definition 2 (Weakly lower semi-continuous [30]). *Let X be a linear normed space, and f be a function. The function f is said to be weakly lower semi-continuous if and only if for any sequence $x_n \to x_0$, we have $f(x_0) \leq \underline{\lim}_{n\to\infty} f(x_n)$.*

Theorem 2 ([30]). *Suppose X is the dual space of a separable Banach space, and let $E \subset X$ be a weak * sequentially closed, non-empty subset. If $f : E \to R^1$ is weak * lower semi-continuous and coercive (i.e., $\forall x \in E$, $f(x) \to +\infty$ as $\|x\| \to \infty$), then f attains its minimum on E.*

As Theorem 2 implies, to establish the existence of the optimal repair rate $\mu^*(x) \in U_{ad}$, it is necessary to investigate the state space of the controlled system (9), which satisfies weak closedness and weak continuity.

3.1. State Space Description of the Controlled System (9)

In this subsection, we investigate the state space of the controlled system (9). To this end, we first give the concept of standard Sobolev space.

For any positive integer m and $1 \leq p \leq \infty$, we consider the standard Sobolev space $W^{m,p}(0,T) \equiv \{u | D^\alpha u \in L^p(0,T) \; for \; 0 \leq |\alpha| \leq m\}$ as follows:

$$\|u\|_{m,p} = \left(\sum_{0\leq|\alpha|\leq m} \|D^\alpha u\|_p^p \right)^{1/p}, \quad \text{if} \quad 1 \leq p < \infty; \tag{12}$$

$$\|u\|_{m,\infty} = \max_{0\leq|\alpha|\leq m} \|D^\alpha u\|_\infty, \tag{13}$$

where $D^\alpha u$ is a weak partial derivative of u. Let $L_t^1(0,T)$ and $L_x^1(0,T)$ refer to the L^1-space, with respect to t and x.

Theorem 3. *The software system (9) has a unique non-negative dynamical solution*

$$\begin{aligned}\vec{P}(t) =&(p_0(t),\ p_1(t),\ p_2(t,x))^{\mathrm{T}}\\ =&\begin{cases}\begin{pmatrix} \mathrm{e}^{-(\alpha+\gamma)t}+\int_0^t \mathrm{e}^{-(\alpha+\gamma)(t-s)}\int_0^s \mu(x)p_2(s,x)\mathrm{d}x\mathrm{d}s \\ \alpha\mathrm{e}^{-\beta t}\int_0^t p_0(s)\mathrm{e}^{\beta s}\mathrm{d}s \\ (\gamma p_0(t-x)+\beta p_1(t-x))\mathrm{e}^{-\int_0^x \mu(s)\mathrm{d}s}\end{pmatrix}, & x<t,\\ \begin{pmatrix} \mathrm{e}^{-(\alpha+\gamma)t}+\int_0^t \mathrm{e}^{-(\alpha+\gamma)(t-s)}\int_0^s \mu(x)p_2(s,x)\mathrm{d}x\mathrm{d}s \\ \alpha\mathrm{e}^{-\beta t}\int_0^t p_0(s)\mathrm{e}^{\beta s}\mathrm{d}s \\ 0\end{pmatrix}, & x\geq t,\end{cases}\end{aligned} \tag{14}$$

which satisfies

$$p_0(t)+p_1(t)+\int_0^T p_2(t,x)\mathrm{d}x=1,\quad \forall t\geq 0 \tag{15}$$

and

$$\begin{aligned}(p_0(t),\ p_1(t),\ p_2(t,x))^{\mathrm{T}}\in& W^{1,\infty}(0,T)\times W^{1,\infty}(0,T)\\ &\times(W^{1,\infty}(0,T;L^1(0,T))\cap L^{\infty}(0,T;W^{1,1}(0,T))).\end{aligned} \tag{16}$$

Proof. Firstly, we derive (14). Substituting the dynamical solution (8) of $p_2(t,x)$ into the first and second equations of the system (9) and combining the initial values $p_0(0)=1$, $p_1(0)=p_2(0,x)=0$ shows that when $x<t$ or $x\geq t$,

$$\begin{aligned}p_0(t)&=\mathrm{e}^{-(\alpha+\gamma)t}+\int_0^t \mathrm{e}^{-(\alpha+\gamma)(t-s)}\int_0^s \mu(x)p_2(s,x)\mathrm{d}x\mathrm{d}s,\\ p_1(t)&=\alpha\mathrm{e}^{-\beta t}\int_0^t p_0(s)\mathrm{e}^{\beta s}\mathrm{d}s;\end{aligned}$$

Obviously, the solution $\vec{P}(t)$ is non-negative. This completes the proof of (14).

Secondly, we derive (15). Taking the integral of the third partial differential equation of (9) with respect to x from 0 to T, and then adding it to the first and second differential equation of (9) to obtain the following:

$$\frac{\mathrm{d}p_0(t)}{\mathrm{d}t}+\frac{\mathrm{d}p_1(t)}{\mathrm{d}t}+\frac{\mathrm{d}}{\mathrm{d}t}\int_0^T p_2(t,x)\mathrm{d}x=0,\forall 0\leq t\leq T,$$

which implies the following:

$$p_0(t)+p_1(t)+\int_0^T p_2(t,x)\mathrm{d}x=p_0(0)+p_1(0)+\int_0^T p_2(0,x)\mathrm{d}x=1,\forall 0\leq t\leq T, \tag{17}$$

As a consequence, the system is conservative as follows:

$$\sup_{t\in[0,T]}|p_0|\leq 1,\quad \sup_{t\in[0,T]}|p_1|\leq 1,\quad \text{and}\quad \sup_{t\in[0,T]}\|p_2\|_{L^1_x(0,T)}\leq 1. \tag{18}$$

Then based on (9), (10), and (18), we have the following:

$$\sup_{t\in[0,T]}\left|\frac{\mathrm{d}p_0(t)}{\mathrm{d}t}\right|\leq(\alpha+\gamma)\|p_0\|_{L^1_t(0,T)}+\sup_{t\in[0,T]}\int_0^T \mu(x)p_2(t,x)\mathrm{d}x\leq\alpha+\gamma+\bar{\mu}; \tag{19}$$

$$\sup_{t\in[0,T]}\left|\frac{\mathrm{d}p_1(t)}{\mathrm{d}t}\right|\leq\alpha\|p_0\|_{L^1_t(0,T)}+\beta\|p_1\|_{L^1_t(0,T)}\leq\alpha+\beta, \tag{20}$$

which yields $p_0(t) \in W^{1,\infty}(0,T)$ and $p_1(t) \in W^{1,\infty}(0,T)$. Moreover, for $x < t$, from (14) and the third partial differential equation of (9) is as follows:

$$\frac{\partial p_2(t,x)}{\partial x} = -(\gamma\frac{\mathrm{d}p_0(t-x)}{\mathrm{d}t} + \beta\frac{\mathrm{d}p_1(t-x)}{\mathrm{d}t})\mathrm{e}^{-\int_0^x \mu(s)\mathrm{d}s} - \mu(x)(\gamma p_0(t-x) + \beta p_1(t-x))\mathrm{e}^{-\int_0^x \mu(s)\mathrm{d}s}. \tag{21}$$

Combining (21) with (18)–(20) obtains the following:

$$\begin{aligned}
\sup_{t\in[0,T]}\int_0^T \left|\frac{\partial p_2(t,x)}{\partial x}\right|\mathrm{d}x \leq & \sup_{t\in[0,T]}\int_0^T ((\gamma|\frac{\mathrm{d}p_0(t-x)}{\mathrm{d}t}| + \beta|\frac{\mathrm{d}p_1(t-x)}{\mathrm{d}t}|)\mathrm{e}^{-\int_0^x \mu(s)\mathrm{d}s}\mathrm{d}x \\
& + \sup_{t\in[0,T]}\int_0^T (\gamma p_0(t-x) + \beta p_1(t-x))\mathrm{e}^{-\int_0^x \mu(s)\mathrm{d}s}\mu(x)\mathrm{d}x \\
\leq & (\gamma \sup_{t\in[0,T]}|\frac{\mathrm{d}p_0(t-x)}{\mathrm{d}t}| + \beta \sup_{t\in[0,T]}|\frac{\mathrm{d}p_1(t-x)}{\mathrm{d}t}|)\int_0^T \mathrm{e}^{-\int_0^x \mu(s)\mathrm{d}s}\mathrm{d}x \\
& + (\gamma \sup_{t\in[0,T]} p_0(t) + \beta \sup_{t\in[0,T]} p_1(t))\int_0^T \mathrm{e}^{-\int_0^x \mu(s)\mathrm{d}s}\mu(x)\mathrm{d}x \\
\leq & [\gamma(\alpha+\gamma+\bar{\mu}) + \beta(\alpha+\beta)]T + (\gamma+\beta),
\end{aligned}$$

which yields the following:

$$\begin{aligned}
\sup_{t\in[0,T]}\int_0^T \left|\frac{\partial p_2(t,x)}{\partial t}\right|\mathrm{d}x \leq & \sup_{t\in[0,T]}\int_0^T \left|\frac{\partial p_2(t,x)}{\partial x}\right|\mathrm{d}x + \sup_{t\in[0,T]}\int_0^T |\mu(x)p_2(t,x)|\mathrm{d}x \\
\leq & [\gamma(\alpha+\gamma+\bar{\mu}) + \beta(\alpha+\beta)]T + (\gamma+\beta) + \bar{\mu}.
\end{aligned}$$

Therefore, $p_2(t,x) \in W^{1,\infty}(0,T;L^1(0,T)) \cap L^\infty(0,T;W^{1,1}(0,T)))$.

Finally, due to the Rellich–Kondrachov embedding theorem, for any that give $T(0 < T < \infty)$, we have the following [30]:

$$W^{1,\infty}(0,T) \hookrightarrow C[0,T]; W^{1,1}(0,T) \hookrightarrow L^1(0,T)$$

are compact; moreover, according to the Aubin–Lions–Simon Lemma [31], we know that

$$W^{1,\infty}(0,T;L^1(0,T)) \cap L^\infty(0,T;W^{1,1}(0,T)) \hookrightarrow C\Big([0,T];L^1(0,T)\Big)$$

is compact. This finishes the proof for Theorem 3. □

Based on Theorem 3, we can derive the following corollary:

Corollary 1. *$W^{1,\infty}(0,T) \times W^{1,\infty}(0,T) \times (W^{1,\infty}(0,T;L^1(0,T)) \cap L^\infty(0,T;W^{1,1}(0,T))) \hookrightarrow C[0,T] \times C[0,T] \times C\Big([0,T];L^1(0,T)\Big)$ is compact.*

3.2. Existence of an Optimal Repair Rate

In this subsection, we investigate the existence of an optimal repair rate of (11).

Theorem 4. *There exists an optimal repair rate $\mu^*(x) \in U_{ad}$ such that $J(\mu^*) = \inf_{\mu\in U_{ad}} J(\mu)$.*

Proof. Since $J(\mu)$ is bounded from below, we choose $\{\mu_n\} \in U_{ad}$ as a minimizing sequence for (10) such that

$$\lim_{n\to\infty} J(\mu_n) = \inf_{\mu\in U_{ad}} J(\mu), \tag{22}$$

that is, for any given $\varepsilon > 0$, there exists a $N(\varepsilon) > 0$, when $n \geq N(\varepsilon)$, we have the following:

$$J(\mu_n) < \inf_{\mu\in U_{ad}} J(\mu) + \varepsilon \equiv M > 0,$$

while (11) shows that $\|\mu_n\|_{L^\infty(0,T)} < J(\mu_n) < M$. Thus, we derive that $\{\mu_n(x)\}$ is uniformly bounded in $L^\infty(0,T)$; that is,

$$\|\mu_n(x)\|_{L^\infty(0,T)} \leq C_1,\ n = 1, 2, \cdots, \tag{23}$$

here, C_1 is a positive constant that is independent of n.

For convenience, we complete the remainder proof of this theorem in **Steps 1–3**.

Step 1: Prove that there exists $\mu^*(x)$ such that $\mu_n(x) \rightharpoonup^* \mu^*(x)$ and $\mu^* \in U_{ad}$.

By the Theorem 1 and the uniformly boundedness of $\mu_n(x)$ in $L^\infty(0,T)$, it follows that there exists a convergent subsequence $\{u_{n_k}(x)\}$, still denoted $\{\mu_n(x)\}$, such that in $L^\infty(0,T)$,

$$\mu_n(x) \rightharpoonup^* \mu^*(x), \tag{24}$$

as $n \to \infty$. Moreover, from (10), it follows that U_{ad} is closed under the weak * topology of $L^\infty(0,T)$, i.e., $\mu^* \in U_{ad}$.

Step 2: Prove that there exists $\vec{P}^*(t) \in C[0,T] \times C[0,T] \times C\Big([0,T]; L^1(0,T)\Big)$ such that the solution sequence $\vec{P}_n(t)$ of the controlled system (9) corresponds to $\{\mu_n(x)\}$, converges to $\vec{P}^*(t)$, and $\vec{P}^*(t)$ is the solution corresponding to $\mu^*(x)$.

Let sequence $\{\vec{P}_n(t)\} = \{(p_{0n}(t),\ p_{1n}(t),\ p_{2n}(t,x))^{\mathrm{T}}\}$ be the solution corresponding to $\{\mu_n(x)\}$, we have

$$\begin{cases} \dfrac{\mathrm{d}p_{0n}(t)}{\mathrm{d}t} = -(\alpha+\gamma)p_{0n}(t) + \displaystyle\int_0^\infty \mu_n(x)p_{2n}(t,x)\mathrm{d}x \\ \dfrac{\mathrm{d}p_{1n}(t)}{\mathrm{d}t} = \alpha p_0(t) - \beta p_{1n}(t) \\ \dfrac{\partial p_{2n}(t,x)}{\partial t} + \dfrac{\partial p_{2n}(t,x)}{\partial x} = -\mu_n(x)p_{2n}(t,x) \\ p_{2n}(t,0) = \gamma p_{0n}(t) + \beta p_{1n}(t) \\ p_{0n}(0) = 1,\ p_{1n}(0) = p_{2n}(0,x) = 0. \end{cases} \tag{25}$$

With Theorem 3, there exists a convergent subsequence $\{\vec{P}_{n_k}(t)\}$, still denoted $\{\vec{P}_n(t)\}$, such that

$$\vec{P}_n(t) \to \vec{P}^*(t) \quad \text{strongly in } C[0,T] \times C[0,T] \times C\Big([0,T]; L^1(0,T)\Big). \tag{26}$$

Next we verify that $\vec{P}^*(t)$ is the solution corresponding to $\mu^*(x)$ based on (14) by showing that

$$\sup_{t\in[0,T]} |p_{0n} - p_0^*| \to 0,\ \sup_{t\in[0,T]} |p_{1n} - p_1^*| \to 0,\ \sup_{t\in[0,T]} \|p_{2n} - p_2^*\|_{L_x^1(0,T)} \to 0.$$

In fact, as indicated by (14), we can obtain that

$$\begin{aligned}
&\sup_{t\in[0,T]}|p_{0n}-p_0^*|\\
&=\sup_{t\in[0,T]}\left|\int_0^t \mathrm{e}^{-(\alpha+\gamma)(t-\tau)}\int_0^\tau \mu_n(x)p_{2n}(\tau,x)\mathrm{d}x\mathrm{d}\tau-\int_0^t \mathrm{e}^{-(\alpha+\gamma)(t-\tau)}\int_0^\tau \mu(x)p_2^*(\tau,x)\mathrm{d}x\mathrm{d}\tau\right|\\
&\le\sup_{t\in[0,T]}\left|\int_0^t \mathrm{e}^{-(\alpha+\gamma)(t-\tau)}\int_0^\tau \mu_n(x)(p_{2n}(\tau,x)-p_2^*(\tau,x))\mathrm{d}x\mathrm{d}\tau\right|\\
&\quad+\sup_{t\in[0,T]}\left|\int_0^t \mathrm{e}^{-(\alpha+\gamma)(t-\tau)}\int_0^\tau (\mu_n(x)-\mu^*(x))p_2^*(\tau,x)\mathrm{d}x\mathrm{d}\tau\right|\\
&\le\bar{\mu}T\sup_{t\in[0,T]}\|p_{2n}-p_2^*\|_{L_x^1(0,T)}+\sup_{t\in[0,T]}\left|\int_0^t(\mu_n(x)-\mu^*(x))\int_x^t \mathrm{e}^{-(\alpha+\gamma)(t-\tau)}p_2^*(\tau,x)\mathrm{d}\tau\mathrm{d}x\right|\\
&\to 0.
\end{aligned}$$

$$\sup_{t\in[0,T]}|p_{1n}-p_1^*|=\sup_{t\in[0,T]}\left|\alpha\mathrm{e}^{-\beta t}\int_0^t \mathrm{e}^{\beta s}(p_{0n}(s)-p_0^*(s))\mathrm{d}s\right|\to 0 \tag{27}$$

and

$$\begin{aligned}
&\sup_{t\in[0,T]}\|p_{2n}-p_2^*\|_{L_x^1(0,T)}\\
&=\sup_{t\in[0,T]}\int_0^T\left|(\gamma p_{0n}(t-x)+\beta p_{1n}(t-x))\mathrm{e}^{-\int_0^x\mu_n(s)\mathrm{d}s}-(\gamma p_0^*(t-x)+\beta p_1^*(t-x))\mathrm{e}^{-\int_0^x\mu^*(s)\mathrm{d}s}\right|\\
&\le\gamma\sup_{t\in[0,T]}\int_0^T\left|(p_{0n}(t-x)-p_0^*(t-x))\mathrm{e}^{-\int_0^x\mu_n(s)\mathrm{d}s}\mathrm{d}x\right|\\
&\quad+\gamma\sup_{t\in[0,T]}\int_0^T\left|p_0^*(t-x)(\mathrm{e}^{-\int_0^x\mu_n(s)\mathrm{d}s}-\mathrm{e}^{-\int_0^x\mu^*(s)\mathrm{d}s})\mathrm{d}x\right|\\
&\quad+\beta\sup_{t\in[0,T]}\int_0^T\left|(p_{1n}(t-x)-p_1^*(t-x))\mathrm{e}^{-\int_0^x\mu_n(s)\mathrm{d}s}\mathrm{d}x\right|\\
&\quad+\beta\sup_{t\in[0,T]}\int_0^T\left|p_1^*(t-x)(\mathrm{e}^{-\int_0^x\mu_n(s)\mathrm{d}s}-\mathrm{e}^{-\int_0^x\mu^*(s)\mathrm{d}s})\mathrm{d}x\right|\\
&\le\gamma\sup_{t\in[0,T]}\int_0^T|p_{0n}(t)-p_0^*(t)\mathrm{d}x|\int_0^t\mathrm{e}^{-\int_0^x\mu_n(s)\mathrm{d}s}\mathrm{d}x+\beta\sup_{t\in[0,T]}\int_0^T|p_{1n}(t)-p_1^*(t)\mathrm{d}x|\int_0^t\mathrm{e}^{-\int_0^x\mu_n(s)\mathrm{d}s}\mathrm{d}x\\
&\quad+(\gamma+\beta)\sup_{t\in[0,T]}\int_0^t\left|\mathrm{e}^{-\int_0^x\mu_n(s)\mathrm{d}s}-\mathrm{e}^{-\int_0^x\mu^*(s)\mathrm{d}s}\mathrm{d}x\right|\\
&\le\gamma T\sup_{t\in[0,T]}|p_{0n}(t)-p_0^*(t)|+\beta T\sup_{t\in[0,T]}|p_{1n}(t)-p_1^*(t)|+(\gamma+\beta)T\sup_{t\in[0,T]}\left|\mathrm{e}^{-\int_0^x\mu_n(s)\mathrm{d}s}-\mathrm{e}^{-\int_0^x\mu^*(s)\mathrm{d}s}\right|\\
&\to 0.
\end{aligned}$$

Therefore, $\vec{P}^*(t)$ is the solution corresponding to $\mu^*(x)$ in view of (10).

Step 3: Prove that $J(\mu^*)=\inf_{\mu\in U_{ad}}J(\mu)$, $\mu^*\in U_{ad}$.

The functional defined on $L^1(0,T)$ is given as follows:

$$\mu_n(h)=\int_0^T\mu_n(x)h(x)\mathrm{d}x,$$

then $\mu_n(h) \in L^\infty(0,T)$, and from (24), we know that as $n \to \infty$,

$$\mu_n(h) = \int_0^T \mu_n(x)h(x)\mathrm{d}x \to \int_0^T \mu^*(x)h(x)\mathrm{d}x = \mu^*(h) \tag{28}$$

and $\mu^*(x) \in L^\infty(0,T)$. Because $|\mu_n(h)| \le \|\mu_n\|\|h\|_{L^1(0,T)}$, we have the following:

$$\inf_{k\ge n} |\mu_k(h)| \le \inf_{k\ge n} \|\mu_k\|\|h\|_{L^1(0,T)}.$$

Taking $n \to \infty$, we obtain the following:

$$\varliminf_{n\to\infty} |\mu_n(h)| \le (\varliminf_{n\to\infty} \|\mu_n\|)\|h\|_{L^1(0,T)}. \tag{29}$$

Combining (29) with (28) derives the following:

$$|\mu^*(h)| = \varliminf_{n\to\infty} |\mu_n(h)| = \varliminf_{n\to\infty} |\mu_n(h)| \le (\varliminf_{n\to\infty} \|\mu_n\|)\|h\|_{L^1(0,T)},$$

thus

$$\|\mu^*\| \le \varliminf_{n\to\infty} \|\mu_n\|_{L^\infty(0,T)}, \tag{30}$$

which means that $\|\mu\|_{L^\infty(0,T)}$ is weak * lower semi-continuous at $\mu^* \in L^\infty(0,T)$.

From (27), it follows that for $\mu_n(x) \rightharpoonup^* \mu^*(x)$,

$$\|p_2^*\|_{L_x^1(0,T)} \le \varliminf_{n\to\infty} \inf_{k\ge n} \|p_{2k}(t,x)\|_{L_x^1(0,T)}, \tag{31}$$

meaning that the function $\mu \to \|p_2(t,x)\|_{L_x^1(0,T)}$ is weak* lower semi-continuous.

Combining (30), (31) and $J(\mu)$ given in (11), we have that $J(\mu)$ is weak-* lower semi-continuous at $\mu^* \in U_{ad}$, i.e., as follows:

$$J(\mu^*) \le \varliminf_{n\to\infty} J(\mu_n).$$

At the same time, from (22) and the definition of infimum, we know that

$$\varliminf_{n\to\infty} J(\mu_n) = \varliminf_{n\to\infty} \inf_{k\ge n} J(\mu_k) \le \inf_{\mu\in U_{ad}} J(\mu),$$

thus we have the following:

$$J(\mu^*) \le \inf_{\mu\in U_{ad}} J(\mu). \tag{32}$$

On the other hand, based on the definition of infimum, we have the following:

$$J(\mu^*) \ge \inf_{\mu\in U_{ad}} J(\mu),\ \mu^* \in U_{ad}. \tag{33}$$

From (32) and (33), we obtain the following:

$$J(\mu^*) = \inf_{\mu\in U_{ad}} J(\mu),\ \mu^* \in U_{ad},$$

which implies that $\mu^*(x)$ is an optimal repair rate of problem (11). □

4. First-Order Necessary Optimality Conditions

In this section, we investigate the first-order necessary conditions for an optimal control problem (11) by applying a variational inequality [32]; that is, if $\mu^*(x)$ is an optimal solution of the controlled system (9), then

$$J'(\mu^*) \cdot (\nu - \mu^*) \ge 0,\ \forall \nu \in U_{ad}. \tag{34}$$

If $\mu^*(x)$ denotes the optimal repair rate control for the controlled system (9), and for any given $\nu \in U_{ad}$, let $\nu - \mu^* = h, z_0^* = p_0'(\mu^*)h,\ z_1^* = p_1'(\mu^*)h,\ z_2^* = p_2'(\mu^*)h$ for $\nu \in U_{ad}$, which satisfies the following:

$$\begin{cases} \dfrac{\mathrm{d}z_0^*(t)}{\mathrm{d}t} = -(\alpha+\gamma)z_0^*(t) + \displaystyle\int_0^T \mu^*(x)z_2^*(t,x)\mathrm{d}x + \int_0^T h(x)p_2^*(t,x)\mathrm{d}x \\ \dfrac{\mathrm{d}z_1^*(t)}{\mathrm{d}t} = \alpha z_0^*(t) - \beta z_1^*(t) \\ \dfrac{\partial z_2^*(t,x)}{\partial t} + \dfrac{\partial z_2^*(t,x)}{\partial x} = -\mu^*(x)z_2^*(t,x) - h(x)p_2^*(t,x) \end{cases} \tag{35}$$

with the following boundary condition:

$$z_2^*(t,0) = \gamma z_0^*(t) + \beta z_1^*(t) \tag{36}$$

and the following initial values:

$$z_0^*(0) = 0,\ z_1^*(0) = z_2^*(0,x) = 0. \tag{37}$$

According to the definition of J in (5), we have the following:

$$J'(\mu^*)\cdot(\nu-\mu^*) = a\int_0^T\int_0^T z_2^*(t,x)\mathrm{d}x\mathrm{d}t + b\int_0^T m(x)(\nu(x)-\mu^*(x))\mathrm{d}x. \tag{38}$$

To obtain the first-order necessary conditions of (38), we define the adjoint space $X^* = R\times R\times L_x^\infty(0,T)$ of $X = R\times R\times L_x^1(0,T)$ for system (9) with the following relationship:

$$(\vec{P}^*, \vec{Q}^*)_{X,X^*} = p_0^*q_0^* + p_1^*q_1^* + \int_0^T p_2^*q_2^*\mathrm{d}x \tag{39}$$

between $\vec{P}^*(t) = (p_0^*(t),\ p_1^*(t),\ p_2^*(t,x))^{\mathrm{T}} \in X$ and its adjoint state $\vec{Q}^*(t) = (q_0^*(t),\ q_1^*(t),\ q_2^*(t,x))^{\mathrm{T}}$ satisfying the following adjoint system:

$$\begin{cases} \dfrac{\mathrm{d}q_0^*(t)}{\mathrm{d}t} = (\alpha+\gamma)q_0^*(t) - \alpha q_1^*(t) - \gamma q_2^*(t,0), \\ \dfrac{\mathrm{d}q_1^*(t)}{\mathrm{d}t} = \beta q_1^*(t) - \beta q_2^*(t,0), \\ \dfrac{\partial q_2^*(t,x)}{\partial t} + \dfrac{\partial q_2^*(t,x)}{\partial x} = \mu^*(x)q_2^*(t,x) - \mu^*(x)q_0^*(t) + a, \\ q_0^*(T) = q_1^*(T) = q_2^*(T,x) = 0, \end{cases} \tag{40}$$

By solving (40) using the method of characteristics, we obtain the following:

$$\begin{cases} q_0^*(t) = \displaystyle\int_t^T \mathrm{e}^{-(\alpha+\gamma)(s-t)}(\alpha q_1^*(s) + \gamma q_2^*(s,0))\mathrm{d}s, \\ q_1^*(t) = \displaystyle\int_t^T \mathrm{e}^{-\beta(s-t)}\beta q_2^*(s,0)\mathrm{d}s, \\ q_2^*(t,x) = \displaystyle\int_t^T \mathrm{e}^{-\int_x^{x+s-t}\mu^*(\rho)\mathrm{d}\rho}(\mu^*(x+s-t)q_0^*(s) - a)\mathrm{d}s. \end{cases} \tag{41}$$

Theorem 5. *Let $\mu^*(x) \in U_{ad}$ be an optimal bilinear control of (11), whose corresponding solutions $(p_0^*,\ p_1^*,\ p_2^*(x))^{\mathrm{T}}$ and $(q_0^*,\ q_1^*,\ q_2^*(x))^{\mathrm{T}}$ are given by (14) and (41), then*

$$\mu^*(x) = \begin{cases} 0, & \dfrac{1}{b}\displaystyle\int_0^T p_2^*(t,x)(q_0^*(t) - q_2^*(t,x))\mathrm{d}t < m(x), \\ \bar{\mu}, & \dfrac{1}{b}\displaystyle\int_0^T p_2^*(t,x)(q_0^*(t) - q_2^*(t,x))\mathrm{d}t > m(x), \end{cases} \tag{42}$$

and if there exists $x^* \in [0,T]$ *such that* $\frac{1}{b}\int_0^T p_2^*(t,x^*)(q_0^*(t)-q_2^*(t,x))\mathrm{d}t = m(x^*)$, $\mu^*(x^*)$ *can take any value within* $[0,\bar{\mu}]$.

Proof. With (35)–(40) and integration by parts, the first part of (38) can be calculated as follows:

$$\begin{aligned}
a\int_0^T\int_0^T z_2^*(t,x)\mathrm{d}x\mathrm{d}t &= \int_0^T\int_0^T (\frac{\partial q_2(t,x)}{\partial t}+\frac{\partial q_2^*(t,x)}{\partial x}-\mu^*(x)q_2^*(t,x)+\mu^*(x)q_0^*(t))z_2^*(t,x)\mathrm{d}x\mathrm{d}t\\
&= -\int_0^T(\gamma z_0^*(t)+\beta z_1^*(t))q_2^*(t,0)\mathrm{d}t-\int_0^T\int_0^T q_2^*(t,x)(\frac{\partial z_2^*(t,x)}{\partial t}+\frac{\partial z_2^*(t,x)}{\partial x})\mathrm{d}x\mathrm{d}t\\
&\quad -\int_0^T\int_0^T \mu(x)q_2^*(t,x)z_2^*(t,x)\mathrm{d}x\mathrm{d}t+\int_0^T\int_0^T \mu(x)q_0^*(t)z_2^*(t,x)\mathrm{d}x\mathrm{d}t\\
&= -\int_0^T(\gamma z_0^*(t)+\beta z_1^*(t))q_2^*(t,0)\mathrm{d}t\\
&\quad +\int_0^T\int_0^T q_2^*(t,x)h(x)p_2^*(t,x)\mathrm{d}x\mathrm{d}t\\
&\quad +\int_0^T\int_0^T \mu^*(x)q_0^*(t)z_2^*(t,x)\mathrm{d}x\mathrm{d}t\\
&= -\int_0^T \beta z_1^*(t)q_2^*(t,0)\mathrm{d}t+\int_0^T\int_0^T q_2^*(t,x)h(x)p_2^*(t,x)\mathrm{d}x\mathrm{d}t\\
&\quad +\int_0^T \alpha q_1^*(t)z_0^*(t)\mathrm{d}t-\int_0^T\int_0^T h(x)p_2^*(t,x)\mathrm{d}xq_0^*(t)\mathrm{d}t\\
&= -\int_0^T \beta z_1^*(t)q_2^*(t,0)\mathrm{d}t+\int_0^T\int_0^T q_2^*(t,x)h(x)p_2^*(t,x)\mathrm{d}x\mathrm{d}t\\
&\quad +\int_0^T(\frac{\mathrm{d}z_1^*(t)}{\mathrm{d}t}+\beta z_1^*(t))q_1^*(t)\mathrm{d}t-\int_0^T\int_0^T h(x)p_2^*(t,x)\mathrm{d}xq_0^*(t)\mathrm{d}t\\
&= \int_0^T\int_0^T (q_2^*(t,x)-q_0^*(t))h(x)p_2^*(t,x)\mathrm{d}x\mathrm{d}t
\end{aligned} \tag{43}$$

Since $\mu^*(x) \in U_{ad}$ is an optimal solution of (11), we have the following:

$$\begin{aligned}
&J'(\mu^*)\cdot(\nu-\mu^*)\\
&= a\int_0^T\int_0^T z_2^*(t,x)\mathrm{d}x\mathrm{d}t+b\int_0^T m(x)(\nu(x)-\mu^*(x))\mathrm{d}x\\
&= \int_0^T\int_0^T (q_2^*(t,x)-q_0^*(t))(\nu(x)-\mu^*(x))p_2^*(t,x)\mathrm{d}x\mathrm{d}t+b\int_0^T m(x)(\nu(x)-\mu^*(x))\mathrm{d}x\\
&\geq 0,
\end{aligned} \tag{44}$$

which implies that the (42) holds. Moreover, if there exists $x^* \in [0,T]$ such that $\frac{1}{b}\int_0^T p_2^*(t,x^*)(q_0^*(t)-q_2^*(t,x))\mathrm{d}t = m(x^*)$, $\mu^*(x^*)$ can take any value within $[0,\bar{\mu}]$. □

Remark 6. *The optimal repair rate control exhibits the Bang-Bang property: when* $\frac{1}{b}\int_0^T p_2^*(t,x)(q_0^*(t)-q_2^*(t,x))\mathrm{d}t < m(x)$, *no maintenance is required to maximize the interval availability and minimize maintenance cost of system (9) over the time interval ; when* $\frac{1}{b}\int_0^T p_2^*(t,x)(q_0^*(t)-q_2^*(t,x))\mathrm{d}t > m(x)$, *the maintenance rate is set to to maximize the interval availability and minimize maintenance cost of system (9) over the time interval; when* $\frac{1}{b}\int_0^T p_2^*(t,x)(q_0^*(t)-q_2^*(t,x))\mathrm{d}t = m(x)$, *regardless of the value of chosen from the interval, system (9) can achieve maximum interval availability and minimum maintenance cost over the time interval.These theoretical results provide valuable guidance for optimizing the availability and maintenance costs of software aging systems.*

5. Conclusions

This paper has proposed a corrective rejuvenation policy for software aging systems, with the aim of finding the optimal repair rate that maximizes the interval availability of the system and minimizes the repair cost within a given time interval $[0, T]$. Firstly, the corrective rejuvenation policy was described as an optimal repair rate control problem. Secondly, the existence of the optimal repair rate $\mu^*(x) \in U_{ad}$ was proven using the theory of functional extreme values. Finally, the first-order necessity condition of the optimal repair rate was derived from the variational inequality, and the Bang-Bang property of the optimal repair rate control was obtained. These theoretical results have offered valuable direction for enhancing the availability and managing the maintenance expenses of software systems.

In the future, I will focus on conducting sensitivity analysis on the parameters of the software aging system and discussing how changes in these parameters affect the optimal maintenance strategy.

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