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# The Computational Testing Procedure for the Comprehensive Lifetime Performance Index of Burr XII Products in Multiple Production Lines

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**Abstract:** Extending from a single production line to multiple production lines, a comprehensive life performance index is proposed for evaluating the quality of lifetime products. The connection between the comprehensive lifetime performance index and the individual lifetime performance index is explored. For products with a lifetime following Burr XII distribution for the *ith* production line, the maximum likelihood estimation method and the corresponding asymptotic distribution for all lifetime performance indices are derived. Checking whether the comprehensive lifetime performance index has achieved the target value is essentially the same as testing whether each individual lifetime performance index has reached its corresponding target value. A testing procedure is proposed for a given significance level using the maximum likelihood estimator as the test statistic, and the power analysis is presented through graphical representations. For the power analysis, the impacts of sample size, the number of inspection intervals, the removal probability, the level of significance, and the number of production lines on the test power are analyzed, and the results show that there is a monotonic relationship between the test power and the above five impact factors. To illustrate how to apply the proposed testing procedure, we give one practical example with two production lines to test whether the comprehensive production process is capable.

**Keywords:** Burr XII distribution; multiple production lines; progressive type I interval censoring; maximum likelihood estimator; lifetime performance index; testing procedure

**MSC:** 62P30



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## 1. Introduction

In this investigation, we explore the durations of components across various production lines, with a specific focus on the "larger-the-better" characteristic observed in product lifetimes. Our approach involves employing the lifetime performance index $C_L$, as introduced by Montgomery [1], with unilateral tolerance. While many process capability indices (PCIs) commonly assume a normal distribution for quality characteristics, product lifetimes often follow distributions such as exponential, gamma, Weibull distributions, or Burr XII distributions (as referenced in Johnson et al. [2], Anderson et al. [3], Meyer [4], Epstein and Sobel [5]). In this study, we assume a Burr XII distribution for the product lifetimes. Given the preference for longer product lifetimes, we adopt the lifetime performance index recommended by Montgomery [1] as the focal point of this paper.

For products produced in a single production line, Tong et al. [6] developed the uniformly minimum variance unbiased estimator for the index of $\mathrm{C_L}$ and established a hypothesis testing procedure assuming a one-parameter exponential distribution based on the complete sample. Nevertheless, practical constraints in real-world situations may hinder experimenters from observing the lifetimes of all tested items. Factors like time limitations, financial and material constraints, and difficulties in the experimental process can contribute to incomplete data collection, such as progressive censoring of data. Please refer to Aggarwala [7], Balakrishnan and Aggarwala [8], Wu et al. [9], Sanjel and Balakrishnan [10], Lee et al. [11], Hanan et al. [12] for the application of the progressive censoring data. For the step-stress accelerated life-testing data, Lee et al. [13] conducted the assessment on the lifetime performance index assuming the exponential distribution. For the progressive type I interval censored sample and one single production line, Wu et al. [14] proposed a testing procedure for the lifetime of the product following a Burr XII distribution. Wu et al. [15] studied the experimental design for this testing procedure to attain the given test power or to find the minimum total experimental cost for Burr XII products. Wu and Song [16] investigated the optimal sampling design for the testing procedure for the lifetime performance index to reach the given test power or to minimize the total experimental cost for Chen products. All these research studies related to the inferences on the lifetime performance index are based on the manufacturing process with one single production line. Since many manufacturing processes encompass multiple production lines, we introduce a comprehensive lifetime performance index for products manufactured across multiple production lines, where their lifetimes follow the Burr XII distributions. This research developed a comprehensive lifetime performance index testing procedure using the maximum likelihood estimator as the test statistic to assess whether the comprehensive lifetime performance index reaches the target level for the progressive type I interval censored sample. Regarding the likelihood-based inferences, Guillermo et al. [17] found the maximum likelihood estimators for the parameters of the asymmetric beta-skew alpha-power distribution and derived the Fisher information matrix. Seung and Gareth [18] examined several different maximum likelihood estimators and their asymptotic and finite-sample properties for the dynamic panel data models. They focus on the analysis of the panel data with a large number (N) of cross-sectional units and a small number (T) of repeated time series observations for each cross-sectional unit. Phaphan et al. [19] investigated the maximum likelihood estimation of the weighted mixture generalized gamma distribution.

Our emphasis is on the scenario of progressive type I interval censoring. The censoring scheme is delineated as follows: Suppose that a life test is initiated at time 0 for a total of $n$ products. Let $(t_1, \ldots, t_m)$ be the inspection time points, where $t_m$ is the termination time of the experiment. Therefore, we have $m$ inspection intervals in this life test. At the time point of $t_i$, $X_i$ failed units are recorded in the $i$*th* time interval $(t_{\mathrm{i}-1}, t_{\mathrm{i}})$, and then $R_i$ number of products are progressively removed from the rest of the products. Continue the same process until $X_m$ failed units are recorded at the time point of $t_m$, and then all the remaining products are removed and terminated in this experiment. The sample $(X_1, \ldots, X_m)$ we collect is the progressive type I interval censored sample under the censoring scheme of $(R_1, \ldots, R_m)$.

The remaining sections of this paper are organized as follows: Section 2 introduces the comprehensive lifetime performance index for products manufactured across multiple production lines, elucidating the relationship between the conforming rate and individual lifetime performance index. In Section 3.1, we outline the derivation of the maximum likelihood estimator and the asymptotic distribution for both comprehensive and individual lifetime performance indices, considering a progressive type I interval censored sample under the assumption of a Burr XII distribution. Section 3.2 proposes the testing procedure for the comprehensive lifetime performance index, encompassing all individual lifetime performance indices. This section also examines the impact of various parameter configurations, particularly the number of production lines, on the test power. Section 3.3 includes a numerical example to illustrate the proposed testing procedure. The conclusive findings are summarized in Section 4.

The contributions of this work are articulated as follows:

1. We extend the research on the inferences of the lifetime performance index for single production lines to multiple production lines for products with lifetimes following a Burr XII distribution based on the progressive type I interval censored samples.
2. We introduce the comprehensive lifetime performance index and explore the relationship between the comprehensive lifetime performance index and individual lifetime performance index.
3. We propose a testing procedure to test if the lifetime performance of the manufacturing process comprised of multiple production lines has reached the desired target value.
4. We analyze the impact of sample size, the number of inspection intervals, the removal probability, the level of significance, and the number of production lines on the test power.
5. We give a numerical example with two production lines to illustrate how to apply the proposed testing procedure to determine whether the comprehensive production process is capable.

## 2. The Conforming Rate and the Comprehensive Lifetime Performance Index

Suppose that there are $d$ production lines producing products with lifetimes following a Burr XII distribution. Suppose that the lifetime $U_i$ of products in the i*th* production line has a Burr XII distribution a the probability density function (PDF), cumulative distribution function (CDF), and hazard function (HF), as follows:

$$f_{U_i}(u) = \delta_i k_i u^{\delta_i - 1}\left(1 + u^{\delta_i}\right)^{-(k_i+1)},\ u > 0,\ \delta_i > 0,\ \ k_i > 0, \tag{1}$$

$$F_{U_i}(u) = 1 - \left(1 + u^{\delta_i}\right)^{-k_i},\ u > 0,\ \delta_i > 0,\ k_i > 0 \tag{2}$$

and

$$h_{U_i}(u) = \frac{f_{U_i}(u)}{1 - F_{U_i}(u)} = \frac{\delta_i k_i u^{\delta_i - 1}}{1 + u^{\delta_i}}, \tag{3}$$

where $k_i$ is the scale parameter, and $\delta_i$ is the shape parameter, $i$ = 1,...,$d$. The Burr XII distribution can exhibit a variety of shapes, including skewed, heavy-tailed, and flexible distributions, depending on the specific values of the scale parameter and shape parameter. It is commonly used to model data with long tails and diverse failure rates. The larger-the-better type lifetime performance index $C_L$ proposed by Montgomery [1] is given by

$$C_L = \frac{\mu - L}{\sigma}, \tag{4}$$

where $\mu$ represents the process mean, σ is the process standard deviation, and $L$ is the specified lower specification limit. Consider the transformation of $Y_i = log(1 + U_i^{\delta_i})$, then we have the new lifetime $Y_i$ following a one-parameter exponential distribution with PDF and CDF as

$$f_{Y_i}(y) = k_i \exp\{-k_i y\},\ y > 0, k_i > 0, \tag{5}$$

$$F_{Y_i}(y) = 1 - exp(-k_i y), y > 0, k_i > 0. \tag{6}$$

The mean and standard deviation of the new lifetime of products are obtained as $\mu_i = E(Y_i) = \frac{1}{k_i},\ \sigma_i = \sqrt{Var(Y_i)} = \frac{1}{k_i}$. If $L_{U_i} = log\left(1 + L_i^{\delta_i}\right)$ is the lower specification limit for $U_i$, then the lower specification limit for $Y_i$ isgivenby $L_i$. The lifetime performance index for the ith production line is reduced to

$$C_{L_i} = \frac{\mu_i - L_i}{\sigma_i} = \frac{\frac{1}{k_i} - L_i}{\frac{1}{k_i}} = 1 - k_i L_i. \tag{7}$$

The lifetime performance index $C_{Li}$ can accurately assess the lifetime performance of products since the larger the expected value of $Y_i$, the larger the lifetime performance index $C_{Li}$.

The conforming rate $P_{ri}$ for the i*th* production line is the probability that the lifetime of an item of product exceeds the lower specification limit (i.e.,$U_i \geq L_{U_i}$) and it can be obtained as

$$P_{r_i} = P(U_i \geq L_{U_i}) = P(Y_i \geq L_i) = e^{-k_i L_i} = e^{C_{L_i}-1}, -\infty < C_{L_i} < 1. \tag{8}$$

From Equation (8), it can be observed that the conforming rate for the i*th* production line is an increasing function of the lifetime performance index $C_{Li}$. If the experimenter desired $P_{ri}$ to be greater than 0.9048, then he can obtain the value of $C_{Li}$ to exceed 0.9 from Equation (8).

Suppose that the manufacturing process consists of $d$ independent production lines for products. The overall conforming rate denoted by $P_r$ can be obtained as

$$P_r = P(Y_i \geq L_i\,, i = 1, \cdots, d) = \exp\left\{-\sum_{i=1}^{d} k_i L_i\right\} = \exp\left\{\sum_{i=1}^{d} C_{L_i} - d\right\}, -\infty < C_{L_i} < 1.$$

Denote the comprehensive lifetime performance index as $C_T$, which is satisfying

$$P_r = \exp\left\{\sum_{i=1}^{d} C_{L_i} - d\right\} = \exp\{C_T - 1\}, -\infty < C_T < 1 \tag{9}$$

From Equation (9), we can see that $C_T$ is an increasing function of $P_r$ and solve this equation to yield the relationship between the comprehensive lifetime performance index and individual lifetime performance index as follows

$$C_T = \sum_{i=1}^{d} C_{L_i} - (d-1), -\infty < C_T < 1. \tag{10}$$

Under the reasonable setup of equal individual lifetime performance indices, as $C_{L_1} = \ldots = C_{L_d} = C_L$, solve Equation (10) to yield the relationship between $C_L$ and $C_T$ as follows

$$C_L = \frac{C_T + d - 1}{d}, -\infty < C_T < 1. \tag{11}$$

Suppose that the desired target value for the comprehensive lifetime performance index is given by $C_T = c_0$; we can obtain the target value for the individual lifetime performance index $C_L$ as $C_L = \frac{c_0+d-1}{d}$ for each production line by solving Equation (11). The corresponding values of $C_L$ for a given value of $C_T$ are listed in Table 1 under $d$ = 2, 3, 4, 5, 6. For example, the experimenter wants to have the overall conforming rate as $P_r$ = 0.9048, then we have the comprehensive lifetime performance index $C_T$ = 0.90. From Table 1, we can see that for a given value of $C_T$ = 0.90, we can find the desired target value for each production line as $C_L$ = 0.9500, 0.9667, 0.9750, 0.9800, and 0.9833 for $d$ = 2, 3, 4, 5, 6.

**Table 1.** The corresponding values of $C_L$ for a given value of $C_T$.

| | $C_L$ | | | | |
|---|---|---|---|---|---|
| $C_T\backslash k$ | 2 | 3 | 4 | 5 | 6 |
| 0.525 | 0.762500 | 0.841667 | 0.881250 | 0.905000 | 0.920833 |
| 0.550 | 0.775000 | 0.850000 | 0.887500 | 0.910000 | 0.925000 |
| 0.575 | 0.787500 | 0.858333 | 0.893750 | 0.915000 | 0.929167 |
| 0.600 | 0.800000 | 0.866667 | 0.900000 | 0.920000 | 0.933333 |
| 0.625 | 0.812500 | 0.875000 | 0.906250 | 0.925000 | 0.937500 |
| 0.650 | 0.825000 | 0.883333 | 0.912500 | 0.930000 | 0.941667 |
| 0.675 | 0.837500 | 0.891667 | 0.918750 | 0.935000 | 0.945833 |
| 0.700 | 0.850000 | 0.900000 | 0.925000 | 0.940000 | 0.950000 |
| 0.725 | 0.862500 | 0.908333 | 0.931250 | 0.945000 | 0.954167 |
| 0.750 | 0.875000 | 0.916667 | 0.937500 | 0.950000 | 0.958333 |
| 0.775 | 0.887500 | 0.925000 | 0.943750 | 0.955000 | 0.962500 |
| 0.800 | 0.900000 | 0.933333 | 0.950000 | 0.960000 | 0.966667 |
| 0.825 | 0.912500 | 0.941667 | 0.956250 | 0.965000 | 0.970833 |
| 0.850 | 0.925000 | 0.950000 | 0.962500 | 0.970000 | 0.975000 |
| 0.875 | 0.937500 | 0.958333 | 0.968750 | 0.975000 | 0.979167 |
| 0.900 | 0.950000 | 0.966667 | 0.975000 | 0.980000 | 0.983333 |
| 0.925 | 0.962500 | 0.975000 | 0.981250 | 0.985000 | 0.987500 |
| 0.950 | 0.975000 | 0.983333 | 0.987500 | 0.990000 | 0.991667 |
| 0.975 | 0.987500 | 0.991667 | 0.993750 | 0.995000 | 0.995833 |
| 1.000 | 1.000000 | 1.000000 | 1.000000 | 1.000000 | 1.000000 |

## 3. The Algorithmic Testing Procedure for the Comprehensive Lifetime Performance Index

In this section, we derive the maximum likelihood estimator for the individual lifetime performance index and the comprehensive lifetime performance index in Section 3.1. Their asymmetric distributions are also derived. Section 3.2 developed the testing procedure to test if the comprehensive lifetime performance index attains the specified target level given by the experimenter. The power analysis and all findings are summarized in this section. Section 3.3 gives one numerical example to illustrate how to apply our proposed testing procedure.

### *3.1. Maximum Likelihood Estimator of the Lifetime Performance Indices*

We collect the progressive type I interval censored sample $X_{i1}, \ldots, X_{im}$ at the observation time points $t_1, \ldots, t_m$ for the $i$*th* production line. The likelihood function for the censored sample $X_{i1}, \ldots, X_{im}$, is

$$\begin{aligned} L(k_i) &\propto \prod_{j=1}^{m} \left(F\left(t_j\right) - F\left(t_{j-1}\right)\right)^{x_{ij}} \left(1 - F\left(t_j\right)\right)^{R_{ij}} \\ &\propto \prod_{i=1}^{m} \left(1 - e^{-k_i\left(log\left(1+t_j^{\delta_i}\right) - log\left(1+t_{j-1}^{\delta_i}\right)\right)}\right)^{X_{ij}} \left(e^{-k_i\left(log\left(1+t_j^{\delta_i}\right)R_{ij} + log\left(1+t_{j-1}^{\delta_i}\right)X_{ij}\right)}\right). \end{aligned} \tag{12}$$

The log-likelihood function is

$$\log L(k_i) \propto \sum_{j=1}^{m} \left\{ X_{ij} log\left(1 - e^{-k_i log\left(\frac{1+t_j^{\delta_i}}{1+t_j^{\delta_i}}\right)}\right) - k_i\left(log\left(1+t_j^{\delta_i}\right)R_{ij} + log\left(1+t_{j-1}^{\delta_i}\right)X_{ij}\right) \right\}. \tag{13}$$

Take the derivative of the log-likelihood function in Equation (13) with respect to parameter $k_i$ and set it to zero. We can obtain the log-likelihood equation as

$$\frac{d}{dk_i}\log L(k_i) = \sum_{j=1}^{m} X_{ij}\frac{log\left(\frac{1+t_j^{\delta_i}}{1+t_{j-1}^{\delta_i}}\right)e^{-k_i log\left(\frac{1+t_j^{\delta_i}}{1+t_{j-1}^{\delta_i}}\right)}}{\left(1-e^{-k_i log\left(\frac{1+t_j^{\delta_i}}{1+t_{j-1}^{\delta_i}}\right)}\right)} - \left(log\left(1+t_j^{\delta_i}\right)R_{ij} + log\left(1+t_{j-1}^{\delta_i}\right)X_{ij}\right) = 0. \tag{14}$$

Solving Equation (14) numerically, we can find the maximum likelihood estimator of $k_i$ denoted by $\hat{k}_i$.

From Casella and Berger [20], we need to find Fisher's information number, defined as $I_i(k_i) = -\mathrm{E}\left[\frac{d^2 logL(k_i)}{dk_i^2}\right]$ in order to obtain the asymptotic variance of the distribution of the maximum likelihood estimator. Taking the second derivative of the log-likelihood function in (13) with respect to parameter $k_i$, then we have

$$\begin{aligned}\frac{d^2 logL(k_i)}{dk_i^2} &= \sum_{j=1}^{m}\left\{X_{ij}\frac{-log^2\left(\frac{1+t_j^{\delta_i}}{1+t_{j-1}^{\delta_i}}\right)e^{-k_i log\left(\frac{1+t_j^{\delta_i}}{1+t_{j-1}^{\delta_i}}\right)}\left(1-e^{-k_i log\left(\frac{1+t_j^{\delta_i}}{1+t_{j-1}^{\delta_i}}\right)}\right)-log^2\left(\frac{1+t_j^{\delta_i}}{1+t_{j-1}^{\delta_i}}\right)e^{-2k_i log\left(\frac{1+t_j^{\delta_i}}{1+t_{j-1}^{\delta_i}}\right)}}{\left(1-e^{-k_i log\left(\frac{1+t_j^{\delta_i}}{1+t_{j-1}^{\delta_i}}\right)}\right)^2}\right\} \\ &= \sum_{j=1}^{m} X_{ij}\frac{-log^2\left(\frac{1+t_j^{\delta_i}}{1+t_{j-1}^{\delta_i}}\right)e^{-k_i log\left(\frac{1+t_j^{\delta_i}}{1+t_{j-1}^{\delta_i}}\right)}}{\left(1-e^{-k_i log\left(\frac{1+t_j^{\delta_i}}{1+t_{j-1}^{\delta_i}}\right)}\right)^2}.\end{aligned} \tag{15}$$

It is observed that

$$X_{ij}\middle|X_{i(j-1)},\ldots,X_{i1} \sim \text{Binomial}\left(n-\sum_{l=1}^{j-1}X_{il}, q_{ij}\right), \tag{16}$$

where

$$q_{ij} = \frac{F(t_j)-F(t_{j-1})}{1-F(t_{j-1})} = 1-e^{-k_i log\left(\frac{1+t_j^{\delta_i}}{1+t_{j-1}^{\delta_i}}\right)}, \; j = 1,\cdots,m.$$

Hence, we have

$$E\left(X_{ij}\middle|X_{i(j-1)},\cdots,X_{i1},R_{i,j-1},\cdots,R_{i1}\right) == nq_{ij}\prod_{l=1}^{j-1}(1-p_l)(1-q_{il}),\; j = 1,\ldots,m. \tag{17}$$

Using Equation (17), the Fisher's information becomes

$$I_i(k_i) = -E\left(\sum_{j=1}^{m} X_{ij}\frac{-\frac{1}{k_i^2}(1-q_{ij})log^2(1-q_{ij})}{q_{ij}^2}\right) = \frac{n}{k_i^2}\sum_{j=1}^{m}\frac{log^2(1-q_{ij})}{q_{ij}}\prod_{l=1}^{j-1}(1-p_{il})(1-q_{il}). \quad (18)$$

The asymptotic variance is denoted as $V_i\left(\hat{k}_i\right) = I_i^{-1}\left(\hat{k}_i\right)$. Then we have $\hat{k}_i \xrightarrow[n\to\infty]{d} N\left(k_i, I_i^{-1}\left(\hat{k}_i\right)\right)$.

In the special case, we consider the case of equal interval lengths as $t_j - t_{j-1} = \mathrm{t}$ and $q_{ij} = 1 - e^{-k_i log\left(\frac{1+(jt)^{\delta_i}}{1+((j-1)t)^{\delta_i}}\right)}, j = 1, \cdots, m$. We also considered $p_j = p, j = 1, \cdots, m$. The Equation (14) can be simplified as

$$\sum_{j=1}^{m}\left[x_{ij}\frac{log\left(\frac{1+(jt)^{\delta_i}}{1+((j-1)t)^{\delta_i}}\right)exp\left\{-k_i log\left(\frac{1+(jt)^{\delta_i}}{1+((j-1)t)^{\delta_i}}\right)\right\}}{1-exp\left\{-k_i log\left(\frac{1+(jt)^{\delta_i}}{1+((j-1)t)^{\delta_i}}\right)\right\}} - \left(log\left(1+(jt)^{\delta_i}\right)R_{ij} + log\left(1+((j-1)t)^{\delta_i}\right)X_{ij}\right)\right] \equiv 0. \quad (19)$$

Solving for $k_i$ numerically, we can obtain the maximum likelihood estimator for $k_i$ denoted by $\hat{k}_i$. Furthermore, the asymptotic variance of $\hat{k}_i$ can be expressed as

$$V\left(\hat{k}_i\right) = I_i^{-1}\left(\hat{k}_i\right) = \frac{\hat{k}_i^2}{n}\left[\sum_{j=1}^{m}\frac{log^2(1-q_{ij})}{q_{ij}}\prod_{l=1}^{j-1}(1-p_l)(1-q_{il})\right]^{-1} \quad (20)$$

By the invariance property of the maximum likelihood estimator, the maximum likelihood estimator of $C_{Li}$ can be found as

$$\hat{C}_{L_i} = 1 - \hat{k}_i L_i. \quad (21)$$

Making use of the Delta method in Casella and Berger [20], we can find that

$$\hat{C}_{L_i} \xrightarrow[n\to\infty]{d} N\left(C_{L_i}, V\left(\hat{C}_{L_i}\right)\right), \quad (22)$$

where $V\left(\hat{C}_{L_i}\right) = {L_i}^2 V\left(\hat{k}_i\right)$.

By the invariance property, we can find the maximum likelihood estimator of $C_T$ as

$$\hat{C}_T = \sum_{i=1}^{d}\hat{C}_{L_i} - (d-1) \quad (23)$$

Its asymptotic distribution is $\hat{C}_T \xrightarrow[n\to\infty]{d} N\left(C_T, \sum_{i=1}^{d}\hat{V}\left(\hat{C}_{L_i}\right)\right)$, where $\sum_{i=1}^{d}\hat{V}\left(\hat{C}_{Li}\right)$ is the estimate of the variance of $\hat{C}_T$ given by $\sum_{i=1}^{d}V\left(\hat{C}_{Li}\right)$.

### *3.2. The Algorithmic Testing Procedure for the Comprehensive Lifetime Performance Index*

If the experimenter wants to test whether the comprehensive lifetime performance index exceeds the desired target value $c_0$, the null and alternative hypotheses are setupas$H_0 : C_T \le c_0$ vs. $H_a : C_T > c_0$. From Equation (11), we can find the corresponding target value $c_0^*$ for each individual lifetime performance index as $c_0^* = \frac{c_0+d-1}{d}$. To test whether the comprehensive lifetime performance index exceeds the desired target value $c_0$ is equivalent to testing the statistical hypothesis of $H_0 : C_{L_i} \le c_0^*$ for some $i$ vs. $H_a : C_{L_i} > c_0^*$, $i = 1, \cdots, d$, we use the maximum likelihood estimator of $C_{L_i}$ given by $\hat{C}_{L_i}$ as the test statistic and define the critical value as $C_{Li}^0$. The critical value $C_{L_i}^0$ is determined by controlling the overall error rate to be at most the level of significance $\alpha$ in the multiple hypothesis testing. The probability of a type I error (false positive) is

$$
\begin{aligned}
&P\left(\hat{C}_{L_i} > C^0_{L_i}, \middle| C_{L_i} = c^*_0,\ i = 1,\ldots,d\right) = P\left(1-\hat{k}_i L_i > C^0_{L_i} \middle| C_{L_i} = c^*_0,\ i = 1,\ldots,d\right) \\
&= P\left(1-\hat{k}_i L_i > C^0_{L_i} \middle| k_i = k_{i0},\ i = 1,\ldots,d\right),\ \text{where } k_{i0} = \tfrac{1-c^*_0}{L_i} \\
&= P\left(\hat{k}_i < \frac{1-C^0_{L_i}}{L_i} \middle| k_i = k_{i0},\ i = 1,\ldots,d\right) \\
&= P\left(Z < \frac{\frac{1-C^0_{L_i}}{L_i} - k_{i0}}{\sqrt{I_i^{-1}(k_{i0})}},\ i = 1,\ldots,d\right),\ \text{due to } \hat{k}_i \underset{n\to\infty}{\overset{d}{\to}} N\left(k_{i0}, I_i^{-1}(k_{i0})\right). \\
&= \prod_{i=1}^{d} \Phi\left(\frac{\frac{1-C^0_{L_i}}{L_i} - k_{i0}}{\sqrt{I_i^{-1}(k_{i0})}}\right) \le \alpha,
\end{aligned}
$$

where $\Phi(\cdot)$ is the cumulative distribution function for the standard normal distribution.

Set $\Phi\left(\frac{\frac{1-C^0_{L_i}}{L_i} - k_{i0}}{\sqrt{I_i^{-1}(k_{i0})}}\right) = \alpha^{\frac{1}{d}} = \alpha'$. Then, the probability of a type I error has reached the level of significance $\alpha$. To solve the above equation, the critical value $C^0_{L_i}$ is determined as $C^0_{L_i} = 1-\left(Z_{1-\alpha'}\sqrt{I_i^{-1}(k_{i0})}+k_{i0}\right)L_i$, where $Z_{1-\alpha'}$ is the $\alpha'$ percentile for the standard normal distribution. The overall rejection region is determined as $R = \bigcap_{i=1}^{d}\left\{\hat{C}_{L_i} \,\middle|\, \hat{C}_{L_i} > C^0_{L_i}\right\}$ such that the level of this test is still $\alpha$.

The algorithmic testing procedure for $C_T$ is constructed as follows:

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**Algorithm 1: Testing procedure for the comprehensive lifetime performance index $C_T$**

Step 1: For a known lower specification $L_i$, we collect the progressive type I interval censored sample $X_{i1},\ldots,X_{im}$ at the observation time points $t_1,\ldots,t_m$ with censoring schemes of $R_1,\ldots,R_m$ from the Burr XII distribution.
Step 2: From Equation (9), determine the desired level $c_0$ for a pre-assigned conforming rate $P_r$. Then, the required level $c^*_0$ for each production line can be determined from Table 1. Thus, the testing hypothesis $H_0 : C_{L_i} \le c^*_0$ for some $i$ v.s. $H_a : C_{L_i} > c^*_0, i = 1,\cdots,d$ is set up.
Step 3: Compute the value of the test statistic $\hat{C}_{L_i} = 1-\hat{k}_i L_i$.
Step 4: For the level of significance $\alpha$, find the critical value of $C^0_{L_i} = 1-\left(Z_{1-\alpha'}\sqrt{I_i^{-1}(k_{i0})}+k_{i0}\right)L_i$ where $k_{i0} = \frac{1-c^*_0}{L_i}$, $\alpha' = \alpha^{1/d}$ and $I_i^{-1}(k_{i0})$ is defined in Equation (18).
Step 5: Compare $\hat{C}_{L_i}$ with $C^0_{L_i}$.If$\hat{C}_{L_i} > C^0_{L_i}$ sustained for $i = 1,\ldots,d$, we can infer that the comprehensive lifetime performance index of the products has reached the required level $c_0$.

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For this test, the power function $h(c_1)$ at the point of $C_T = dC_L-(d-1) = c_1 > c_0$ or $C_L = \frac{c_1+d-1}{d} = c^*_1$ is derived as follows:

$$\begin{aligned}
h(c_1) &= P\left(\hat{C}_{Li} > C^0_{Li}, i = 1, \cdots, d \mid c_1^* = 1 - k_{i1}L_i, i = 1, \cdots, d\right), \\
&= P\left(1 - \hat{k}_i L_i > 1 - \left(Z_{1-\alpha'}\sqrt{I_i^{-1}(k_{i0})} + k_{i0}\right)L_i, i = 1, \cdots, d \middle| k_{i1} = \frac{1-c_1^*}{L_i}, k_{i0} = \frac{1-c_0^*}{L_i}, i = 1, \cdots, d\right) \\
&= P\left(\hat{k}_i < k_{i0} + Z_{1-\alpha'}\sqrt{I_i^{-1}(k_{i0})}, i = 1, \cdots, d \middle| k_{i1} = \frac{1-c_1^*}{L_i}, k_{i0} = \frac{1-c_0^*}{L_i}, i = 1, \cdots, d\right) \\
&= \prod_{i=1}^{d} P\left(\hat{k}_i < k_{i0} + Z_{1-\alpha'}\sqrt{I_i^{-1}(k_{i0})} \middle| k_{i1} = \frac{1-c_1^*}{L_i}, k_{i0} = \frac{1-c_0^*}{L_i}\right) \\
&= \prod_{i=1}^{d} P\left(\frac{\hat{k}_i - k_{i1}}{\sqrt{I_i^{-1}(k_{i1})}} < \frac{k_{i0} - k_{i1} + Z_{1-\alpha'}\sqrt{I_i^{-1}(k_{i0})}}{\sqrt{I_i^{-1}(k_{i1})}} \middle| k_{i1} = \frac{1-c_1^*}{L_i}, k_{i0} = \frac{1-c_0^*}{L_i}\right) \\
&= \prod_{i=1}^{d}\left(\Phi\left(\frac{k_{i0} - k_{i1} + Z_{1-\alpha'}\sqrt{I^{-1}(k_{i0})}}{\sqrt{I_i^{-1}(k_{i1})}} \middle| k_{i1} = \frac{1-c_1^*}{L_i}, k_{i0} = \frac{1-c_0^*}{L_i}\right)\right),
\end{aligned} \tag{24}$$

Under $L_1 = \ldots = L_d = L$, the power function is reduced to

$$h(c_1) = \prod_{i=1}^{d}\left(\Phi\left(\frac{k_0 - k_1 + Z_{1-\alpha'}\sqrt{I_i^{-1}(k_0)}}{\sqrt{I_i^{-1}(k_1)}} \middle| k_1 = \frac{1-c_1^*}{L}, k_0 = \frac{1-c_0^*}{L}\right)\right)$$

The powers $h(c_1)$ for testing $H_0 : C_T \leq 0.8$ are tabulated in Tables A1–A9 for $d$ = 2, 3, 4 with $\alpha$ = 0.01, 0.05, 0.1 for $c_1$ = 0.80, 0.825, 0.85, 0.875, 0.90, 0.925, $m$ = 5,6,7, $n$ = 25, 50, 75 and $p$ = 0.01, 0.05, 0.1 with $L$ = 0.05, $T$ = 0.5. The power values are illustrated in Figures 1–5, demonstrating various standard scenarios. We have the following six findings from Tables A1–A9 and Figures 1–5. (1) From Figure 1, the power increases when $d$ increases under $n$ = 25, $m$ = 5, $p$ = 0.05 and $\alpha$ = 0.05. However, the difference is not significant, especially when $c_1$ is getting closer to 1 (the same pattern is observed with alternative combinations of $n$, $m$, $p$, and $\alpha$ as well). (2) From Figure 2, the power value is an increasing function of $n$ for fixed $d$ = 3, $\alpha$ = 0.05, $m$ = 5, and $p$ = 0.05 (the same pattern is observed with alternative combinations of $d$, $m$, $p$ and $\alpha$ as well). (3) From Figure 3, the power value is a decreasing function of $m$ for fixed $d$ = 3, $n$ = 25, $p$ = 0.05, and $\alpha$ = 0.05 (the same pattern is observed with alternative combinations of $d$, $n$, $p$, and $\alpha$ as well). However, the difference is not significant. Therefore, the users do not need to increase the number of inspection intervals in order to yield higher test power. (4) From Figure 4, the power increases when $p$ increases for fixed $d$ = 3, $n$ = 25, $m$ = 5, and $\alpha$ = 0.05 (the same pattern is observed with alternative combinations of $d$, $n$, $m$, and $\alpha$ as well). (5) From Figure 5, the power value is an increasing function of $\alpha$ under $d$ = 3, $n$ = 25, $m$ = 5, and $p$ = 0.05 (the same pattern is observed with alternative combinations of $d$, $n$, $m$, and $p$ as well). (6) From Figures 1–5, the power increases when $c_1$ increases for any combinations of $d$, $n$, $m$, $p$, and $\alpha$.

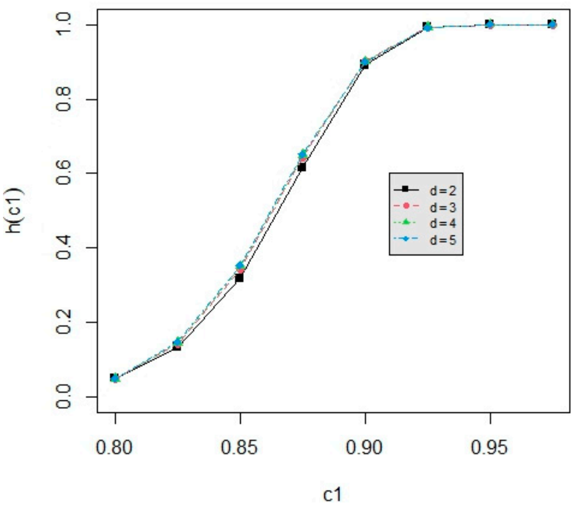


**Figure 1.** The power curve for $d$ = 2, 3, 4, 5 with $n$ = 25, $m$ = 5, $p$ = 0.05, $\alpha$ = 0.05.

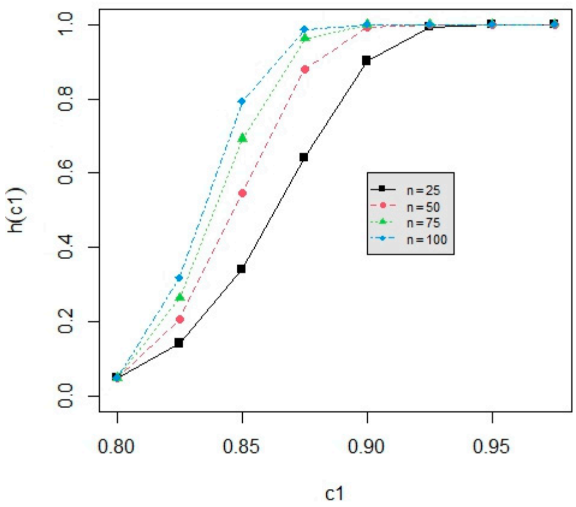


**Figure 2.** The power curve for $n = 25, 50, 75, 100$ with $d = 3$, $\alpha = 0.05$, $m = 5$, $p = 0.05$.

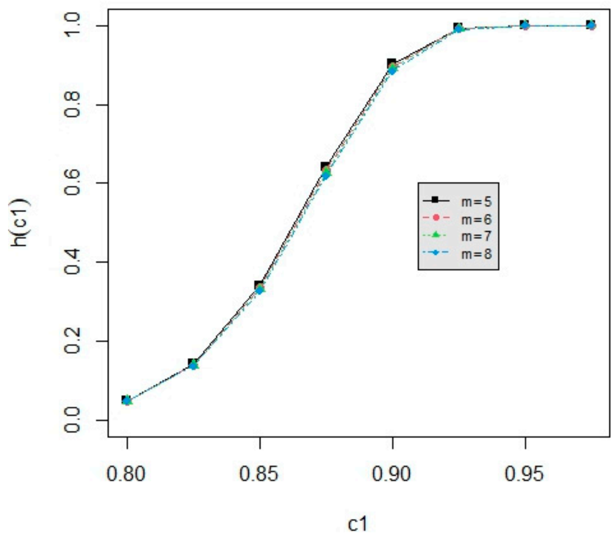


**Figure 3.** The power curve for $m = 5, 6, 7, 8$ with $d = 3$, $\alpha = 0.05$, $n = 25$, $p = 0.05$.

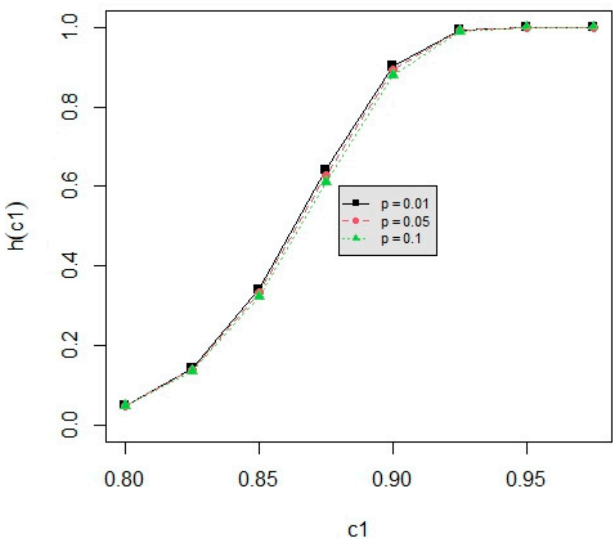


**Figure 4.** The power curve for $p = 0.01, 0.05, 0.1$ with $d = 3$, $\alpha = 0.05$, $n = 25$, $m = 5$.

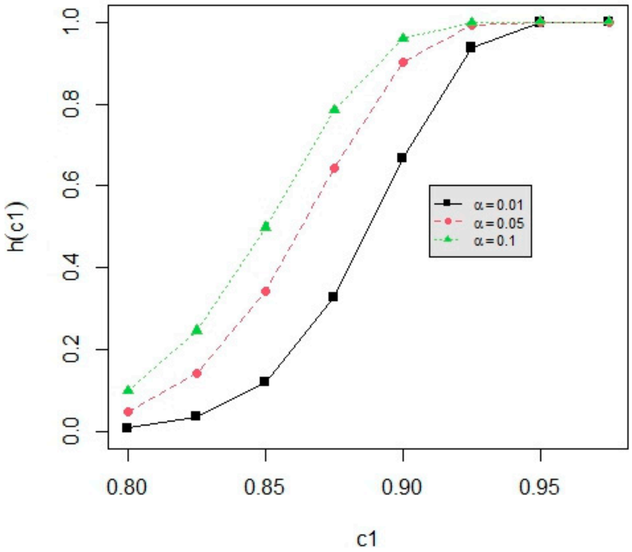


**Figure 5.** The power curve for $\alpha = 0.01, 0.05, 0.1$ with $d = 3$, $n = 25$, $m = 5$, $p = 0.05$.

*3.3. Numerical Example*

We consider a manufacturing process comprising two production lines ($d = 2$). For the first production line, the data in Lawless [21] is comprised of the failure times of $n = 36$ electrical appliances. The data of 36 failure times $U_{1j}$, j = 1, ..., 36, is listed as follows: 0.0011, 0.0035, 0.0049, 0.017, 0.0329, 0.0381, 0.0708, 0.0958, 0.1062, 0.1167, 0.1594, 0.1925, 0.199, 0.2223, 0.2327, 0.24, 0.2451, 0.2471, 0.2551, 0.2565, 0.2568, 0.2702, 0.2761, 0.2831, 0.3034, 0.3034, 0.3059, 0.3112, 0.3214, 0.3478, 0.3504, 0.4329, 0.6367, 0.6976, 0.7846, 1.3403. We use the G test based on the Gini statistic from Gail and Gastwirth [22] to test if the data fits a Burr XII distribution or not. The $p$-value of this test is a function of the shape parameter $\delta_1$. From Figure 6, we can see that the maximum $p$-value of $0.9834 > 0.05$ occurred when the value of $\delta_1 = 1.37$. The large $p$-value indicates that the data fits the Burr XII distribution very well. Therefore, the value of $\delta_1$ is determined as $\delta_1 = 1.37$. For the second production line, the data in Lai et al. [23] is comprised of the failure times of $n = 20$ components. The data of 20 failure times $U_{2j}$, j = 1, ..., 20, is listed as follows: 0.0481, 0.1196, 0.1438, 0.1797, 0.1811, 0.1831, 0.1885, 0.2104, 0.2133, 0.2144, 0.2282, 0.2322, 0.2334, 0.2341, 0.2428, 0.2447, 0.2511, 0.2593, 0.2715, 0.3218. Using the G test, the $p$-value of this test is a function of the shape parameter $\delta_2$, and it is plotted in Figure 7. From Figure 7, we can see that the maximum $p$-value of $0.9183 > 0.05$ occurred when the value of $\delta_2 = 4.62$. Likewise, the large $p$-value indicates that the data fits the Burr XII distribution very well. Thus, the value of $\delta_2$ is determined as $\delta_2 = 4.62$.

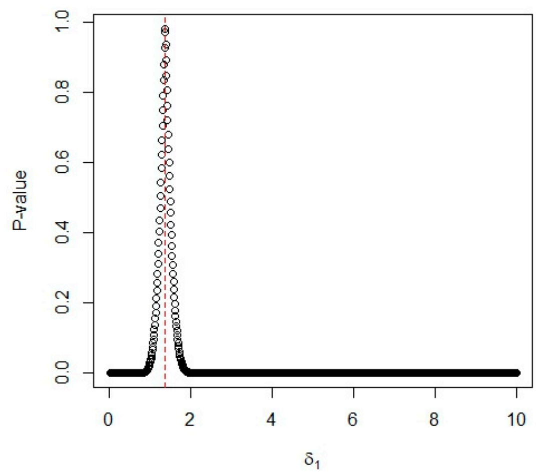


**Figure 6.** The $p$-values vs. the values of $\delta_1$.

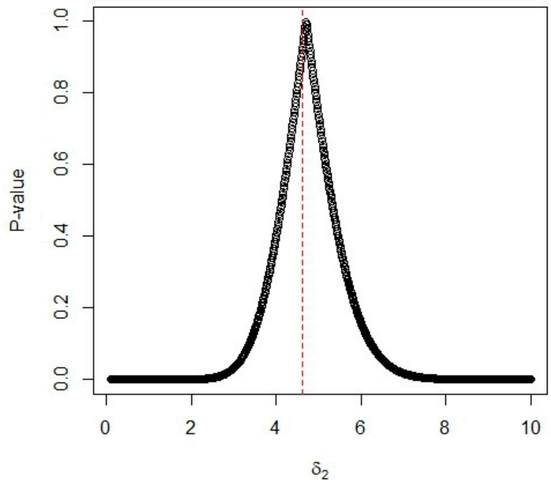


**Figure 7.** The $p$-values vs. the values of $\delta_2$.

Suppose that we want to test $H_0$: $\leq 0.80$. We create the progressive type I interval censored sample for the failure times of products from two production lines. Let the termination time be $T = 0.25$ with the number of inspections $m = 5$; the equal length of the inspection interval $t = 0.05$ and the pre-specified removal percentages of the remaining survival units given are by $(p_1, p_2, p_3, p_4, p_5) = (0.05, 0.05, 0.05, 0.05, 1.0)$.

Applying Algorithm 1, the testing procedure is implemented as follows:

Step 1: Given the known lower specification $L_1 = L_2 = 0.005$, collect the progressive type I interval censored sample $(X_{11}, X_{12}, X_{13}, X_{14}, X_{15}) = (6, 1, 2, 3, 3)$ and $(X_{21}, X_{22}, X_{23}, X_{24}, X_{25}) = (1, 0, 2, 4, 7)$ for each production line at the pre-set times $(t_1, \ldots, t_5) = (0.05, 0.1, 0.15, 0.2, 0.25)$ with censoring schemes of $(R_{11}, R_{12}, R_{13}, R_{14}, R_{15}) = (2, 2, 2, 1, 14)$ and $(R_{21}, R_{22}, R_{23}, R_{24}, R_{25}) = (1, 1, 1, 1, 2)$.

Step 2: Under the conforming rate of$P_r = 0.818731$, therequired level for $C_T$ can be found to be $c_0 = 0.80$. Then, the required target level $c_0^* = \frac{c_0+d-1}{d} = \frac{0.85+2-1}{2} = 0.925$ is determined for each production line. Then, the testing hypothesis $H_0 : C_{L_i} \leq c_0^*$ for some $i$ vs. $H_a : C_{L_i} > c_0^*$, $i = 1,2$ is set up.

Step 3: Obtain the maximum likelihood estimators $\hat{k}_1 = 4.7847$ and $\hat{k}_2 = 9.9999$ for two production lines. Compute the values of the test statistics $\hat{C}_{L_1} = 1 - 4.7847 \times 0.005 \approx 0.9761$ and $\hat{\mathrm{C}}_{\mathrm{L}_2} = 1 - 9.9999 \times 0.005 = 0.9500$ respectively.

Step 4: For the level of significance $\alpha = 0.05$, we have $\alpha' = (0.05)^{\frac{1}{2}} = 0.2236$ and $\mathrm{k}_{10} = \mathrm{k}_{20} = \mathrm{k}_0 = \frac{1-\mathrm{c}_0^*}{\mathrm{L}_2} = \frac{1-0.9}{0.005} = 20$ Then, we can compute the critical values $\mathrm{C}^0_{\mathrm{L}_1} = 0.9137$ and $C^0_{L_2} = 0.9210$ for two production lines.

Step 5: Since $\hat{C}_{L_1} > C^0_{L_1}$ and $\hat{C}_{L_2} > C^0_{L_2}$, we can deduce that the individual lifetime performance indices meet the specified target values, resulting in the comprehensive lifetime performance index reaching the target level.

## 4. Conclusions

In diverse manufacturing fields, analyzing the lifetime performance indices is crucial, especially when a product's lifespan follows a Burr XII distribution with progressive type I interval censored samples. Our study extends beyond a single production line to encompass multiple lines, introducing an overarching lifetime performance index applicable to these varied production lines. We investigate the interplay between this comprehensive lifetime performance index and individual lifetime performance indices, delving into deriving maximum likelihood estimators and asymptotic distributions for both based on progressive type I interval censored samples. We devise a computational algorithm for hypothesis testing at a specified significance level to assess whether the comprehensive lifetime performance index meets the target. This involves testing each individual lifetime performance index using the maximum likelihood estimator as the test statistic. We scrutinize the impact of the number of production lines ($d$), the sample size ($n$), the number of inspection intervals ($m$), removal probability ($p$), and the significance level ($\alpha$) on the test power using visual illustrations, Figures 1–5. All findings reveal that the test power is an increasing function of $n$, $\alpha$, and $d$ when the other parameters are fixed. However, it is a decreasing function of $p$ or $m$ when the other parameters are fixed. To demonstrate the practical application of our testing algorithm for the comprehensive lifetime performance index, we provide a numerical example involving two production lines in the concluding section.

In the future, we will focus on the investigation of an experimental design for the proposed testing procedure for the comprehensive lifetime performance index in this research to attain the given test power or to find the minimum total experimental cost under a specific cost structure. We can also consider other types of censoring schemes other than the progressive type I interval censoring.

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**Data Availability Statement:** Data are available in a publicly accessible repository. The data presented in this study are openly available in Lawless [21] and Lai et al. [23].

**Conflicts of Interest:** The authors declare no conflict of interest.

## Appendix A

**Table A1.** The power $h(c_1)$ at $d = 2$, $\alpha = 0.01$.

| | | | $c_1$ | | | | | |
|---|---|---|---|---|---|---|---|---|
| $m$ | $n$ | $p$ | 0.80 | 0.825 | 0.85 | 0.875 | 0.90 | 0.925 |
| 5 | 25 | 0.01 | 0.0100 | 0.0329 | 0.1031 | 0.2848 | 0.6159 | 0.9214 |
| | | 0.05 | 0.0100 | 0.0321 | 0.0986 | 0.2706 | 0.5915 | 0.9072 |
| | | 0.1 | 0.0100 | 0.0313 | 0.0945 | 0.2572 | 0.5673 | 0.8916 |
| | 50 | 0.01 | 0.0100 | 0.0574 | 0.2432 | 0.6376 | 0.9477 | 0.9994 |
| | | 0.05 | 0.0100 | 0.0556 | 0.2322 | 0.6155 | 0.9379 | 0.9991 |
| | | 0.1 | 0.0100 | 0.0538 | 0.2217 | 0.5935 | 0.9271 | 0.9987 |
| | 75 | 0.01 | 0.0100 | 0.0843 | 0.3942 | 0.8486 | 0.9951 | 1.0000 |
| | | 0.05 | 0.0100 | 0.0813 | 0.3777 | 0.8317 | 0.9935 | 1.0000 |
| | | 0.1 | 0.0100 | 0.0784 | 0.3618 | 0.8140 | 0.9914 | 1.0000 |
| 6 | 25 | 0.01 | 0.0100 | 0.0325 | 0.1009 | 0.2777 | 0.6038 | 0.9145 |
| | | 0.05 | 0.0100 | 0.0315 | 0.0955 | 0.2606 | 0.5734 | 0.8958 |
| | | 0.1 | 0.0100 | 0.0305 | 0.0906 | 0.2447 | 0.5438 | 0.8750 |
| | 50 | 0.01 | 0.0100 | 0.0565 | 0.2377 | 0.6267 | 0.9430 | 0.9993 |
| | | 0.05 | 0.0100 | 0.0543 | 0.2244 | 0.5991 | 0.9299 | 0.9988 |
| | | 0.1 | 0.0100 | 0.0522 | 0.2120 | 0.5721 | 0.9153 | 0.9981 |
| | 75 | 0.01 | 0.0100 | 0.0828 | 0.3859 | 0.8403 | 0.9944 | 1.0000 |
| | | 0.05 | 0.0100 | 0.0792 | 0.3658 | 0.8186 | 0.9920 | 1.0000 |
| | | 0.1 | 0.0100 | 0.0758 | 0.3467 | 0.7959 | 0.9890 | 1.0000 |

**Table A1.** *Cont.*

| | | | | | $c_1$ | | | |
|---|---|---|---|---|---|---|---|---|
| 7 | 25 | 0.01 | 0.0100 | 0.0321 | 0.0987 | 0.2708 | 0.5916 | 0.9073 |
| | | 0.05 | 0.0100 | 0.0309 | 0.0925 | 0.2509 | 0.5556 | 0.8836 |
| | | 0.1 | 0.0100 | 0.0299 | 0.0869 | 0.2328 | 0.5208 | 0.8572 |
| | 50 | 0.01 | 0.0100 | 0.0556 | 0.2322 | 0.6156 | 0.9380 | 0.9991 |
| | | 0.05 | 0.0100 | 0.0530 | 0.2168 | 0.5829 | 0.9214 | 0.9984 |
| | | 0.1 | 0.0100 | 0.0506 | 0.2027 | 0.5511 | 0.9025 | 0.9973 |
| | 75 | 0.01 | 0.0100 | 0.0813 | 0.3778 | 0.8318 | 0.9935 | 1.0000 |
| | | 0.05 | 0.0100 | 0.0771 | 0.3542 | 0.8052 | 0.9903 | 1.0000 |
| | | 0.1 | 0.0100 | 0.0732 | 0.3323 | 0.7774 | 0.9860 | 1.0000 |

**Table A2.** The power $h(c_1)$ at $d = 2$, $\alpha = 0.05$.

| | | | | | $c_1$ | | | |
|---|---|---|---|---|---|---|---|---|
| $m$ | $n$ | $p$ | 0.80 | 0.825 | 0.85 | 0.875 | 0.90 | 0.925 |
| 5 | 25 | 0.01 | 0.0500 | 0.1344 | 0.3194 | 0.6160 | 0.8921 | 0.9931 |
| | | 0.05 | 0.0500 | 0.1320 | 0.3104 | 0.6000 | 0.8802 | 0.9912 |
| | | 0.1 | 0.0500 | 0.1296 | 0.3019 | 0.5841 | 0.8676 | 0.9889 |
| | 50 | 0.01 | 0.0500 | 0.1981 | 0.5323 | 0.8804 | 0.9939 | 1.0000 |
| | | 0.05 | 0.0500 | 0.1936 | 0.5183 | 0.8686 | 0.9923 | 1.0000 |
| | | 0.1 | 0.0500 | 0.1894 | 0.5046 | 0.8564 | 0.9903 | 1.0000 |
| | 75 | 0.01 | 0.0500 | 0.2570 | 0.6892 | 0.9661 | 0.9997 | 1.0000 |
| | | 0.05 | 0.0500 | 0.2507 | 0.6741 | 0.9606 | 0.9996 | 1.0000 |
| | | 0.1 | 0.0500 | 0.2446 | 0.6591 | 0.9547 | 0.9994 | 1.0000 |
| 6 | 25 | 0.01 | 0.0500 | 0.1332 | 0.3149 | 0.6080 | 0.8863 | 0.9922 |
| | | 0.05 | 0.0500 | 0.1302 | 0.3040 | 0.5882 | 0.8709 | 0.9895 |
| | | 0.1 | 0.0500 | 0.1274 | 0.2937 | 0.5688 | 0.8547 | 0.9862 |
| | 50 | 0.01 | 0.0500 | 0.1959 | 0.5253 | 0.8746 | 0.9931 | 1.0000 |
| | | 0.05 | 0.0500 | 0.1904 | 0.5081 | 0.8595 | 0.9909 | 1.0000 |
| | | 0.1 | 0.0500 | 0.1853 | 0.4915 | 0.8439 | 0.9881 | 1.0000 |
| | 75 | 0.01 | 0.0500 | 0.2538 | 0.6817 | 0.9634 | 0.9996 | 1.0000 |
| | | 0.05 | 0.0500 | 0.2461 | 0.6629 | 0.9562 | 0.9994 | 1.0000 |
| | | 0.1 | 0.0500 | 0.2388 | 0.6444 | 0.9482 | 0.9992 | 1.0000 |
| 7 | 25 | 0.01 | 0.0500 | 0.1320 | 0.3105 | 0.6000 | 0.8802 | 0.9912 |
| | | 0.05 | 0.0500 | 0.1285 | 0.2978 | 0.5765 | 0.8613 | 0.9876 |
| | | 0.1 | 0.0500 | 0.1253 | 0.2859 | 0.5538 | 0.8414 | 0.9831 |
| | 50 | 0.01 | 0.0500 | 0.1937 | 0.5184 | 0.8687 | 0.9923 | 1.0000 |
| | | 0.05 | 0.0500 | 0.1873 | 0.4981 | 0.8502 | 0.9893 | 1.0000 |
| | | 0.1 | 0.0500 | 0.1814 | 0.4788 | 0.8311 | 0.9855 | 0.9999 |
| | 75 | 0.01 | 0.0500 | 0.2507 | 0.6742 | 0.9607 | 0.9996 | 1.0000 |
| | | 0.05 | 0.0500 | 0.2417 | 0.6518 | 0.9515 | 0.9993 | 1.0000 |
| | | 0.1 | 0.0500 | 0.2333 | 0.6299 | 0.9413 | 0.9989 | 1.0000 |

**Table A3.** The power $h(c_1)$ at $d = 2$, $\alpha = 0.1$.

| | | | $c_1$ | | | | | |
|---|---|---|---|---|---|---|---|---|
| $m$ | $n$ | $p$ | 0.80 | 0.825 | 0.85 | 0.875 | 0.90 | 0.925 |
| 5 | 25 | 0.01 | 0.1000 | 0.2370 | 0.4800 | 0.7724 | 0.9580 | 0.9987 |
| | | 0.05 | 0.1000 | 0.2335 | 0.4700 | 0.7597 | 0.9522 | 0.9983 |
| | | 0.1 | 0.1000 | 0.2301 | 0.4603 | 0.7469 | 0.9459 | 0.9977 |
| | 50 | 0.01 | 0.1000 | 0.3219 | 0.6877 | 0.9460 | 0.9985 | 1.0000 |
| | | 0.05 | 0.1000 | 0.3162 | 0.6753 | 0.9396 | 0.9980 | 1.0000 |
| | | 0.1 | 0.1000 | 0.3106 | 0.6630 | 0.9327 | 0.9975 | 1.0000 |
| | 75 | 0.01 | 0.1000 | 0.3937 | 0.8141 | 0.9876 | 1.0000 | 1.0000 |
| | | 0.05 | 0.1000 | 0.3862 | 0.8027 | 0.9852 | 0.9999 | 1.0000 |
| | | 0.1 | 0.1000 | 0.3789 | 0.7912 | 0.9826 | 0.9999 | 1.0000 |
| 6 | 25 | 0.01 | 0.1000 | 0.2353 | 0.4750 | 0.7661 | 0.9552 | 0.9985 |
| | | 0.05 | 0.1000 | 0.2310 | 0.4627 | 0.7502 | 0.9476 | 0.9979 |
| | | 0.1 | 0.1000 | 0.2269 | 0.4509 | 0.7343 | 0.9393 | 0.9970 |
| | 50 | 0.01 | 0.1000 | 0.3190 | 0.6815 | 0.9429 | 0.9983 | 1.0000 |
| | | 0.05 | 0.1000 | 0.3120 | 0.6661 | 0.9345 | 0.9976 | 1.0000 |
| | | 0.1 | 0.1000 | 0.3053 | 0.6510 | 0.9255 | 0.9968 | 1.0000 |
| | 75 | 0.01 | 0.1000 | 0.3900 | 0.8085 | 0.9864 | 0.9999 | 1.0000 |
| | | 0.05 | 0.1000 | 0.3808 | 0.7941 | 0.9833 | 0.9999 | 1.0000 |
| | | 0.1 | 0.1000 | 0.3720 | 0.7797 | 0.9797 | 0.9998 | 1.0000 |
| 7 | 25 | 0.01 | 0.1000 | 0.2335 | 0.4700 | 0.7597 | 0.9522 | 0.9983 |
| | | 0.05 | 0.1000 | 0.2285 | 0.4556 | 0.7407 | 0.9427 | 0.9974 |
| | | 0.1 | 0.1000 | 0.2238 | 0.4419 | 0.7218 | 0.9322 | 0.9962 |
| | 50 | 0.01 | 0.1000 | 0.3162 | 0.6753 | 0.9396 | 0.9980 | 1.0000 |
| | | 0.05 | 0.1000 | 0.3080 | 0.6570 | 0.9292 | 0.9972 | 1.0000 |
| | | 0.1 | 0.1000 | 0.3002 | 0.6392 | 0.9180 | 0.9960 | 1.0000 |
| | 75 | 0.01 | 0.1000 | 0.3863 | 0.8028 | 0.9852 | 0.9999 | 1.0000 |
| | | 0.05 | 0.1000 | 0.3755 | 0.7855 | 0.9812 | 0.9999 | 1.0000 |
| | | 0.1 | 0.1000 | 0.3652 | 0.7682 | 0.9765 | 0.9998 | 1.0000 |

**Table A4.** The power $h(c_1)$ at $d = 3$, $\alpha = 0.01$.

| | | | $c_1$ | | | | | |
|---|---|---|---|---|---|---|---|---|
| $m$ | $n$ | $p$ | 0.80 | 0.825 | 0.85 | 0.875 | 0.90 | 0.925 |
| 5 | 25 | 0.01 | 0.0100 | 0.0366 | 0.1207 | 0.3290 | 0.6682 | 0.9375 |
| | | 0.05 | 0.0100 | 0.0357 | 0.1154 | 0.3135 | 0.6452 | 0.9264 |
| | | 0.1 | 0.0100 | 0.0347 | 0.1104 | 0.2987 | 0.6222 | 0.9140 |
| | 50 | 0.01 | 0.0100 | 0.0628 | 0.2663 | 0.6624 | 0.9489 | 0.9991 |
| | | 0.05 | 0.0100 | 0.0607 | 0.2544 | 0.6409 | 0.9398 | 0.9988 |
| | | 0.1 | 0.0100 | 0.0587 | 0.2430 | 0.6195 | 0.9297 | 0.9982 |
| | 75 | 0.01 | 0.0100 | 0.0910 | 0.4145 | 0.8514 | 0.9935 | 1.0000 |
| | | 0.05 | 0.0100 | 0.0876 | 0.3974 | 0.8351 | 0.9916 | 1.0000 |
| | | 0.1 | 0.0100 | 0.0844 | 0.3809 | 0.8181 | 0.9892 | 1.0000 |

**Table A4.** *Cont.*

| | | | | | $c_1$ | | | |
|---|---|---|---|---|---|---|---|---|
| 6 | 25 | 0.01 | 0.0100 | 0.0361 | 0.1180 | 0.3212 | 0.6567 | 0.9321 |
| | | 0.05 | 0.0100 | 0.0350 | 0.1117 | 0.3024 | 0.6281 | 0.9173 |
| | | 0.1 | 0.0100 | 0.0339 | 0.1058 | 0.2848 | 0.5997 | 0.9008 |
| | 50 | 0.01 | 0.0100 | 0.0617 | 0.2603 | 0.6516 | 0.9445 | 0.9990 |
| | | 0.05 | 0.0100 | 0.0592 | 0.2458 | 0.6249 | 0.9324 | 0.9984 |
| | | 0.1 | 0.0100 | 0.0568 | 0.2323 | 0.5985 | 0.9189 | 0.9975 |
| | 75 | 0.01 | 0.0100 | 0.0893 | 0.4059 | 0.8433 | 0.9926 | 1.0000 |
| | | 0.05 | 0.0100 | 0.0852 | 0.3850 | 0.8225 | 0.9899 | 1.0000 |
| | | 0.1 | 0.0100 | 0.0814 | 0.3652 | 0.8007 | 0.9864 | 1.0000 |
| 7 | 25 | 0.01 | 0.0100 | 0.0357 | 0.1154 | 0.3135 | 0.6453 | 0.9264 |
| | | 0.05 | 0.0100 | 0.0343 | 0.1081 | 0.2917 | 0.6112 | 0.9077 |
| | | 0.1 | 0.0100 | 0.0330 | 0.1015 | 0.2716 | 0.5776 | 0.8865 |
| | 50 | 0.01 | 0.0100 | 0.0607 | 0.2544 | 0.6409 | 0.9398 | 0.9988 |
| | | 0.05 | 0.0100 | 0.0578 | 0.2376 | 0.6091 | 0.9245 | 0.9979 |
| | | 0.1 | 0.0100 | 0.0551 | 0.2222 | 0.5779 | 0.9072 | 0.9967 |
| | 75 | 0.01 | 0.0100 | 0.0876 | 0.3974 | 0.8351 | 0.9916 | 1.0000 |
| | | 0.05 | 0.0100 | 0.0829 | 0.3731 | 0.8096 | 0.9879 | 1.0000 |
| | | 0.1 | 0.0100 | 0.0785 | 0.3502 | 0.7829 | 0.9832 | 0.9999 |

**Table A5.** The power $h(c_1)$ at $d = 3$, $\alpha = 0.05$.

| | | | | | $c_1$ | | | |
|---|---|---|---|---|---|---|---|---|
| $m$ | $n$ | $p$ | 0.80 | 0.825 | 0.85 | 0.875 | 0.90 | 0.925 |
| 5 | 25 | 0.01 | 0.0500 | 0.1425 | 0.3421 | 0.6430 | 0.9015 | 0.9929 |
| | | 0.05 | 0.0500 | 0.1398 | 0.3327 | 0.6274 | 0.8908 | 0.9911 |
| | | 0.1 | 0.0500 | 0.1372 | 0.3235 | 0.6119 | 0.8796 | 0.9890 |
| | 50 | 0.01 | 0.0500 | 0.2065 | 0.5458 | 0.8803 | 0.9921 | 1.0000 |
| | | 0.05 | 0.0500 | 0.2017 | 0.5317 | 0.8689 | 0.9903 | 0.9999 |
| | | 0.1 | 0.0500 | 0.1971 | 0.5179 | 0.8570 | 0.9881 | 0.9999 |
| | 75 | 0.01 | 0.0500 | 0.2647 | 0.6921 | 0.9610 | 0.9994 | 1.0000 |
| | | 0.05 | 0.0500 | 0.2580 | 0.6771 | 0.9553 | 0.9991 | 1.0000 |
| | | 0.1 | 0.0500 | 0.2516 | 0.6621 | 0.9489 | 0.9988 | 1.0000 |
| 6 | 25 | 0.01 | 0.0500 | 0.1411 | 0.3374 | 0.6352 | 0.8962 | 0.9921 |
| | | 0.05 | 0.0500 | 0.1378 | 0.3258 | 0.6159 | 0.8825 | 0.9896 |
| | | 0.1 | 0.0500 | 0.1347 | 0.3148 | 0.5969 | 0.8680 | 0.9866 |
| | 50 | 0.01 | 0.0500 | 0.2041 | 0.5387 | 0.8746 | 0.9912 | 1.0000 |
| | | 0.05 | 0.0500 | 0.1982 | 0.5214 | 0.8601 | 0.9887 | 0.9999 |
| | | 0.1 | 0.0500 | 0.1927 | 0.5046 | 0.8449 | 0.9857 | 0.9999 |
| | 75 | 0.01 | 0.0500 | 0.2613 | 0.6846 | 0.9582 | 0.9993 | 1.0000 |
| | | 0.05 | 0.0500 | 0.2532 | 0.6659 | 0.9506 | 0.9989 | 1.0000 |
| | | 0.1 | 0.0500 | 0.2454 | 0.6475 | 0.9422 | 0.9985 | 1.0000 |

**Table A5.** *Cont.*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | $c_1$ | | | |
| 7 | 25 | 0.01 | 0.0500 | 0.1398 | 0.3327 | 0.6274 | 0.8908 | 0.9911 |
| | | 0.05 | 0.0500 | 0.1360 | 0.3192 | 0.6045 | 0.8740 | 0.9879 |
| | | 0.1 | 0.0500 | 0.1324 | 0.3065 | 0.5822 | 0.8561 | 0.9839 |
| | 50 | 0.01 | 0.0500 | 0.2017 | 0.5317 | 0.8689 | 0.9903 | 0.9999 |
| | | 0.05 | 0.0500 | 0.1949 | 0.5113 | 0.8511 | 0.9870 | 0.9999 |
| | | 0.1 | 0.0500 | 0.1885 | 0.4918 | 0.8326 | 0.9829 | 0.9998 |
| | 75 | 0.01 | 0.0500 | 0.2580 | 0.6771 | 0.9553 | 0.9991 | 1.0000 |
| | | 0.05 | 0.0500 | 0.2485 | 0.6549 | 0.9457 | 0.9987 | 1.0000 |
| | | 0.1 | 0.0500 | 0.2396 | 0.6331 | 0.9351 | 0.9980 | 1.0000 |

**Table A6.** The power $h(c_1)$ at $d = 3$, $\alpha = 0.1$.

| | | | | | $c_1$ | | | |
|---|---|---|---|---|---|---|---|---|
| $m$ | $n$ | $p$ | 0.80 | 0.825 | 0.85 | 0.875 | 0.90 | 0.925 |
| 5 | 25 | 0.01 | 0.1000 | 0.2461 | 0.4980 | 0.7844 | 0.9584 | 0.9983 |
| | | 0.05 | 0.1000 | 0.2424 | 0.4878 | 0.7723 | 0.9530 | 0.9978 |
| | | 0.1 | 0.1000 | 0.2387 | 0.4779 | 0.7602 | 0.9471 | 0.9972 |
| | 50 | 0.01 | 0.1000 | 0.3293 | 0.6920 | 0.9418 | 0.9976 | 1.0000 |
| | | 0.05 | 0.1000 | 0.3232 | 0.6796 | 0.9352 | 0.9970 | 1.0000 |
| | | 0.1 | 0.1000 | 0.3174 | 0.6674 | 0.9283 | 0.9963 | 1.0000 |
| | 75 | 0.01 | 0.1000 | 0.3987 | 0.8096 | 0.9838 | 0.9999 | 1.0000 |
| | | 0.05 | 0.1000 | 0.3909 | 0.7982 | 0.9811 | 0.9998 | 1.0000 |
| | | 0.1 | 0.1000 | 0.3834 | 0.7867 | 0.9780 | 0.9997 | 1.0000 |
| 6 | 25 | 0.01 | 0.1000 | 0.2442 | 0.4929 | 0.7784 | 0.9557 | 0.9981 |
| | | 0.05 | 0.1000 | 0.2396 | 0.4804 | 0.7633 | 0.9487 | 0.9974 |
| | | 0.1 | 0.1000 | 0.2353 | 0.4684 | 0.7481 | 0.9410 | 0.9965 |
| | 50 | 0.01 | 0.1000 | 0.3262 | 0.6858 | 0.9385 | 0.9973 | 1.0000 |
| | | 0.05 | 0.1000 | 0.3189 | 0.6705 | 0.9301 | 0.9965 | 1.0000 |
| | | 0.1 | 0.1000 | 0.3119 | 0.6555 | 0.9211 | 0.9954 | 1.0000 |
| | 75 | 0.01 | 0.1000 | 0.3948 | 0.8039 | 0.9825 | 0.9998 | 1.0000 |
| | | 0.05 | 0.1000 | 0.3853 | 0.7896 | 0.9788 | 0.9997 | 1.0000 |
| | | 0.1 | 0.1000 | 0.3761 | 0.7752 | 0.9747 | 0.9996 | 1.0000 |
| 7 | 25 | 0.01 | 0.1000 | 0.2424 | 0.4878 | 0.7723 | 0.9530 | 0.9978 |
| | | 0.05 | 0.1000 | 0.2370 | 0.4732 | 0.7543 | 0.9442 | 0.9969 |
| | | 0.1 | 0.1000 | 0.2320 | 0.4592 | 0.7363 | 0.9346 | 0.9957 |
| | 50 | 0.01 | 0.1000 | 0.3232 | 0.6797 | 0.9352 | 0.9970 | 1.0000 |
| | | 0.05 | 0.1000 | 0.3147 | 0.6615 | 0.9248 | 0.9958 | 1.0000 |
| | | 0.1 | 0.1000 | 0.3065 | 0.6438 | 0.9136 | 0.9943 | 1.0000 |
| | 75 | 0.01 | 0.1000 | 0.3909 | 0.7982 | 0.9811 | 0.9998 | 1.0000 |
| | | 0.05 | 0.1000 | 0.3798 | 0.7810 | 0.9764 | 0.9997 | 1.0000 |
| | | 0.1 | 0.1000 | 0.3692 | 0.7638 | 0.9711 | 0.9995 | 1.0000 |

**Table A7.** The power $h(c_1)$ at $d = 4$, $\alpha = 0.01$.

| | | | $c_1$ | | | | | |
|---|---|---|---|---|---|---|---|---|
| $m$ | $n$ | $p$ | 0.80 | 0.825 | 0.85 | 0.875 | 0.90 | 0.925 |
| 5 | 25 | 0.01 | 0.0100 | 0.0386 | 0.1293 | 0.3482 | 0.6861 | 0.9408 |
| | | 0.05 | 0.0100 | 0.0375 | 0.1238 | 0.3326 | 0.6645 | 0.9308 |
| | | 0.1 | 0.0100 | 0.0365 | 0.1185 | 0.3176 | 0.6429 | 0.9198 |
| | 50 | 0.01 | 0.0100 | 0.0649 | 0.2729 | 0.6634 | 0.9442 | 0.9987 |
| | | 0.05 | 0.0100 | 0.0627 | 0.2609 | 0.6426 | 0.9351 | 0.9982 |
| | | 0.1 | 0.0100 | 0.0606 | 0.2494 | 0.6219 | 0.9251 | 0.9975 |
| | 75 | 0.01 | 0.0100 | 0.0929 | 0.4150 | 0.8422 | 0.9911 | 1.0000 |
| | | 0.05 | 0.0100 | 0.0894 | 0.3981 | 0.8261 | 0.9887 | 1.0000 |
| | | 0.1 | 0.0100 | 0.0861 | 0.3818 | 0.8092 | 0.9859 | 0.9999 |
| 6 | 25 | 0.01 | 0.0100 | 0.0380 | 0.1265 | 0.3403 | 0.6754 | 0.9359 |
| | | 0.05 | 0.0100 | 0.0368 | 0.1198 | 0.3214 | 0.6484 | 0.9227 |
| | | 0.1 | 0.0100 | 0.0356 | 0.1136 | 0.3035 | 0.6216 | 0.9080 |
| | 50 | 0.01 | 0.0100 | 0.0638 | 0.2668 | 0.6530 | 0.9398 | 0.9984 |
| | | 0.05 | 0.0100 | 0.0612 | 0.2523 | 0.6272 | 0.9278 | 0.9977 |
| | | 0.1 | 0.0100 | 0.0587 | 0.2387 | 0.6016 | 0.9144 | 0.9966 |
| | 75 | 0.01 | 0.0100 | 0.0911 | 0.4065 | 0.8342 | 0.9899 | 1.0000 |
| | | 0.05 | 0.0100 | 0.0869 | 0.3859 | 0.8136 | 0.9867 | 0.9999 |
| | | 0.1 | 0.0100 | 0.0830 | 0.3663 | 0.7921 | 0.9827 | 0.9999 |
| 7 | 25 | 0.01 | 0.0100 | 0.0375 | 0.1238 | 0.3326 | 0.6646 | 0.9308 |
| | | 0.05 | 0.0100 | 0.0361 | 0.1161 | 0.3107 | 0.6325 | 0.9142 |
| | | 0.1 | 0.0100 | 0.0347 | 0.1091 | 0.2902 | 0.6007 | 0.8954 |
| | 50 | 0.01 | 0.0100 | 0.0627 | 0.2609 | 0.6427 | 0.9352 | 0.9982 |
| | | 0.05 | 0.0100 | 0.0597 | 0.2441 | 0.6120 | 0.9200 | 0.9971 |
| | | 0.1 | 0.0100 | 0.0568 | 0.2285 | 0.5818 | 0.9030 | 0.9956 |
| | 75 | 0.01 | 0.0100 | 0.0894 | 0.3981 | 0.8261 | 0.9887 | 1.0000 |
| | | 0.05 | 0.0100 | 0.0845 | 0.3741 | 0.8009 | 0.9844 | 0.9999 |
| | | 0.1 | 0.0100 | 0.0801 | 0.3516 | 0.7746 | 0.9789 | 0.9998 |

**Table A8.** The power $h(c_1)$ at $d = 4$, $\alpha = 0.05$.

| | | | $c_1$ | | | | | |
|---|---|---|---|---|---|---|---|---|
| $m$ | $n$ | $p$ | 0.80 | 0.825 | 0.85 | 0.875 | 0.90 | 0.925 |
| 5 | 25 | 0.01 | 0.0500 | 0.1460 | 0.3506 | 0.6502 | 0.9013 | 0.9920 |
| | | 0.05 | 0.0500 | 0.1432 | 0.3411 | 0.6352 | 0.8912 | 0.9902 |
| | | 0.1 | 0.0500 | 0.1405 | 0.3320 | 0.6203 | 0.8806 | 0.9881 |
| | 50 | 0.01 | 0.0500 | 0.2084 | 0.5445 | 0.8729 | 0.9898 | 0.9999 |
| | | 0.05 | 0.0500 | 0.2035 | 0.5307 | 0.8616 | 0.9876 | 0.9999 |
| | | 0.1 | 0.0500 | 0.1989 | 0.5171 | 0.8499 | 0.9852 | 0.9998 |
| | 75 | 0.01 | 0.0500 | 0.2645 | 0.6833 | 0.9538 | 0.9989 | 1.0000 |
| | | 0.05 | 0.0500 | 0.2579 | 0.6685 | 0.9475 | 0.9985 | 1.0000 |
| | | 0.1 | 0.0500 | 0.2514 | 0.6538 | 0.9408 | 0.9981 | 1.0000 |

**Table A8.** *Cont.*

| | | | | | $c_1$ | | | |
|---|---|---|---|---|---|---|---|---|
| 6 | 25 | 0.01 | 0.0500 | 0.1446 | 0.3458 | 0.6427 | 0.8963 | 0.9911 |
| | | 0.05 | 0.0500 | 0.1412 | 0.3343 | 0.6241 | 0.8834 | 0.9886 |
| | | 0.1 | 0.0500 | 0.1380 | 0.3233 | 0.6058 | 0.8698 | 0.9857 |
| | 50 | 0.01 | 0.0500 | 0.2059 | 0.5375 | 0.8673 | 0.9887 | 0.9999 |
| | | 0.05 | 0.0500 | 0.2001 | 0.5206 | 0.8529 | 0.9858 | 0.9998 |
| | | 0.1 | 0.0500 | 0.1945 | 0.5042 | 0.8380 | 0.9824 | 0.9998 |
| | 75 | 0.01 | 0.0500 | 0.2612 | 0.6759 | 0.9507 | 0.9987 | 1.0000 |
| | | 0.05 | 0.0500 | 0.2531 | 0.6575 | 0.9426 | 0.9982 | 1.0000 |
| | | 0.1 | 0.0500 | 0.2454 | 0.6394 | 0.9337 | 0.9975 | 1.0000 |
| 7 | 25 | 0.01 | 0.0500 | 0.1432 | 0.3412 | 0.6352 | 0.8913 | 0.9902 |
| | | 0.05 | 0.0500 | 0.1393 | 0.3277 | 0.6132 | 0.8754 | 0.9869 |
| | | 0.1 | 0.0500 | 0.1356 | 0.3151 | 0.5918 | 0.8587 | 0.9830 |
| | 50 | 0.01 | 0.0500 | 0.2035 | 0.5307 | 0.8617 | 0.9876 | 0.9999 |
| | | 0.05 | 0.0500 | 0.1967 | 0.5107 | 0.8441 | 0.9839 | 0.9998 |
| | | 0.1 | 0.0500 | 0.1903 | 0.4916 | 0.8260 | 0.9793 | 0.9997 |
| | 75 | 0.01 | 0.0500 | 0.2579 | 0.6685 | 0.9476 | 0.9985 | 1.0000 |
| | | 0.05 | 0.0500 | 0.2484 | 0.6467 | 0.9374 | 0.9978 | 1.0000 |
| | | 0.1 | 0.0500 | 0.2396 | 0.6254 | 0.9262 | 0.9969 | 1.0000 |

**Table A9.** The power $h(c_1)$ at $d = 4$, $\alpha = 0.1$.

| | | | | | $c_1$ | | | |
|---|---|---|---|---|---|---|---|---|
| $m$ | $n$ | $p$ | 0.80 | 0.825 | 0.85 | 0.875 | 0.90 | 0.925 |
| 5 | 25 | 0.01 | 0.1000 | 0.2494 | 0.5028 | 0.7846 | 0.9558 | 0.9978 |
| | | 0.05 | 0.1000 | 0.2457 | 0.4928 | 0.7730 | 0.9505 | 0.9973 |
| | | 0.1 | 0.1000 | 0.2420 | 0.4831 | 0.7614 | 0.9448 | 0.9966 |
| | 50 | 0.01 | 0.1000 | 0.3295 | 0.6862 | 0.9349 | 0.9965 | 1.0000 |
| | | 0.05 | 0.1000 | 0.3235 | 0.6741 | 0.9282 | 0.9957 | 1.0000 |
| | | 0.1 | 0.1000 | 0.3178 | 0.6622 | 0.9210 | 0.9948 | 1.0000 |
| | 75 | 0.01 | 0.1000 | 0.3959 | 0.7987 | 0.9792 | 0.9997 | 1.0000 |
| | | 0.05 | 0.1000 | 0.3882 | 0.7874 | 0.9760 | 0.9996 | 1.0000 |
| | | 0.1 | 0.1000 | 0.3808 | 0.7759 | 0.9725 | 0.9994 | 1.0000 |
| 6 | 25 | 0.01 | 0.1000 | 0.2475 | 0.4978 | 0.7788 | 0.9532 | 0.9976 |
| | | 0.05 | 0.1000 | 0.2429 | 0.4856 | 0.7644 | 0.9463 | 0.9968 |
| | | 0.1 | 0.1000 | 0.2385 | 0.4738 | 0.7499 | 0.9389 | 0.9958 |
| | 50 | 0.01 | 0.1000 | 0.3265 | 0.6801 | 0.9316 | 0.9961 | 1.0000 |
| | | 0.05 | 0.1000 | 0.3192 | 0.6652 | 0.9229 | 0.9950 | 1.0000 |
| | | 0.1 | 0.1000 | 0.3122 | 0.6506 | 0.9137 | 0.9937 | 1.0000 |
| | 75 | 0.01 | 0.1000 | 0.3920 | 0.7931 | 0.9777 | 0.9996 | 1.0000 |
| | | 0.05 | 0.1000 | 0.3827 | 0.7788 | 0.9735 | 0.9995 | 1.0000 |
| | | 0.1 | 0.1000 | 0.3737 | 0.7645 | 0.9688 | 0.9993 | 1.0000 |

**Table A9.** *Cont.*

| | | | | | $c_1$ | | | |
|---|---|---|---|---|---|---|---|---|
| 7 | 25 | 0.01 | 0.1000 | 0.2457 | 0.4928 | 0.7731 | 0.9505 | 0.9973 |
| | | 0.05 | 0.1000 | 0.2403 | 0.4785 | 0.7558 | 0.9420 | 0.9962 |
| | | 0.1 | 0.1000 | 0.2352 | 0.4648 | 0.7386 | 0.9327 | 0.9949 |
| | 50 | 0.01 | 0.1000 | 0.3236 | 0.6742 | 0.9282 | 0.9957 | 1.0000 |
| | | 0.05 | 0.1000 | 0.3150 | 0.6565 | 0.9175 | 0.9942 | 1.0000 |
| | | 0.1 | 0.1000 | 0.3070 | 0.6392 | 0.9062 | 0.9924 | 0.9999 |
| | 75 | 0.01 | 0.1000 | 0.3882 | 0.7874 | 0.9760 | 0.9996 | 1.0000 |
| | | 0.05 | 0.1000 | 0.3773 | 0.7703 | 0.9707 | 0.9994 | 1.0000 |
| | | 0.1 | 0.1000 | 0.3669 | 0.7533 | 0.9648 | 0.9990 | 1.0000 |

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