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Novel Robust Stability Criteria for Lur'e Systems with Time-Varying Delay

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Abstract: This paper focuses on addressing the issue of absolute stability for uncertain Lur'e systems with time-varying delay using a delay-segmentation approach. The approach involves decomposing the delay interval into two distinct subintervals of unequal lengths. This allows for the introduction of a delay-segmentation-based augmented Lyapunov–Krasovskii functional that ensures piecewise continuity at the partition points. By selecting two sets of Lyapunov matrices for the time-varying delay in each interval, the obtained results are less conservative, providing a more accurate assessment of absolute stability. Finally, a numerical example is given to demonstrate the superiority of the delay-segmentation approach.

Keywords: Lur'e system; absolute stability; Lyapunov–Krasovskii functional; time-varying delay; delay-segmentation approach

MSC: 93D05

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1. Introduction

A Lur'e system is a type of nonlinear system in which the nonlinear component satisfies specific sector constraints. Since their introduction in the 20th century, Lur'e systems and their absolute stability have garnered significant research interest. Numerous scholars have conducted in-depth studies on various aspects of Lur'e systems [1–9]. On the other hand, time delay is a prevalent phenomenon that arises during the implementation process of actual systems [10–12]. The presence of time delay can significantly impact a system's performance and efficiency during normal operation, potentially leading to performance deterioration or even system collapse [13–18]. Therefore, it is crucial to determine the range of time delay within which a system's stability can be ensured [19–25]. To address this issue, numerous studies have been conducted on the problem of the delay-dependent absolute stability of Lur'e systems, resulting in a variety of notable stability criteria [26–28].

As is widely recognized, the most common approaches to analyzing the absolute stability of Lur'e systems with constant delays are the so-called frequency-domain and time-domain approaches [29]. The interconnection between these two approaches was established by the Kalman–Yakubovich–Popov (KYP) lemma, which has greatly contributed to the development of relevant research [30]. Using the KYP lemma, it has been shown that the classical circle criterion and a linear matrix inequality can be transformed equivalently. For the robust stability of uncertain Lur'e systems with time-varying delay, the Lyapunov approach is considered one of the most effective methods. This approach involves using the Lyapunov method to establish stability criteria, typically in the form of sufficient conditions. One way to measure the conservativeness of these criteria is based on the Maximum Allowable Delay Bound (MADB) [31].

The construction of an appropriate Lyapunov functional plays a crucial role in obtaining stability criteria with reduced conservativeness. Substantial efforts have been dedicated to developing Lyapunov functionals aimed at mitigating the conservativeness of derived stability conditions. These Lyapunov functionals can be broadly classified into two categories: state-augmentation-based Lyapunov functionals and delay-decomposition-based Lyapunov functionals. In a recent work by Han et al. [27], a discretized Lyapunov functional was constructed, leading to the derivation of delay-dependent absolute stability conditions. Another notable contribution by Gou et al. [32] introduced a novel Lyapunov functional based on the delay-segmentation approach. This approach significantly reduces the conservativeness of the stability criterion for linear systems with constant delay. Building upon this foundation, the delay-segmentation approach was further extended to systems with time-varying delay in order to enhance delay-dependent stability; see [31] and the references therein. In another related study, Wang et al. [28] proposed a novel complete delay-segmentation Lyapunov functional by decomposing the delay in all integral terms. However, it is worth noting that in the constructed Lyapunov functional [28], only the integral terms were decomposed, while the term $x^{\mathrm{T}}(t)Px(t)$ remained independent of the delay interval. Consequently, it is necessary to investigate how to decompose the term $x^{\mathrm{T}}(t)Px(t)$ and assess whether the conservativeness of the proposed results can be further reduced by decomposing $x^{\mathrm{T}}(t)Px(t)$ based on the delay interval.

Motivated by the aforementioned considerations, this paper focuses on investigating the robust absolute stability of uncertain Lur'e systems with time-varying delay using a delay-segmentation approach. The approach involves introducing a segmentation point, denoted as h_α, within the delay interval $[0, h]$. Thus, the original interval is divided into two distinct sub-intervals, namely $[0, h_\alpha]$ and $[h_\alpha, h]$. To analyze the stability, two separate Lyapunov functionals are constructed for each of these sub-intervals. By employing different pairs of Lyapunov matrices for the delay on each sub-interval, the conservatism of the resulting condition is significantly reduced. This approach offers enhanced effectiveness and merits in addressing robust absolute stability. The validity and advantages of the proposed methodology are confirmed through a numerical example, which serves to illustrate its practical applicability.

Notation: In this paper, $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$, and $\mathbb{S}^n_+$ denote the n-dimensional Euclidean space, the set of $n \times m$ real matrices, and the set of $n \times n$ symmetric positive definite matrices, respectively. $\mathbb{N}$ represents the set of positive integers, and the superscripts 'T' and '-1' stand for transpose and the inverse of a matrix, respectively. The symbol $diag\{\cdots\}$ denotes the block-diagonal matrix. 0 and I represent zero matrices and identity matrices with appropriate dimensions, respectively. The notation $*$ represents the symmetric term in a symmetric matrix, and the notation $\mathrm{Sym}\{\Omega\} = \Omega + \Omega^T$.

2. Problem Statement and Preliminaries

Let us consider the following time-varying delay Lur'e system:

$$\begin{cases} \dot{x}(t) = \hat{A}x(t) + \hat{A}_h x(t-h_t) + \hat{B}\hat{\omega}(t) \\ p(t) = \hat{L}_1 x(t) + \hat{L}_2 x(t-h_t) \\ \hat{\omega}(t) = -\Upsilon(t, p(t)) \\ x(t) = \phi(t), t \in [-\bar{h}, 0] \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $p(t) \in \mathbb{R}^p$, and $\hat{\omega}(t) \in \mathbb{R}^p$ are used to represent the Lur'e system state and output and input vectors, respectively. Constant matrices $\hat{A}$, $\hat{A}_h$, $\hat{B}$, $\hat{L}_1$, and $\hat{L}_2$ are given. $\phi(t)$ defined in $[-h, 0]$ is the initial condition. For brevity, the time-varying delay $h(t)$ is denoted by h_t and satisfies

$$0 < h_t < \bar{h} \tag{2}$$

$$|\dot{h}_t| \leq \bar{\tau} \tag{3}$$

where $\bar{h}$ and $\bar{\tau}$ are constants. The nonlinear function $\Upsilon(t, p(t)) \in \mathbb{R}^p$, Lipschitz in $p(t)$, is continuous in t and belongs to the sector $[\mathscr{K}_1, \mathscr{K}_2]$. Note that real matrices $\mathscr{K}_1$ and $\mathscr{K}_2$ are given, and we denote $\mathscr{K}_{21} = \mathscr{K}_2 - \mathscr{K}_1$. $\Upsilon(t, p(t))$ globally satisfies

$$[\mathscr{K}_1 p(t) - \Upsilon(t, p(t))]^{\mathrm{T}}[\mathscr{K}_2 p(t) - \Upsilon(t, p(t))] \leq 0 \tag{4}$$

and $\forall t \geq 0,\ \Upsilon(t, 0) = 0$.

Remark 1. *By using the loop transformation* [33], *the absolute stability of Lur'e System* (1) *in the sector* $[\mathscr{K}_1, \mathscr{K}_2]$ *is equivalent to that of the following system in the sector* $[0, \mathscr{K}_{21}]$*:*

$$\begin{cases} \dot{x}(t) =(\hat{A} - \hat{B}\mathscr{K}_1\hat{L}_1)x(t) + (\hat{A}_h - \hat{B}\mathscr{K}_1\hat{L}_2)x(t-h_t) + \hat{B}\hat{\omega}(t) \\ p(t) =\hat{L}_1 x(t) + \hat{L}_2 x(t-h_t) \\ \hat{\omega}(t) = -\Upsilon(t, p(t)) \end{cases} \tag{5}$$

Prior to presenting our main results, we will introduce the definition of absolute stability and two essential lemmas that play a crucial role in the proof process.

Definition 1 ([27])**.** *The systems described by* (1) *are said to be absolutely stable within the sector* $[\mathscr{K}_1, \mathscr{K}_2]$ *if it demonstrates global uniform asymptotic stability for any nonlinear function* $\Upsilon(t, p(t))$ *that satisfies Condition* (4).

Lemma 1 ([34])**.** *Consider a continuous differentiable function* x: $[\delta_1, \delta_2] \to \mathbb{R}^n$. *If there exist any matrices* N *and* $Z > 0$ *with appropriate dimensions, the following condition is true:*

$$-\int_{\delta_1}^{\delta_2} \dot{x}^{\mathrm{T}}(\vartheta) Z \dot{x}(\vartheta) \mathrm{d}\vartheta \leq Sym\{NM\} + (\delta_2 - \delta_1) N \tilde{Z}^{-1} N^{\mathrm{T}} \tag{6}$$

where

$$M =[x^{\mathrm{T}}(\delta_2) - x^{\mathrm{T}}(\delta_1) \quad x^{\mathrm{T}}(\delta_2) + x^{\mathrm{T}}(\delta_1) - \frac{2}{\delta_2 - \delta_1}\int_{\delta_1}^{\delta_2} x^{\mathrm{T}}(\vartheta)\mathrm{d}\vartheta]^{\mathrm{T}}$$
$$\tilde{Z} =diag\{Z, 3Z\}.$$

Lemma 2 ([35])**.** *For a given scalar* $\varepsilon > 0$ *and matrices* X, Y, $\mathcal{Z}(t)$ *with appropriate dimensions and* $\mathcal{Z}^{\mathrm{T}}(t)\mathcal{Z}(t) \leq I$, *the following inequality is satisfied:*

$$X\mathcal{Z}(t)Y + (X\mathcal{Z}(t)Y)^{\mathrm{T}} \leq \varepsilon^{-1}XX^{\mathrm{T}} + \varepsilon Y^{\mathrm{T}}Y. \tag{7}$$

3. Main Results

In this section, we will present two stability criteria. To facilitate clarity and brevity in our presentation, we need the following notation:

$$h_\alpha = \alpha\bar{h}\ (\alpha \in [0,1]),\ h_\beta = (1-\alpha)\bar{h},\ \bar{h}_t = \bar{h} - h_t, h_d = 1 - \dot{h}_t,$$
$$\hat{h}_d = h_\alpha - h_t, \bar{h}_d = h_t - h_\alpha, \bar{A} = \hat{A} - \hat{B}\mathscr{K}_1\hat{L}_1, \bar{A}_h = \hat{A}_h - \hat{B}\mathscr{K}_1\hat{L}_2,$$
$$\nu_0(t) = [x^{\mathrm{T}}(t)\ \ x^{\mathrm{T}}(t-h_t)\ \ x^{\mathrm{T}}(t-h_\alpha)\ \ x^{\mathrm{T}}(t-\bar{h})]^{\mathrm{T}},$$
$$\nu_1(t) = \int_{t-h_t}^{t} x(\vartheta)\mathrm{d}\vartheta,\ \ \nu_2(t) = \int_{t-h_\alpha}^{t} x(\vartheta)\mathrm{d}\vartheta,\ \ \nu_3(t) = \int_{t-\bar{h}}^{t-h_t} x(\vartheta)\mathrm{d}\vartheta,$$

$$\nu_4(t) = \int_{t-h_\alpha}^{t-h_t} x(\vartheta)\mathrm{d}\vartheta,\ \ \nu_5(t) = \int_{t-h_t}^{t-h_\alpha} x(\vartheta)\mathrm{d}\vartheta,\ \ \nu_6(t) = \int_{t-\bar{h}}^{t-h_\alpha} x(\vartheta)\mathrm{d}\vartheta,$$
$$\eta_1(t) = [\nu_0^{\mathrm{T}}(t)\ \nu_1^{\mathrm{T}}(t)\ \nu_3^{\mathrm{T}}(t)\ \nu_2^{\mathrm{T}}(t)\ \nu_6^{\mathrm{T}}(t)]^{\mathrm{T}},\ \eta_2(t,s) = [x^{\mathrm{T}}(s)\ \dot{x}^{\mathrm{T}}(s)\ \nu_0^{\mathrm{T}}(t)]^{\mathrm{T}},$$
$$\xi_1(t) = [\nu_0^{\mathrm{T}}(t)\ \ \dot{x}^{\mathrm{T}}(t)\ \ \dot{x}^{\mathrm{T}}(t-h_t)\ \ \dot{x}^{\mathrm{T}}(t-h_\alpha)\ \ \dot{x}^{\mathrm{T}}(t-\bar{h})]^{\mathrm{T}},$$
$$\xi_2(t) = [\frac{1}{h_t}\nu_1^{\mathrm{T}}(t)\ \ \frac{1}{\hat{h}_d}\nu_4^{\mathrm{T}}(t)\ \ \frac{1}{h_\beta}\nu_6^{\mathrm{T}}(t)\ \ \nu_1^{\mathrm{T}}(t)\ \ \nu_3^{\mathrm{T}}(t)\ \ \nu_2^{\mathrm{T}}(t)\ \ \hat{\omega}^{\mathrm{T}}(t)]^{\mathrm{T}},$$
$$\xi_3(t) = [\frac{1}{h_\alpha}\nu_2^{\mathrm{T}}(t)\ \ \frac{1}{\bar{h}_d}\nu_5^{\mathrm{T}}(t)\ \ \frac{1}{h_t}\nu_3^{\mathrm{T}}(t)\ \ \nu_1^{\mathrm{T}}(t)\ \ \nu_3^{\mathrm{T}}(t)\ \ \nu_6^{\mathrm{T}}(t)\ \ \hat{\omega}^{\mathrm{T}}(t)]^{\mathrm{T}},$$
$$\xi_e(t) = [\xi_1^{\mathrm{T}}(t)\ \xi_2^{\mathrm{T}}(t)]^{\mathrm{T}},\ \ \ \xi_f(t) = [\xi_1^{\mathrm{T}}(t)\ \ \xi_3^{\mathrm{T}}(t)]^{\mathrm{T}},$$
$$c_i = [0_{n\times(i-1)n}\ I_n\ 0_{n\times(14-i)n}\ 0_{n\times p}],\ \ i = 1,2,\cdots,14,\ \ \ c_{15} = [0_{p\times 14n}\ I_p].$$

Next, an improved absolute stability criterion is derived via a delay-segmentation approach.

Theorem 1. *For given scalars $\bar{h} > 0$ and $\bar{\tau} \in [0,1)$, Lur'e System (5) is absolutely stable on condition that there exist matrices $P_i \in \mathbb{S}_+^{8n}$, $Q_i \in \mathbb{S}_+^{6n}$, $\bar{Z}_i \in \mathbb{S}_+^{n}$, $i \in \{1,2,3\}$, R_1, $R_2 \in \mathbb{S}_+^{6n}$, Y_1, $Y_2 \in \mathbb{S}_+^{n}$, $\check{N}_j \in \mathbb{R}^{(14n+p)\times 2n}$, $S_{1j}, S_{2j} \in \mathbb{R}^{(14n+p)\times n}$, $j \in \{1,2,\cdots,6\}$, U_1, U_2, U_3, $U_4 \in \mathbb{R}^n$ and two scalars ε_1, $\varepsilon_2 > 0$ such that, for $h_t \in [0,\bar{h}], \dot{h}_t \in [-\bar{\tau},\bar{\tau}]$,*

$$\begin{bmatrix} \aleph(0,\dot{h}_t) & \sqrt{\bar{h}_d}\check{N}_2 & \sqrt{h_\beta}\check{N}_3 \\ * & -\check{Z}_2 & 0 \\ * & * & -\check{Z}_3 \end{bmatrix} < 0 \tag{8}$$

$$\begin{bmatrix} \aleph(h_\alpha,\dot{h}_t) & \sqrt{h_t}\check{N}_1 & \sqrt{h_\beta}\check{N}_3 \\ * & -\check{Z}_1(\dot{h}_t) & 0 \\ * & * & -\check{Z}_3 \end{bmatrix} < 0 \tag{9}$$

$$\begin{bmatrix} \bar{\aleph}(h_\alpha,\dot{h}_t) & \sqrt{\bar{h}_t}\check{N}_6 & \sqrt{h_\alpha}\check{N}_4 \\ * & -\check{Z}_3 & 0 \\ * & * & -\check{Z}_4(\dot{h}_t) \end{bmatrix} < 0 \tag{10}$$

$$\begin{bmatrix} \bar{\aleph}(\bar{h},\dot{h}_t) & \sqrt{\bar{h}_d}\check{N}_5 & \sqrt{h_\alpha}\check{N}_4 \\ * & -\check{Z}_5(\dot{h}_t) & 0 \\ * & * & -\check{Z}_4(\dot{h}_t) \end{bmatrix} < 0 \tag{11}$$

$$Y_1 + h_d\bar{Z}_1 + \dot{h}_t\bar{Z}_3 > 0 \tag{12}$$

$$Y_1 + \bar{Z}_1 - \dot{h}_t\bar{Z}_2 + \dot{h}_t\bar{Z}_3 > 0 \tag{13}$$

where

$$\begin{aligned}\aleph(h_t,\dot{h}_t) = {} & Sym\{\Pi_1^{\mathrm{T}}(h_tP_1 + \hat{h}_dP_2 + h_\beta P_3)\Pi_2 + \Pi_7^{\mathrm{T}}R_1\Pi_{10} + \Pi_8^{\mathrm{T}}R_2\Pi_{10} + \Pi_9^{\mathrm{T}}Q_1\Pi_{10} \\ & + \Pi_{11}^{\mathrm{T}}Q_3\Pi_{10} + \Pi_{12}^{\mathrm{T}}\Pi_{13} + \check{N}_1\check{M}_1 + \check{N}_2\check{M}_2 + \check{N}_3\check{M}_3 + (S_{11} + \dot{h}_tS_{21})(h_tc_9 - c_{12}) \\ & + (S_{12} + \dot{h}_tS_{22})(\hat{h}_dc_{10} + h_\beta c_{11} - c_{13}) + (S_{13} + \dot{h}_tS_{23})(h_tc_9 + \hat{h}_dc_{10} - c_{14}) \\ & - \varepsilon_1(c_{15}^{\mathrm{T}}c_{15} + c_{15}^{\mathrm{T}}\mathscr{K}_{21}\hat{L}_1c_1 + c_{15}^{\mathrm{T}}\mathscr{K}_{21}\hat{L}_2c_2)\} + \dot{h}_t\Pi_1^{\mathrm{T}}(P_1 - P_2)\Pi_1 \\ & + \Pi_3^{\mathrm{T}}(R_1 + Q_1)\Pi_3 + \Pi_4^{\mathrm{T}}(R_2 - R_1)\Pi_4 - \Pi_5^{\mathrm{T}}(R_2 + Q_3)\Pi_5 \\ & + h_d\Pi_6^{\mathrm{T}}(Q_3 - Q_1)\Pi_6 + c_5^{\mathrm{T}}(h_\alpha Y_1 + h_\beta Y_2 + h_t\bar{Z}_1 + \bar{h}_t\bar{Z}_3)c_5,\end{aligned}$$

$$\begin{aligned}\bar{\aleph}(h_t,\dot{h}_t) = {} & Sym\{\bar{\Pi}_1^{\mathrm{T}}(h_\alpha P_1+\bar{h}_d P_2+\bar{h}_t P_3)\bar{\Pi}_2+\bar{\Pi}_7^{\mathrm{T}}R_1\bar{\Pi}_{10}+\bar{\Pi}_8^{\mathrm{T}}R_2\bar{\Pi}_{10}+\bar{\Pi}_7^{\mathrm{T}}Q_1\bar{\Pi}_{10}\\ & +\bar{\Pi}_9^{\mathrm{T}}Q_2\bar{\Pi}_{10}+\bar{\Pi}_{11}^{\mathrm{T}}Q_3\bar{\Pi}_{10}+\bar{\Pi}_{12}^{\mathrm{T}}\Pi_{13}+\check{N}_4\check{M}_4+\check{N}_5\check{M}_5+\check{N}_6\check{M}_6\\ & +(S_{14}+\dot{h}_t S_{24})(\bar{h}_t c_{11}-c_{13})+(S_{15}+\dot{h}_t S_{25})(h_\alpha c_9+\bar{h}_d c_{10}-c_{12})\\ & +(S_{16}+\dot{h}_t S_{26})(\bar{h}_d c_{10}+\bar{h}_t c_{11}-c_{14})-\varepsilon_2(c_{15}^{\mathrm{T}}c_{15}+c_{15}^{\mathrm{T}}\mathscr{K}_{21}\hat{L}_1 c_1+c_{15}^{\mathrm{T}}\mathscr{K}_{21}\hat{L}_2 c_2)\}\\ & +\dot{h}_t\bar{\Pi}_1^{\mathrm{T}}(P_2-P_3)\bar{\Pi}_1+\bar{\Pi}_3^{\mathrm{T}}R_1\bar{\Pi}_3+\bar{\Pi}_4^{\mathrm{T}}(R_2-R_1)\bar{\Pi}_4-\bar{\Pi}_5^{\mathrm{T}}(R_2+Q_3)\bar{\Pi}_5\\ & +\bar{\Pi}_3^{\mathrm{T}}Q_1\bar{\Pi}_3+\bar{\Pi}_4^{\mathrm{T}}(Q_2-Q_1)\bar{\Pi}_4+h_d\bar{\Pi}_6^{\mathrm{T}}(Q_3-Q_2)\bar{\Pi}_6\\ & +c_5^{\mathrm{T}}(h_\alpha Y_1+h_\beta Y_2+h_\alpha\bar{Z}_1+\bar{h}_d\bar{Z}_2+\bar{h}_t\bar{Z}_3)c_5\end{aligned}$$

with

$$\begin{aligned}
\Pi_1 &= \left[c_1^{\mathrm{T}}\ c_2^{\mathrm{T}}\ c_3^{\mathrm{T}}\ c_4^{\mathrm{T}}\ c_{12}^{\mathrm{T}}\ c_{13}^{\mathrm{T}}\ c_{14}^{\mathrm{T}}\ h_\beta c_{11}^{\mathrm{T}}\right]^{\mathrm{T}}\\
\Pi_2 &= \left[c_5^{\mathrm{T}}\ h_d c_6^{\mathrm{T}}\ c_7^{\mathrm{T}}\ c_8^{\mathrm{T}}\ c_1^{\mathrm{T}}-h_d c_2^{\mathrm{T}}\ h_d c_2^{\mathrm{T}}-c_4^{\mathrm{T}}\ c_1^{\mathrm{T}}-c_3^{\mathrm{T}}\ c_3^{\mathrm{T}}-c_4^{\mathrm{T}}\right]^{\mathrm{T}}\\
\Pi_3 &= \left[c_1^{\mathrm{T}}\ c_5^{\mathrm{T}}\ c_1^{\mathrm{T}}\ c_2^{\mathrm{T}}\ c_3^{\mathrm{T}}\ c_4^{\mathrm{T}}\right]^{\mathrm{T}}\\
\Pi_4 &= \left[c_3^{\mathrm{T}}\ c_7^{\mathrm{T}}\ c_1^{\mathrm{T}}\ c_2^{\mathrm{T}}\ c_3^{\mathrm{T}}\ c_4^{\mathrm{T}}\right]^{\mathrm{T}}\\
\Pi_5 &= \left[c_4^{\mathrm{T}}\ c_8^{\mathrm{T}}\ c_1^{\mathrm{T}}\ c_2^{\mathrm{T}}\ c_3^{\mathrm{T}}\ c_4^{\mathrm{T}}\right]^{\mathrm{T}}\\
\Pi_6 &= \left[c_2^{\mathrm{T}}\ c_6^{\mathrm{T}}\ c_1^{\mathrm{T}}\ c_2^{\mathrm{T}}\ c_3^{\mathrm{T}}\ c_4^{\mathrm{T}}\right]^{\mathrm{T}}\\
\Pi_7 &= \left[c_{14}^{\mathrm{T}}\ c_1^{\mathrm{T}}-c_3^{\mathrm{T}}\ h_\alpha c_1^{\mathrm{T}}\ h_\alpha c_2^{\mathrm{T}}\ h_\alpha c_3^{\mathrm{T}}\ h_\alpha c_4^{\mathrm{T}}\right]^{\mathrm{T}}\\
\Pi_8 &= \left[h_\beta c_{11}^{\mathrm{T}}\ c_3^{\mathrm{T}}-c_4^{\mathrm{T}}\ h_\beta c_1^{\mathrm{T}}\ h_\beta c_2^{\mathrm{T}}\ h_\beta c_3^{\mathrm{T}}\ h_\beta c_4^{\mathrm{T}}\right]^{\mathrm{T}}\\
\Pi_9 &= \left[h_t c_9^{\mathrm{T}}\ c_1^{\mathrm{T}}-c_2^{\mathrm{T}}\ h_t c_1^{\mathrm{T}}\ h_t c_2^{\mathrm{T}}\ h_t c_3^{\mathrm{T}}\ h_t c_4^{\mathrm{T}}\right]^{\mathrm{T}}\\
\Pi_{10} &= \left[0\ 0\ c_5^{\mathrm{T}}\ h_d c_6^{\mathrm{T}}\ c_7^{\mathrm{T}}\ c_8^{\mathrm{T}}\right]^{\mathrm{T}}\\
\Pi_{11} &= \left[c_{13}^{\mathrm{T}}\ c_2^{\mathrm{T}}-c_4^{\mathrm{T}}\ \bar{h}_t c_1^{\mathrm{T}}\ \bar{h}_t c_2^{\mathrm{T}}\ \bar{h}_t c_3^{\mathrm{T}}\ \bar{h}_t c_4^{\mathrm{T}}\right]^{\mathrm{T}}\\
\Pi_{12} &= \left[c_1^{\mathrm{T}}U_1+c_5^{\mathrm{T}}U_2\right]^{\mathrm{T}}\\
\Pi_{13} &= \left[c_1^{\mathrm{T}}\bar{A}^{\mathrm{T}}+c_2^{\mathrm{T}}\bar{A}_h^{\mathrm{T}}+c_{15}^{\mathrm{T}}\hat{B}^{\mathrm{T}}-c_5^{\mathrm{T}}\right]^{\mathrm{T}}\\
\bar{\Pi}_1 &= \left[c_1^{\mathrm{T}}\ c_2^{\mathrm{T}}\ c_3^{\mathrm{T}}\ c_4^{\mathrm{T}}\ c_{12}^{\mathrm{T}}\ c_{13}^{\mathrm{T}}\ h_\alpha c_9^{\mathrm{T}}\ c_{14}^{\mathrm{T}}\right]^{\mathrm{T}}\\
\bar{\Pi}_2 &= \left[c_5^{\mathrm{T}}\ h_d c_6^{\mathrm{T}}\ c_7^{\mathrm{T}}\ c_8^{\mathrm{T}}\ c_1^{\mathrm{T}}-h_d c_2^{\mathrm{T}}\ h_d c_2^{\mathrm{T}}-c_4^{\mathrm{T}}\ c_1^{\mathrm{T}}-c_3^{\mathrm{T}}\ c_3^{\mathrm{T}}-c_4^{\mathrm{T}}\right]^{\mathrm{T}}\\
\bar{\Pi}_3 &= \left[c_1^{\mathrm{T}}\ c_5^{\mathrm{T}}\ c_1^{\mathrm{T}}\ c_2^{\mathrm{T}}\ c_3^{\mathrm{T}}\ c_4^{\mathrm{T}}\right]^{\mathrm{T}}\\
\bar{\Pi}_4 &= \left[c_3^{\mathrm{T}}\ c_7^{\mathrm{T}}\ c_1^{\mathrm{T}}\ c_2^{\mathrm{T}}\ c_3^{\mathrm{T}}\ c_4^{\mathrm{T}}\right]^{\mathrm{T}}\\
\bar{\Pi}_5 &= \left[c_4^{\mathrm{T}}\ c_8^{\mathrm{T}}\ c_1^{\mathrm{T}}\ c_2^{\mathrm{T}}\ c_3^{\mathrm{T}}\ c_4^{\mathrm{T}}\right]^{\mathrm{T}}\\
\bar{\Pi}_6 &= \left[c_2^{\mathrm{T}}\ c_6^{\mathrm{T}}\ c_1^{\mathrm{T}}\ c_2^{\mathrm{T}}\ c_3^{\mathrm{T}}\ c_4^{\mathrm{T}}\right]^{\mathrm{T}}\\
\bar{\Pi}_7 &= \left[h_\alpha c_9^{\mathrm{T}}\ c_1^{\mathrm{T}}-c_3^{\mathrm{T}}\ h_\alpha c_1^{\mathrm{T}}\ h_\alpha c_2^{\mathrm{T}}\ h_\alpha c_3^{\mathrm{T}}\ h_\alpha c_4^{\mathrm{T}}\right]^{\mathrm{T}}\\
\bar{\Pi}_8 &= \left[c_{14}^{\mathrm{T}}\ c_3^{\mathrm{T}}-c_4^{\mathrm{T}}\ h_\beta c_1^{\mathrm{T}}\ h_\beta c_2^{\mathrm{T}}\ h_\beta c_3^{\mathrm{T}}\ h_\beta c_4^{\mathrm{T}}\right]^{\mathrm{T}}\\
\bar{\Pi}_9 &= \left[\bar{h}_d c_{10}^{\mathrm{T}}\ c_3^{\mathrm{T}}-c_2^{\mathrm{T}}\ \bar{h}_d c_1^{\mathrm{T}}\ \bar{h}_d c_2^{\mathrm{T}}\ \bar{h}_d c_3^{\mathrm{T}}\ \bar{h}_d c_4^{\mathrm{T}}\right]^{\mathrm{T}}\\
\bar{\Pi}_{10} &= \left[0\ 0\ c_5^{\mathrm{T}}\ h_d c_6^{\mathrm{T}}\ c_7^{\mathrm{T}}\ c_8^{\mathrm{T}}\right]^{\mathrm{T}}\\
\bar{\Pi}_{11} &= \left[c_{13}^{\mathrm{T}}\ c_2^{\mathrm{T}}-c_4^{\mathrm{T}}\ \bar{h}_t c_1^{\mathrm{T}}\ \bar{h}_t c_2^{\mathrm{T}}\ \bar{h}_t c_3^{\mathrm{T}}\ \bar{h}_t c_4^{\mathrm{T}}\right]^{\mathrm{T}}\\
\bar{\Pi}_{12} &= \left[c_1^{\mathrm{T}}U_3+c_5^{\mathrm{T}}U_4\right]^{\mathrm{T}}\\
\check{M}_1 &= \left[c_1^{\mathrm{T}}-c_2^{\mathrm{T}}\ \ c_1^{\mathrm{T}}+c_2^{\mathrm{T}}-2c_9^{\mathrm{T}}\right]^{\mathrm{T}}\\
\check{M}_2 &= \left[c_2^{\mathrm{T}}-c_3^{\mathrm{T}}\ \ c_2^{\mathrm{T}}+c_3^{\mathrm{T}}-2c_{10}^{\mathrm{T}}\right]^{\mathrm{T}}\\
\check{M}_3 &= \left[c_3^{\mathrm{T}}-c_4^{\mathrm{T}}\ \ c_3^{\mathrm{T}}+c_4^{\mathrm{T}}-2c_{11}^{\mathrm{T}}\right]^{\mathrm{T}}\\
\check{M}_4 &= \left[c_1^{\mathrm{T}}-c_3^{\mathrm{T}}\ \ c_1^{\mathrm{T}}+c_3^{\mathrm{T}}-2c_9^{\mathrm{T}}\right]^{\mathrm{T}}\\
\check{M}_5 &= \left[c_3^{\mathrm{T}}-c_2^{\mathrm{T}}\ \ c_3^{\mathrm{T}}+c_2^{\mathrm{T}}-2c_{10}^{\mathrm{T}}\right]^{\mathrm{T}}\\
\check{M}_6 &= \left[c_2^{\mathrm{T}}-c_4^{\mathrm{T}}\ \ c_2^{\mathrm{T}}+c_4^{\mathrm{T}}-2c_{11}^{\mathrm{T}}\right]^{\mathrm{T}}\\
\check{Z}_1(\dot{h}_t) &= diag(Y_1+h_d\bar{Z}_1+\dot{h}_t\bar{Z}_3, 3(Y_1+h_d\bar{Z}_1+\dot{h}_t\bar{Z}_3))\\
\check{Z}_2 &= diag(Y_1+\bar{Z}_3, 3(Y_1+\bar{Z}_3))\\
\check{Z}_3 &= diag(Y_2+\bar{Z}_3, 3(Y_2+\bar{Z}_3))
\end{aligned}$$

$$\check{Z}_4(\dot{h}_t) = diag(Y_1 + \bar{Z}_1 - \dot{h}_t\bar{Z}_2 + \dot{h}_t\bar{Z}_3, 3(Y_1 + \bar{Z}_1 - \dot{h}_t\bar{Z}_2 + \dot{h}_t\bar{Z}_3))$$
$$\check{Z}_5(\dot{h}_t) = diag(Y_2 + h_d\bar{Z}_2 + \dot{h}_t\bar{Z}_3, 3(Y_2 + h_d\bar{Z}_2 + \dot{h}_t\bar{Z}_3)).$$

Proof. Firstly, we decompose the time delay interval $[0, \bar{h}]$ into two unequal sub-intervals $[0, h_\alpha]$ and $[h_\alpha, \bar{h}]$. For $h_t \in [0, h_\alpha]$, we construct a Lyapunov functional as follows:

$$V_a(t) = \sum_{i=1}^{3} V_i(t) \tag{14}$$

with

$$\begin{aligned}
V_1(t) =& h_t\eta_1^{\mathrm{T}}(t)P_1\eta_1(t) + \hat{h}_d\eta_1^{\mathrm{T}}(t)P_2\eta_1(t) + h_\beta\eta_1^{\mathrm{T}}(t)P_3\eta_1(t),\\
V_2(t) =& \int_{t-h_t}^{t} \eta_2^{\mathrm{T}}(t,\vartheta)Q_1\eta_2(t,\vartheta)\mathrm{d}\vartheta + \int_{t-\bar{h}}^{t-h_t} \eta_2^{\mathrm{T}}(t,\vartheta)Q_3\eta_2(t,\vartheta)\mathrm{d}\vartheta,\\
V_3(t) =& \int_{-h_t}^{0}\int_{t+\vartheta}^{t} \dot{x}^{\mathrm{T}}(\theta)\bar{Z}_1\dot{x}(\theta)\mathrm{d}\theta\mathrm{d}\vartheta + \int_{-\bar{h}}^{-h_t}\int_{t+\vartheta}^{t} \dot{x}^{\mathrm{T}}(\theta)\bar{Z}_3\dot{x}(\theta)\mathrm{d}\theta\mathrm{d}\vartheta.
\end{aligned}$$

For $h_t \in [h_\alpha, \bar{h}]$, we construct another Lyapunov functional as follows:

$$V_b(t) = \sum_{j=1}^{3} \bar{V}_j(t) \tag{15}$$

with

$$\begin{aligned}
\bar{V}_1(t) =& h_\alpha\eta_1^{\mathrm{T}}(t)P_1\eta_1(t) + \bar{h}_d\eta_1^{\mathrm{T}}(t)P_2\eta_1(t) + \bar{h}_t\eta_1^{\mathrm{T}}(t)P_3\eta_1(t),\\
\bar{V}_2(t) =& \int_{t-h_\alpha}^{t} \eta_2^{\mathrm{T}}(t,\vartheta)Q_1\eta_2(t,\vartheta)\mathrm{d}\vartheta + \int_{t-h_t}^{t-h_\alpha} \eta_2^{\mathrm{T}}(t,\vartheta)Q_2\eta_2(t,\vartheta)\mathrm{d}\vartheta\\
&+ \int_{t-\bar{h}}^{t-h_t} \eta_2^{\mathrm{T}}(t,\vartheta)Q_3\eta_2(t,\vartheta)\mathrm{d}\vartheta,\\
\bar{V}_3(t) =& \int_{-h_\alpha}^{0}\int_{t+\vartheta}^{t} \dot{x}^{\mathrm{T}}(\theta)\bar{Z}_1\dot{x}(\theta)\mathrm{d}\theta\mathrm{d}\vartheta + \int_{-h_t}^{-h_\alpha}\int_{t+\vartheta}^{t} \dot{x}^{\mathrm{T}}(\theta)\bar{Z}_2\dot{x}(\theta)\mathrm{d}\theta\mathrm{d}\vartheta\\
&+ \int_{-\bar{h}}^{-h_t}\int_{t+\vartheta}^{t} \dot{x}^{\mathrm{T}}(\theta)\bar{Z}_3\dot{x}(\theta)\mathrm{d}\theta\mathrm{d}\vartheta.
\end{aligned}$$

Then, the following Lyapunov functional candidate is chosen for System (5):

$$V_\Sigma(t) = \begin{cases} V_c(t) + V_a(t)\,, & h_t \in [0, h_\alpha],\\ V_c(t) + V_b(t), & h_t \in [h_\alpha, \bar{h}], \end{cases} \tag{16}$$

where

$$\begin{aligned}
V_c(t) =& \int_{t-h_\alpha}^{t} \eta_2^{\mathrm{T}}(t,\vartheta)R_1\eta_2(t,\vartheta)d\vartheta + \int_{t-\bar{h}}^{t-h_\alpha} \eta_2^{\mathrm{T}}(t,\vartheta)R_2\eta_2(t,\vartheta)d\vartheta\\
&+ \int_{-h_\alpha}^{0}\int_{t+\vartheta}^{t} \dot{x}^{\mathrm{T}}(\theta)Y_1\dot{x}(\theta)\mathrm{d}\theta\mathrm{d}\vartheta + \int_{-\bar{h}}^{-h_\alpha}\int_{t+\vartheta}^{t} \dot{x}^{\mathrm{T}}(\theta)Y_2\dot{x}(\theta)\mathrm{d}\theta\mathrm{d}\vartheta
\end{aligned}$$

with $R_1, R_2 \in \mathbb{S}_+^{6n}$ and $Y_1, Y_2 \in \mathbb{S}_+^{n}$.

First, consider the case of $h_t \in [0, h_\alpha]$. Taking the derivative of the functionals $V_c(t)$ and $V_a(t)$ along the trajectories of Lur'e System (5) yields

$$\begin{aligned}\dot{V}_c(t) =&\eta_2^{\mathrm{T}}(t,t)R_1\eta_2(t,t) + \eta_2^{\mathrm{T}}(t,t-h_\alpha)(R_2-R_1)\eta_2(t,t-h_\alpha)\\ &-\eta_2^{\mathrm{T}}(t,t-\bar{h})R_2\eta_2(t,t-\bar{h}) + \dot{x}^{\mathrm{T}}(t)(h_\alpha Y_1 + h_\beta Y_2)\dot{x}(t) + \hat{J}_1 + \hat{J}_2\\ &+2\int_{t-h_\alpha}^{t}\eta_2^{\mathrm{T}}(t,\vartheta)R_1\frac{\partial\eta_2(t,\vartheta)}{\partial t}\mathrm{d}\vartheta + 2\int_{t-\bar{h}}^{t-h_\alpha}\eta_2^{\mathrm{T}}(t,\vartheta)R_2\frac{\partial\eta_2(t,\vartheta)}{\partial t}\mathrm{d}\vartheta, \qquad (17)\\ \dot{V}_1(t) =&\dot{h}_t\eta_1^{\mathrm{T}}(t)(P_1-P_2)\eta_1(t) + 2h_t\eta_1^{\mathrm{T}}(t)P_1\dot{\eta}_1(t) + 2\hat{h}_d\eta_1^{\mathrm{T}}(t)P_2\dot{\eta}_1(t)\\ &+2h_\beta\eta_1^{\mathrm{T}}(t)P_3\dot{\eta}_1(t), \qquad (18)\\ \dot{V}_2(t) =&\eta_2^{\mathrm{T}}(t,t)Q_1\eta_2(t,t) + h_d\eta_2^{\mathrm{T}}(t,t-h_t)(Q_3-Q_1)\eta_2(t,t-h_t)\\ &-\eta_2^{\mathrm{T}}(t,t-\bar{h})Q_3\eta_2(t,t-\bar{h}) + 2\int_{t-h_t}^{t}\eta_2^{\mathrm{T}}(t,\vartheta)Q_1\frac{\partial\eta_2(t,\vartheta)}{\partial t}\mathrm{d}\vartheta\\ &+2\int_{t-\bar{h}}^{t-h_t}\eta_2^{\mathrm{T}}(t,\vartheta)Q_3\frac{\partial\eta_2(t,\vartheta)}{\partial t}\mathrm{d}\vartheta, \qquad (19)\\ \dot{V}_3(t) =&\dot{x}^{\mathrm{T}}(t)(h_t\bar{Z}_1 + \bar{h}_t\bar{Z}_3)\dot{x}(t) + \hat{J}_3 + \hat{J}_4 + \hat{J}_5 \qquad (20)\end{aligned}$$

with

$$\begin{aligned}\hat{J}_1 =& -\int_{t-h_\alpha}^{t}\dot{x}^{\mathrm{T}}(\vartheta)Y_1\dot{x}(\vartheta)\mathrm{d}\vartheta\\ \hat{J}_2 =& -\int_{t-\bar{h}}^{t-h_\alpha}\dot{x}^{\mathrm{T}}(\vartheta)Y_2\dot{x}(\vartheta)\mathrm{d}\vartheta\end{aligned}$$

$$\begin{aligned}\hat{J}_3 =& -h_d\int_{t-h_t}^{t}\dot{x}^{\mathrm{T}}(\vartheta)\bar{Z}_1\dot{x}(\vartheta)\mathrm{d}\vartheta\\ \hat{J}_4 =& -\dot{h}_t\int_{t-h_t}^{t}\dot{x}^{\mathrm{T}}(\vartheta)\bar{Z}_3\dot{x}(\vartheta)\mathrm{d}\vartheta\\ \hat{J}_5 =& -\int_{t-\bar{h}}^{t-h_t}\dot{x}^{\mathrm{T}}(\vartheta)\bar{Z}_3\dot{x}(\vartheta)\mathrm{d}\vartheta\end{aligned}$$

It follows from (12) and Lemma 1 that

$$\begin{aligned}\sum_{i=1}^{5}\hat{J}_i =& -\int_{t-h_t}^{t}\dot{x}^{\mathrm{T}}(\vartheta)(Y_1 + h_d\bar{Z}_1 + \dot{h}_t\bar{Z}_3)\dot{x}(\vartheta)\mathrm{d}\vartheta\\ &-\int_{t-h_\alpha}^{t-h_t}\dot{x}^{\mathrm{T}}(\vartheta)(Y_1+\bar{Z}_3)\dot{x}(\vartheta)\mathrm{d}\vartheta\\ &-\int_{t-\bar{h}}^{t-h_\alpha}\dot{x}^{\mathrm{T}}(\vartheta)(Y_2+\bar{Z}_3)\dot{x}(\vartheta)\mathrm{d}\vartheta\\ \leq&\xi_e^{\mathrm{T}}(t)[Sym\{\sum_{l=1}^{3}\check{N}_l\check{M}_l\} + \Omega_1(h_t)]\xi_e(t) \qquad (21)\end{aligned}$$

where

$$\Omega_1(h_t) = h_t\check{N}_1\check{Z}_1^{-1}(\dot{h}_t)\check{N}_1^{\mathrm{T}} + \hat{h}_d\check{N}_2\check{Z}_2^{-1}\check{N}_2^{\mathrm{T}} + h_\beta\check{N}_3\check{Z}_3^{-1}\check{N}_3^{\mathrm{T}}. \qquad (22)$$

For matrices U_1 and U_2 with appropriate dimensions, the following condition holds:

$$\begin{aligned}0 =&2[x^{\mathrm{T}}(t)U_1 + \dot{x}^{\mathrm{T}}(t)U_2][\bar{A}x(t) + \bar{A}_h x(t-h_t) + B\hat{\omega}(t) - \dot{x}(t)]\\ =&2\xi_e^{\mathrm{T}}(t)\Pi_{12}^{\mathrm{T}}\Pi_{13}\xi_e(t). \qquad (23)\end{aligned}$$

For $\varepsilon_1 > 0$, it follows from (4) that

$$
\begin{aligned}
0 \leq & -2\varepsilon_1(\hat{\omega}^{\mathrm{T}}(t)\hat{\omega}(t)+\hat{\omega}^{\mathrm{T}}(t)\mathscr{K}_{21}[\hat{L}_1x(t)+\hat{L}_2x(t-h_t)]) \\
= & -2\varepsilon_1\xi_e^{\mathrm{T}}(t)[c_{15}^{\mathrm{T}}c_{15}+c_{15}^{\mathrm{T}}\mathscr{K}_{21}\hat{L}_1c_1+c_{15}^{\mathrm{T}}\mathscr{K}_{21}\hat{L}_2c_2]\xi_e(t).
\end{aligned} \tag{24}
$$

In addition, for any matrices $S_{ij} \in \mathbb{R}^{(14n+p)\times n}, i \in \{1,2\}, j \in \{1,2,3\}$, it follows from the definition of $\xi_e(t)$ that the following zero equations are true:

$$2\xi_e(t)[S_{11}+\dot{h}_tS_{21}][h_tc_9-c_{12}]\xi_e(t)=0, \tag{25}$$

$$2\xi_e(t)[S_{12}+\dot{h}_tS_{22}][\hat{h}_dc_{10}+h_\beta c_{11}-c_{13}]\xi_e(t)=0, \tag{26}$$

$$2\xi_e(t)[S_{13}+\dot{h}_tS_{23}][h_tc_9+\hat{h}_dc_{10}-c_{14}]\xi_e(t)=0. \tag{27}$$

To sum up, we then obtain

$$\dot{V}_c(t)+\dot{V}_a(t) \leq \xi_e^{\mathrm{T}}(t)\big(\aleph(h_t,\dot{h}_t)+\Omega_1(h_t)\big)\xi_e(t) \tag{28}$$

with $\aleph(h_t,\dot{h}_t)$ being defined in Theorem 1.

On condition that linear matrix inequalities (8) and (9) are satisfied, $\dot{V}_c(t)+\dot{V}_a(t) < -\epsilon_1\|x(t)\|^2$ for a sufficiently small scalar $\epsilon_1 > 0$.

For the case of $h_t \in [h_\alpha,\bar{h}]$, applying a familiar computational approach, we obtain

$$\dot{V}_c(t)+\dot{V}_b(t) \leq \xi_f^{\mathrm{T}}(t)\big(\bar{\aleph}(h_t,\dot{h}_t)+\Omega_2(h_t)\big)\xi_f(t) \tag{29}$$

with $\bar{\aleph}(h_t,\dot{h}_t)$ being defined in Theorem 1, and

$$\Omega_2(h_t) = h_d\check{N}_4\check{Z}_4^{-1}(\dot{h}_t)\check{N}_4^{\mathrm{T}}+\bar{h}_d\check{N}_5\check{Z}_5^{-1}(\dot{h}_t)\check{N}_5^{\mathrm{T}}+\bar{h}_t\check{N}_6\check{Z}_3^{-1}\check{N}_6^{\mathrm{T}}. \tag{30}$$

If the linear matrix inequalities (10) and (11) are satisfied, then it follows that $\dot{V}_c(t)+\dot{V}_b(t) < -\epsilon_2\|x(t)\|^2$ for a sufficiently small scalar $\epsilon_2 > 0$. To summarize, there exists a scalar $\epsilon_m = min\{\epsilon_1,\epsilon_2\}$ such that $\dot{V}_\Sigma(t) < -\epsilon_m\|x(t)\|^2$ for $h_t \in [0,h]$. Thus, Lur'e System (5) with time-varying delay satisfying (2) and (3) is absolutely stable. This completes the proof. □

Remark 2. *In [28], the time delay is uniformly divided into m segments, where m is the number of segments. By contrast, the time delay is non-uniformly divided into two segments by setting the segmentation point parameter* α*. Note that at* $h_t = h_\alpha$*, the Lyapunov functionals* $V_a(t)$ *and* $V_b(t)$ *become equal, resulting in the continuity of the overall Lyapunov functional* $V_\Sigma(t)$ *over time. By utilizing two distinct pairs of Lyapunov matrices for the intervals* $h_t \in [0,h_\alpha]$ *and* $h_t \in [h_\alpha,\bar{h}]$*, relaxed conditions can be derived, which leads to less conservative results compared with some existing methods.*

Remark 3. *In previous works [36,37], the interval* $[-\bar{h},0]$ *of the integral term in the derivative of the Lyapunov functional was usually separated into* $[-\bar{h},-h_t]$ *and* $[-h_t,0]$*. By contrast, in this paper, the interval is separated into* $[-\bar{h},-h_\alpha]$*,* $[-h_\alpha,-h_t]$ *and* $[-h_t,0]$ *for* $h_t \in [0,h_\alpha]$*, or* $[-\bar{h},-h_t]$ $[-h_t,-h_\alpha]$ *and* $[-h_\alpha,0]$ *for* $h_t \in [h_\alpha,\bar{h}]$*.*

Remark 4. *There are a series of sub-vectors* $\nu_1(t)$*,* $\nu_2(t)$*,* $\nu_3(t)$*,*$\nu_4(t)$*,* $\nu_5(t)$*,* $\nu_6(t)$*,* $\frac{1}{h_t}\nu_1(t)$*,* $\frac{1}{h_\alpha}\nu_2(t)$*,* $\frac{1}{h_t}\nu_3(t)$*,* $\frac{1}{\hat{h}_d}\nu_4(t)$*,* $\frac{1}{\bar{h}_d}\nu_5(t)$*, and* $\frac{1}{h_\beta}\nu_6(t)$ *contained in* $\xi_e(t)$ *and* $\xi_f(t)$ *in this paper. Inspired by [34,38,39], six zero-equalities (25)–(27) with delay-derivative-dependent free matrices are given to consider the connections between* $\nu_1(t)$ *and* $\frac{1}{h_t}\nu_1(t)$*,* $\nu_2(t)$ *and* $\frac{1}{h_\alpha}\nu_2(t)$*,* $\nu_3(t)$ *and* $\frac{1}{\bar{h}_t}\nu_3(t)$*,* $\nu_4(t)$ *and* $\frac{1}{\hat{h}_d}\nu_4(t)$*,* $\nu_5(t)$ *and* $\frac{1}{\bar{h}_d}\nu_5(t)$*, and* $\nu_6(t)$ *and* $\frac{1}{h_\beta}\nu_6(t)$*, respectively. This paper uses this approach to avoid the appearance of a quadratic term related to* h_t *and considers more information about the derivative of the time-varying delay.*

Remark 5. *To simplify the calculation process, only first-order integral terms are considered in the assessment of the integral inequality. By adding the second-order terms* $\frac{1}{h_t}\int_{t-h_t}^{t}\int_{\vartheta}^{t}x(\theta)d\theta d\vartheta$, $\frac{1}{h_\alpha-h_t}\int_{t-h_\alpha}^{t-h_t}\int_{\vartheta}^{t-h_t}x(\theta)d\theta d\vartheta$, $\frac{1}{\bar{h}-h_\alpha}\int_{t-h}^{t-h_\alpha}\int_{\vartheta}^{t-h_\alpha}x(\theta)d\theta d\vartheta$, $\frac{1}{(h_\alpha-h_t)^2}\int_{t-h_\alpha}^{t-h_t}\int_{\vartheta}^{t-h_t}x(\theta)d\theta d\vartheta$, $\frac{1}{h_t^2}\int_{t-h_t}^{t}\int_{\vartheta}^{t}x(\theta)d\theta d\vartheta$, *and* $\frac{1}{(\bar{h}-h_\alpha)^2}\int_{t-h}^{t-h_\alpha}\int_{\vartheta}^{t-h_\alpha}x(\theta)d\theta d\vartheta$ *to* $\xi_e(t)$ *and using the second-order integral inequality to bound the integral terms in the derivative of the Lyapunov functional, less conservative stability conditions are expected.*

Due to the complex and uncertain nature of actual systems, we extend Theorem 1 to the following Lur'e system with time-varying parameter uncertainty:

$$\begin{cases} \dot{x}(t) = (\hat{A}+\Delta\hat{A}(t))x(t) + (\hat{A}_h+\Delta\hat{A}_h(t))x(t-h_t) + \hat{B}\hat{\omega}(t) \\ p(t) = \hat{L}_1 x(t) + \hat{L}_2 x(t-h_t) \\ \hat{\omega}(t) = -\Upsilon(t,p(t)) \\ x(t) = \phi(t), t\in[-\bar{h},0] \end{cases} \tag{31}$$

where the uncertainties are represented in the form

$$[\Delta\hat{A}(t)\ \Delta\hat{A}_h(t)] = X\mathcal{Z}(t)[Y_a\ Y_b] \tag{32}$$

Here, X, Y_a and Y_b are given matrices, and $\mathcal{Z}(t)$ is an unknown time-varying matrix satisfying

$$\mathcal{Z}^{\mathrm{T}}(t)\mathcal{Z}(t) \le I, \forall t. \tag{33}$$

For the uncertain Lur'e system in (31), replacing $\hat{A}+X\mathcal{Z}(t)Y_a$ and $\hat{A}_h+X\mathcal{Z}(t)Y_b$, respectively, with $\hat{A}$ and $\hat{A}_h$, and using Lemma 2 and the Schur complement [40], the following condition is derived to ensure its absolute stability.

Theorem 2. *For given scalars* $\bar{h}>0$ *and* $\bar{\tau}\in[0,1)$, *System* (31) *is robustly stable on condition that there exist matrices* $P_i\in\mathbb{S}_+^{8n}$, $Q_i\in\mathbb{S}_+^{6n}$, $\bar{Z}_i\in\mathbb{S}_+^{n}$, $i\in\{1,2,3\}$, R_1, $R_2\in\mathbb{S}_+^{6n}$, Y_1, $Y_2\in\mathbb{S}_+^{n}$, $\breve{N}_j\in\mathbb{R}^{(14n+p)\times 2n}$, S_{1j}, $S_{2j}\in\mathbb{R}^{(14n+p)\times n}$, $j\in\{1,2,\cdots,6\}$, $U_1,U_2,U_3,U_4\in\mathbb{R}^n$, *and scalars* ε_1, ε_2,σ_1, $\sigma_2>0$ *such that, for* $h_t\in[0,\bar{h}],\dot{h}_t\in[-\bar{\tau},\bar{\tau}]$,

$$\begin{bmatrix} \aleph(0,\dot{h}_t)+\sigma_1\chi_2^{\mathrm{T}}\chi_2 & \sqrt{\hat{h}_d}\breve{N}_2 & \sqrt{h_\beta}\breve{N}_3 & \chi_1^{\mathrm{T}} \\ * & -\breve{Z}_2 & 0 & 0 \\ * & * & -\breve{Z}_3 & 0 \\ * & * & * & -\sigma_1 I \end{bmatrix} < 0 \tag{34}$$

$$\begin{bmatrix} \aleph(h_\alpha,\dot{h}_t)+\sigma_1\chi_2^{\mathrm{T}}\chi_2 & \sqrt{h_t}\breve{N}_1 & \sqrt{h_\beta}\breve{N}_3 & \chi_1^{\mathrm{T}} \\ * & -\breve{Z}_1(\dot{h}_t) & 0 & 0 \\ * & * & -\breve{Z}_3 & 0 \\ * & * & * & -\sigma_1 I \end{bmatrix} < 0 \tag{35}$$

$$\begin{bmatrix} \bar{\aleph}(h_\alpha,\dot{h}_t)+\sigma_2\chi_2^{\mathrm{T}}\chi_2 & \sqrt{\bar{h}_t}\breve{N}_6 & \sqrt{h_\alpha}\breve{N}_4 & \chi_3^{\mathrm{T}} \\ * & -\breve{Z}_3 & 0 & 0 \\ * & * & -\breve{Z}_4(\dot{h}_t) & 0 \\ * & * & * & -\sigma_2 I \end{bmatrix} < 0 \tag{36}$$

$$\begin{bmatrix} \bar{\aleph}(\bar{h},\dot{h}_t)+\sigma_2\chi_2^{\mathrm{T}}\chi_2 & \sqrt{\bar{h}_d}\breve{N}_5 & \sqrt{h_\alpha}\breve{N}_4 & \chi_3^{\mathrm{T}} \\ * & -\breve{Z}_5(\dot{h}_t) & 0 & 0 \\ * & * & -\breve{Z}_4(\dot{h}_t) & 0 \\ * & * & * & -\sigma_2 I \end{bmatrix} < 0 \tag{37}$$

where

$$\chi_1 = [c_1^{\mathrm{T}}U_1X + c_5^{\mathrm{T}}U_2X]^{\mathrm{T}}$$

$$\chi_2 = [c_1^T Y_a^T + c_2^T Y_b]^T$$
$$\chi_3 = [c_1^T U_3 X + c_5^T U_4 X]^T$$

and $\aleph(h_t, \dot{h}_t)$, $\bar{\aleph}(h_t, \dot{h}_t)$ *are defined in Theorem 1.*

4. Specific Example Analysis

In this section, a numerical example is presented to illustrate the improvement and superiority of the proposed criteria.

Consider System (31) with

$$\hat{A} = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad \hat{A}_h = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$
$$\hat{B} = \begin{bmatrix} -0.2 \\ -0.3 \end{bmatrix}, \; \hat{L}_1 = \begin{bmatrix} 0.3 & 0.1 \end{bmatrix}, \; \hat{L}_2 = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix},$$
$$\mathscr{K}_1 = 0.2, \quad \mathscr{K}_2 = 0.5, \quad X = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$
$$Y_a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Y_b = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

To verify the effectiveness of the delay-segmentation approach, the MADBs of time delay calculated by Theorem 2 and the approach proposed in [7,28,36,37,41] are summarized in Table 1. It is observed in Table 1 that, when $\alpha = 0$ and $\alpha = 1$, i.e., $h_\alpha = 0$ and $h_\alpha = \bar{h}$, which means that the delay interval was not decomposed, the MADBs were still higher than those in the existing literature. It is also noted that the obtained MADBs increased as α grew. It is shown that uniformly dividing the time delay intervals in [28] is not optimal. However, the optimal value of α is uncertain and may be related to specific numerical examples. Even when the time delay was evenly divided into three segments in [28], the obtained MADBs were still lower than that obtained in this paper. The superiority of the proposed delay-segmentation-based augmented Lyapunov–Krasovskii functional approach was thus demonstrated.

Table 1. MADB for different $\bar{\tau}$ values.

$\bar{\tau}$	0.3	0.6	0.9
[7]	2.0787	1.4195	0.9228
[41]	2.2262	1.7409	1.4682
[28] (m = 2)	2.4660	1.8787	1.7190
[28] (m = 3)	2.5164	1.9147	1.7923
[36]	2.6873	2.2021	1.9897
[37]	2.9358	2.4721	2.2356
Theorem 2 (α = 0.00)	3.0236	2.5522	2.2949
Theorem 2 (α = 0.25)	3.0912	2.5875	2.3205
Theorem 2 (α = 0.50)	3.1102	2.6069	2.3503
Theorem 2 (α = 0.75)	3.1185	2.6466	2.4079
Theorem 2 (α = 0.90)	3.1429	2.6886	2.4353
Theorem 2 (α = 1.0)	3.0236	2.5522	2.2949

5. Conclusions

This paper addressed the issue of the absolute stability of uncertain Lur'e systems with time-varying delay. A novel approach was proposed, which utilizes a delay-segmentation-based augmented Lyapunov–Krasovskii functional. The functional was designed to be piecewise continuous at the segmentation point. By employing this functional, improved absolute stability conditions were derived by utilizing a generalized free-matrix-based integral inequality to evaluate the integral terms in the functional derivatives. Finally, a numerical example was presented to demonstrate the effectiveness of our delay-segmentation approach. It was observed from the numerical example that the obtained MADBs are dependent on the segment point parameter α. How to optimize the parameter α needs to be further investigated. Increasing the number of segments can further reduce conservatism, but optimizing the segment points is a more challenging task. In addition, the computational complexity will increase sharply with the increase in segments.

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