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Apriorics: Information and Graphs in the Description of the Fundamental Particles—A Mathematical Proof

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Abstract: In our earlier work, we suggested an axiomatic framework for deducing the fundamental entities which constitute the building block of the elementary particles in physics. The basic concept of this theory, named apriorics, is the ontological structure (OS)—an undirected simple graph satisfying specified conditions. The vertices of this graph represent the fundamental entities (FEs), its edges are binary compounds of the FEs (which are the fundamental bosons and fermions), and the structures constituting more than two connected vertices are composite particles. The objective of this paper is to focus the attention on several mathematical theorems and ideas associated with such graphs of order n, including their enumeration, showing what is the information content of apriorics.

Keywords: graph theory; fundamental particles

MSC: 05C90; 81V25

1. Introduction: The Use of Simple Graphs in the Description of the Fundamental Entities

Apriorics is an attempt to describe the fundamental building block of physical reality in terms of graph theory and information. The philosophical basis of apriorics was dealt with in [1], in which apriorics was connected to structuralism, a philosophical approach denying inherent meaning to any notion, and interprets each and everything solely in terms of its connections to other things. Here, we are more concerned with the mathematical aspects of the theory and less with the philosophical aspects. Hence, we focus our attention on proofs and rigorous mathematical derivations.

To be sure, also the General Theory of Information, as a multidisciplinary framework, seeks to understand and quantify information in various contexts. Proposed by physicist Giancarlo Ruffo, it builds on the foundational work of Claude Shannon's information theory. This theory goes beyond traditional concepts of information, extending to complex systems, thermodynamics, biology, and even the quantum realm. It explores the role of information in shaping the behavior and organization of systems, emphasizing the link between entropy, complexity, and information. By doing so, the General Theory of Information provides a comprehensive perspective on how information is a fundamental element in the fabric of our universe, impacting diverse scientific domains [2–11]. Notice, however, that the General Theory of Information cannot predict the correct number of fundamental fermions and fundamental gauge bosons and the three fundamental interactions in the way that apriorics does, as is described below.

Category theory is a branch of abstract mathematics that focuses on studying relationships and structures in a generalized way. It deals with objects and morphisms (arrows) between them, emphasizing their properties and how they interact, rather than specific elements. Categories provide a framework for understanding diverse mathematical and scientific concepts, including algebra, topology, and logic. By abstracting the commonalities

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between different mathematical structures, category theory promotes a deeper understanding of mathematical concepts and their universal principles. Its broad applicability extends beyond mathematics, finding use in computer science, physics, and other fields for modeling and reasoning about complex systems [12–31]. However, category theory does not predict the correct number of fundamental fermions and fundamental gauge bosons and the three fundamental interactions in the way that apriorics does.

Of course, graph theory has a much wider application domain with respect to apriorics. Graph theory, being a branch of mathematics, has numerous real-life applications across various fields. Here are some examples:

1. Social networks: Graph theory is extensively used in social network analysis to model relationships between individuals or entities. It helps analyze connectivity patterns, identify influential nodes (people), and understand information flow within networks like Facebook, Twitter, or LinkedIn.
2. Transportation networks: Graphs model transportation systems such as road networks, railway lines, and airline routes. Graph algorithms optimize routes, schedule public transportation, and analyze traffic flow, contributing to urban planning and logistics management.
3. Telecommunications: Graph theory aids in designing communication networks, including the internet, mobile networks, and satellite communication systems. It optimizes data routing, analyzes network robustness, and ensures efficient transmission of information.
4. Epidemiology: Graphs model disease spread within populations, with nodes representing individuals and edges representing contacts or interactions. Graph analysis helps track and predict disease outbreaks, assess vaccination strategies, and identify key factors influencing transmission.
5. Biology and genetics: Graph theory is used to model biological systems such as protein–protein interaction networks, metabolic pathways, and gene regulatory networks. It aids in understanding biological processes, identifying essential genes or proteins, and predicting drug targets.
6. Computer networks: Graphs represent computer networks, with nodes as devices (computers and routers) and edges as connections (links and cables). Graph algorithms optimize network routing, detect network intrusions, and ensure robustness and reliability of communication systems.
7. Recommendation systems: Graph theory underlies recommendation algorithms in e-commerce, streaming services, and social media platforms. It models user–item interactions and generates personalized recommendations based on similarities between users or items.
8. Operations research: Graph theory is applied in operations research to model complex systems; optimize resource allocation; and solve logistical problems such as the traveling salesman problem, network flow problems, and facility location problems.
9. Chemistry and molecular biology: Graph theory represents molecular structures and chemical compounds, aiding in drug discovery, molecular modeling, and understanding chemical reactions. It helps analyze molecular interactions, predict compound properties, and design novel drugs.
10. Image processing and computer vision: Graph-based techniques are used in image segmentation, object recognition, and pattern recognition. Graph algorithms extract features from images, analyze image similarity, and enhance image understanding in various applications, including medical imaging and surveillance systems.

These examples demonstrate the versatility and applicability of graph theory in addressing complex real-world problems across diverse domains.

Attack graphs are graphical representations used in cybersecurity to model and analyze potential threats and vulnerabilities in a computer network. Here is how one can model threats using attack graphs:

1. Identify assets and vulnerabilities:
   - Begin by identifying the assets within the network that need to be protected, such as servers, databases, and sensitive data.
   - Identify potential vulnerabilities in the network, including software vulnerabilities, misconfigurations, weak passwords, and insecure network protocols.
2. Construct the attack graph:
   - Represent the network as a graph, where nodes represent network components (e.g., hosts, routers, and switches) and edges represent possible attack paths between them.
   - Start with an initial node representing the attacker's entry point into the network.
   - Add nodes and edges to the graph to represent the attacker's progression through the network, exploiting vulnerabilities and gaining access to different components.
   - Each node in the graph represents a state of compromise (e.g., gaining unauthorized access to a server), and each edge represents a specific attack path or exploitation technique.
3. Assign attack costs and probabilities:
   - Assign costs or probabilities to edges in the graph to represent the difficulty or likelihood of a successful attack along that path.
   - Factors influencing costs or probabilities may include the complexity of the attack, the strength of security measures in place, and the attacker's skill level.
4. Analyze attack paths:
   - Analyze the attack graph to identify critical paths that attackers could take to compromise sensitive assets.
   - Identify common attack patterns and techniques used by adversaries to exploit vulnerabilities and move laterally within the network.
5. Mitigation and remediation:
   - Use the attack graph analysis to prioritize security measures and allocate resources for mitigating vulnerabilities and reducing the risk of successful attacks.
   - Implement security controls, such as firewalls, intrusion detection systems, and access controls, to disrupt or block potential attack paths identified in the graph.
   - Continuously monitor and update the attack graph as new vulnerabilities are discovered or as the network configuration changes.

By modeling threats using attack graphs, organizations can gain insights into potential attack scenarios, prioritize security efforts, and enhance the resilience of their networks against cyber threats.

The computational complexity of using graphs in real-life applications can vary widely depending on the specific problem being addressed, the size and complexity of the graph, and the algorithms and techniques employed. Here are some considerations:

1. Graph representation: The complexity of representing a graph in memory depends on the number of vertices (nodes) and edges, as well as the data structures used. For sparse graphs, where the number of edges is relatively low compared to the number of vertices, adjacency lists may be more efficient than adjacency matrices.
2. Graph traversal: Algorithms for traversing graphs, such as depth-first search (DFS) or breadth-first search (BFS), have time complexities of $O(V + E)$, where V is the number of vertices, and E is the number of edges. Traversal can become computationally expensive for large graphs or graphs with dense connectivity.
3. Shortest paths: Finding the shortest path between two vertices in a graph, using algorithms like Dijkstra's algorithm or the Bellman–Ford algorithm, typically has a time complexity of $O((V + E)\log V)$ or $O(VE)$, depending on the algorithm used and whether negative edge weights are allowed.
4. Graph connectivity: Determining graph connectivity, such as whether the graph is strongly connected or whether there exists a path between every pair of vertices, can be

computationally intensive. Algorithms like Tarjan's strongly connected components algorithm have a time complexity of O(V + E).

5. Graph coloring: Problems such as graph coloring, where vertices must be assigned colors such that no adjacent vertices share the same color, are known to be NP-hard. Finding an optimal coloring for an arbitrary graph can be computationally intractable, and heuristic approaches are often used in practice.
6. Maximum flow: Computing maximum flow in a graph, such as in network flow problems, can be solved efficiently using algorithms like the Ford–Fulkerson algorithm or the Edmonds–Karp algorithm, with a time complexity of $O(V E^2)$.
7. Graph clustering and community detection: Identifying clusters or communities within a graph involves partitioning the vertices into groups based on their connectivity patterns. Various algorithms exist for this task, such as the Louvain method or spectral clustering, with varying computational complexities depending on the approach.

In summary, the computational complexity of using graphs in real-life applications can range from linear time complexity for simple traversal algorithms to exponential or NP-hard complexity for more complex problems such as graph coloring or finding optimal solutions. Efficient algorithms, careful problem formulation, and leveraging problem-specific properties of the graph are essential for managing computational complexity in practical applications.

In previous works [1,32–36], a theory for the description of the fundamental entities, named apriorics, was suggested and elaborated. The predictions of this theory were compatible with experimental findings in elementary particle physics. In this theory, the fundamental entities (FEs) are the vertices in a simple undirected graph [37–40] called an ontological structure (OS), and they can be described by their internal connections defining the graph, without any needed reference to anything else. These nodes must satisfy the following five axioms (which are explained and justified in [32,33]):

(a) Every node (vertex) must be connected to one other vertex (at least).
(b) The connection between two nodes (vertices) is non-reflective (a node cannot connect to itself) and commutative (if node F_1 is connected to node F_2, it necessarily follows that F_2 is connected to F_1), but not necessarily transitive (if F_1 is connected to F_2 and F_2 is connected to F_3, it does not mean that F_1 and F_3 are connected).
(c) Two nodes (vertices) can be connected by not more than one connection.
(d) If two nodes (vertices), F_1 and F_2, are connected, then there must exist an additional third node (vertex), F_3, connected to F_1 and F_2. F_3 will be denoted the "mediator" between the nodes F_1 and F_2.
(e) It is required that every two nodes (vertices) are different (the meaning of the word "different" is to be explained below) in terms of their connections to the other nodes (vertices).

In this paper, we show through mathematical proof that there is no ontological structure with less than seven nodes. We also show through construction that there are at least two distinct ontological structures with seven nodes. In addition, we show how the connections (edges) of those graphs can be mapped into the elementary particles (basic fermions and bosons) and how the properties of those connections (elementary particles) can be derived.

Thus, the information contained in the above set of axioms is equivalent to the information of the required particles (fundamental fermions and gauge bosons) in the standard model of elementary particles, as we show below.

The fifth axiom (axiom e) requires criteria to distinguish between two nodes (vertices) in the ontological structure (OS) graph by their connections to other nodes (vertices). The valency (number of connections) alone cannot serve to uniquely distinguish between two nodes (vertices) because, according to the equivalence theorem, every simple graph has at least two vertices of the same valency, so that axiom e will not be satisfied. An additional criterion, which might serve the purpose of distinction between two nodes (vertices), is given below:

Definition 1. *The significance set (SS) of vertex P is the set of all the valences of the nodes connected to P.*

Figure 1 is given as an example. Let us look at SS(P_1) =2,3,3,5, since P_1 is connected to P_2, P_4, P_5, and P_6; and the valences of these nodes (vertices), which are given in parentheses in the figure, are, respectively, 3, 3, 2, and 5. By the same method, we derive SS (P_5) = 4,5.

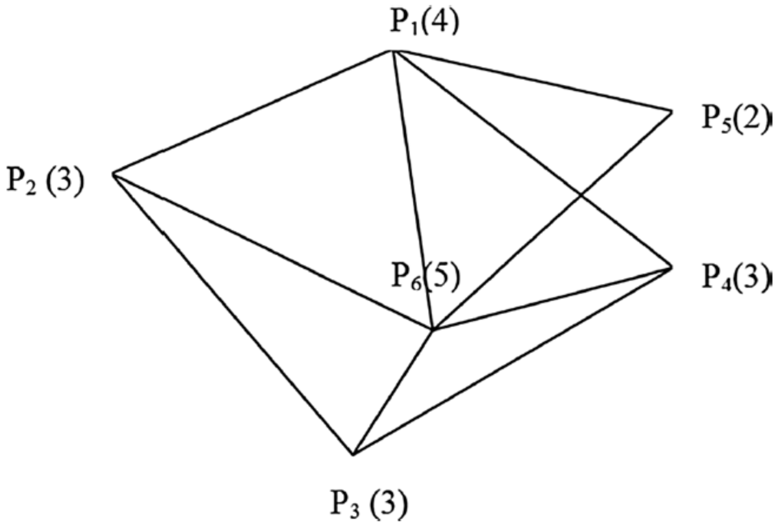

Figure 1. The significance set. $P_i(k)$ means the node P_i is connected to k edges.

Definition 2. *The vertices P_1 and P_2 are considered equal; $P_1 = P_2$ if, and only if, they have the same valency and SS (P_1) = SS (P_2).*

For example, in Figure 1, $P_3 \neq P_2$. Both vertices have the same number of connections, 3, but they have different significance sets. SS (P_2) = 3,4,5; and SS (P_3) = 3,3,5.

Definition 3. *A simple, undirected graph respecting all five axioms, a to e, is to be named ontological structure (OS).*

In the OS graph, every node corresponds to an FE, and every node connection (edge) corresponds to a binary combination of FEs which will be mapped to an elementary particle. The *order* of an OS is the number of nodes in the graph. Since each node corresponds to an FE, it is expected that the simplest description of physical reality will correspond to the OS with the lowest number of nodes.

2. Examples

It is shown in the subsequent section that the lowest-order OS graphs each contain seven vertices: two of them, designated $O_1(7)$ and $O_2(7)$, are shown in Figures 2 and 3, respectively. The valency (V) and the significance sets (SSs) of the vertices included in these graphs are listed in Tables 1 and 2.

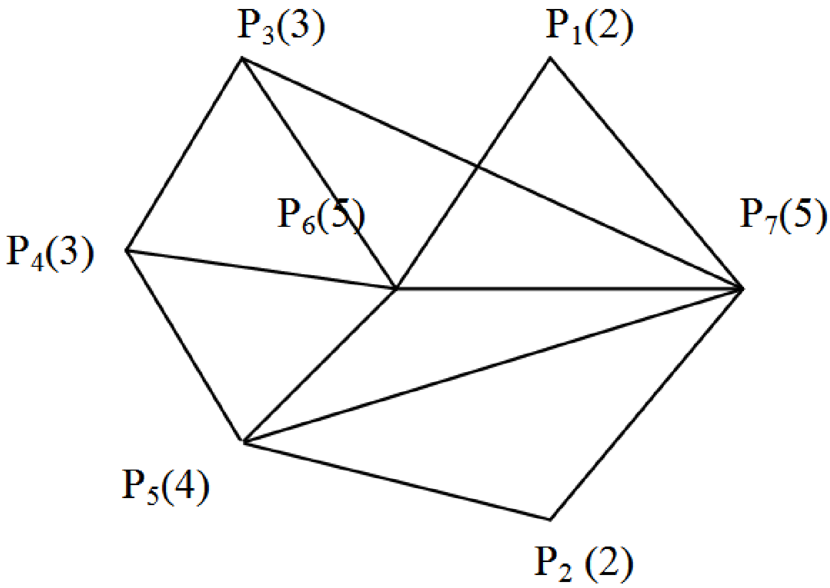

Figure 2. The OS $O_1(7)$.

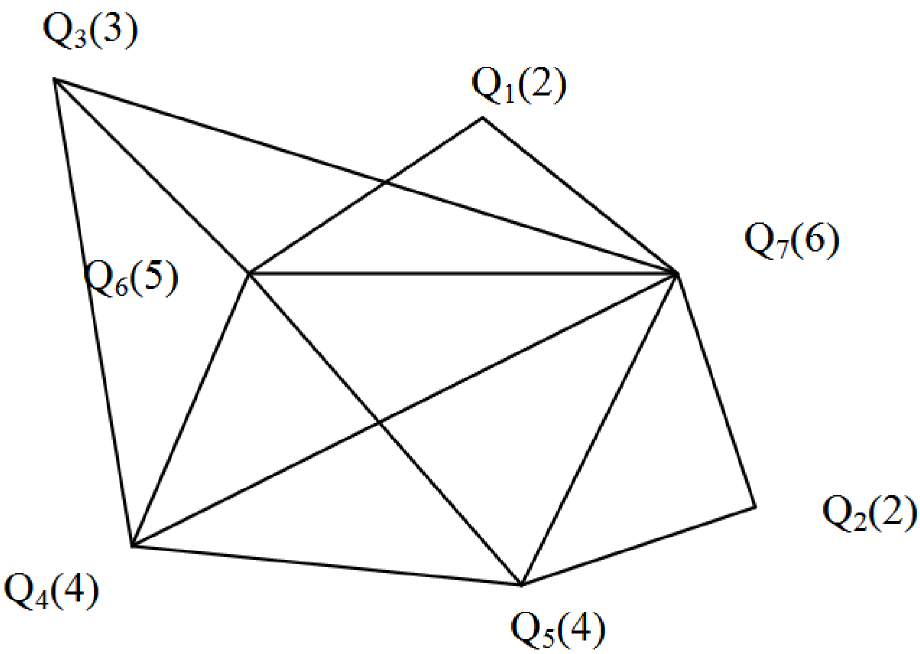

Figure 3. The OS $O_2(7)$.

Table 1. The OS O_1 (7).

Vertex	V	SS
P_1	2	5,5
P_2	2	4,5
P_3	3	3,5,5
P_4	3	3,4,5
P_5	4	2,3,5,5
P_6	5	2,3,3,4,5
P_7	5	2,2,3,4,5

Table 2. The OS $O_2(7)$.

	V	SS
Q_1	2	5,6
Q_2	2	4,6
Q_3	3	4,5,6
Q_4	4	3,4,5,6
Q_5	4	2,4,5,6
Q_6	5	2,3,4,4,6
Q_7	6	2,2,3,4,4,5

In this paper, aprioris is shown to predict that there is no ontological structure (graph) with less than seven nodes (FEs). It is also shown through construction that there are at least two different graphs of order seven. Those are two of the simplest sub-universes, which are denoted by $O_1(7)$ and $O_2(7)$. The number of edges in $O_1(7)$ is 12, so that the universe containing this OS has 12 fermion elementary particles. The number of connections in $O_2(7)$ is thirteen, so that the universe containing this OS has thirteen elementary particles (bosons). This result does not contradict the known elementary particle physics. No other axiomatic theory in physics predicts the observed numbers of particles. Of course, one may claim it to be a pure coincidence; however, we show later that other characteristics of elementary particles can also find parallels in apriorics.

Larger graphs of the OS type can be easily constructed. For example, an OS of order eight can be obtained from O_1 (7) (or $O_2(7)$) by adding the eight^h vertex, P_8 (or Q_8), and joining it to several nodes of O_1 (7) (or $O_2(7)$). Of course, the set of nodes of $O_1(7)$ connected to P_8 must be selected such that axioms a to e are satisfied. An example of such a procedure is $O_1(7{\rightarrow}8;\, 1{,}6)$, for which $O_1(7)$ is enlarged to a graph of order eight by adding an eight^h FE (node) and connecting it to the nodes P_1 and P_6. Table 3 gives the V and the SS of the vertices of this OS.

Table 3. The V and SS of the vertices of $O_1(7{\rightarrow}8;\, 1{,}6)$.

Vertex	V	SS
P_1	3	2,5,6
P_2	2	4,5
P_3	3	3,5,6
P_4	3	3,4,6
P_5	4	2,3,5,6
P_6	6	2,3,3,3,4,5
P_7	5	2,3,3,4,6
P_8	2	3,6

Each of such augmented graphs shows the way to a universe containing multiple sub-graphs of seven elements representing our own universe. One can further continue this enlargement process by creating ontological structures of order nine from those of order eight and so on.

The adjacency matrix, A, (defined as $a_{ij} = 1$ iff vertex i is connected to vertex j; otherwise, $a_{ij} = 0$) can be used to calculate the number of the binary compounds (elementary particles) and triple compounds (composite particles) of these and other possible OS graphs. In particular, since the sum of diagonal elements of the square of the A matrix A^2 are the number of connections of the nodes, the number of edges of the graph (number of binary compounds or elementary particles) is half of trace(A^2). The number of triangles (that is, composite particles combined from three nodes) is sixth of trace(A^3).

An ontological structure graph of significance is the one with an infinite number of nodes—O_∞. It is created by the following method: we denote the nodes of this graph by A_1,A_2,A_3, ...A_n, . . ., and we define the number of connections (valency) of these nodes by $V(A_i) = i + 1$. According to Definition 2, all the vertices are different from one another since they have different valences (SN). An example of a graph satisfying this requirement and axioms a to e is depicted in Table 4 and in Figure 4.

Table 4. Connectivities of the vertices belonging to O_∞.

Vertex No.	Connected to Vertices
1	2,3
2	1,3,4
3	1,2,4,5
4	2,3,5,6,7
5	3,4,6,7,8,9
6	4,5,7,8,9,10,11
7	4,5,6,8,9,10,11,12
8	5,6,7,9,10,11,12,13,14
9	5,6,7,8,10,11,12,13,14,15

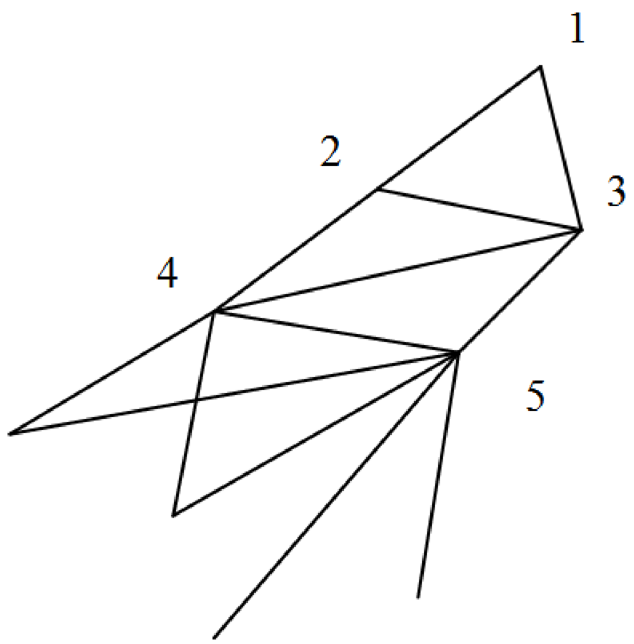

Figure 4. Connectivities of the nodes belonging to O $_\infty$.

Table 4 was calculated as follows. Node k = 1 is connected to nodes 2 and 3. Let j be the number of the row in which the vertex $k \geq 2$ appears in the first time. The vertex k is connected to all the vertices between j and k + j + 1, except the vertex k (since a vertex cannot be connected to itself, as the connections are nonreflective). Thus, the 5th node (k = 5) is joined to all the nodes between 3 and 9 (3, 4, 6, 7, 8, and 9), since the row number in which node 5 appears initially would be j = 3.

It was demonstrated [35] that the infinite set of nodes, A_1, A_2, . . ., A_n, respects axioms a to e and is thus an OS.

It was argued [35] that this ontological structure imparts a novel description of the space–time manifold (which only appears continuous on a scale much larger than the ontological structure scale) that yields interesting derivations, such as the exponential expansion of a matter-less universe.

For the convenience of the reader, Table 5 summarizes the mentioned terms in apriorics and their corresponding terms in the theory of graphs.

Table 5. Comparison of the terms used in apriorics and the corresponding terms in graph theory.

Terms in Apriorics	Terms in Graph Theory
Ontological structure, OS	Undirected simple graph satisfying axioms a to e
Fundamental entity (FE)	Vertex
B compound (BC)	Edge
Order of OS	Order of the graph
Number of BCs	Size of the graph
Significance set (SS)	See Definition 1
Equality between vertices	See Definition 2

The notion of scientific data described in terms of structures was suggested by Worrall [41] and further elaborated by Psillos [42]. Kantorovich [43] emphasized the importance of describing elementary particle physic symmetries in terms of structures, and French and Ladyman [44] argued that the structuralism notion always involves a shift in attention away from entities to the structures in which they exist. According to this point of view, the structure is ontologically prior to the entities (substructures) which are a part of it.

3. Characteristics of OS Graphs

An interesting problem is the construction of the graphs of all possible OSs of a given order, n. The goal of obtaining a general solution to this problem has not been achieved yet, and we hope that experts in the enumeration of graphs satisfying specified conditions [15] will be able to help in this endeavor. We believe that the following concepts and theorems have some potential in resolving this problem.

In the subsequent theorems and proofs, we use the following notations:

V(P) denotes the valency of the vertex P.

O(n)—OS graph of order n satisfying axioms a to e.

The multiplicity of k, m_k, is the number of vertices having valency k. (See Definition 4).

The index I of O(n) is the number of vertices in the OS having different valency. (See Definition 5).

Theorem 1. *Let the vertices of O(n) be $P_1, P_2, \ldots, P_n$. Let $K_i = V(P_i)$ $(1 \le i \le n)$. If $\max(K_i) < n - 1$, then it is possible to enlarge the O(n) to O(n + 1) by adding the vertex $P_{n+1}{}^*$ and by connecting it to all the n vertices of O(n).*

Proof. Let us denote by $P_1{}^*, P_2{}^*, \ldots, P_n{}^*$ the vertices $P_1, \ldots, P_n$ when they are connected to $P_{n+1}{}^*$. It is clear that $V(P_i{}^*) = V(P_i) + 1$ for $1 \le i \le n$. We have to show that the set of the vertices $P_1{}^*, P_2{}^*, \ldots P_{n+1}{}^*$ satisfies the basic axioms, a to e. The proof of the validity of axioms a to c is trivial. Since $P_{n+1}{}^*$ is connected to all the n vertices, axiom d is satisfied for $P_{n+1}{}^*$. The fact that the set $P_{1,\ldots,} P_n$ is an OS guarantees that axiom d is satisfied also for the vertices $P_1{}^*, \ldots, P_n{}^*$. Now, let us turn to axiom e. It is clear that $V(P_{n+1}{}^*) \neq V(P_i{}^*)$ for every $i \le n$. Since we assumed that $\max\{ V(P_i) \} < n - 1$, then $\max\{ V(P_i{}^*) \} < n$. It is easy to show that, for every two vertices, $P_i{}^*$ and $P_j{}^*$ in O(n + 1) $(1 \le i \neq j \le n)$ $P_i{}^* \neq P_j{}^*$ (because $P_i \neq P_j$ in O(n).), so that axiom e is satisfied for all the vertices of O(n + 1). □

As a result of Theorem 1, we obtain the following theorem:

Theorem 2. *In O(n), if max{V(P_i)} = n − 1, (1 ≤ i ≤ n), then by adding the vertex P_{n+1}* and by connecting it to all the other vertices, we do not obtain an OS.*

Proof. Suppose that V(P_k) = n − 1; then, V(P_k*) = n = V(P_{n+1}*). Also, since P_k* and P_{n+1}* are connected to all the other vertices, they have the same SS, so that P_k* = P_{n+1}*, and axiom e is not satisfied. □

A magnitude of some use in the analysis of OS is the multiplicity of a valency, V, defined as follows.

Definition 4. *In O(n), the vertices having the same valency shall be called equivalent vertices. The number of vertices, m_k, having V = k shall be called the multiplicity of k.*

Example 1. *The multiplicities of the vertices of the OS O_1(7) and O_2(7) discussed in the previous section are given in Table 6.*

Table 6. Multiplicities of the OS O_1(7) and O_2(7).

Valency	2	3	4	5	6
O_1(7)	2	2	1	2	0
O_2(7)	2	1	2	1	1

Definition 5. *The index of O(n), I, is the number of vertices in the OS with a different valency.*

Example 2. *In 0_1(7), I = 4 since there are 4 different equivalent subsets corresponding to the valencies 2, 3, 4, and 5. In O_2(7), I = 5.*

One direction towards the solution of the problem formulated above is to find analytical expressions of m_k as a function of k and n. Let us turn now to a few theorems concerning this issue.

Theorem 3. *(a) $\Sigma_i m_i$ = n. (b) 1/2·Σ_i m_i·i = N. Here, N is the number of edges in the OS considered*.

The proof of this theorem emerges directly from Definitions 4 and 5.

Example 3. *For O_1(7), 1/2·$\sum_i m_i$·i = 1/2·(2·2 + 2·3 + 1·4 + 2·5 + 0·6) = 12. For O_2(7), 1/2·$\sum_i m_i$·i = 1/2·(2·2 + 1·3 + 2·4 + 1·5 + 1·6) = 13 (see Tables 1 and 2).*

Theorem 4. *In O(n), $m_{n-1} \leq 1$.*

Proof. The theorem states that, in the OS of order n, the maximum number of vertices having n − 1 connections (i.e., connected to all the other vertices) is 1. Suppose that for the two vertices, P_k and P_i, we have V(P_k) = V(P_i) = n − 1. Then, since both P_k and P_i are connected to all the other vertices, they have the same SS, such that P_k = P_i, in contradiction to axiom e. □

Theorem 5. *In O(n), if V(P_k) = V(P_i) = n − 2, then P_k must be connected to P_i.*

Proof. If P_k is not connected to P_i, then it must be connected to all the other n − 2 vertices. The same goes for the vertex P_i. (Since P_k and P_i have valency equal to n − 2.) So, both SS(P_k) = SS(P_i) and P_k = P_i are in contradiction to axiom e. □

Theorem 6. *In O(n) $m_{n-2} \leq n/2$.*

Proof. Let us divide the set of the n vertices $P_1, \ldots, P_n$ into two subsets: (a) the set containing all the m_{n-2} interconnected vertices having valency equal to $n - 2$ and (b) the set of all the other $n - m_{n-2}$ vertices. Each vertex of set (a) has to be connected to $n - 2 - (m_{n-2} - 1)$ $=n - m_{n-2} - 1$ vertices of set (b) (since it is connected to the $m_{n-2} - 1$ valency equal to $n - 2$, according to Theorem 5). Hence, each vertex of set (a) has to be connected to all the $n - m_{n-2}$ vertices of set (b), except one (the "unpaired" one). In order that all the vertices of set (a) will have different SSs (otherwise, part of them would be equal, in contradiction to axiom e), they must be different in the vertex to which they are not connected (the "unpaired"). Hence, the number of different vertices in set (a) cannot be greater than the number of possible "unpaired" vertices, $n - m_{n-2}$. We then have $m_{n-2} \leq n - m_{n-2}$ or $m_{n-2} \leq n/2$. □

Example 4. *In $O_1(7)$ and in $O_2(7)$, n = 7, so that $m_5 \leq 3$.*

Theorem 7. *In O(n), if $m_{n-1} = 1$, then $m_{n-2} \leq (n - 1)/2$.*

Proof. According to Theorem 6, every vertex of set (a) (defined in the proof of the previous theorem) has to be connected to the other $m_{n-2} - 1$ vertices of this set, and, since $m_{n-1} = 1$, it also has to be connected to the vertex having $V = n - 1$. Hence, every vertex of set (a) is connected to those m_{n-2} vertices. Since its $V = n - 2$, it has to be connected to $n - 2\text{-}m_{n-2}$ vertices having a valency different from $n - 2$ or $n - 1$. The number of vertices of set (b) is $n - 1 - m_{n-2}$. Therefore, every vertex of set (a) has to be connected to all the $n - m_{n-2} - 1$ vertices of set (b), except one (the "unpaired" one). In order that all the vertices of set (a) will be different (according to axiom e), they must be different in the vertex to which they are not connected (the "unpaired"). Hence, the number of different vertices in set (a) cannot be greater than the number of possible "unpaired" vertices, $n - m_{n-2} - 1$. We then have $m_{n-2} \leq n - m_{n-2} - 1$ or $m_{n-2} \leq (n - 1)/2$. □

Theorem 8. *In O(n), if $V(P_k) = V(P_i) = 2$, then P_k cannot be connected to P_i.*

Proof. If P_k is connected to P_i, then, according to axiom d, there exists a mediator vertex, P_m, connected to both P_k and P_i. Let $V(P_m) = x$; then, $SS(P_k) = 2,x$. $SS(P_i) = 2,x.$, so that $P_k = P_i$, in contradiction to axiom e. □

Theorem 9. *If $m_{n-1} = 1$, then $m_2 \leq \min\{(n - 1)/2, I - 2\}$.*

Proof. Let set (a) be the set of all the m_2 vertices having V = 2 and (b) be the set of all the other n- m_2 vertices. Every vertex of (a) has to be connected to 2 vertices of (b) according to Theorem 8. Now, if $m_{n-1} = 1$, then every vertex of (a) must be connected to the vertex P_{n-1}, having $V = n - 1$, so that the number of possibilities of choosing the second vertex of the pair is $n - m_2 - 1$, and so that $m_2 \leq n - m_2 - 1$ or $m_2 \leq (n - 1)/2$. Also, every vertex of set (a) must be connected to a vertex with different valency of set (b) (since it is already connected to the vertex having valency equal to $n - 1$). The number of such subsets of vertices with different valency, excluding P_{n-1}, which is a subset of its own and the subset of vertices having V = 2, is $I - 2$, so that $m_2 \leq I - 2$. Hence, $m_2 \leq \min\{(n - 1)/2, I - 2\}$. □

Theorem 10. *In O(n), if V(P) = 3 and P is connected to the vertices P_1, P_2, and P_3, then one of these vertices, say P_2, has to be connected to the other two and satisfy the condition $v(P_2) > 3$.*

Proof. According to axiom d, if P_1 is connected to P, then there must be another vertex connected to P and P_1. Let this vertex be P_2. The same goes for P_3, which must be connected either to P_1 or to P_2. Hence, there is a vertex (from P_1, P_2, and P_3) connected to the other two. Now, if this vertex is P_2 and $V(P_2) = 3$, then $SS(P) = SS(P_2) = 3,x,y$, where $x = V(P_1)$ and $y = V(P_3)$, and $P = P_2$. So, there exists at least one other vertex connected to P_2; hence, $V(P_2) > 3$. □

Theorem 11. *In O(n), if V(P) = 3, then P cannot be connected to more than one vertex having V = 2.*

Proof. Suppose that P is connected to P_1, P_2, and P_3 and let $V(P_1) = V(P_3) = 2$ and $V(P_2) = x > 3$ (according to the previous theorem). According to Theorem 10, the only possible connection of P to P_1, P_2, and P_3 satisfies $SS(P_1) = SS(P_3) = 3,x$. So, $P_1 = P_3$, in contradiction to axiom e. □

Theorem 12. *If V(P) = 2, then P cannot be connected to more than one vertex having V = 3.*

Proof. Let P be connected to P_1 and P_2, and $V(P_1) = V(P_2) = 3$. According to axiom d, P_1 and P_2 have to be connected to each other, since if P and P_1 are connected, there must be a vertex connected to both, but since V(P) = 2, this third vertex must be P_2. According to Theorem 10, P_1 has to be connected to P_3, having $V = x > 3$. However, since P_1 and P_3 are connected, then according to axiom d, there must be a third vertex connected to both. Now, since P_1 already has three connections (P, P_2, and P_3), no additional vertex can be connected to P_1; hence, this mediator can be either P or P_2. However, P has two connections already and can have no more. Thus, P_3 must be connected to P_2 so that $SS(P_1) = SS(P_2) = 2,3,x$, in contradiction to axiom e. □

Theorem 13. *There cannot exist an OS of the order n = 3.*

Proof. According to axioms a and d every vertex has to be connected to at least two other vertices so that all the three vertices in the graph have to be connected to each other. Hence the three vertices have the same valency = 2, and the same SS = 2,2. Thus, they are equal in contradiction to axiom e. □

Theorem 14. *There cannot exist an OS of order n = 4.*

Proof. In this case, the valency of the vertices can be 2 or 3. According to Theorem 10, the existence of a vertex having V = 3 is impossible since it necessitates the existence of a vertex having V greater than 3. Hence, all vertices must have V = 2. But according to Theorem 8, the connections between these vertices cannot be realized. □

Theorem 15. *There cannot exist an OS of order n = 5.*

Proof. Case 1: $m_4 = 0$. According to the Theorem 6, $m_3 \leq 2$. Also, $m_2 \leq 1$ because the only available connections for a vertex having V = 2 are to two vertices having V = 3 (see Theorem 8). So, $m_2 + m_3 + m_4 \leq 3$ is in contradiction to Theorem 3a, claiming that this sum should be equal to n = 5.
Case 2: $m_4 = 1$. According to the Theorem 7, $m_3 \leq 2$. According to Theorem 9, $m_2 \leq I - 2 = 1$. (In this case I = 3 since there are three equivalent sets of vertices having V = 2,3,4.) Hence, $m_2 + m_3 + m_4 \leq 1 + 2 + 1 = 4 < 5$, in contradiction to Theorem 3(a). □

Theorem 16. *There cannot exist an OS of the order n = 6.*

Proof. According to Theorem 4, m_5 = 0,1. Let us examine these two possibilities.

Case A. $m_5 = 0$.

According to Theorems 8 and 12, the only possible SS for the vertices having V = 2 are 3,4 and 4,4, so that $m_2 \leq 2$.

Let us show now that $m_4 \neq 0$. If $m_4 = 0$, then $m_2 = 0$ because none of the two possibilities discussed above for the vertices having V = 2 can exist. Hence, according to 3(a), $m_3 = 6$. However, this is impossible due to Theorem 10, which prevents m_3 vertices from being connected among themselves. Therefore, $m_4 \neq 0$.

Let us show now that $m_4 \neq 1$. If $m_4 = 1$, then there is one possible SS for the vertices having V = 2, namely SS = 3,4, so that $m_2 = 1$. According to Theorem 3(a), we obtain $m_3 = 6 - 1 - 1 = 4$. This result is impossible since, according to Theorems 10 to 12, there are only two possible SSs for the vertices having V = 3: 2,3,4 and 3,3,4, so that m_3 must be equal to 2 and not to 4. Hence, $m_4 \neq 1$.

Since $m_4 \leq 3$ (Theorem 6), let us consider the case in which $m_4 = 3$. According to Theorem 5, the only possible SS for the vertices having V = 4 are 4,4,2,2; 4,4,3,2; and 4,4,3,3. However, these three possibilities cannot be realized since, in this case, the OS will include seven vertices (having valences equal to 2, 2, 3, 3, 4, 4, and 4) instead of 6. Therefore, $m_4 \leq 2$. Since we proved that $m_4 \neq 0,1$, there is only one possible value to m_4, namely $m_4 = 2$.

Let us show now that $m_2 \neq 0$. If $m_2 = 0$, since $m_4 = 2$, then $m_3 = 4$ (Theorem 3(a)). But this is impossible since, according to Theorem 10, there are only three possible SSs for the vertices having V = 3: 4,4,4; 3,4,4; and 3,3,4. Therefore, since $m_2 \leq 2$, we obtain $m_2 = 1,2$.

We now show that $m_2 \neq 1$, so that $m_2 = 2$

If $m_2 = 1$, then (since we found that $m_4 = 2$) $m_3 = 6 - 2 - 1 = 3$. The handshaking lemma is a consequence of the degree sum formula, according to which the sum of the degrees (the numbers of times each vertex is touched) equals twice the number of edges N in the graph. Specifically, this means that, in the apriorics n = 6 case, we have the following (Theorem 3(b)):

$$\sum_{i=2}^{5} i\, m_i = 2N$$

Since according to axiom d, $m_1 = 0$ and according to axiom a, $m_0 = 0$ this implies the following:

$$3\, m_3 + 5\, m_5 = \text{even}$$

Now, this is in clear contradiction to $m_3 = 3$, $m_5 = 0$; thus, this case cannot occur.

So that $m_2 \neq 1$. Since we found that $m_2 \leq 2$ and $m_2 \neq 0$ we obtain $m_2 = 2$.

We have then to find an OS of order 6 having the following multiplicities: $m_5 = 0$, $m_4 = 2$, $m_2 = 2$ and $m_3 = 6 – 2 - 2 = 2$. Let the vertices of this OS be denoted as follows: $P_1(4)$, $P_2(4)$, $P_3(3)$, $P_4(3)$, $P_5(2)$, and $P_6(2)$—see Figure 3-1c. According to Theorem 5 $P_1(4)$ is connected to $P_2(4)$. The possible SS values for these 2 vertices are, respectively: SS = 2,2,3,4, and SS = 2,3,3,4. Hence $P_1(4)$ is connected to the 3 vertices—$P_3(3)$, $P_5(2)$ and $P_6(2)$. And $P_2(4)$ is connected to three vertices say $P_6(2)$ and $P_3(3)$, $P_4(3)$.

Now $P_5(2)$ and $P_6(2)$ cannot be connected according to theorem 8. Hence according to axiom d at least one of the vertices having V = 2, say $P_5(2)$, must be connected to $P_3(3)$ (The mediator between $P_5(2)$ and $P_1(4)$ must be one of the vertices already connected to $P_1(4)$ as all its connections are already allocated. It cannot be $P_6(2)$ due to Theorem 8 and it cannot be $P_2(4)$ which has only one V = 2 vertex in its SS structure and this is $P_6(2)$). Since $P_2(4)$ is connected to $P_4(3)$ (circled in the figure), then according to axiom d they both have to be connected to their mediator, but this is impossible since the structure includes already six vertices and all the other vertices, except $P_4(3)$, are fully connected in accordance with their valence.

We see then that the attempt to design an OS satisfying the above-mentioned conditions ($m_2 = m_3 = m_4 = 2$) results in contradiction.

Case B. $m_5 = 1$.

In this case, $m_4 \leq 2$ (Theorem 7). Also, $m_2 \leq 2$ because the only possible SSs for the vertex having V = 2 are 5,4 and 5,3 (since these vertices must be connected to the vertex having V = 5, and according to Theorem 8, they cannot be connected to each other). Hence, if $m_4 = 0$, then $m_2 = 1$, and according to Theorem 3, $m_3 = 4$. But this result is impossible since, according to Theorems 10 and 12, there are only two possible SSs for the vertices having V = 3; SS = 5,3,3; and SS = 5,3,2. Hence, $m_4 \neq 0$, so that $m_4 = 1,2$.

Let us show now that $m_2 \neq 0$. Suppose that $m_2 = 0$, since $m_4 \leq 2$ (Theorem 7) and $m_3 + m_4 + 1 = 6$ (Theorem 3) -> $m_3 = 5 - m_4$; then, $m_3 \geq 3$. The only possible SSs for the vertex having V = 3 are 5,3,3; 5,4,3; and 5,4,4. But the case where SS = 5,3,3 is impossible. To show this, let us try to form an OS satisfying this condition. Let $V(P_1) = 3$ and $SS(P_1) = 5,3,3$. Let P_1 be connected to P_2, P_3, and P_4, where $V(P_2) = 3$, $V(P_3) = 3$, and $V(P_4) = 5$. P_2 and P_3 have to be connected to P_4 (having V = 5). Since $V(P_2) = 3$, and it is connected only to two vertices (P_1 and P_4), it has to be connected to an additional vertex, P_5 (not to P_3 because, in this case, $SS(P_3) = 5,3,3$, the same as $SS(P_1)$), and P_5 has to be connected to P_4. By the same token, P_3 and P_4 have to be connected to P_6. Now, the only possible values of the V of P_5 and P_6 is 4 (since $m_2 = 0$, and there are already three vertices having V = 3 and one vertex having V = 5). Therefore, $SS(P_2) = SS(P_3) = 3,4,5$. Hence, $P_2 = P_3$, in contradiction to axiom e.

We are left then with the following four possibilities: $m_2 = 1,2$ and $m_4 = 1,2$. For each of these cases, $m_3 = 5 - m_2 - m_4$, as $3\ m_3 + 5\ m_5 = \text{even}$. It follows for $m_5 = 1$ that

$$3\ m_3 = \text{even} - 5 = \text{odd}$$

Thus, $m_3 \neq 2$. The remaining cases are given in Table 7.

Table 7. The possible multiplicities of the V = 2,3,4.

V = 2	V = 3	V = 4
1	3	1
2	1	2

We subsequently show that the attempt to design an OS satisfying the conditions of each row in this table always results in the contradiction of one of the axioms, a to e. Pay attention to the fact that, in all the cases discussed below, all the vertices have to be connected to $P_6(5)$.

Let us begin with the first row of Table 7, in which $m_2 = 1$, $m_3 = 3$, $m_4 = 1$, and $m_5 = 1$. Let us denote the vertices of the possible OS by $P_1(2)$, $P_2(3)$, $P_3(3)$, $P_4(3)$, $P_5(4)$, and $P_6(5)$. (The numbers in parentheses are the valences of the vertices). The possible SSs of $P_1(2)$ can be either SS = 4,5 or SS = 3,5. Let us discuss each of these cases separately.

Case 1. *SS(P_1(2)) = 4,5.*

In this case (see Figure 5d), $P_1(2)$ is connected to $P_5(4)$ and $P_6(5)$. Then, $P_5(4)$ is connected to $P_2(3)$, $P_3(3)$, and $P_6(5)$. Now, $P_2(3)$ and $P_3(3)$ have two connections (to $P_5(4)$ and $P_6(5)$) instead of three, and the only available connection is to a vertex having V = 3. But such connections will yield that the SS of these vertices (circled in the figure) will be SS = 3,4,5, so that these two vertices will be equal, in contradiction to axiom e.

Case 2. *SS(P_1(2)) = 3,5.*

In this case (see Figure 5e), $P_1(2)$ is connected to $P_2(3)$ and $P_6(5)$. Then, $P_5(4)$ is connected to $P_2(3)$, $P_3(3)$, $P_4(3)$, and $P_6(5)$. (As noted earlier, all the vertices have to be connected to $P_6(5)$.) As a result, all the vertices are fully connected in accordance with their V, except the vertices having V = 3. $P_3(3)$ and $P_4(3)$ are circled in the figure. They each have only two connections instead of three. Trying to form a connection between these vertices will result in making them equal (they will both have SS = 3,4,5), in contradiction to axiom e.

The analysis of these two cases shows that it is impossible to design an OS satisfying the conditions of the first row of Table 7.

Let us examine now the second row in which $m_2 = 2$, $m_3 = 1$, $m_4 = 2$, and $m_5 = 1$. Let us denote the vertices of the possible OS by $P_1(2)$, $P_2(2)$, $P_3(3)$, $P_4(4)$, $P_5(4)$, and $P_6(5)$—see Figure 5f. The possible SSs of $P_1(2)$ and $P_2(2)$ are, respectively, SS = 3,5, and SS = 4,5 since, as was indicated previously, they cannot be connected to each other. Hence, $P_1(2)$ has to be connected to $P_3(3)$ and $P_6(5)$, and $P_2(2)$ has to be connected to $P_4(4)$ and $P_6(5)$. At this stage, $P_5(4)$ has only two connections (to $P_6(5)$ and to $P_4(4)$), and it cannot be connected to $P_1(2)$ and $P_2(2)$, which do not have any more available connections. Therefore, only $P_3(3)$ is available for connection. Hence, $P_5(4)$ and $P_4(4)$ (circled in the figure) each have only three connections instead of four. Hence, it is impossible to design an OS satisfying the conditions of the fourth row. □

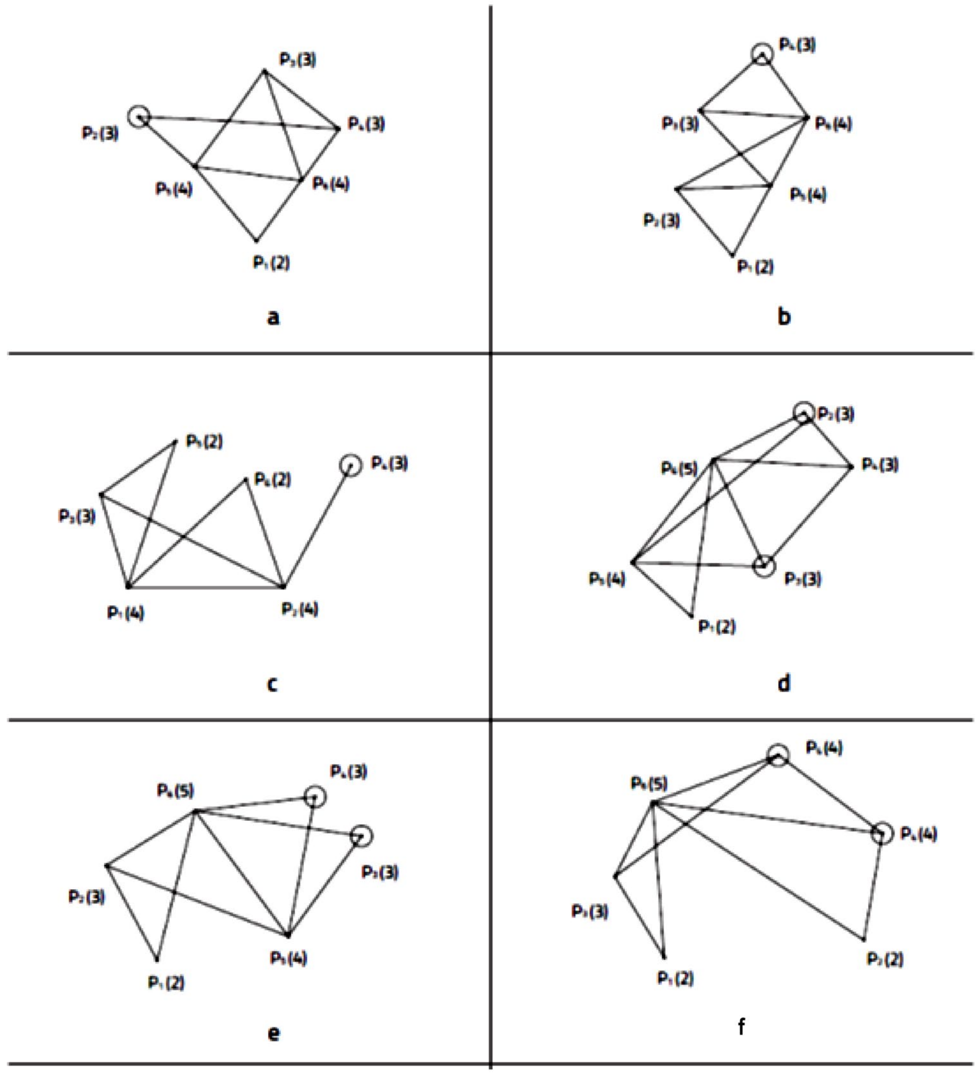

Figure 5. Graphs corresponding to the proof of Theorem 16. The subfigures demonstrate the impossibility of various ontological structures of order 6 as described in the text.

Remark 1. *We described two OSs of order 7 in Section 2—O_1(7) and O_2(7)—satisfying the axioms of apriorics. There may be other OSs of order 7. The enumeration of all the OSs of order 7 and of higher orders is still an open problem in apriorics.*

4. Properties of the Vertices of OS Graph

4.1. Introduction

In this section, we propose the notion of specifying properties to the nodes of the ontological structure (graph) determined uniquely by the nodes to which they are joined. This is demonstrated by analyzing the property combinations of the FE's corresponding to the nodes of O_1(7) and O_2(7), which are discussed in Section 2.

How can the properties of a node (FE) be defined through its connections to the other nodes? According to the intuitive meaning of the relationship between the concept of a common property of two vertices and their connection, we can stipulate the following rules:

(a) If two connected nodes in an ontological structure, V_1 and V_2, have a common property, f, then their mediator vertex connected to V_1 and V_2 (obligatory by axiom d) will be assigned the property f^ (complementary property). Remark: The empirical meaning of f^ is that it facilitates the detection of f.
(b) Two nodes of an ontological structure are joined if and only if either they have at least a common single property or one node has a complementary property of the other node.
(c) A node in an ontological structure cannot have both a property and its complementary.
(d) The set of properties of every node in an ontological structure must differ in at least a single property from the set of properties of any other node which is part of the same ontological structure.

Remark 2. *This rule is derived from axiom e, which requires that every two nodes be different.*

4.2. Example: Intrinsic Properties of the Nodes of $O_1(7)$ and $O_2(7)$

Following the rules specified above, the properties of the nodes belonging to $O_1(7)$ are as follows: (a) The first property, f_1, is assigned to the fundamental entity P_1, as shown in Figure 6. (b) Since P_2 is not joined to P_1, it cannot be given the same property, so that we assign to it the different property, f_2. (c) P_4 is not joined to P_1 or to P_2. Thus, an additional property, f_3, is obligatory for its depiction. (d) P_5 is joined to P_2 and P_4 so that it possesses their complementary properties, denoted as f_2^ and f_3^. Thus, we should note that there is freedom in this assignment of properties according to the rules elaborated above. However, this freedom will not affect the conclusions. (e) P_7 is joined to P_1 and P_2 so that it is depicted by f_1 and f_2. Following the same procedure, we assign properties to other nodes, as clarified in Figure 6.

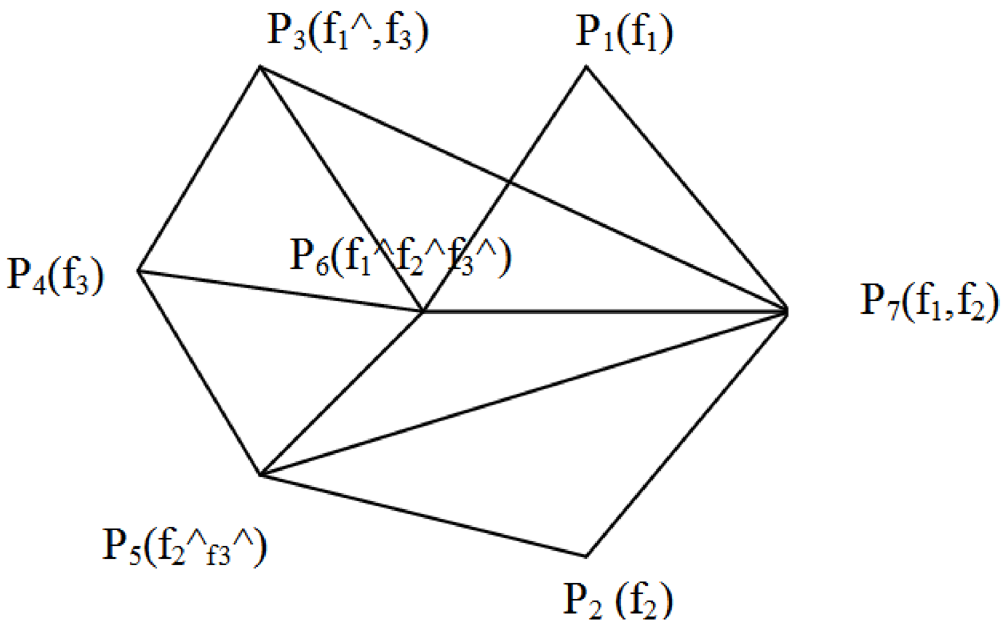

Figure 6. Properties of the vertices of $O_1(7)$.

We observe that the minimal number of properties necessary for the ontological structure $O_1(7)$ is three. It can easily be demonstrated that this prediction is valid also for $O_2(7)$. The properties of $O_2(7)$ nodes are specified by g_1, g_2, and g_3. To each of these properties, there is a complementary property, i.e., g_1^, g_2^, and g_3^.

Table 8 and Figure 6 give the properties of the nodes of $O_1(7)$, and Table 9 and Figure 7 give those of the nodes of $O_2(7)$.

Table 8. Properties of the vertices of $O_1(7)$.

	Properties
P_1	f_1
P_2	f_2
P_3	f_1ˆ,f_3
P_4	f_3
P_5	f_2ˆ,f_3ˆ
P_6	f_1ˆ,f_2ˆ,f_3ˆ
P_7	f_1,f_2

Table 9. Properties of the vertices of $O_2(7)$.

Vertex	Properties
Q_1	g_1
Q_2	g_2
Q_3	g_3
Q_4	g_1ˆ,g_3
Q_5	g_2ˆ,g_3ˆ
Q_6	g_1ˆ,g_3ˆ
Q_7	g_1,g_2,g_3

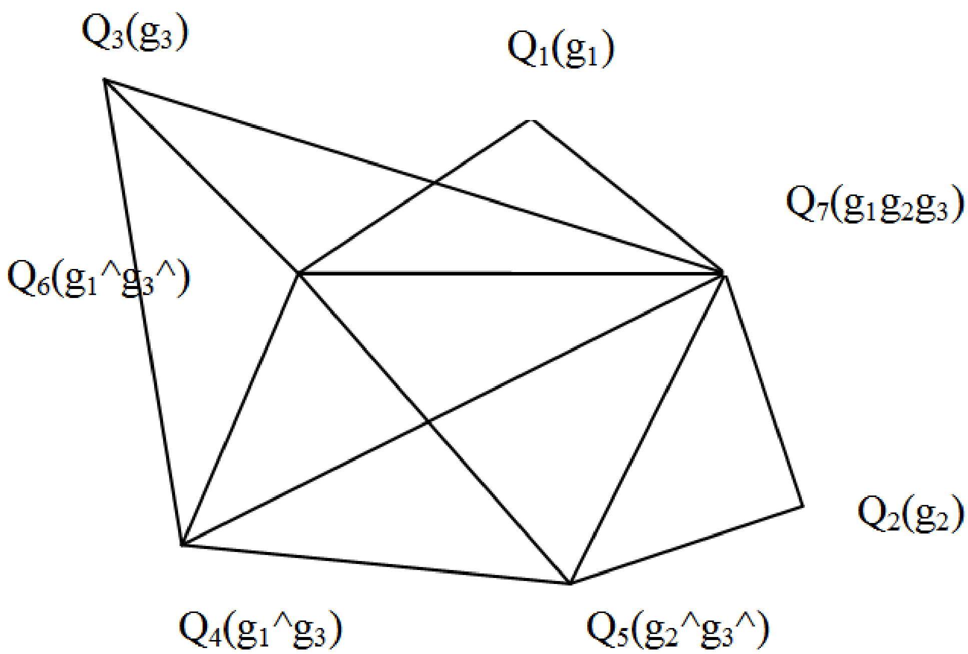

Figure 7. Properties of the vertices of $O_2(7)$.

Aprioric predicts the three basic interactions corresponding to the three properties given in Table 8. This derivation is compatible with the existence of three fundamental interactions—electromagnetic, nuclear strong, and nuclear weak. These interactions bind the fundamental entities, thus forming the elementary and more complex particles.

Using Table 8, it is possible to give properties to the twelve binary compounds of $O_1(7)$ that are the twelve fermion elementary particles of the standard model. This mapping enables the classification of these particles according to their properties. These predictions of apriorics do not contradict the known empirical standard model of particle physics.

Using Table 9, it is also possible to assign properties to the thirteen binary compounds of $O_2(7)$ that are mapped to the thirteen known mediator particles (gauge bosons) that transfer the interaction between twelve elementary fermions. This designation enables the classification of these thirteen particles according to their properties. This prediction is again not contradicted by known empirical data [36].

5. Discussion: OS Graphs Used for Displaying Coreless Languages

In this paper, we described and analyzed a specified undirected simple OS graph and used it for the investigation of the fundamental entities and elementary and composite particles. We believe that these graphs can also be used in the holistic theory of knowledge [45,46] expressed in a certain language in which the meaning of every word and sentence is specified by the way it is joined to other words and sentences. Thus, it can be displayed as a graph of interconnected nodes corresponding to these words and sentences. Such a graph will have "core words" and "core sentences", whose interpretation and truth are decided by elements which are beyond language. In our daily language, we have a multitude of core words representing objects existing in the real world. In the language used in empirical sciences, there exist a significant number of core sentences that have meaning dependent on observations and experiments. Mathematical theories depend on a core of fundamental notions joined via axioms whose justification lies externally. In each and every language, the core words and core sentences serve as means for specifying meaning to the other words and sentences of the language.

Can we contemplate a language without a core? Is it possible to imagine a language in which the meaning of its words (and sentences) will be specified solely by their interconnections?

We believe that the ontological structure is appropriate for displaying such a language, and the seven vertices of $O_1(7)$ and $O_2(7)$ can be thought of as the fundamental words in these simplest possible languages. The compounds of these nodes, which can be characterized by their properties, represent complex words and sentences.

In fact, every OS of order n represents a coreless language, including n basic words whose significance is derived solely from their interconnections. The compounds of these vertices define composite words and sentences in this language.

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