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# Kropina Metrics with Isotropic Scalar Curvature via Navigation Data

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**Abstract:** Through an interesting physical perspective and a certain contraction of the Ricci curvature tensor in Finsler geometry, Akbar-Zadeh introduced the concept of scalar curvature for the Finsler metric. In this paper, we show that the Kropina metric is of isotropic scalar curvature if and only if $F$ is an Einstein metric according to the navigation data. Moreover, we obtain the three-dimensional rigidity theorem for an Einstein–Kropina metric.

**Keywords:** Kropina metrics; scalar curvature; Einstein metrics

**MSC:** 53C30; 53C60

## 1. Introduction

Over the past several years, rapid development in Finsler geometry has been observed. A significant class of Finsler metrics, referred to as $(\alpha, \beta)$-metrics, has gained attention for its special characteristics. These metrics are expressed in the form $F = \alpha\phi(s)$, where $\alpha := \alpha(x,y) = \sqrt{a_{ij}(x)y^i y^j}$ denotes a Riemannian metric, $\beta := \beta(y) = b_i(x)y^i$ represents a 1-form on $M$, and $\phi(s)$ is a smooth positive function defined on a specific open interval and $s = \frac{\beta}{\alpha}$. In particular, the Finsler metric $F = \frac{\alpha^2}{\beta}$ is called a Kropina metric when $\phi(s) = \frac{1}{s}$. Kropina metrics were initially innovated by Berwald in relation to a two-dimensional Finsler space with rectilinear extremal, and later studied by Kropina [1]. Recently, geometers discovered important geometric properties of Kropina metrics, which have diverse and significant applications [2–8].

Consider an object moving in a metric space, such as Euclidean space, driven by an interval force and an external force field. The shortest time problem aims to determine a curve from one point to another in the space, along which it takes the least time for the object to travel. This is called the Zermelo navigation problem [9]. Later, Randers spaces were viewed from a new perspective by Shen [10]. He identified these metrics with the solution of Zermelo's navigation problem on some Riemannian spaces and described a Randers metric by a new Riemannian metric $h$ and a vector field $W$ with $\|W\|_h < 1$. Based on this, in 2004, Bao, Robles, and Shen [11] established the necessary and sufficient conditions for a Randers metric to be of constant flag curvature. Additionally, they obtained the classification of a Randers metric with constant flag curvature using the navigation method. Kropina metrics can be easily treated as the limit of the navigation problem for Randers metrics as $\|W\|_h \to 1$ [2,4,12,13]. Zhang and Shen [4] obtained the expression of Ricci curvature for Kropina metrics and certified that a non-Riemannian Kropina metric, which has a constant Killing 1-form $\beta$, is an Einstein metric in the same way that $\alpha$ is an Einstein metric. More generally, Xia [5] classified Kropina metrics with weakly isotropic flag curvature via the navigation data. Based on these results, Yoshikawa and Sabau [6] obtained the classification theorem for Kropina metrics with constant flag curvature via the navigation data. Cheng, Li, and Yin [7] characterized conformal vector fields on Kropina



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manifolds by the navigation data, and fully ascertained conformal vector fields on Kropina manifolds which have weakly isotropic flag curvature.

Ricci curvature in Finsler geometry is the natural generalization of that in Riemannian geometry. Nevertheless, there is no unified definition of Ricci curvature tensor in Finsler geometry. Therefore, several different versions of definitions of scalar curvature can be found in Finsler geometry. Here, we take the notion of scalar curvature introduced by Akbar-Zadeh [14]. The scalar curvature $R$ of a Finsler metric $F$ is defined as

$$R := g^{ij} Ric_{ij},$$

where $Ric_{ij} := \frac{1}{2} Ric_{y^i y^j}$. Tayebi [15] studied general fourth-root manifolds with isotropic scalar curvature. Also, he characterized Bryant metrics with isotropic scalar curvature. Meanwhile, Chen and Xia [16] explored an $(\alpha, \beta)$-metric that is conformally flat and of weakly isotropic scalar curvature. They proved that its scalar curvature must vanish. Cheng and Gong [17] obtained that a Randers metric with weakly isotropic scalar curvature must have isotropic $S$-curvature. They claimed that a conformally flat Randers metric, which is of weakly isotropic scalar curvature, must be either Minkowskian or Riemannian. Recently, in Zhu and Song's manuscript [18], they proved that a Kropina metric is of weakly isotropic scalar curvature if and only if it is an Einstein metric. Further, they gave a negative answer to the Yamabe problem on Kropina metrics with isotropic $S$-curvature.

Moreover, Li and Shen [19] introduced a new notion of Ricci curvature tensor

$$\widetilde{Ric}_{ij} := \frac{1}{2}(R_{i\ mj}^{\ m} + R_{j\ mi}^{\ m}).$$

Sevim, Shen, and Ulgen [20] discussed several Ricci curvature tensors and their relationship with the Ricci curvature to provide a better understanding of non-Riemannian quantities. Liu, Zhang, and Zhao [8] obtained expressions of Ricci curvature tensor $\widetilde{Ric}_{ij}$ and scalar curvature $\tilde{R} := g^{ij}\widetilde{Ric}_{ij}$ of Kropina metrics. And they characterized Kropina metrics with isotropic scalar curvature $\tilde{R}$.

In this paper, we mainly focus on the scalar curvature introduced by Akbar-Zadeh [14], study Kropina metrics with isotropic scalar curvature via the navigation data, and obtain the following results.

**Theorem 1.** *Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric on $M$. Then $F$ is of isotropic scalar curvature if and only if $F$ is an Einstein metric.*

Although our conclusion is the special case of Zhu and Song's [18], the methods of the proof are different. Here, we use the navigation method, which can simplify the proof process, instead of divisibility analysis.

The flag curvature in Finsler geometry generalizes the sectional curvature in Riemannian geometry. If the flag curvature $K = constant$, then a Finsler metric $F$ is said to be of constant flag curvature. Xia [5] classified Kropina metrics with constant flag curvature in three dimensions as follows.

**Lemma 1** ([5]). *(Three-dimensional rigidity.) Let $F$ be a Kropina metric in three dimensions. Then, $F$ is an Einstein metric if and only if it has non-negative constant flag curvature.*

In virtue of Lemma 1 and Theorem 1, we directly obtain the following result.

**Theorem 2.** *(Three-dimensional rigidity). Let $F$ be a Kropina metric in three dimensions. Then, $F$ is of isotropic scalar curvature if and only if it has a non-negative constant flag curvature.*

## 2. Preliminaries

For the sake of simplicity, we always set the dimension of Kropina metrics as $n \geq 2$ in the following unless otherwise specified.

Let $M$ be an $n(\geq 2)$-dimensional smooth manifold. A Finsler structure of $M$ is a function

$$F : TM \to [0, \infty)$$

with the following properties:
(1) Regularity: $F$ is smooth on the entire slit tangent bundle $TM\backslash\{0\}$;
(2) Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y)$, $\forall \lambda > 0$;
(3) Strong convexity:

$$g_{ij}(x,y) := \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x,y) = \frac{1}{2}F^2_{y^i y^j}$$

is positive-definite at every point of $TM\backslash\{0\}$. A smooth manifold $M$ endowed with a Finsler structure $F$ is called a Finsler manifold, which is denoted by $(M, F)$.

Let $(M, F)$ be a Finsler manifold. The geodesics of a Finsler metric $F$ on $M$ are classified by the following ODEs:

$$\frac{d^2 x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where

$$G^i := \frac{1}{4} g^{il}\left[\left(F^2\right)_{x^j y^l} y^j - F^2_{x^l}\right],$$

$\left(g^{ij}\right) := \left(g_{ij}\right)^{-1}$. We call the local functions $G^i = G^i(x,y)$ geodesic coefficients (or spray coefficients).

For $x \in M$ and $y \in T_x M\backslash\{0\}$, Riemann curvature $R_y := R^i{}_k(x,y)\frac{\partial}{\partial x^i} \otimes dx^k$ is defined by

$$R^i{}_k := 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

The Ricci curvature of $F$ is the trace of the Riemann curvature, i.e., $Ric := R^k{}_k$.

The Hessian of the Ricci curvature also gives rise to a Ricci curvature tensor

$$Ric_{ij} := \frac{1}{2}(Ric)_{y^i y^j}.$$

Then, one can define the notion of scalar curvature

$$R := g^{ij} Ric_{ij}.$$

If $R = n(n-1)\left(\frac{\theta}{F} + \kappa\right)$, where $\kappa = \kappa(x)$ is a scalar function and $\theta = \theta_i(x) y^i$ is a 1-form on $M$, then it is said that $F$ has weak isotropic scalar curvature. In particular, when $\theta = 0$, i.e.,

$$R = n(n-1)\kappa(x),$$

it is said that $F$ is of isotropic scalar curvature.

Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric. Its fundamental tensor $g = g_{ij} dx^i \otimes dx^j$ is given by [4]

$$g_{ij} = \frac{F}{\beta}\left[2a_{ij} + \frac{3F}{\beta} b_i b_j - \frac{4}{\beta}(b_i y_j + b_j y_i) + \frac{4 y_i y_j}{F\beta}\right],$$

where $y_i := a_{ij} y^j$. Moreover,

$$g^{ij} = \frac{\beta}{2F}\left[a^{ij} - \frac{b^i b^j}{b^2} + \frac{2}{b^2 F}\left(b^i y^j + b^j y^i\right) + 2\left(1 - \frac{2\beta}{b^2 F}\right)\frac{y^i y^j}{F\beta}\right],$$

where $b := \parallel \beta \parallel_\alpha$, $\left(a^{ij}\right) := \left(a_{ij}\right)^{-1}$ and $b^i := a^{ij}b_j$.

Let $\nabla\beta = b_{i|j}y^i dx^j$ denote the covariant derivative of $\beta$ with respect to $\alpha$. Set

$$r_{ij} := \frac{1}{2}\left(b_{i|j} + b_{j|i}\right),\ s_{ij} := \frac{1}{2}\left(b_{i|j} - b_{j|i}\right),\ r_{00} := r_{ij}y^i y^j,\ r_i := b^j r_{ij},\ r_0 := r_i y^i,$$
$$r^i := a^{ij}r_j,\ r := b^i r_i,\ s^i{}_0 := a^{ij}s_{jk}y^k,\ s_i := b^j s_{ji},\ s_0 := s_i y^i,\ s^i := a^{ij}s_j.$$

The Ricci curvature of Kropina metrics is presented as follows.

**Lemma 2** ([4]). *Let $F$ be a Kropina metric on $M$. Then, the Ricci curvature of $F$ is given by*

$$Ric = {}^\alpha Ric + T, \tag{1}$$

*where ${}^\alpha Ric$ is the Ricci curvature of $\alpha$, and*

$$\begin{aligned} T =& \frac{3(n-1)}{b^4F^2}r_{00}^2 + \frac{n-1}{b^2F}r_{00|0} - \frac{4(n-1)}{b^4F}r_{00}r_0 + \frac{2(n-1)}{b^4F}r_{00}s_0 - \frac{1}{b^4}rr_{00} + \frac{1}{b^2}r^k{}_k r_{00} \\ &+ \frac{2n}{b^2}r_{0k}s^k{}_0 + \frac{1}{b^2}b^k r_{00|k} + \frac{1}{b^4}r_0^2 - \frac{1}{b^2}r_{0|0} - \frac{2(2n-3)}{b^4}r_0 s_0 + \frac{n-2}{b^2}s_{0|0} - \frac{n-2}{b^4}s_0^2 \\ &- \frac{1}{b^2}r_{0k}s^k F - \frac{1}{b^2}r_k s^k{}_0 F - \frac{1}{b^4}rs_0F + \frac{1}{b^2}r^k{}_k s_0 F + \frac{n-1}{b^2}s^k{}_0 s_k F - s^k{}_{0|k}F + \frac{1}{b^2}b^k s_{0|k}F \\ &- \frac{1}{4}s^j{}_k s^k{}_j F^2 - \frac{1}{2b^2}s^k s_k F^2. \end{aligned}$$

A Kropina metric can also be characterized by a Riemannian metric $\bar{\alpha}$ and a 1-form $\bar{\beta}$ with $\parallel \bar{\beta} \parallel_{\bar{\alpha}} = 1$. Between $F = \frac{\alpha^2}{\beta}$ and a pair $(\bar{\alpha}, \bar{\beta})$, there exists a one-to-one correspondence with

$$F = \frac{\bar{\alpha}^2}{2\bar{\beta}}, \tag{2}$$

where $\bar{\alpha} = \frac{2}{b}\alpha$ and $\bar{\beta} = \frac{1}{b}\beta$ (see [5]). We call this pair $(\bar{\alpha}, \bar{\beta})$ the navigation data of a Kropina metric.

**Note 1** ([4]). *Let $F$ be an $(\alpha,\ \beta)$-metric. Then, $\parallel \beta \parallel_\alpha = 1$ if and only if $r_k + s_k = 0$.*

**Note 2** ([4]). *For a 1-form $\beta = b_i y^i$ on $M$, we say that $\beta$ is a conformal 1-form with respect to $\alpha$ if it satisfies $b_{i|j} + b_{j|i} = \rho a_{ij}$, where $\rho = \rho(x)$ is a function on $M$. If $\rho = 0$, we say that $\beta$ is a Killing 1-form. Furthermore, $\beta$ is said to be a constant Killing 1-form if it is a Killing 1-form and has constant length, which is equivalent to $r_{ij} = 0,\ s_k = 0$.*

**Note 3** ([4]). *Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric given by (2) with the navigation data $(\bar{\alpha}, \bar{\beta})$. Thus, $\beta$ is a conformal 1-form with respect to $\alpha$ if and only if $\bar{\beta}$ is a Killing 1-form with respect to $\bar{\alpha}$.*

## 3. Scalar Curvature of Kropina Metrics

In this section, we shall present an expression of the scalar curvature of Kropina metrics. Then, we will obtain the necessary condition for the Kropina metric with isotropic scalar curvature.

Firstly, using the notion of Ricci curvature tensor and Lemma 2, we can derive the expression of Ricci curvature tensor for Kropina metrics.

**Proposition 1.** *Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric on M. Then, the Ricci curvature tensor of F is given by*

$$
\begin{aligned}
Ric_{ij} = {}& {}^{\alpha}Ric_{ij} \\
&+ \frac{1}{2}F_{.i.j}\Big[-\frac{6(n-1)}{b^4F^3}r_{00}^2 - \frac{2(n-1)}{b^4F^2}r_{00}s_0 + \frac{4(n-1)}{b^4F^2}r_{00}r_0 - \frac{n-1}{b^2F^2}r_{00|0} + \frac{n-1}{b^2}s_ks^k{}_0 \\
&+ \frac{1}{b^2}r^k{}_ks_0 - \frac{1}{b^4}rs_0 + \frac{1}{b^2}b^ks_{0|k} - \frac{1}{b^2}r_ks^k{}_0 - \frac{1}{b^2}s^kr_{0k} - s^k{}_{0|k} - \frac{F}{b^2}s^ks_k - \frac{F}{2}s^k{}_ls^l{}_k\Big] \\
&+ \frac{1}{2}F_{.i}F_{.j}\Big[\frac{18(n-1)}{b^4F^4}r_{00}^2 - \frac{8(n-1)}{b^4F^3}r_{00}r_0 + \frac{4(n-1)}{b^4F^3}r_{00}s_0 + \frac{2(n-1)}{b^2F^3}r_{00|0} \\
&- \frac{1}{b^2}s^ks_k - \frac{1}{2}s^k{}_ls^l{}_k\Big] \\
&+ \frac{1}{2}F_{.i}\Big[-\frac{24(n-1)}{b^4F^3}r_{00}r_{0j} + \frac{4(n-1)}{b^4F^2}r_{00}r_j - \frac{2(n-1)}{b^4F^2}r_{00}s_j + \frac{8(n-1)}{b^4F^2}r_0r_{0j} \\
&- \frac{4(n-1)}{b^4F^2}s_0r_{0j} - \frac{n-1}{b^2F^2}r_{00|j} - \frac{2(n-1)}{b^2F^2}r_{0j|0} + \frac{1}{b^2}r^k{}_ks_j - \frac{1}{b^2}r_ks^k{}_j + \frac{n-1}{b^2}s_ks^k{}_j \\
&- \frac{1}{b^2}s^kr_{kj} + \frac{1}{b^2}b^ks_{j|k} - \frac{1}{b^4}rs_j - s^k{}_{j|k}\Big] \\
&+ \frac{1}{2}F_{.j}\Big[-\frac{24(n-1)}{b^4F^3}r_{00}r_{0i} + \frac{4(n-1)}{b^4F^2}r_{00}r_i - \frac{2(n-1)}{b^4F^2}r_{00}s_i + \frac{8(n-1)}{b^4F^2}r_0r_{0i} \\
&- \frac{4(n-1)}{b^4F^2}s_0r_{0i} - \frac{n-1}{b^2F^2}r_{00|i} - \frac{2(n-1)}{b^2F^2}r_{0i|0} + \frac{1}{b^2}r^k{}_ks_i - \frac{1}{b^2}r_ks^k{}_i + \frac{n-1}{b^2}s_ks^k{}_i \\
&- \frac{1}{b^2}s^kr_{ki} + \frac{1}{b^2}b^ks_{i|k} - \frac{1}{b^4}rs_i - s^k{}_{i|k}\Big] + \frac{6(n-1)}{b^4F^2}(r_{00}r_{ij} + 2r_{0i}r_{0j}) \\
&+ \frac{n-1}{b^4F}\Big[b^2\Big(r_{ij|0} + r_{0i|j} + r_{0j|i}\Big) - 4\big(r_0r_{ij} + r_ir_{0j} + r_jr_{0i}\big) + 2\big(s_0r_{ij} + s_jr_{0i} + s_ir_{0j}\big)\Big] \\
&+ \frac{1}{2}\Big[-\frac{2}{b^4}rr_{ij} + \frac{2}{b^2}b^kr_{ij|k} + \frac{n-2}{b^2}\Big(s_{i|j} + s_{j|i}\Big) + \frac{2}{b^2}r^k{}_kr_{ij} - \frac{1}{b^2}\Big(r_{i|j} + r_{j|i}\Big) \\
&- \frac{2(2n-3)}{b^4}(r_is_j + r_js_i) - \frac{2(n-2)}{b^4}s_is_j + \frac{2n}{b^2}\Big(s^k{}_ir_{jk} + s^k{}_jr_{ik}\Big) + \frac{2}{b^4}r_ir_j\Big],
\end{aligned}
$$

*where ${}^{\alpha}Ric_{ij}$ denotes the Ricci curvature tensor of α.*

**Proof.** From Lemma 2, one can obtain the proposition by a direct computation. □

Using $g^{ij}$ to contract the Ricci curvature tensor, we can directly obtain the expression of the scalar curvature $R$ for Kropina metrics as follows.

**Proposition 2.** *Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric on M. Then, the scalar curvature of F is given by*

$$
\begin{aligned}
R =& -\frac{36(n-1)}{b^6F^5}r_{00}^2\beta \\
&+\frac{3(n-1)}{b^4F^4}\left[-(n-2)r_{00}^2+\frac{20}{b^2}r_{00}r_0\beta-\frac{4}{b^2}r_{00}s_0\beta-2r_{00|0}\beta\right] \\
&+\frac{1}{F^3}\left[-\frac{2}{b^2}{}^{\alpha}Ric\,\beta+\frac{2(n-1)(n-3)}{b^4}r_{00}r_0-\frac{(n-1)(n-3)}{b^4}r_{00}s_0-\frac{11n-13}{b^6}rr_{00}\beta\right. \\
&-\frac{(n-1)(n-3)}{b^2}r_{00|0}+\frac{3n-5}{b^4}r^k{}_kr_{00}\beta+\frac{2(n-1)}{b^4}r_{0k}r^k{}_0\beta-\frac{4(2n-1)}{b^4}r_{0k}s^k{}_0\beta \\
&+\frac{2(n-2)}{b^4}b^kr_{00|k}\beta-\frac{2(11n-10)}{b^6}r_0^2\beta+\frac{4(4n-5)}{b^6}r_0s_0\beta+\frac{2(2n-1)}{b^4}r_{0|0}\beta \\
&\left.+\frac{2(n-2)}{b^6}s_0^2\beta-\frac{2(n-2)}{b^4}s_{0|0}\beta\right] \\
&+\frac{1}{F^2}\left[{}^{\alpha}Ric+\frac{2}{b^2}b^iy^j\,{}^{\alpha}Ric_{ij}\,\beta+\frac{1}{b^2}r^k{}_kr_{00}-\frac{1}{b^4}rr_{00}+\frac{2n}{b^2}r_{0k}s^k{}_0+\frac{1}{b^2}b^kr_{00|k}+\frac{1}{b^4}r_0^2\right. \\
&-\frac{n-2}{b^4}s_0^2-\frac{2(2n-3)}{b^4}r_0s_0-\frac{1}{b^2}r_{0|0}+\frac{n-2}{b^2}s_{0|0}-\frac{2(n-2)}{b^4}r^k{}_kr_0\beta+\frac{n-1}{b^4}r^k{}_ks_0\beta \\
&-\frac{2n-1}{b^4}r_kr^k{}_0\beta+\frac{3n}{b^4}r_ks^k{}_0\beta-\frac{2n-3}{b^4}s_kr^k{}_0\beta+\frac{n-1}{2b^2}r^k{}_{k|0}\beta+\frac{n-1}{b^2}r^k{}_{0|k}\beta-\frac{5n-7}{b^6}rs_0\beta \\
&\left.+\frac{6(n-1)}{b^6}rr_0\beta-\frac{n-2}{b^4}s_ks^k{}_0\beta-\frac{n-2}{b^4}b^kr_{0|k}\beta+\frac{n-2}{b^4}b^ks_{0|k}\beta-\frac{n+1}{2b^4}r_{|0}\beta\right] \\
&+\frac{1}{F}\left[\frac{1}{2}{}^{\alpha}R\,\beta-\frac{1}{2b^2}b^ib^j\,{}^{\alpha}Ric_{ij}\,\beta+\frac{n+1}{2b^2}r^k{}_ks_0-\frac{n+1}{2b^2}s^kr_{0k}-\frac{n+1}{2b^2}r^ks_{0k}-\frac{n+1}{2b^4}rs_0\right. \\
&+\frac{n+1}{2b^2}b^ks_{0|k}-\frac{n+1}{2}s^k{}_{0|k}+\frac{n^2-1}{2b^2}s_ks^k{}_0+\frac{1}{2b^2}(r^k{}_k)^2\beta-\frac{1}{b^4}rr^k{}_k\beta+\frac{n}{b^2}r^l{}_ks^k{}_l\beta \\
&\left.+\frac{1}{2b^2}b^kr^l{}_{l|k}\beta+\frac{1}{b^4}r_kr^k\beta-\frac{1}{2b^2}r^k{}_{|k}\beta-\frac{n-3}{2b^4}r_ks^k\beta-\frac{n-2}{b^4}s_ks^k\beta+\frac{n-2}{2b^2}s^k{}_{|k}\beta\right] \\
&-\frac{n}{2}\left(\frac{1}{b^2}s^ks_k+\frac{1}{2}s^k{}_ls^l{}_k\right),
\end{aligned}
\tag{3}
$$

*where ${}^{\alpha}R$ denotes the scalar curvature of $\alpha$.*

This is also calculated by Zhu and Song [18], and there is a detailed calculation process.

**Lemma 3.** *Let $F=\frac{\alpha^2}{\beta}$ be a Kropina metric on $M$. Assume $F$ is of isotropic scalar curvature. Then, $\beta$ is a conformal 1-form with respect to $\alpha$.*

**Proof.** Assume the Kropina metric $F$ has isotropic scalar curvature. Then, $R=n(n-1)\kappa(x)$ holds for some scalar function $\kappa(x)$. Plugging (3) into it, we have

$$\alpha^{10}\Gamma_{10}+\alpha^8\Gamma_8+\alpha^6\Gamma_6+\alpha^4\Gamma_4+\alpha^2\Gamma_2+\Gamma_0=0, \tag{4}$$

where

$$
\begin{aligned}
\Gamma_{10} =& nb^4\left[\frac{1}{2}s^k s_k + \frac{1}{2}b^2 s^k{}_i s^i{}_k + (n-1)cb^2\right],\\
\Gamma_8 =& b^2\Big[-\frac{1}{2}b^4 R\beta + \frac{1}{2}b^2 b^i b^j\,{}^{\alpha}Ric_{ij}\beta - \frac{n+1}{2}b^2 r^k{}_k s_0 + \frac{n+1}{2}b^2 s^k r_{0k} + \frac{n+1}{2}b^2 r_k s^k{}_0 - r_k r^k\beta\\
&+ \frac{n+1}{2}rs_0 - \frac{n+1}{2}b^2 b^k s_{0|k} + \frac{n+1}{2}b^4 s^k{}_{0|k} - \frac{n^2-1}{2}b^2 s_k s^k{}_0 - \frac{1}{2}b^2(r^k{}_k)^2\beta + rr^k{}_k\beta\\
&- nb^2 r^i{}_k s^k{}_i\beta - \frac{1}{2}b^2 b^k r^i{}_{i|k}\beta + \frac{n-3}{2}r_k s^k\beta + \frac{1}{2}b^2 r^k{}_{|k}\beta + (n-2)s^k s_k\beta - \frac{n-2}{2}b^2 s^k{}_{|k}\beta\Big]\beta,\\
\Gamma_6 =& \Big[-b^6\,{}^{\alpha}Ric - 2b^4 b^i y^j\,{}^{\alpha}Ric_{ij}\beta - b^4 r^k{}_k r_{00} + b^2 rr_{00} - 2nb^4 r_{0k}s^k{}_0 - b^4 b^k r_{00|k} - b^2 r_0^2\\
&+ (n-2)b^2 s_0^2 + 2(2n-3)b^2 r_0 s_0 + b^4 r_{0|0} - (n-2)b^4 s_{0|0} + 2(n-2)b^2 r^k{}_k r_0\beta\\
&- (n-1)b^2 r^k{}_k s_0\beta - \frac{(n-1)}{2}b^4 r^k{}_{k|0}\beta + (2n-1)b^2 r_k r^k{}_0\beta + (2n-3)b^2 s_k r^k{}_0\beta\\
&- (n-1)b^4 r^k{}_{0|k}\beta - 3nb^2 r_k s^k{}_0\beta - 6(n-1)rr_0\beta + (5n-7)rs_0\beta + (n-2)b^2 s_k s^k{}_0\beta\\
&+ (n-2)b^2 b^k r_{0|k}\beta - (n-2)b^2 b^k s_{0|k}\beta + \frac{(n+1)}{2}b^2 r_{|0}\beta\Big]\beta^2,\\
\Gamma_4 =& \Big[2b^4\,{}^{\alpha}Ric\,\beta - 2(n-1)(n-3)b^2 r_{00}r_0 + (n-1)(n-3)b^2 r_{00}s_0 + (11n-13)rr_{00}\beta\\
&+ (n-1)(n-3)b^4 r_{00|0} - (3n-5)b^2 r^k{}_k r_{00}\beta - 2(n-1)b^2 r_{0k}r^k{}_0\beta - 2(2n-1)b^2 r_{0|0}\beta\\
&- 2(n-2)s_0^2\beta - 2(n-2)b^2 b^k r_{00|k}\beta + 2(11n-10)r_0^2\beta - 4(4n-5)r_0 s_0\beta\\
&+ 4(2n-1)b^2 r_{0k}s^k{}_0\beta + 2(n-2)b^2 s_{0|0}\beta\Big]\beta^3,\\
\Gamma_2 =& 3(n-1)\Big[(n-2)b^2 r_{00}^2 - 20r_{00}r_0\beta + 4r_{00}s_0\beta + 2b^2 r_{00|0}\beta\Big]\beta^4,\\
\Gamma_0 =& 36(n-1)r_{00}^2\beta^6.
\end{aligned}
$$

By (4), $\Gamma_0$ can be divided by $\alpha^2$. Hence, $r_{00} = f(x)\alpha^2$ holds for some scalar function $f = f(x)$. It means that $\beta$ is a conformal 1-form with respect to $\alpha$. This completes the proof of Lemma 3. □

## 4. Isotropic Scalar Curvature via the Navigation Data

In this section, we study Kropina metrics by the navigation data. We obtain an equivalent characterization for a Kropina metric with isotropic scalar curvature.

Let $F$ and $\bar{F}$ be two Finsler metrics on $M$. If

$$\bar{F}(x,y) = e^{\sigma(x)}F(x,y)$$

holds for some smooth function $\sigma(x)$ on $M$, then $F$ and $\bar{F}$ are said to be locally conformally related. And we call the smooth function $\sigma(x)$ the conformal factor.

For conformally related Finsler metrics, B*á*cs*ó* and Cheng [21] gave some transformation conclusions. Here are some related results.

**Proposition 3.** *Let $\bar{F} = 2F$ be two Finsler metrics on $M$. Then*
*(1) $\bar{g}_{ij} = 4g_{ij}$, $\bar{g}^{ij} = \frac{1}{4}g^{ij}$, $\bar{G}^i = G^i$, $\overline{Ric} = Ric$, $\overline{Ric}_{ij} = Ric_{ij}$ and $\bar{R} = \frac{1}{4}R$;*
*(2) $F$ is an Einstein metric in the same way that $\bar{F}$ is also an Einstein metric;*
*(3) $F$ is of isotropic scalar curvature if and only if $\bar{F}$ is of isotropic scalar curvature.*

**Lemma 4** ([12])**.** *Let $F$ be an $(\alpha,\beta)$-metric on $M$. Assume that $\alpha$ is an Einstein metric and $\beta$ is a conformal 1-form with respect to $\alpha$, i.e., ${}^{\alpha}Ric = g\alpha^2,\ r_{00} = f\alpha^2$ for some scalar functions $g = g(x)$ and $f = f(x)$. Then the following hold:*

$$\begin{cases} s^k{}_{0|k} = (n-1)f_0 + g\beta, \\ \quad 0 = (n-1)b^k f_k + gb^2 + s^k{}_{|k} + s^k{}_j s^j{}_k, \end{cases}$$

*where $f_k := \frac{\partial f}{\partial x^k}$ and $f_0 := f_k y^k$.*

To prove the main theorem, we shall express the Kropina metric $F = \frac{\alpha^2}{\beta}$ using the navigation data $(\bar{\alpha},\bar{\beta})$ as Formula (2). Let $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$. Then, $\bar{F} = 2F$.

**Theorem 3.** *Let $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ be a Kropina metric on $M$. Assume $(\bar{\alpha},\bar{\beta})$ is the navigation data of the Kropina metric $F = \frac{\alpha^2}{\beta}$. Then, $\bar{F}$ is of isotropic scalar curvature if and only if $\bar{\alpha}$ is an Einstein metric and $\bar{\beta}$ is a constant Killing 1-form with respect to $\bar{\alpha}$.*

**Proof.** *Necessity.* Assume $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ has isotropic scalar curvature. Then, $\bar{R} = n(n-1)\bar{\kappa}(x)$ holds for some scalar function $\bar{\kappa}(x)$. By Proposition 3, $F = \frac{1}{2}\bar{F}$ must be also of isotropic scalar curvature. Thus, $\beta$ is a conformal 1-form with respect to $\alpha$ by Lemma 3. Furthermore, based on Note 1, Note 2, and Note 3, $\bar{\beta}$ is a constant Killing 1-form, namely, $\bar{r}_{00} = 0$ and $\bar{s}_k = 0$. Substituting $\bar{r}_{00} = 0$, $\bar{s}_k = 0$ into (3), we obtain

$$\begin{aligned}\bar{R} = &-\frac{\bar{\beta}^4}{2\bar{\alpha}^6}\,{}^{\bar{\alpha}}Ric + \frac{\bar{\beta}^2}{4\bar{\alpha}^4}\left({}^{\bar{\alpha}}Ric + 2\bar{b}^i\bar{y}^j\,{}^{\bar{\alpha}}Ric_{ij}\bar{\beta}\right) + \frac{\bar{\beta}}{8\bar{\alpha}^2}\left[{}^{\bar{\alpha}}R\bar{\beta} - \bar{b}^i\bar{b}^j\,{}^{\bar{\alpha}}Ric_{ij}\bar{\beta} - (n+1)\bar{s}^k{}_{0\|k}\right] \\ &- \frac{n}{16}\bar{s}^k{}_i\,\bar{s}^i{}_k, \end{aligned} \tag{5}$$

where " $\|$ " denotes the covariant derivative with respect to the Levi-Civita connection of $\bar{\alpha}$. Multiplying both sides of (5) by $8\bar{\alpha}^6$ yields

$$\begin{aligned}0 = &n\left[\frac{1}{2}\bar{s}^k{}_i\,\bar{s}^i{}_k + 8(n-1)\bar{\kappa}\right]\bar{\alpha}^6 - \left[{}^{\bar{\alpha}}R\bar{\beta} - b^i b^j\,{}^{\bar{\alpha}}Ric_{ij}\bar{\beta} - (n+1)\bar{s}^k{}_{0\|k}\right]\bar{\beta}\bar{\alpha}^4 \\ &- 2\left({}^{\bar{\alpha}}Ric + 2b^i y^j\,{}^{\bar{\alpha}}Ric_{ij}\bar{\beta}\right)\bar{\beta}^2\bar{\alpha}^2 + 4\,{}^{\bar{\alpha}}Ric\bar{\beta}^4. \end{aligned} \tag{6}$$

The above equation shows that $\bar{\alpha}^2$ can divide ${}^{\bar{\alpha}}Ric$. Thus, there exists a scalar function $\bar{g}(x)$ such that

$${}^{\bar{\alpha}}Ric = \bar{g}(x)\bar{\alpha}^2,$$

which means that $\bar{\alpha}$ is an Einstein metric. Thus, ${}^{\bar{\alpha}}Ric_{ij} = \bar{g}\bar{a}_{ij}$ and ${}^{\bar{\alpha}}R = n\bar{g}$. Since ${}^{\bar{\alpha}}Ric = g\bar{\alpha}^2$ and $\bar{r}_{00} = 0$ hold, we have $\bar{s}^k{}_{0\|k} = \bar{g}\bar{\beta}$ and $\bar{s}^k{}_j\bar{s}^j{}_k = -\bar{g}$ by Lemma 4. Substituting all of these into (6) yields

$$0 = n\left[-\frac{1}{2}\bar{g} + 8(n-1)\bar{\kappa}\right]\bar{\alpha}^4.$$

It implies that $\bar{\kappa} = \frac{\bar{g}}{16(n-1)}$.

*Sufficiency.* Suppose that $\bar{r}_{00} = 0$, $\bar{s}_k = 0$ and ${}^{\bar{\alpha}}Ric = \bar{g}\bar{\alpha}^2$ hold. Then we have $\bar{s}^k{}_{0\|k} = \bar{g}\bar{\beta}$ and $\bar{s}^k{}_j\bar{s}^j{}_k = -\bar{g}$ by Lemma 4. Plugging them into (3) yields $\bar{R} = \frac{1}{16}n\bar{g}$, which means that $\bar{F}$ is of isotropic scalar curvature. It completes the proof of Theorem 3. □

The following lemma is necessary for the proof of the main theorem.

**Lemma 5** ([4])**.** *Let $F = \frac{\alpha^2}{\beta}$ be a non-Riemannian Kropina metric with constant Killing 1-form $\beta$ on $M$. Then, $F$ is an Einstein metric if and only if $\alpha$ is also an Einstein metric. In this case, $\sigma = \frac{1}{4}\lambda b^2 \geq 0$, where $\sigma = \sigma(x)$ and $\lambda = \lambda(x)$ are Einstein scalars of $F$ and $\alpha$, respectively. Moreover, $F$ is Ricci-constant when $n \geq 3$.*

Now we are in the position to give a proof of Theorem 1.

**Proof of Theorem 1.** Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric on a manifold $M$ given by (2) with the navigation data $(\bar{\alpha}, \bar{\beta})$. We claim that $\bar{F}$ is of isotropic scalar curvature if and only if $\bar{F}$ is an Einstein metric. Let $(\bar{\alpha}, \bar{\beta})$ be the navigation data of the Kropina metric $F = \frac{\alpha^2}{\beta}$. Suppose $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ is of isotropic scalar curvature. Then, we have that $\bar{\alpha}$ is an Einstein metric and $\bar{\beta}$ is a constant Killing 1-form by Theorem 3. Hence, $\bar{F}$ is an Einstein metric, as stated in Lemma 5. Conversely, if a Finsler metric $\bar{F}$ is an Einstein metric, then $\bar{F}$ is of isotropic scalar curvature by the definition of scalar curvature.

Since $F = \frac{1}{2}\bar{F}$, we conclude that $F$ is of isotropic scalar curvature in the same way that $F$ is an Einstein metric by Proposition 3. This completes the proof of Theorem 1. □

Let $(M, F)$ be a Finsler manifold. Express the volume form of $F$ by $dV_F = \sigma(x)dx^1 \dots dx^n$. For a non-zero vector $y \in T_pM$, the $S$-curvature $S(y)$ is defined by

$$S(x,y) := \frac{\partial G^i}{\partial y^i}(x,y) - y^i \frac{\partial \ln \sigma(x)}{\partial x^i}.$$

And the non-Riemannian quantity $\chi$-curvature on the tangent bundle $TM$ is defined by

$$\chi_i := S_{\cdot i;m} y^m - S_{;i},$$

where " ; " and " . " denote the horizontal and vertical covariant derivatives with respect to the Chern connection, respectively. Further, $H$-curvature can be expressed by

$$H_{ij} := \frac{1}{4}(\chi_{i\cdot j} + \chi_{j\cdot i}).$$

The notion of projective Ricci curvature $PRic$ is given by

$$PRic := Ric + \frac{n-1}{n+1} S_{;m} y^m + \frac{n-1}{(n+1)^2} S^2.$$

**Lemma 6** ([5])**.** *For a Kropina metric $F$ on $M$, the following are equivalent: (1) $F$ has isotropic $S$-curvature; (2) $S$-curvature vanishes; and (3) $\beta$ is a conformal 1-form with respect to $\alpha$.*

**Proposition 4.** *Let $F$ be a Kropina metric on $M$. If $F$ is of isotropic scalar curvature, then (1) $S$-curvature vanishes; (2) $\chi$-curvature vanishes; (3) $H$-curvature vanishes; and (4) $PRic = Ric$.*

**Proof.** Assume a Kropina metric $F = \frac{\alpha^2}{\beta}$ is of isotropic scalar curvature. We have that $\beta$ is a conformal 1-form with respect to $\alpha$ by Lemma 3. Thus, $S = 0$ by Lemma 6. By definitions of $\chi$-curvature, $H$-curvature, and projective Ricci curvature, we have $\chi = 0$, $H = 0$, and $PRic = Ric$. This completes the proof. □

## 5. Conclusions

In this paper, we discuss Kropina metrics with isotropic scalar curvature $R$. We present expressions of Ricci curvature tensor $Ric_{ij}$ and scalar curvature of Kropina metrics. Using the navigation method, we conclude that Kropina metrics are of isotropic scalar curvature if and only if they are Einstein metrics in Theorem 1.

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