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# Chern Flat and Chern Ricci-Flat Twisted Product Hermitian Manifolds

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**Abstract:** Let $(M_1, g)$ and $(M_2, h)$ be two Hermitian manifolds. The twisted product Hermitian manifold $(M_1 \times {}_f M_2, G)$ is the product manifold $M_1 \times M_2$ endowed with the Hermitian metric $G = g + f^2 h$, where $f$ is a positive smooth function on $M_1 \times M_2$. In this paper, the Chern curvature, Chern Ricci curvature, Chern Ricci scalar curvature and holomorphic sectional curvature of the twisted product Hermitian manifold are derived. The necessary and sufficient conditions for the compact twisted product Hermitian manifold to have constant holomorphic sectional curvature are obtained. Under the condition that the logarithm of the twisted function is pluriharmonic, it is proved that the twisted product Hermitian manifold is Chern flat or Chern Ricci-flat, if and only if $(M_1, g)$ and $(M_2, h)$ are Chern flat or Chern Ricci-flat, respectively.

**Keywords:** Hermitian manifold; twisted product; holomorphic sectional curvature; Chern flat; Chern Ricci-flat

**MSC:** 53C55

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## 1. Introduction

Warped product and twisted product are important methods used to construct new classes of geometric spaces, and these models are widely applied in theoretical physics. In 1969, warped product was firstly introduced by O′Neill and Bishop to construct Riemannian manifolds with negative sectional curvature [1]. In 2001, Kozma, Peter and Varga [2] extended the warped product to real Finsler manifolds. Asanov [3,4] obtained some models of relativity theory by studying the warped product Finsler metric. In 2018, the notion of warped product was extended to Hermitian geometry by the work of He and Zhang [5], and they obtained the necessary and sufficient conditions for the compact nontrivial doubly warped product (abbreviated as DWP) Hermitian manifold to have constant holomorphic sectional curvature.

The notion of twisted product, as a generalization of warped product, was first introduced by Chen [6]. In 1993, Ponge and Reckziegel [7] extended twisted product to pseudo-Riemannian manifolds. Then, Fernández-López showed that a mixed Ricci-flat twisted product semi-Riemannian manifold can be expressed as a warped product semi-Riemannian manifold [8]. In 2017, Kazan and Sahin [9] deeply investigated the twisted product and multiply twisted product semi-Riemannian manifolds, which further promoted the development of twisted product in Riemannian geometry. Kozma, Peter and Shimada [10] extended the twisted product to real Finsler manifolds and studied some geometric properties relating to Cartan connection, geodesic and completeness. Recently, Xiao and He [11] extended the twisted product to complex Finsler manifolds and gave the formulae of holomorphic curvature and Ricci scalar curvature of the doubly twisted product (abbreviated as DTP) complex Finsler manifold. In light of the above results, we shall extend the twisted product to Hermitian manifold, and attempt to derive the Chern curvature, Chern Ricci curvature, Chern Ricci scalar curvature and holomorphic sectional curvature of the twisted product Hermitian manifold. In addition, we intend to find the necessary and sufficient conditions for the compact Hermitian manifold to have constant holomorphic sectional curvature.

One of the most important problems in geometry is to characterize Chern flat or Chern Ricci-flat manifolds. In 1967, Tani [12] firstly gave the definition of Ricci-flat space in Riemannian geometry. Later, Bando and Kobayashi [13] constructed Ricci-flat metrics on Einstein–Kähler manifolds. Liu and Yang [14] obtained the sufficient and necessary conditions for the Hopf manifold to be Levi-Civita Ricci-flat. Recently, Ni and He [15] gave the necessary and sufficient conditions for DWP-Hermitian manifold to be Levi-Civita Ricci-flat. In 2012, Di Scala [16] showed that quasi-Kähler Chern flat almost Hermitian structures on compact manifolds correspond to complex parallelizable Hermitian structures satisfying the second Gray identity. Wu and Zheng [17] proved that the compact Hermitian manifold with complex dimension 3, having vanishing real bisectional curvature, must be Chern flat. Based on the above mentioned studies, we are interested in the condition under which the twisted product Hermitian manifold is Chern flat or Chern Ricci-flat.

The structure of this paper is as follows. In Section 2, we briefly recall some basic concepts of Hermitian geometry and related symbolic conventions. In Section 3, we shall extend the concept of twisted product to Hermitian geometry, and derive the Chern connection coefficients of a twisted product Hermitian manifold. In Section 4, we shall give the formulae of Chern curvature, Chern Ricci curvature and Chern Ricci scalar curvature of the twisted product Hermitian manifold. In Section 5, we focus on investigating the twisted product Hermitian manifold with constant holomorphic sectional curvature. In Section 6, under the condition that the logarithm of the twisted function is pluriharmonic, we shall show that the twisted product Hermitian manifold is Chern flat or Chern Ricci-flat if and only if $(M_1, g)$ and $(M_2, h)$ are Chern flat or Chern Ricci-flat, respectively.

## 2. Preliminary

In this section, we briefly introduce the definitions and notations which we need in this paper.

Let $(M, J, G)$ be a n-dimensional Hermitian manifold with complex structure $J$ and Hermitian metric $G$. Let $T^{\mathbb{C}}M$ denote the complexified tangent bundle of $M$, which can be decomposed as

$$T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M,$$

where $T^{1,0}M$ and $T^{0,1}M$ are eigenspaces of $J$ corresponding to eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively.

Let $z = (z^1, \cdots, z^n)$ denote the local holomorphic coordinates on $M$, then vector fields $\{\partial_\alpha\}$ and $\{\partial_{\bar{\alpha}}\}$ form the basis of $T^{1,0}M$ and $T^{0,1}M$, respectively, where $\partial_\alpha = \frac{\partial}{\partial z^\alpha}, \partial_{\bar{\alpha}} = \frac{\partial}{\partial \bar{z}^\alpha}$. On the Hermitian holomorphic tangent bundle $T^{1,0}M$, the coefficients of Chern connection $\nabla$ are [18]

$$\Gamma^\beta_{\gamma\alpha} = G^{\beta\overline{\sigma}}\partial_\gamma G_{\alpha\overline{\sigma}}, \tag{1}$$

and their complex conjugate.

**Definition 1** ([18])**.** *Let $\nabla$ be the Chern connection, its Chern curvature tensor $K$ on the Hermitian manifold $(M, J, G)$ is defined by*

$$K = K_{\alpha\overline{\beta}\gamma\overline{\sigma}} dz^\alpha \otimes d\bar{z}^\beta \otimes dz^\gamma \otimes d\bar{z}^\sigma, \tag{2}$$

*where*

$$K_{\alpha\overline{\beta}\gamma\overline{\sigma}} = -G_{\epsilon\overline{\beta}} K^\epsilon_{\alpha\gamma\overline{\sigma}}, \tag{3}$$
$$K^\epsilon_{\alpha\gamma\overline{\sigma}} = -\partial_{\overline{\sigma}}\Gamma^\epsilon_{\gamma\alpha}. \tag{4}$$

**Definition 2** ([14])**.** *The first and the second Chern Ricci curvature on the Hermitian manifold $(M, J, G)$ are defined by*

$$K^{(1)} = -\sqrt{-1}K^{(1)}_{\alpha\overline{\beta}}dz^{\alpha} \wedge d\overline{z}^{\beta}, \tag{5}$$

$$K^{(2)} = -\sqrt{-1}K^{(2)}_{\alpha\overline{\beta}}dz^{\alpha} \wedge d\overline{z}^{\beta},$$

*respectively, where*

$$K^{(1)}_{\alpha\overline{\beta}} = G^{\gamma\overline{\sigma}}K_{\alpha\overline{\beta}\gamma\overline{\sigma}}, \tag{6}$$

$$K^{(2)}_{\alpha\overline{\beta}} = G^{\gamma\overline{\sigma}}K_{\gamma\overline{\sigma}\alpha\overline{\beta}}. \tag{7}$$

**Definition 3** ([14])**.** *The Chern Ricci scalar curvature on the Hermitian manifold $(M, J, G)$ is defined by*

$$S_G = G^{\alpha\overline{\beta}}K^{(1)}_{\alpha\overline{\beta}} = G^{\alpha\overline{\beta}}K^{(2)}_{\alpha\overline{\beta}}. \tag{8}$$

For research purposes, we introduce the following two definitions.

**Definition 4** ([19])**.** *Let $D$ be open in $C^n$. A function $f \in C^2(D)$ is said to be pluriharmonic if it satisfies the differential equations*

$$\frac{\partial^2 f}{\partial z^{\alpha}\partial\overline{z}^{\beta}} = 0. \tag{9}$$

**Definition 5** ([20])**.** *The complex Laplace operator*

$$L = G^{\overline{\beta}\alpha}\frac{\partial^2}{\partial z^{\alpha}\partial\overline{z}^{\beta}} \tag{10}$$

*is a second-order elliptic partial differential operator with smooth coefficients.*

Clearly, if $f$ is a pluriharmonic function, then $L(f) = 0$.

## 3. Twisted Product Hermitian Manifold

Let $(M_1, g)$ and $(M_2, h)$ be two Hermitian manifolds with $dim_{\mathbb{C}}M_1 = m$ and $dim_{\mathbb{C}}M_2 = n$, respectively, then $M = M_1 \times M_2$ is a Hermitian manifold with $dim_{\mathbb{C}}M = m + n$.

We denote $z_1 = (z^1, \ldots, z^m) \in M_1$ and $z_2 = (z^{m+1}, \ldots, z^{m+n}) \in M_2$, so $z = (z_1, z_2) \in M$. Let $\pi_1 : M_1 \times M_2 \to M_1, \pi_2 : M_1 \times M_2 \to M_2$ be the natural projection maps, then $\pi_1(z) = z_1, \pi_2(z) = z_2$.

Let $T^{1,0}M_1$ and $T^{1,0}M_2$ be the holomorphic tangent bundle of $M_1$ and $M_2$, respectively. Denote $v_1 = (v^1, \cdots, v^m) \in T^{1,0}M_1$ and $v_2 = (v^{m+1}, \cdots, v^{m+n}) \in T^{1,0}M_2$, then $v = (v_1, v_2) \in T^{1,0}M$. Let $d\pi_1 : T^{1,0}(M_1 \times M_2) \to T^{1,0}M_1, d\pi_2 : T^{1,0}(M_1 \times M_2) \to T^{1,0}M_2$ be the holomorphic tangent maps induced by $\pi_1$ and $\pi_2$, then $d\pi_1(z, v) = (z_1, v_1), d\pi_2(z, v) = (z_2, v_2)$, where $z$ is called the base coordinates (or points) on $M$ and $v$ is called the fiber coordinates (or tangent directions).

For the reader′s convenience, the lowercase Greek indices like $\alpha, \beta, \gamma, \cdots$ run from 1 to $m + n$, the lowercase Latin indices like $i, j, k, s, t, \cdots$ run from 1 to m, while the lowercase Latin indices with a prime like $i', j', k', s', t', \cdots$ run from $m + 1$ to $m + n$. Quantities associated with $(M_1, g)$ and $(M_2, h)$ are denoted with upper indices 1 and 2, respectively; for example, $\overset{1}{\Gamma}{}^{i}_{jk}, \overset{2}{\Gamma}{}^{i'}_{j'k'}$ are Chern connection coefficients of $(M_1, g)$ and $(M_2, h)$, respectively. In the following, we use the Einstein summation convention.

**Definition 6.** *Let $(M_1, g)$ and $(M_2, h)$ be two Hermitian manifolds. Let $f : M_1 \times M_2 \to (0, +\infty)$ be a positive smooth function. The twisted product Hermitian manifold $(M_1 \times {}_f M_2, G)$ is the product manifold $M = M_1 \times M_2$ endowed with the Hermitian metric $G : TM \to (0, +\infty)$:*

$$G(z, v) = g(\pi_1(z), d\pi_1(v)) + f^2 h(\pi_2(z), d\pi_2(v)), \tag{11}$$

*for $z = (z_1, z_2) \in M$ and $v = (v_1, v_2) \in T^{1,0}M$. The function $f$ is called the twisted function and $G$ is called the twisted product Hermitian metric for simplicity.*

In particular, if $f$ only depends on $M_1$, then $(M_1 \times {}_f M_2, G)$ is a warped product Hermitian manifold. If $f$ only depends on $M_2$, then $(M_1 \times {}_f M_2, G)$ is the product Hermitian manifold.

Denote

$$g_{i\bar{j}} = \frac{\partial^2 g}{\partial v^i \partial \overline{v}^j}, \quad h_{i'\overline{j'}} = \frac{\partial^2 h}{\partial v^{i'} \partial \overline{v}^{j'}}. \tag{12}$$

Then, the fundamental tensor matrix $(G_{\alpha\overline{\beta}})$ of $G$ has the following forms

$$(G_{\alpha\overline{\beta}}) = \begin{pmatrix} g_{i\bar{j}} & 0 \\ 0 & f^2 h_{i'\overline{j'}} \end{pmatrix}, \tag{13}$$

its inverse matrix $(G^{\bar{\beta}\alpha})$ is also given by

$$(G^{\overline{\beta}\alpha}) = \begin{pmatrix} g^{\bar{j}i} & 0 \\ 0 & f^{-2} h^{\overline{j'}i'} \end{pmatrix}. \tag{14}$$

**Proposition 1.** *Let $(M_1 \times {}_f M_2, G)$ be a twisted product Hermitian manifold. Then, the Chern connection coefficients associated with $G$ are given by*

$$\Gamma^i_{jk} = \overset{1}{\Gamma}{}^i_{jk}, \quad \Gamma^{i'}_{j'k'} = 2f^{-1}\delta^{i'}_{k'}\partial_{j'} f + \overset{2}{\Gamma}{}^{i'}_{j'k'}, \tag{15}$$

$$\Gamma^{i'}_{jk'} = 2f^{-1}\delta^{i'}_{k'}\partial_j f, \quad \Gamma^i_{j'k} = \Gamma^i_{jk'} = \Gamma^i_{j'k'} = \Gamma^{i'}_{jk} = \Gamma^{i'}_{j'k} = 0. \tag{16}$$

**Proof.** By putting $\alpha = k', \beta = i', \gamma = j'$ in (1), we have

$$\Gamma^{i'}_{j'k'} = G^{i'\overline{\sigma}}\partial_{j'} G_{k'\overline{\sigma}} = G^{i'\bar{s}}\partial_{j'} G_{k'\bar{s}} + G^{i'\overline{s'}}\partial_{j'} G_{k'\overline{s'}}. \tag{17}$$

Plunging (13) and (14) into (17), we can obtain

$$\begin{aligned} \Gamma^{i'}_{j'k'} &= f^{-2} h^{\overline{s'}i'}\partial_{j'}(f^2 h_{k'\overline{s'}}) \\ &= 2f^{-1}\delta^{i'}_{k'}\partial_{j'} f + h^{\overline{s'}i'}\partial_{j'} h_{k'\overline{s'}} \\ &= 2f^{-1}\delta^{i'}_{k'}\partial_{j'} f + \overset{2}{\Gamma}{}^{i'}_{j'k'}. \end{aligned}$$

Similarly, the other equalities of Proposition 1 can be deduced. □

## 4. Curvatures of Twisted Product Hermitian Manifold

In this section, we shall derive the Chern curvature, Chern Ricci curvature and Chern Ricci scalar curvature of the twisted product Hermitian manifold.

**Proposition 2.** *Let* $(M_1 \times {}_f M_2, G)$ *be a twisted product Hermitian manifold. Then, the coefficients of Chern curvature tensor* $K^{\epsilon}_{\alpha\gamma\bar{\sigma}}$ *are given by*

$$K^t_{kj\bar{s}} = \overset{1}{K}{}^t_{kj\bar{s}}, \quad K^{t'}_{k'j'\bar{s}'} = -2\frac{\partial^2 \ln f}{\partial z^{j'}\partial \bar{z}^{s'}}\delta^{t'}_{k'} + \overset{2}{K}{}^{t'}_{k'j'\bar{s}'}, \tag{18}$$

$$K^{t'}_{k'j\bar{s}} = -2\frac{\partial^2 \ln f}{\partial z^{j}\partial \bar{z}^{s}}\delta^{t'}_{k'}, \quad K^{t'}_{k'j'\bar{s}} = -2\frac{\partial^2 \ln f}{\partial z^{j'}\partial \bar{z}^{s}}\delta^{t'}_{k'}, \quad K^{t'}_{k'j\bar{s}'} = -2\frac{\partial^2 \ln f}{\partial z^{j}\partial \bar{z}^{s'}}\delta^{t'}_{k'}, \tag{19}$$

$$K^t_{k'j\bar{s}} = K^t_{kj'\bar{s}} = K^t_{kj\bar{s}'} = K^t_{k'j'\bar{s}} = K^t_{k'j\bar{s}'} = K^t_{kj'\bar{s}'} = K^t_{k'j'\bar{s}'} = 0, \tag{20}$$

$$K^{t'}_{kj\bar{s}} = K^{t'}_{kj'\bar{s}} = K^{t'}_{kj\bar{s}'} = K^{t'}_{kj'\bar{s}'} = 0. \tag{21}$$

**Proof.** By putting $\alpha = k', \gamma = j', \sigma = s', \epsilon = t'$ in (4), we have

$$K^{t'}_{k'j'\bar{s}'} = -\partial_{\bar{s}'}\Gamma^{t'}_{j'k'}. \tag{22}$$

Substituting the second equality of (15) into (22), and using (4), we have

$$\begin{aligned} K^{t'}_{k'j'\bar{s}'} &= -\partial_{\bar{s}'}\big(2f^{-1}\delta^{t'}_{k'}\partial_{j'}f + \overset{2}{\Gamma}{}^{t'}_{j'k'}\big) \\ &= 2f^{-2}\delta^{t'}_{k'}(\partial_{\bar{s}'}f)(\partial_{j'}f) - 2f^{-1}\frac{\partial^2 f}{\partial z^{j'}\partial \bar{z}^{s'}}\delta^{t'}_{k'} - \partial_{\bar{s}'}\overset{2}{\Gamma}{}^{t'}_{j'k'} \\ &= -2\frac{\partial^2 \ln f}{\partial z^{j'}\partial \bar{z}^{s'}}\delta^{t'}_{k'} + \overset{2}{K}{}^{t'}_{k'j'\bar{s}'}. \end{aligned}$$

Similarly, we can obtain other equalities of Proposition 2. □

**Proposition 3.** *Let* $(M_1 \times {}_f M_2, G)$ *be a twisted product Hermitian manifold. Then,*

$$K_{k\bar{i}j\bar{s}} = \overset{1}{K}_{k\bar{i}j\bar{s}}, \tag{23}$$

$$K_{k'\bar{i}'j\bar{s}} = 2f^2\frac{\partial^2 \ln f}{\partial z^{j}\partial \bar{z}^{s}}h_{k'\bar{i}'}, \tag{24}$$

$$K_{k'\bar{i}'j'\bar{s}} = 2f^2\frac{\partial^2 \ln f}{\partial z^{j'}\partial \bar{z}^{s}}h_{k'\bar{i}'}, \tag{25}$$

$$K_{k'\bar{i}'j\bar{s}'} = 2f^2\frac{\partial^2 \ln f}{\partial z^{j}\partial \bar{z}^{s'}}h_{k'\bar{i}'}, \tag{26}$$

$$K_{k'\bar{i}'j'\bar{s}'} = 2f^2\frac{\partial^2 \ln f}{\partial z^{j'}\partial \bar{z}^{s'}}h_{k'\bar{i}'} + f^2\overset{2}{K}_{k'\bar{i}'j'\bar{s}'}, \tag{27}$$

$$K_{k'\bar{i}j\bar{s}} = K_{k\bar{i}'j\bar{s}} = K_{k\bar{i}j'\bar{s}} = K_{k\bar{i}j\bar{s}'} = K_{k'\bar{i}j'\bar{s}'} = K_{k\bar{i}'j'\bar{s}'} = 0, \tag{28}$$

$$K_{k'\bar{i}j'\bar{s}} = K_{k'\bar{i}j\bar{s}'} = K_{k\bar{i}'j'\bar{s}} = K_{k\bar{i}'j\bar{s}'} = K_{k\bar{i}j'\bar{s}'} = 0. \tag{29}$$

**Proof.** By putting $\alpha = k', \beta = i', \gamma = j', \sigma = s'$ in (3), we have

$$K_{k'\bar{i}'j'\bar{s}'} = -G_{\epsilon\bar{i}'}K^{\epsilon}_{k'j'\bar{s}'} = -G_{t\bar{i}'}K^t_{k'j'\bar{s}'} - G_{t'\bar{i}'}K^{t'}_{k'j'\bar{s}'}. \tag{30}$$

Plunging (13) and the second equality of (18) into (30), a trivial caculation yields

$$
\begin{aligned}
K_{k'\bar{i}'j'\bar{s}'} &= -f^2 h_{t'\bar{i}'}\Big(-2\frac{\partial^2 \ln f}{\partial z^{j'}\partial \bar{z}^{s'}}\delta^{t'}_{k'} + \overset{2}{K}{}^{t'}_{k'j'\bar{s}'}\Big) \\
&= 2f^2\frac{\partial^2 \ln f}{\partial z^{j'}\partial \bar{z}^{s'}} h_{k'\bar{i}'} - f^2 h_{t'\bar{i}'}\overset{2}{K}{}^{t'}_{k'j'\bar{s}'} \\
&= 2f^2\frac{\partial^2 \ln f}{\partial z^{j'}\partial \bar{z}^{s'}} h_{k'\bar{i}'} + f^2 \overset{2}{K}_{k'\bar{i}'j'\bar{s}'}.
\end{aligned}
$$

Similarly, we can obtain other equalities of Proposition 3. □

**Proposition 4.** *Let $(M_1 \times {}_f M_2, G)$ be a twisted product Hermitian manifold. Then, the coefficients of the first and the second Chern Ricci curvature tensor are given by*

$$
K^{(1)}_{k\bar{i}} = \overset{1}{K}{}^{(1)}_{k\bar{i}}, \qquad K^{(1)}_{k'\bar{i}} = K^{(1)}_{k\bar{i}'} = 0, \tag{31}
$$

$$
K^{(1)}_{k'\bar{i}'} = 2f^2 \overset{1}{L}(\ln f) h_{k'\bar{i}'} + 2\overset{2}{L}(\ln f) h_{k'\bar{i}'} + \overset{2}{K}{}^{(1)}_{k'\bar{i}'}, \tag{32}
$$

*and*

$$
K^{(2)}_{k\bar{i}} = \overset{1}{K}{}^{(2)}_{k\bar{i}} + 2\frac{\partial^2 \ln f}{\partial z^{k}\partial \bar{z}^{i}}, \quad K^{(2)}_{k'\bar{i}} = 2\frac{\partial^2 \ln f}{\partial z^{k'}\partial \bar{z}^{i}}, \tag{33}
$$

$$
K^{(2)}_{k'\bar{i}'} = 2\frac{\partial^2 \ln f}{\partial z^{k'}\partial \bar{z}^{i'}} + \overset{2}{K}{}^{(2)}_{k'\bar{i}'}, \quad K^{(2)}_{k\bar{i}'} = 2\frac{\partial^2 \ln f}{\partial z^{k}\partial \bar{z}^{i'}}. \tag{34}
$$

**Proof.** Letting $\alpha = k', \beta = i'$ in (6), we have

$$
K^{(1)}_{k'\bar{i}'} = G^{\gamma\bar{\sigma}} K_{k'\bar{i}'\gamma\bar{\sigma}} = G^{j\bar{s}} K_{k'\bar{i}'j\bar{s}} + G^{j'\bar{s}} K_{k'\bar{i}'j'\bar{s}} + G^{j\bar{s}'} K_{k'\bar{i}'j\bar{s}'} + G^{j'\bar{s}'} K_{k'\bar{i}'j'\bar{s}'}. \tag{35}
$$

Substituting (14), (24) and (27) into (35), and noticing that (10), we can obtain

$$
\begin{aligned}
K^{(1)}_{k'\bar{i}'} &= 2f^2 g^{j\bar{s}}\frac{\partial^2 \ln f}{\partial z^{j}\partial \bar{z}^{s}} h_{k'\bar{i}'} + f^{-2}h^{j'\bar{s}'}\Big(2f^2\frac{\partial^2 \ln f}{\partial z^{j'}\partial \bar{z}^{s'}} h_{k'\bar{i}'} + f^2 \overset{2}{K}_{k'\bar{i}'j'\bar{s}'}\Big) \\
&= 2f^2 \overset{1}{L}(\ln f) h_{k'\bar{i}'} + 2\overset{2}{L}(\ln f) h_{k'\bar{i}'} + h^{j'\bar{s}'}\overset{2}{K}_{k'\bar{i}'j'\bar{s}'} \\
&= 2f^2 \overset{1}{L}(\ln f) h_{k'\bar{i}'} + 2\overset{2}{L}(\ln f) h_{k'\bar{i}'} + \overset{2}{K}{}^{(1)}_{k'\bar{i}'}.
\end{aligned}
$$

Similarly, we can obtain other equalities of Proposition 4. □

**Theorem 1.** *Let $(M_1 \times {}_f M_2, G)$ be a twisted product Hermitian manifold. Then, the Chern Ricci scalar curvature of $G$ along a nonzero vector $v = \left(v^i, v^{i'}\right) \in T^{1,0}M$ is given by*

$$
S_G(v) = S_g(v_1) + f^{-2}S_h(v_2) + 2\overset{1}{L}(\ln f) + 2f^{-2}\overset{2}{L}(\ln f). \tag{36}
$$

**Proof.** According to (8), we have

$$
S_G(v) = G^{k\bar{i}}K^{(1)}_{k\bar{i}} + G^{k'\bar{i}}K^{(1)}_{k'\bar{i}} + G^{k\bar{i}'}K^{(1)}_{k\bar{i}'} + G^{k'\bar{i}'}K^{(1)}_{k'\bar{i}'}. \tag{37}
$$

Substituting (14), (31) and (32) into (37), after a straightfoward computation, we see that

$$
\begin{aligned}
S_G(v) &= g^{k\bar{i}}\overset{1}{K}{}^{(1)}_{k\bar{i}} + f^{-2}h^{k'\bar{i}'}\big[2f^2\overset{1}{L}(\ln f)h_{k'\bar{i}'} + 2\overset{2}{L}(\ln f)h_{k'\bar{i}'} + \overset{2}{K}{}^{(1)}_{k'\bar{i}'}\big] \\
&= S_g(v_1) + f^{-2}S_h(v_2) + 2\overset{1}{L}(\ln f) + 2f^{-2}\overset{2}{L}(\ln f).
\end{aligned}
$$

Thus, we complete the proof. □

According to Definitions 4 and 5, we can obtain the following.

**Corollary 1.** *Let $(M_1 \times {}_f M_2, G)$ be a twisted product Hermitian manifold. Suppose $\ln f$ is a pluriharmonic function, then $S_G(v) = S_g(v_1) + f^{-2}S_h(v_2)$.*

## 5. Holomorphic Sectional Curvature of Twisted Product Hermitian Manifold

In this section, we would like to derive the holomorphic sectional curvature of the twisted product Hermitian manifold, and give the necessary and sufficient conditions for the compact twisted product Hermitian manifold to have constant holomorphic sectional curvature.

**Definition 7** ([21]). *Let $(M, G)$ be a Hermitian manifold. Then, the holomorphic sectional curvature of $G$ along a nonzero vector $v = (v^i, v^{i'}) \in T^{1,0}M$ is defined by*

$$K_G(v) = -\frac{1}{G^2(v,\overline{v})} K_{\alpha\overline{\beta}\gamma\overline{\sigma}} v^\alpha \overline{v}^\beta v^\gamma \overline{v}^\sigma. \tag{38}$$

**Theorem 2.** *Let $(M_1 \times {}_f M_2, G)$ be a twisted product Hermitian manifold. Then, the holomorphic sectional curvature of $G$ along a nonzero vector $v = (v^i, v^{i'}) \in T^{1,0}M$ is given by*

$$\begin{aligned} K_G(v) = \frac{1}{G^2(v,\overline{v})}[g^2K_g(v_1) + f^2h^2K_h(v_2) - 2f^2h(\frac{\partial^2 \ln f}{\partial z^j \partial \overline{z}^s} v^j \overline{v}^s \\ + \frac{\partial^2 \ln f}{\partial z^{j'} \partial \overline{z}^s} v^{j'} \overline{v}^s + \frac{\partial^2 \ln f}{\partial z^j \partial \overline{z}^{s'}} v^j \overline{v}^{s'} + \frac{\partial^2 \ln f}{\partial z^{j'} \partial \overline{z}^{s'}} v^{j'} \overline{v}^{s'})]. \end{aligned} \tag{39}$$

**Proof.** According to (28), (29) and (38), we have

$$\begin{aligned} K_G(v) = -\frac{1}{G^2(v,\overline{v})}(K_{k\bar{i}j\bar{s}} v^k \overline{v}^i v^j \overline{v}^s + K_{k'\overline{i'}j\bar{s}} v^{k'} \overline{v}^{i'} v^j \overline{v}^s + K_{k'\overline{i'}j'\bar{s}} v^{k'} \overline{v}^{i'} v^{j'} \overline{v}^s \\ + K_{k'\overline{i'}j\overline{s'}} v^{k'} \overline{v}^{i'} v^j \overline{v}^{s'} + K_{k'\overline{i'}j'\overline{s'}} v^{k'} \overline{v}^{i'} v^{j'} \overline{v}^{s'}). \end{aligned} \tag{40}$$

Using (27) and noting that $h_{k'\overline{i'}} v^{k'} \overline{v}^{i'} = h$, we have

$$\begin{aligned} K_{k'\overline{i'}j'\overline{s'}} v^{k'} \overline{v}^{i'} v^{j'} \overline{v}^{s'} &= (2f^2 \frac{\partial^2 \ln f}{\partial z^{j'} \partial \overline{z}^{s'}} h_{k'\overline{i'}} + f^2 \overset{2}{K}_{k'\overline{i'}j'\overline{s'}}) v^{k'} \overline{v}^{i'} v^{j'} \overline{v}^{s'} \\ &= 2f^2h \frac{\partial^2 \ln f}{\partial z^{j'} \partial \overline{z}^{s'}} v^{j'} \overline{v}^{s'} - f^2h^2K_h(v_2). \end{aligned} \tag{41}$$

Similarly, we can obtain

$$K_{k\bar{i}j\bar{s}} v^k \overline{v}^i v^j \overline{v}^s = \overset{1}{K}_{k\bar{i}j\bar{s}} v^k \overline{v}^i v^j \overline{v}^s = -g^2K_g(v_1), \tag{42}$$

$$K_{k'\overline{i'}j\bar{s}} v^{k'} \overline{v}^{i'} v^j \overline{v}^s = 2f^2 \frac{\partial^2 \ln f}{\partial z^j \partial \overline{z}^s} h_{k'\overline{i'}} v^{k'} \overline{v}^{i'} v^j \overline{v}^s = 2f^2h \frac{\partial^2 \ln f}{\partial z^j \partial \overline{z}^s} v^j \overline{v}^s, \tag{43}$$

$$K_{k'\overline{i'}j'\bar{s}} v^{k'} \overline{v}^{i'} v^{j'} \overline{v}^s = 2f^2 \frac{\partial^2 \ln f}{\partial z^{j'} \partial \overline{z}^s} h_{k'\overline{i'}} v^{k'} \overline{v}^{i'} v^{j'} \overline{v}^s = 2f^2h \frac{\partial^2 \ln f}{\partial z^{j'} \partial \overline{z}^s} v^{j'} \overline{v}^s, \tag{44}$$

$$K_{k'\overline{i'}j\overline{s'}} v^{k'} \overline{v}^{i'} v^j \overline{v}^{s'} = 2f^2 \frac{\partial^2 \ln f}{\partial z^j \partial \overline{z}^{s'}} h_{k'\overline{i'}} v^{k'} \overline{v}^{i'} v^j \overline{v}^{s'} = 2f^2h \frac{\partial^2 \ln f}{\partial z^j \partial \overline{z}^{s'}} v^j \overline{v}^{s'}. \tag{45}$$

Plunging (41)–(45) into (40), we can obtain (39). □

According to Definition 4, we can easily obtain

**Corollary 2.** *Let $(M_1 \times {}_f M_2, G)$ be a twisted product Hermitian manifold. Suppose $\ln f$ is a pluriharmonic function, then*

$$K_G(v) = \frac{1}{G^2(v,\bar{v})}(g^2 K_g(v_1) + f^2 h^2 K_h(v_2)).$$

**Theorem 3** ([21]). *Let $(M, G)$ be a compact Hermitian manifold. Then, $M$ has constant holomorphic sectional curvature $\kappa$ if and only if, at every point of $M$,*

$$\Theta_{\alpha\bar{\beta}\gamma\bar{\sigma}} = -\frac{1}{2}\kappa(G_{\alpha\bar{\beta}}G_{\gamma\bar{\sigma}} + G_{\alpha\bar{\sigma}}G_{\gamma\bar{\beta}}), \tag{46}$$

*where*

$$\Theta_{\alpha\bar{\beta}\gamma\bar{\sigma}} = \frac{1}{4}(K_{\alpha\bar{\beta}\gamma\bar{\sigma}} + K_{\gamma\bar{\sigma}\alpha\bar{\beta}} + K_{\alpha\bar{\sigma}\gamma\bar{\beta}} + K_{\gamma\bar{\beta}\alpha\bar{\sigma}}). \tag{47}$$

**Proposition 5.** *Let $(M_1 \times {}_f M_2, G)$ be a twisted product Hermitian manifold. Then,*

$$\Theta_{k\bar{i}j\bar{s}} = \overset{1}{\Theta}_{k\bar{i}j\bar{s}}, \tag{48}$$

$$\Theta_{k'\bar{i}'j\bar{s}} = \Theta_{k'\bar{s}j\bar{i}'} = \Theta_{j\bar{i}'k'\bar{s}} = \Theta_{j\bar{s}k'\bar{i}'} = \frac{1}{2}f^2\frac{\partial^2 \ln f}{\partial z^j \partial \bar{z}^s}h_{k'\bar{i}'}, \tag{49}$$

$$\Theta_{k'\bar{i}'j'\bar{s}} = \Theta_{k'\bar{s}j'\bar{i}'} = \frac{1}{2}f^2\Big(\frac{\partial^2 \ln f}{\partial z^{k'} \partial \bar{z}^s}h_{j'\bar{i}'} + \frac{\partial^2 \ln f}{\partial z^{j'} \partial \bar{z}^s}h_{k'\bar{i}'}\Big), \tag{50}$$

$$\Theta_{k'\bar{i}'j\bar{s}'} = \Theta_{j\bar{i}'k'\bar{s}'} = \frac{1}{2}f^2\Big(\frac{\partial^2 \ln f}{\partial z^{j} \partial \bar{z}^{i'}}h_{k'\bar{s}'} + \frac{\partial^2 \ln f}{\partial z^{j} \partial \bar{z}^{s'}}h_{k'\bar{i}'}\Big), \tag{51}$$

$$\begin{aligned}\Theta_{k'\bar{i}'j'\bar{s}'} = {}& \frac{1}{2}f^2\Big(\frac{\partial^2 \ln f}{\partial z^{j'} \partial \bar{z}^{s'}}h_{k'\bar{i}'} + \frac{\partial^2 \ln f}{\partial z^{k'} \partial \bar{z}^{i'}}h_{j'\bar{s}'} + \frac{\partial^2 \ln f}{\partial z^{j'} \partial \bar{z}^{i'}}h_{k'\bar{s}'} \\ & + \frac{\partial^2 \ln f}{\partial z^{k'} \partial \bar{z}^{s'}}h_{j'\bar{i}'}\Big) + f^2\overset{2}{\Theta}_{k'\bar{i}'j'\bar{s}'},\end{aligned} \tag{52}$$

$$\Theta_{k'\bar{i}j\bar{s}} = \Theta_{k\bar{i}'j\bar{s}} = \Theta_{j\bar{i}k'\bar{s}} = \Theta_{k\bar{s}j\bar{i}'} = \Theta_{k'\bar{i}j'\bar{s}} = \Theta_{k\bar{i}'j\bar{s}'} = 0. \tag{53}$$

**Proof.** By putting $\alpha = k', \beta = i', \gamma = j', \sigma = s'$ in (47), we have

$$\Theta_{k'\bar{i}'j'\bar{s}'} = \frac{1}{4}(K_{k'\bar{i}'j'\bar{s}'} + K_{j'\bar{s}'k'\bar{i}'} + K_{k'\bar{s}'j'\bar{i}'} + K_{j'\bar{i}'k'\bar{s}'}). \tag{54}$$

By using (27), we obtain

$$\begin{aligned}\Theta_{k'\bar{i}'j'\bar{s}'} &= \frac{1}{4}\Big(2f^2\frac{\partial^2 \ln f}{\partial z^{j'} \partial \bar{z}^{s'}}h_{k'\bar{i}'} + f^2\overset{2}{K}_{k'\bar{i}'j'\bar{s}'} + 2f^2\frac{\partial^2 \ln f}{\partial z^{k'} \partial \bar{z}^{i'}}h_{j'\bar{s}'} + f^2\overset{2}{K}_{j'\bar{s}'k'\bar{i}'} \\ &\quad + 2f^2\frac{\partial^2 \ln f}{\partial z^{j'} \partial \bar{z}^{i'}}h_{k'\bar{s}'} + f^2\overset{2}{K}_{k'\bar{s}'j'\bar{i}'} + 2f^2\frac{\partial^2 \ln f}{\partial z^{k'} \partial \bar{z}^{s'}}h_{j'\bar{i}'} + f^2\overset{2}{K}_{j'\bar{i}'k'\bar{s}'}\Big) \\ &= \frac{1}{2}f^2\Big(\frac{\partial^2 \ln f}{\partial z^{j'} \partial \bar{z}^{s'}}h_{k'\bar{i}'} + \frac{\partial^2 \ln f}{\partial z^{k'} \partial \bar{z}^{i'}}h_{j'\bar{s}'} + \frac{\partial^2 \ln f}{\partial z^{j'} \partial \bar{z}^{i'}}h_{k'\bar{s}'} + \frac{\partial^2 \ln f}{\partial z^{k'} \partial \bar{z}^{s'}}h_{j'\bar{i}'}\Big) + f^2\overset{2}{\Theta}_{k'\bar{i}'j'\bar{s}'}.\end{aligned}$$

Similar calculations give the rest of the equalities of Proposition 5. □

**Theorem 4.** *Let $(M_1 \times {}_f M_2, G)$ be a compact twisted product Hermitian manifold. Then, $G$ has constant holomorphic sectional curvature $\kappa$ if and only if $\kappa = 0$ and the following equalities hold*

$$\begin{cases} \overset{1}{\Theta}_{k\bar{i}j\bar{s}} = 0, & \text{(55a)} \\ \overset{1}{L}(\ln f) = 0, & \text{(55b)} \\ \dfrac{\partial^2 lnf}{\partial z^{j'} \partial \overline{z}^{s}} = 0, & \text{(55c)} \\ \dfrac{\partial^2 lnf}{\partial z^{j} \partial \overline{z}^{s'}} = 0, & \text{(55d)} \\ 2\overset{2}{L}(\ln f) + h^{\overline{s'}j'} h^{\overline{i'}k'} \overset{2}{\Theta}_{k'\overline{i'}j'\overline{s'}} = 0. & \text{(55e)} \end{cases}$$

**Proof.** According to Theorem 3, (13) and (53), $(M_1 \times {}_f M_2, G)$ has constant holomorphic sectional curvature if and only if

$$\begin{cases} \Theta_{k\bar{i}j\bar{s}} = -\dfrac{1}{2}\kappa(G_{k\bar{i}}G_{j\bar{s}} + G_{k\bar{s}}G_{j\bar{i}}), & \text{(56a)} \\ \Theta_{k'\overline{i'}j\bar{s}} = -\dfrac{1}{2}\kappa G_{k'\overline{i'}}G_{j\bar{s}}, & \text{(56b)} \\ \Theta_{k'\overline{i'}j'\bar{s}} = 0, & \text{(56c)} \\ \Theta_{k'\overline{i'}j\overline{s'}} = 0, & \text{(56d)} \\ \Theta_{k'\overline{i'}j'\overline{s'}} = -\dfrac{1}{2}\kappa(G_{k'\overline{i'}}G_{j'\overline{s'}} + G_{k'\overline{s'}}G_{j'\overline{i'}}). & \text{(56e)} \end{cases}$$

Substituting (13) and (48)–(52) into (56a)–(56e), and noticing that $f^2 \neq 0$, (56a)–(56e) are thus equivalent to the following equalities

$$\begin{cases} \overset{1}{\Theta}_{k\bar{i}j\bar{s}} = -\dfrac{1}{2}\kappa(g_{k\bar{i}}g_{j\bar{s}} + g_{k\bar{s}}g_{j\bar{i}}), & \text{(57a)} \\ \dfrac{\partial^2 lnf}{\partial z^{j} \partial \overline{z}^{s}} h_{k'\overline{i'}} = -\kappa h_{k'\overline{i'}} g_{j\bar{s}}, & \text{(57b)} \\ \dfrac{\partial^2 \ln f}{\partial z^{k'} \partial \overline{z}^{s}} h_{j'\overline{i'}} + \dfrac{\partial^2 lnf}{\partial z^{j'} \partial \overline{z}^{s}} h_{k'\overline{i'}} = 0, & \text{(57c)} \\ \dfrac{\partial^2 \ln f}{\partial z^{j} \partial \overline{z}^{i'}} h_{k'\overline{s'}} + \dfrac{\partial^2 lnf}{\partial z^{j} \partial \overline{z}^{s'}} h_{k'\overline{i'}} = 0, & \text{(57d)} \\ \dfrac{\partial^2 \ln f}{\partial z^{j'} \partial \overline{z}^{s'}} h_{k'\overline{i'}} + \dfrac{\partial^2 \ln f}{\partial z^{k'} \partial \overline{z}^{i'}} h_{j'\overline{s'}} + \dfrac{\partial^2 \ln f}{\partial z^{j'} \partial \overline{z}^{i'}} h_{k'\overline{s'}} + \dfrac{\partial^2 \ln f}{\partial z^{k'} \partial \overline{z}^{s'}} h_{j'\overline{i'}} & \\ +2\overset{2}{\Theta}_{k'\overline{i'}j'\overline{s'}} = -\kappa f^2 (h_{k'\overline{i'}}h_{j'\overline{s'}} + h_{k'\overline{s'}}h_{j'\overline{i'}}). & \text{(57e)} \end{cases}$$

The above equalities are equivalent to

$$\begin{cases} \overset{1}{\Theta}_{k\bar{i}j\bar{s}} = -\dfrac{1}{2}\kappa(g_{k\bar{i}}g_{j\bar{s}} + g_{k\bar{s}}g_{j\bar{i}}), & \text{(58a)} \\ \overset{1}{L}(\ln f) = -\kappa, & \text{(58b)} \\ \dfrac{\partial^2 lnf}{\partial z^{j'} \partial \overline{z}^{s}} = 0, & \text{(58c)} \\ \dfrac{\partial^2 lnf}{\partial z^{j} \partial \overline{z}^{s'}} = 0, & \text{(58d)} \\ 2\overset{2}{L}(\ln f) + h^{\overline{s'}j'} h^{\overline{i'}k'} \overset{2}{\Theta}_{k'\overline{i'}j'\overline{s'}} = -\kappa f^2. & \text{(58e)} \end{cases}$$

In fact, contracting (57b) with $h^{\bar{i}'k'}$ and $g^{\bar{s}j}$ successively, and noticing that $\overset{1}{L} = g^{\bar{i}k}\frac{\partial^2}{\partial z^k \partial \bar{z}^i}$, we can obtain (58b). Contracting (57c) and (57d) with $h^{\bar{i}'k'}$, respectively, we can obtain (58c) and (58d). Contracting (57e) with $h^{\bar{i}'k'}$ and $h^{\bar{s}'j'}$ successively, and noticing that $\overset{2}{L} = h^{\bar{i}'k'}\frac{\partial^2}{\partial z^{k'} \partial \bar{z}^{i'}}$, we can obtain (58e).

Proof of the necessity.

Let us suppose that $\kappa \neq 0$, combining (58a) and (58b), we have

$$2\overset{1}{\Theta}_{k\bar{i}j\bar{s}} = \overset{1}{L}(\ln f)(g_{k\bar{i}}g_{j\bar{s}} + g_{k\bar{s}}g_{j\bar{i}}), \tag{59}$$

since $\overset{1}{\Theta}_{k\bar{i}j\bar{s}}$, $g_{k\bar{i}}$ depend only on $z_1$, which says that $f$ only depends on $M_1$. These are contradicted by the fact that $(M_1 \times {}_f M_2, G)$ is a twisted product Hermitian manifold. Thus,

$$\kappa = 0. \tag{60}$$

Plunging (60) into (58a), (58b) and (58e), we can check that (58a)–(58e) can be simplified as (55a)–(55e).

Next, we prove the sufficiency.

Suppose that $\kappa = 0$ and (55a)–(55e) hold; this immediately confirms that (57a)–(57e) hold, i.e., $(M_1 \times {}_f M_2, G)$ has constant holomorphic sectional curvature $\kappa$. Thus, we complete the proof. □

## 6. Chern Flat and Chern Ricci-Flat Twisted Product Hermitian Manifolds

Let $(M_1, g)$ and $(M_2, h)$ be two Chern flat or Chern Ricci-flat Hermitian manifolds, respectively. We would like to know under what conditions the twisted product Hermitian manifold $(M_1 \times {}_f M_2, G)$ is Chern flat or Chern Ricci-flat.

**Definition 8** ([22]). *A Hermitian manifold $(M, G)$ is called Chern flat if*

$$K = 0,$$

*where K is the Chern curvature tensor.*

**Definition 9** ([22]). *A Hermitian manifold $(M, G)$ is called Chern Ricci-flat if*

$$K^{(1)} = 0,$$

*where $K^{(1)}$ is the first Chern Ricci curvature tensor.*

**Theorem 5.** *Let $(M_1 \times {}_f M_2, G)$ be a twisted product Hermitian manifold. Suppose $\ln f$ is pluriharmonic, then $(M_1 \times {}_f M_2, G)$ is Chern flat if and only if $(M_1, g)$ and $(M_2, h)$ are Chern flat.*

**Proof.** Since $\ln f$ is pluriharmonic, then

$$\frac{\partial^2 \ln f}{\partial z^j \partial \bar{z}^s} = 0, \tag{61}$$

$$\frac{\partial^2 \ln f}{\partial z^{j'} \partial \bar{z}^s} = 0, \tag{62}$$

$$\frac{\partial^2 \ln f}{\partial z^j \partial \bar{z}^{s'}} = 0, \tag{63}$$

$$\frac{\partial^2 \ln f}{\partial z^{j'} \partial \bar{z}^{s'}} = 0. \tag{64}$$

According to Definition 8 and (2), $(M_1 \times {}_f M_2, G)$ is Chern flat if and only if

$$K_{\alpha\bar{\beta}\gamma\bar{\sigma}} = 0. \tag{65}$$

Using Proposition 3 and (61)–(64), and noticing that $f^2 \neq 0$, (65) is equivalent to following equalities

$$\begin{cases} \overset{1}{K}_{k\bar{i}j\bar{s}} = 0, \\ \overset{2}{K}_{k'\bar{i}'j'\bar{s}'} = 0. \end{cases} \tag{66}$$

which means that $(M_1, g)$ and $(M_2, h)$ are Chern flat. □

**Theorem 6.** *Let $(M_1 \times {}_f M_2, G)$ be a twisted product Hermitian manifold. Suppose $\ln f$ is pluriharmonic, then $(M_1 \times {}_f M_2, G)$ is Chern Ricci-flat if and only if $(M_1, g)$ and $(M_2, h)$ are Chern Ricci-flat.*

**Proof.** Suppose that $\ln f$ is pluriharmonic, then

$$\overset{1}{L}(\ln f) = \overset{2}{L}(\ln f) = 0. \tag{67}$$

By Definition 9 and (5), $(M_1 \times {}_f M_2, G)$ is Chern Ricci flat if and only if

$$K^{(1)}_{\alpha\bar{\beta}} = 0. \tag{68}$$

Using (31), (32) and (67), (68) is equivalent to the following equalities

$$\begin{cases} \overset{1}{K}{}^{(1)}_{k\bar{i}} = 0, \\ \overset{2}{K}{}^{(1)}_{k'\bar{i}'} = 0. \end{cases} \tag{69}$$

Which means that $(M_1, g)$ and $(M_2, h)$ are Chern Ricci flat. □

## 7. Conclusions

In this paper, we extended the twisted product to Hermitian manifold. Based on this, we confirmed that the compact twisted product Hermitian manifold has constant holomorphic sectional curvature if and only if $\kappa = 0$ and a system of differential equations holds. Under the condition that the logarithm of the twisted function is pluriharmonic, we obtained the necessary and sufficient conditions for the twisted product Hermitian manifold to be Chern flat or Chern Ricci-flat, respectively, so then we gave an effective way to construct Chern flat or Chern Ricci-flat Hermitian manifolds.

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## References


1. Bishop, R.L.; O'Neill, B. Manifolds of negative curvature. *Trans. Am. Math. Soc.* **1969**, *145*, 1–49. [CrossRef]
2. Kozma, L.; Peter, I.R.; Varga, C. Warped product of Finsler manifolds. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **2001**, *44*, 157–170.
3. Asanov, G. Finslerian extension of Schwarzschild metric. *Fortschritte Der Phys. Phys.* **1992**, *40*, 667–693. [CrossRef]
4. Asanov, G. Finslerian metric functions over the product R × M and their potential applications. *Rep. Math. Phys.* **1998**, *41*, 117–132. [CrossRef]
5. He, Y.; Zhang, X. On doubly warped product of Hermitian manifolds. *Acta Math. Sin. Chin. Ser.* **2018**, *61*, 835–842.
6. Chen, B. *Geometry of Submanifolds and Its Applications*; Science University of Tokyo: Tokyo, Japan, 1981.
7. Ponge, R.; Reckziegel, H. Twisted products in pseudo-Riemannian geometry. *Geom. Dedicata* **1993**, *48*, 15–25. [CrossRef]
8. Fernández-López, M.; García-Río, E.; Kupeli, D.; Ünal, B. A curvature condition for a twisted product to be a warped product. *Manuscripta Math.* **2001**, *106*, 213–217. [CrossRef]
9. Kazan, S.; Sahin, B. Characterizations of twisted product manifolds to be warped product manifolds. *Acta Math. Univ. Comen.* **2017**, *82*, 253–263.
10. Kozma, L.; Peter, I.R.; Shimada, H. On the twisted product of Finsler manifolds. *Rep. Math. Phys.* **2006**, *57*, 375–384. [CrossRef]
11. Xiao, W.; He, Y.; Lu, X.; Deng, X. On doubly twisted product of complex Finsler manifolds. *J. Math. Study* **2022**, *55*, 158–179. [CrossRef]
12. Tani, M. On a conformally flat Riemannian space with positive Ricci curvature. *Tohoku Math. J. Second Ser.* **1967**, *19*, 227–231. [CrossRef]
13. Bando, S.; Kobayashi, R. Ricci-flat Kähler metrics on affine algebraic manifolds. II. *Math. Ann.* **1990**, *287*, 175–180. [CrossRef]
14. Liu, K.; Yang, X. Ricci curvatures on Hermitian manifolds. *Trans. Am. Math. Soc.* **2017**, *369*, 5157–5196. [CrossRef]
15. Ni, Q.; He, Y.; Yang, J.; Zhang, H. Levi-Civita Ricci-Flat Doubly Warped Product Hermitian Manifolds. *Adv. Math. Phys.* **2022**, *2022*, 2077040. [CrossRef]
16. Di Scala, A.J.; Lauret, J.; Vezzoni, L. Quasi-Kähler Chern-flat manifolds and complex 2-step nilpotent Lie algebras. *Ann. Della Sc. Norm. Super.-Pisa-Cl. Sci.* **2012**, *11*, 41–60. [CrossRef] [PubMed]
17. Zhou, W.; Zheng, F. Hermitian threefolds with vanishing real bisectional curvature. *arXiv* **2021**, arXiv:2103.04296.
18. Kobayashi, S. *Foundations of Differential Geometry. Vol. II*; Wiley Classics Library, John Wiley & Sons: New York, NY, USA, 1969.
19. Rudin, W. *Function Theory in the Unit Ball of Cn*; Springer Science & Business Media: New York, NY, USA, 2008.
20. Li, H.; Qiu, C.; Zhu, W. A Laplace operator on complex Finsler manifolds. *Differ. Geom. Its Appl.* **2017**, *54*, 437–447. [CrossRef]
21. Balas, A. Compact Hermitian manifolds of constant holomorphic sectional curvature. *Math. Z.* **1985**, *189*, 193–210. [CrossRef]
22. Di Scala, A.J.; Vezzoni, L. Chern-flat and Ricci-flat invariant almost Hermitian structures. *Ann. Glob. Anal. Geom.* **2011**, *40*, 21–45. [CrossRef]