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Existence and Uniqueness of Weak Solutions to Frictionless-Antiplane Contact Problems

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Abstract: We investigate a quasi-static-antiplane contact problem, examining a thermo-electro-visco-elastic material with a friction law dependent on the slip rate, assuming that the foundation is electrically conductive. The mechanical problem is represented by a system of partial differential equations, and establishing its solution involves several key steps. Initially, we obtain a variational formulation of the model, which comprises three systems: a hemivariational inequality, an elliptic equation, and a parabolic equation. Subsequently, we demonstrate the existence of a unique weak solution to the model. The proof relies on various arguments, including those related to evolutionary inequalities, techniques for decoupling unknowns, and certain results from differential equations.

Keywords: electro-visco-elastic materials; antiplane problems; temperature fields; evolution variational inequalities

MSC: 74F15; 74M10; 74M15; 49J40

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1. Introduction

Over the past few decades, phenomena related to the contact between deformable bodies have had a significant impact on society. The contact of brakes with wheels, a ball with the ground, and the application of a force to cutting objects, are just a few everyday instances, among many other examples. Consequently, research in this area is growing across various fields, particularly in engineering and mathematical literature [1,2].

A comprehensive study of mechanical problems involves mathematical modeling. The modeling of mechanical phenomena is determined by a set of hypotheses that influence the system of partial differential equations. These hypotheses cover aspects such as the nature of the mechanical process (static, quasi-static, or dynamic), the behavior of the material (electromechanical, thermomechanical, elastic, etc.), and the boundary conditions on the contact surface (friction, adhesion, etc.) [3,4].

In 1933, notable advancements in the mathematical and mechanical exploration of contact-mechanics problems took place. Signorini seems to have been the first to delve into this subject when he formulated the challenge of contact between a deformable body and a foundation. The solution to this problem eventually emerged in 1964, courtesy of Fichera, who employed emerging mathematical techniques. The initial findings concerning the existence and uniqueness of contact problems were established by Duvaut and Lions. Following this, a multitude of subsequent works have concentrated on resolving these variational problems [5,6].

Mathematics plays a pivotal role in the field of contact mechanics by providing diverse contact conditions. Contemporary research in this field also encompasses behavior laws that establish connections between mechanical and electrical effects, known as piezoelectricity [7,8]. This area is extensively explored in engineering structures, due to its temperature-dependent characteristics. Several articles have tackled the thermo-piezoelectric contact problem with friction [5,9–13], thermo-elasto-visco-plasticity [14–16], and thermo-visco-elasticity [17,18]. These studies are rooted in variations in constitutive laws and contact conditions [19–25].

Here we propose a novel mathematical model designed to address the frictional-antiplane contact problem between a thermo-piezoelectric body and a conductive foundation. The innovation within our model lies in the introduction of a novel frictional contact tailored for such materials. This model incorporates a slip-rate-dependent friction law and introduces modifications to the electrical and thermal conditions used to describe the contact. We specifically focus on the case of antiplane-shear deformation.

The manuscript is organized as follows. In Section 2, we recall some notations and definitions used in the sequel. In Section 3, we introduce our mathematical model for the quasi-static thermo-electro-visco-elastic-antiplane contact problem. Given some consistency assumptions about the data, the variational formulation of the model is then obtained in Section 4. Finally, in Section 5, we prove the existence and uniqueness result of the formulated problem.

2. Preliminaries

In this brief section, we review the fundamental notations and definitions that are employed throughout the work. For more details, we refer the interested reader to [25].

The space of second-order symmetric tensors on $\mathbb{R}^3$ is denoted by $\mathcal{S}^3$. The inner product and the norm on $\mathbb{R}^3$ are given as follows:

$$u^h \cdot v^h = u_i^h v_i^h, \quad \|v^h\| = (v^h \cdot v^h)^{\frac{1}{2}} \quad \text{for all} \quad u^h = (u_i^h), v^h = (v_i^h) \in \mathbb{R}^3,\ 1 \leq i, j \leq 3.$$

The inner product on $\mathcal{S}^3$ is defined by

$$\sigma^h \cdot \tau^h = \sigma_{i,j}^h \tau_{i,j}^h, \quad \text{for all} \quad \sigma^h = (\sigma_{i,j}^h), \quad \tau^h = \tau_{i,j}^h \in \mathcal{S}^3,\ 1 \leq i, j \leq 3,$$

while its corresponding norm is given by

$$\|\tau^h\| = \left(\tau^h \cdot \tau^h\right)^{\frac{1}{2}}.$$

Next, we consider the following function spaces:

$$V^h = \{v^h \in H^1(\Omega^h), v^h = 0 \text{ on } \Gamma_1^h\},$$

$$W^h = \{\psi^h \in H^1(\Omega^h), \psi^h = 0 \text{ on } \Gamma_1^h\},$$

$$E^h = \{\theta^h \in H^1\left(\Omega^h\right),\ \theta^h = 0 \text{ on } \Gamma_1^h \cup \Gamma_2^h\}.$$

It is an established fact that V^h, W^h, and E^h are real Hilbert spaces with the inner products

$$(u^h, v^h)_{V^h} = \int_{\Omega^h} \nabla u^h \cdot \nabla v^h dx, \quad \forall u^h, v^h \in V^h,$$

$$(\varphi^h, \psi^h)_{W^h} = \int_{\Omega^h} \nabla \varphi^h \cdot \nabla \psi^h dx, \quad \forall \varphi^h, \psi^h \in W^h,$$

$$(\theta^h, \phi^h)_{E^h} = \int_{\Omega^h} \nabla \theta^h \cdot \nabla \phi^h dx, \quad \forall \theta^h, \phi^h \in E^h.$$

Furthermore, the associated norms

$$\|v^h\|_{V^h} = \|\nabla v^h\|_{L^2(\Omega^h)^2}, \quad \forall v^h \in V^h, \tag{1}$$

$$\|\psi^h\|_{W^h} = \|\nabla \psi^h\|_{L^2(\Omega^h)^2}, \quad \forall \psi^h \in W^h, \tag{2}$$

$$\|\theta^h\|_{E^h} = \|\nabla \theta^h\|_{L^2(\Omega^h)^2}, \quad \forall \theta^h \in E^h \tag{3}$$

on V^h, W^h, and E^h are equivalent to the usual norm $\|\cdot\|_{H^1(\Omega^h)}$. In the light of Sobolev's trace theorem, we deduce that

$$\exists\, c_{V^h} > 0 : \|v^h\|_{L^2(\Gamma_3^h)} \leq c_V^h \|v^h\|_{V^h}, \quad \forall v^h \in V^h, \tag{4}$$

$$\exists\, c_{W^h} > 0 : \|v^h\|_{L^2(\Gamma_3^h)} \leq c_W^h \|\psi^h\|_{W^h}, \quad \forall \psi \in W^h, \tag{5}$$

$$\exists\, c_{E^h} > 0 : \|v^h\|_{L^2(\Gamma_3^h)} \leq c_E^h \|\theta^h\|_{E^h}, \quad \forall \theta \in E^h. \tag{6}$$

In a real Banach space $(X^h, \|\cdot\|_{X^h})$, we denote the spaces of continuous and continuously differentiable functions on $[0, T]$ into X^h by $C(0, T, X^h)$ and $C^1(0, T, X^h)$, respectively. These spaces are equipped with their respective norms,

$$\|x^h\|_{C(0,T,X^h)} = \max_{t\in[0,T]} \|x^h\|_{X^h}$$

and

$$\|x^h\|_{C^1(0,T,X^h)} = \max_{t\in[0,T]} \|x^h\|_{X^h} + \max_{t\in[0,T]} \|\dot{x}^h\|_{X^h}.$$

Below, we shall use the notations for the Lebesgue space $L^2(0, T, X^h)$ as well as the Sobolev space $W^{1,2}(0, T, X^h)$. Recall that the norm on $L^2(0, T, X^h)$ is given by

$$\|u^h\|^2_{L^2(0,T,X^h)} = \int_0^T \|u^h(t)\|^2_{X^h} dt.$$

Moreover,

$$\|u^h\|^2_{W^{1,2}(0,T,X^h)} = \int_0^T \|u^h(t)\|^2_{X^h} dt + \int_0^T \|\dot{u}^h(t)\|^2_{X^h} dt$$

defines a norm on $W^{1,2}(0, T, X^h)$.

3. Formulation of the Problem

We begin by making explicit the physical setting under investigation. Let us consider a thermo-electro-visco-elastic body $\mathcal{B}^h$ with friction. Assume that $\mathcal{B}^h \in \mathbb{R}^3$ is a cylinder with generators parallel to the x_3^h-axes having a regular region in its cross-section Ω^h in the x_1^h, x_2^h-plane, $Ox_1^h x_2^h x_3^h$ being a Cartesian-coordinate system. The cylinder is presumptively long enough for the end effects in the axial direction to be insignificant. Thus, $\mathcal{B}^h = \Omega^h \times (-\infty, +\infty)$. Let $\partial\Omega^h = \Gamma^h$. We suppose that Γ^h is divided into three disjoint measurable parts: Γ_1^h, Γ_2^h, and Γ_3^h. One has a partition of $\Gamma_1^h \cup \Gamma_2^h$ into two open parts, Γ_a^h and Γ_b^h, such that the measures of Γ_a^h and Γ_b^h are positive. For $T > 0$, we denote by $[0, T]$ the time interval of interest. The $\mathcal{B}^h$ body is affected to time-dependent forces $\mathbf{f_0^h}$ and has a volume with free electric charges of density q_0^h. The body is clamped on $\Gamma_1^h \times (-\infty, +\infty)$ and, therefore, the displacement field disappears there. The surface tractions of density $\mathbf{f_2^h}$ act on $\Gamma_2^h \times (-\infty, +\infty)$. We also assume that the electrical potential vanishes on $\Gamma_a^h \times (-\infty, +\infty)$ and that a surface electrical charge of density q_2^h is prescribed on $\Gamma_b^h \times (-\infty, +\infty)$. The body is in contact with a rigid foundation $\Gamma_3^h \times (-\infty, +\infty)$ [26,27]. We assume that

$$q_0^h = q_0^h(x_1^h, x_2^h, t) : \Omega^h \times [0,T] \longrightarrow \mathbb{R}, \tag{7}$$
$$q_2^h = q_2^h(x_1^h, x_2^h, t) : \Gamma_b^h \times [0,T] \longrightarrow \mathbb{R}, \tag{8}$$
$$\mathbf{f}_0^h = (0,0,f_0^h), \text{ with } f_0^h = f_0^h(x_1^h, x_2^h, t) : \Omega^h \times [0,T] \longrightarrow \mathbb{R}, \tag{9}$$
$$\mathbf{f}_2^h = (0,0,f_2^h), \text{ with } f_2^h = f_2^h(x_1^h, x_2^h, t) : \Gamma_2^h \times [0,T] \longrightarrow \mathbb{R}. \tag{10}$$

The electric charges (7) and (8) and the forces (9) and (10) are expected to give rise to electric charges and to deformations of the piezoelectric cylinder corresponding to an electric-potential field φ^h and to a displacement $\mathbf{u}^h$, which are independent of x_3^h and have the form

$$\varphi^h = \varphi^h(x_1^h, x_2^h, t) : \Omega^h \times [0,T] \longrightarrow \mathbb{R}, \tag{11}$$
$$\mathbf{u}^h = u^h(0,0,u^h), \text{ with } u^h = u^h(x_1^h, x_2^h, t) : \Omega^h \times [0,T] \longrightarrow \mathbb{R}. \tag{12}$$

We denote by θ^h the temperature field, which is of the form

$$\theta^h = \theta^h(x_1^h, x_2^h, t) : \Omega^h \times [0,T] \longrightarrow \mathbb{R}, \tag{13}$$

by $\varepsilon^h(u^h) = (\varepsilon_{i,j}^h(u^h))$ the infinitesimal strain tensor, by $\sigma^h = (\sigma_{ij}^h)$ the stress field, by $E^h(\varphi^h) = (E_i^h(\varphi^h))$ the electric field, and by $D^h = (D_i^h)$ the electric displacement field, where

$$\varepsilon_{i,j}^h(u^h) = \frac{1}{2}(u_{i,j}^h + u_{j,i}^h), \tag{14}$$
$$E_i^h(\varphi^h) = -\varphi_{,i}^h. \tag{15}$$

The material is modeled by a thermo-electro-visco-elastic constitutive law of the form

$$\sigma^h = 2\alpha^h \varepsilon^h(\dot{u}^h) + \zeta^h tr\varepsilon^h(\dot{u^h})I + 2\mu^h \varepsilon^h(u^h) + \lambda^h tr\varepsilon(u^h)I - \mathcal{E}^{h*}E^h(\varphi^h) - M_e^h\theta^h, \tag{16}$$
$$D^h = \mathcal{E}^{h*}\varepsilon^h(u^h) + \beta^h E^h(\varphi^h), \tag{17}$$

where ζ^h and α^h are viscosity coefficients, λ^h and μ^h are the Lame coefficients, $\text{tr}\varepsilon^h(u^h) = \varepsilon_{ii}^h(u^h)$, I is the unit tensor in $\mathbb{R}^3$, β^h is the electric permittivity constant, $\mathcal{E}^h$ represents a third-order piezoelectric tensor, and $\mathcal{E}^{h*}$ is its transpose. We assume that

$$\mathcal{E}^h\varepsilon^h = \begin{pmatrix} e^h(\varepsilon_{13}^h + \varepsilon_{31}^h) \\ e^h(\varepsilon_{23}^h + \varepsilon_{32}^h) \\ e^h(\varepsilon_{33}^h) \end{pmatrix}, \quad \forall \varepsilon^h = (\varepsilon_{i,j}^h) \in \mathcal{S}^3, \tag{18}$$

where e^h is the piezoelectric coefficient. Additionally, $M_e^h = (M_{ij}^h)$ represents the thermal expansion tensor, which takes the form

$$M_e^h = \begin{pmatrix} 0 & 0 & M_{e_1} \\ 0 & 0 & M_{e_2} \\ M_{e_1} & M_{e_2} & 0 \end{pmatrix}. \tag{19}$$

Conversely, we assume that the coefficients M_e^h, α^h, μ^h, β^h, and e^h depend on x_1^h and x_2^h. However, they are independent on x_3^h. Owing to $\mathcal{E}^h\varepsilon^h \cdot v^h = \varepsilon^h \cdot \mathcal{E}^{h*}v^h$ for all $\varepsilon^h \in \mathcal{S}^3$ and $v^h \in \mathbb{R}^3$, it follows from (18) that

$$\mathcal{E}^{h*}v^h = \begin{pmatrix} 0 & 0 & e^h v_1^h \\ 0 & 0 & e^h v_2^h \\ e^h v_1^h & e^h v_2^h & e^h v_3^h \end{pmatrix}, \quad \forall\, v^h = (v_i^h) \in \mathbb{R}^3. \tag{20}$$

In the antiplane context, having in mind (11)–(13), and given the constitutive Equations (16) and (17) and equalities (18) and (20), we obtain the stress field and the electric displacement field as

$$\sigma^h = \begin{pmatrix} 0 & 0 & \sigma^h_{13} \\ 0 & 0 & \sigma^h_{23} \\ \sigma^h_{31} & \sigma^h_{32} & 0 \end{pmatrix}, \ \mathbf{D}^h = \begin{pmatrix} eu^h_{,1} - \beta^h \varphi^h_{,1} \\ eu^h_{,2} - \beta^h \varphi^h_{,2} \\ 0 \end{pmatrix}, \tag{21}$$

where

$$\sigma^h_{13} = \sigma^h_{31} = \alpha^h \dot{u}^h_{,1} + \mu^h u^h_{,1} + e^h \varphi^h_{,1} - M_{e_1}\theta^h,$$
$$\sigma^h_{23} = \sigma^h_{32} = \alpha^h \dot{u}^h_{,2} + \mu^h u^h_{,2} + e^h \varphi^h_{,2} - M_{e_2}\theta^h.$$

We presume that the process is electrically static and mechanically quasi-static. Therefore, the equilibrium equations that govern it are given by

$$\begin{aligned} \mathrm{div}\sigma^h + \mathbf{f}^h_0 &= 0, \\ D^h_{i,i} - q^h_0 &= 0 \quad \text{in } \mathcal{B}^h \times (0,T), \end{aligned} \tag{22}$$

where $\mathrm{div}\sigma^h = (\sigma^h_{ij,j})$ is the divergence of the tensor field σ^h. Accordingly, taking into account (7), (9), (11)–(13), and (21), the equilibrium Equation (22) condenses into the following scalar equations:

$$\mathrm{div}(\alpha^h \nabla \dot{u}^h + \mu^h \nabla u^h) + \mathrm{div}(e^h \nabla \varphi^h) - M^h_e \mathrm{div}(\theta^h) + f^h_0 = 0, \ \text{in} \ \ \Omega^h \times (0,T), \tag{23}$$

$$\mathrm{div}(e^h \nabla u^h) - \mathrm{div}(\beta^h \nabla \varphi^h) = q^h_0, \ \text{in} \ \Omega^h \times (0,T), \tag{24}$$

with

$$M^h_e = \begin{pmatrix} M_{e_1} \\ M_{e_2} \\ 0 \end{pmatrix}. \tag{25}$$

In the sequel, we use the notation

$$\mathrm{div}\tau^h = \tau^h_{1,1} + \tau^h_{1,2} \ \text{ for } \tau^h = (\tau^h_1(x^h_1, x^h_2, t), \tau^h_2(x^h_1, x^h_2, t)),$$

$$\nabla v^h = (v^h_{,1}, v^h_{,2}), \ \partial^h_\nu v^h = v^h_{,1}\nu^h_1 + v^h_{,2}\nu^h_2 \ \text{ for } v^h = v^h(x^h_1, x^h_2, t).$$

Keeping in mind that the cylinder is clamped on $\Gamma^h_1 \times (-\infty, +\infty)$, the electrical potential vanishes on $\Gamma^h_a \times (-\infty, +\infty)$, and from (7) and (12) we find that

$$u^h = 0 \quad \text{on } \Gamma^h_1 \times (0,T), \tag{26}$$

$$\varphi^h = 0 \quad \text{on } \Gamma^h_a \times (0,T). \tag{27}$$

Note that ν^h is the unit normal on $\Gamma^h \times (-\infty, +\infty)$, where

$$\nu^h = (\nu^h_1, \nu^h_2, 0), \text{with } \nu^h_i = \nu^h_i(x^h_1, x^h_2) : \Gamma^h \longrightarrow \mathbb{R}, \quad \text{for } i = 1,2. \tag{28}$$

We denote by v^h_ν and $\mathbf{v}^h_\tau$ the normal and tangential components, respectively, of $\mathbf{v}^h$ on the boundary—that is,

$$v^h_\nu = \mathbf{v}^h \cdot \nu^h, \ \ \mathbf{v}^h_\tau = \mathbf{v}^h - v^h_\nu \nu^h. \tag{29}$$

Therefore, we denote by σ^h_ν and σ^h_τ the normal and the tangential components, respectively, of σ^h on the boundary, meaning that

$$\sigma^h_\nu = (\sigma^h \nu^h) \cdot \nu^h, \quad \sigma^h_\tau = \sigma^h \nu^h - \sigma^h_\nu \nu^h. \tag{30}$$

From (21) and (28), we conclude that the Cauchy stress vector and the normal component of the electric displacement field are given by

$$\sigma^h \nu^h = (0, 0, \alpha^h \partial_{\nu^h} \dot{u}^h + \mu^h \partial_{\nu^h} u^h + \partial_{\nu^h} \varphi^h - M_e^h \theta^h \cdot \nu^h), \;\; D^h \cdot \nu^h = e^h \partial_{\nu^h} u^h - \beta^h \partial_{\nu^h} \varphi^h, \quad (31)$$

respectively. Following this, we utilize the following notations:

$$\partial_{\nu^h} u^h = u_1^h \nu_1^h + u_2^h \nu_2^h,$$

$$\partial_{\nu^h} \varphi^h = \varphi_1^h \nu_1^h + \varphi_2^h \nu_2^h.$$

Taking into account the traction boundary condition on $\Gamma_2^h \times (-\infty, +\infty)$ and the electric condition on $\Gamma_b^h \times (-\infty, +\infty)$, it follows from (8), (10), and (31) that

$$\alpha^h \partial_{\nu^h} \dot{u}^h + \mu^h \partial_{\nu^h} u^h + e^h \partial_{\nu^h} \varphi^h - M_e^h \theta^h \cdot \nu^h = f_2^h \;\text{ on } \Gamma_2^h \times (0, T), \quad (32)$$

$$e^h \partial_{\nu^h} u^h - \beta^h \partial_{\nu^h} \varphi^h = q_2^h \;\text{ on } \Gamma_b^h \times (0, T). \quad (33)$$

We now describe the frictional contact condition on $\Gamma_3^h \times (-\infty, +\infty)$. First, we remark that from (12), (28), and (29), we find that $u_\nu = 0$, which indicates that the contact is bilateral. Thus, the contact is kept during the whole process. Now, using (12) and (28)–(31), we obtain

$$u_\tau^h = (0, 0, u^h), \quad (34)$$

$$\sigma_\tau^h = (0, 0, \sigma_\tau^h), \quad (35)$$

where

$$\sigma_\tau^h = \alpha^h \partial_\nu^h \dot{u}^h + \mu^h \partial_\nu^h u^h + e^h \partial_\nu^h \varphi^h - M_e^h \theta^h \cdot \nu^h. \quad (36)$$

We suppose that the friction is invariant concerning the x_3^h axis, being modeled with a slip-rate-dependent friction law, where the strict inequality is satisfied in the stick zone and the equality in the slip zone—that is,

$$\begin{cases} |\sigma_\tau^h| \le r(|\dot{u}_\tau^h|), \text{ on } \Gamma_3^h \times [0, T], \\ \sigma_\tau^h = -r(|\dot{u}^h|) \frac{\dot{u}_\tau^h}{|\dot{u}_\tau^h|}, \text{ on } \Gamma_3^h \times [0, T]. \end{cases} \quad (37)$$

Here, $r : \Gamma_3^h \times \mathbb{R} \longrightarrow \mathbb{R}^3$ is a given function, the friction bound, and $\dot{u}_\tau^h$ is the tangential velocity on the contact boundary: see [28,29] for details. By (34) and (36), we ascertain that the conditions in (37) are given by

$$\begin{cases} |\alpha^h \partial_\nu^h \dot{u}^h + \mu^h \partial_\nu^h u^h + e^h \partial_\nu^h \varphi^h - M_e^h \theta^h \cdot \nu^h| \le r(|\dot{u}^h|), \\ \alpha^h \partial_\nu^h \dot{u}^h + \mu^h \partial_\nu^h u^h + e^h \partial_\nu^h \varphi^h - M_e^h \theta^h \cdot \nu^h = -r(|\dot{u}^h|) \frac{\dot{u}^h}{|\dot{u}^h|}, \textit{ on } \Gamma_3^h \times [0, T]. \end{cases} \quad (38)$$

Finally, we present the initial conditions:

$$u^h(0) = u_0^h, \quad \varphi^h(0) = \varphi_0^h, \quad \theta^h(0) = \theta_0^h \quad \text{in } \Omega^h, \quad (39)$$

where u_0^h, φ_0^h, and θ_0^h are given functions on Ω^h. Putting this all together, we obtain the problem under investigation.

Problem 1. *Determine a displacement field* $u^h : \Omega^h \times [0, T] \to \mathbb{R}$, *the electric field* $\varphi^h : \Omega^h \times [0, T] \to \mathbb{R}$, *and a temperature field* $\theta^h : \Omega^h \times [0, T] \to \mathbb{R}_+$, *such that*

$$\operatorname{div}(\alpha^h \nabla \dot{u}^h + \mu^h \nabla u^h) + \operatorname{div}(e^h \nabla \varphi^h) - M_e^h \operatorname{div}(\theta^h) + f_0^h = 0, \ in\ \Omega^h \times (0,T), \tag{40}$$

$$\operatorname{div}(e^h \nabla u^h) - \operatorname{div}(\beta^h \nabla \varphi^h) = q_0^h, \ in\ \Omega^h \times (0,T), \tag{41}$$

$$\dot{\theta}^h - \operatorname{div}(k \nabla \theta^h) = -M_e^h \nabla(\dot{u}^h) + p(t), \ in\ \Omega^h \times (0,T), \tag{42}$$

$$u^h = 0\ , \ on\ \Gamma_1^h \times (0,T), \tag{43}$$

$$\alpha^h \partial_{\nu^h} \dot{u}^h + \mu^h \partial_{\nu^h} u^h + e^h \partial_{\nu^h} \varphi^h - M_e^h \partial_{\nu^h} \theta^h = f_2^h, \ on\ \Gamma_2^h \times (0,T), \tag{44}$$

$$\begin{cases} |\alpha^h \partial_{\nu^h} \dot{u}^h + \mu^h \partial_{\nu^h} u^h + \partial_\nu^h \varphi^h - M_e^h \theta^h \cdot \nu^h| \leq r(|\dot{u}^h|), \\ \alpha^h \partial_{\nu^h} \dot{u}^h + \mu^h \partial_{\nu^h} u^h + \partial_\nu^h \varphi^h - M_e^h \theta^h \cdot \nu^h = -r(|\dot{u}^h|)\frac{\dot{u}^h}{|\dot{u}^h|}, \ on\ \Gamma_3^h \times (0,T), \end{cases} \tag{45}$$

$$\varphi^h = 0, \ on\ \Gamma_a^h \times (0,T), \tag{46}$$

$$e^h \partial_{\nu^h} u^h - \beta^h \partial_{\nu^h} \varphi^h = q_2^h, \ on\ \Gamma_b^h \times (0,T), \tag{47}$$

$$\theta_0^h = 0, \ on\ \Gamma_1^h \cup \Gamma_2^h \times (0,T), \tag{48}$$

$$-\mathcal{K}_{i,j}^h \frac{\partial \theta^h}{\partial x_j^h} v_i^h = \mathcal{K}_e^h(\theta^h - \theta_R^h), \ on\ \Gamma_3^h \times (0,T), \tag{49}$$

$$u^h(0) = u_0^h, \quad \varphi^h(0) = \varphi_0^h, \quad \theta^h(0) = \theta_0^h, \ in\ \Omega^h \times (0,T). \tag{50}$$

It should be noted that the problem we consider here is different from that of [20]. Indeed, in [20] they studied electro-visco-elastic material, meaning its behavior was influenced solely by electrical and mechanical factors. The focus was purely on the electrical impact without thermal interference, represented by equation (2.13) of [20]. By contrast, in our paper we add the factor of temperature, which plays a significant role in altering electromechanical properties through heating and cooling. For instance, the electro-rheological frictional force may vary depending on the temperature, thereby impacting the rheological properties of the material. Therefore, we can say that the material's behavior in our study is influenced by mechanical, electrical, and thermal factors, represented by Equation (16), followed by the associated conditions boundary in the Problem 1. On the other hand, the addition of the temperature factor also provides us with an opportunity to explore and search for a different and simpler mathematical-solution method (the decoupling-of-unknowns method). Additionally, there is a difference between the friction conditions. Indeed, in [20] the authors used Tresca's friction law (equation (2.25) there), which is characterized by setting a maximum limit for slip. It relies solely on the maximum force without considering time rates. This was what the equation described there, observing that the friction bound was constant and had one condition, making it easier to solve the posed problem. By contrast, in our paper we use the slip-rate-dependent friction law (37): here, the friction level depends on the rate of slip changes over time. This implies that variations in the slip rate impact the friction level (the friction being higher when the rate of change is greater and lower when the rate of change is smaller). This behavior is described by Equation (37), where we represent friction bound by the variable r.

4. Variational Formulation

To study our Problem 1, we assume that the viscosity coefficient, the electric permittivity coefficient, the Lame coefficient, and the piezoelectric coefficient, satisfy

$$\alpha^h \in L^\infty(\Omega^h) \text{ and there exists } \alpha^{h*} > 0, \text{ such that } \alpha^h(x^h) \geq \alpha^{h*}, \text{ a.e., } x^h \in \Omega^h, \quad (51)$$
$$\beta^h \in L^\infty(\Omega^h) \text{ and there exists } \beta^{h*} > 0, \text{ such that } \beta^h(x^h) \geq \beta^{h*}, \text{ a.e., } x^h \in \Omega^h, \quad (52)$$
$$\mu^h \in L^\infty \text{ and } \mu^h(x^h) > 0, \text{ a.e., } x^h \in \Omega^h, \quad (53)$$
$$e^h \in L^\infty, \quad (54)$$

respectively. We also assume that the thermal tensors $M_e^h = \left(M_{ij}\right) : \Omega^h \to \mathcal{S}^3$ satisfy

$$M_{ij} = M_{ji} \in L^\infty\left(\Omega^h\right), \ 1 \leqslant i, j \leqslant 3. \quad (55)$$

The thermal conductivity tensor $\widetilde{\mathcal{K}^h} = \left(\mathcal{K}_{ij}^h\right) : \Omega^h \to \mathcal{S}^3$ satisfies

$$\begin{cases} \text{(i) } \mathcal{K}_{ij}^h = \mathcal{K}_{ji}^h \in L^\infty\left(\Omega^h\right), \ 1 \leqslant i, j \leqslant 3, \\ \text{(ii) there exists } m_{\mathcal{K}^h} > 0, \text{ such that} \\ \mathcal{K}^h s \cdot s \geq m_{\mathcal{K}^h} |s|^2, \text{ a.e., } x \in \Omega^h, \forall s \in \mathbb{R}^3. \end{cases} \quad (56)$$

The boundary thermal data satisfy

$$p^h \in W^{1,2}(0,T; L^2(\Omega^h)), \ \theta_R^h \in W^{1,2}(0,T; L^2(\Gamma_3^h)), \ K_e^h \in L^\infty\left(\Omega^h, \mathbb{R}_+\right). \quad (57)$$

The forces, tractions, volume, and surface free charge densities have the following regularity:

$$f_0^h \in W^{1,2}(0,T, L^2(\Omega^h)), \ f_2^h \in W^{1,2}(0,T, L^2(\Gamma_2^h)), \quad (58)$$
$$q_0^h \in W^{1,2}(0,T, L^2(\Omega^h)), \quad (59)$$
$$q_2^h \in W^{1,2}(0,T, L^2(\Gamma_b^h)), \ q_2^h = 0, \text{ a.e., } x^h \in \Gamma_b^h, \quad (60)$$

respectively. The friction bound satisfies

$$\begin{cases} a) \ r : \Gamma_3^h \times \mathbb{R} \longrightarrow \mathbb{R}^+, \\ b) \ \exists \, L_r > 0, \ \text{ such that } |r(x^h, s_1) - r(x^h, s_2)| \leq L_r |s_1 - s_2|, \\ c) \ x^h \longrightarrow r(x^h, s) \text{ is Lebesgue-measurable on } \Gamma_3^h, \ \forall \, s \in \mathbb{R}, \\ d) \text{ the mapping } x^h \longrightarrow r(x^h, 0) \text{ belongs to } L^2(\Gamma_3^h). \end{cases} \quad (61)$$

We define the functional $j : v^h \times v^h \longrightarrow \mathbb{R}$ by

$$j(\dot{u}^h, v^h) = \int_{\Gamma_3^h} r(|\dot{u}^h|)|v^h| da, \ \forall v^h \in V^h. \quad (62)$$

Let η_1, η_2, v_1, $v_2 \in V$. By using (61) and (62), we find that

$$j(\eta_1, v_2^h) - j(\eta_1, v_1^h) + j(\eta_2, v_1^h) - j(\eta_2, v_2^h)| \leq L_r \|\eta_1 - \eta_2\|_{L^2(\Gamma_3^h)} \|v_1 - v_2\|_{L^2(\Gamma_3^h)},$$

and applying the norm mentioned in (4), one obtains

$$L_r \|\eta_1 - \eta_2\|_{L^2(\Gamma_3^h)} \|v_1 - v_2\|_{L^2(\Gamma_3^h)} \leq c_V^2 L_r \|\eta_1 - \eta_2\|_{V^h} \|v_1^h - v_2^h\|_{V^h}.$$

Hence, we conclude that

$$j(\eta_1, v_2^h) - j(\eta_1, v_1^h) + j(\eta_2, v_1^h) - j(\eta_2, v_2^h) \leq c_V^2 L_r \|\eta_1 - \eta_2\|_{V^h} \|v_1^h - v_2^h\|_{V^h}. \quad (63)$$

The initial data verify

$$u_0^h \in V^h, \ \theta_0^h \in L^2(\Omega^h). \quad (64)$$

We use functions $f^h : [0, T] \longrightarrow V^h$ and $q^h : [0, T] \longrightarrow W^h$ as

$$(f^h, v^h)_{V^h} = \int_{\Omega^h} f_0^h v^h dx^h + \int_{\Gamma_2^h} f_2^h v^h da, \ \forall v^h \in V^h, \tag{65}$$

$$(q^h, \psi^h)_{V^h} = \int_{\Gamma_b^h} q_2^h \psi^h dx^h + \int_{\Omega^h} q_0^h \psi^h da, \ \forall \psi^h \in W^h. \tag{66}$$

The definitions of f^h and q^h are based on Riesz's representation theorem. Therefore, by using assumptions (65) and (66), we ascertain that the above integrals are well defined and that

$$f^h \in W^{1,2}(0, T, V^h), \tag{67}$$

$$q^h \in W^{1,2}(0, T, W^h). \tag{68}$$

Then, the function $P : [0, T] \longrightarrow (E^h)'$ and the operators $\widetilde{\mathcal{K}^h} : E \longrightarrow (E^h)'$, $\widetilde{\mathcal{M}^h} : V^h \longrightarrow (E^h)'$ are defined by

$$\langle P(t), \mu^h \rangle_{(E^h)' \times E^h} = \int_{\Omega^h} p^h \mu^h dx^h + \int_{\Gamma_3^h} \mathcal{K}_e^h \theta_R^h \mu^h ds,$$

$$\langle \widetilde{\mathcal{K}^h} \tau, \mu^h \rangle_{(E^h)' \times E^h} = \int_{\Gamma_3^h} \mathcal{K}_e^h \tau \mu^h ds + \sum_{i,j=1}^{d} \int_{\Omega^h} \mathcal{K}_{i,j}^h \frac{\partial \mu^h}{\partial x_j^h} \frac{\partial \mu^h}{\partial x_i^h} dx,$$

$$\langle \widetilde{\mathcal{M}^h} v^h, \mu^h \rangle_{(E^h)' \times E^h} = -\int_{\Omega^h} (M_e^h \nabla v^h) \mu^h dx + \int_{\Gamma_3^h} h_\tau(|v_\tau^h|) \mu^h ds$$

for all $v^h \in V^h$, $\tau \in E^h$, $\mu^h \in E^h$. Moreover, we define the bilinear forms $a_{\alpha^h} : V^h \times V^h \longrightarrow \mathbb{R}$, $a_{\mu^h} : V^h \times V^h \longrightarrow \mathbb{R}$, $a_{e^h} : V^h \times W^h \longrightarrow \mathbb{R}$, $a_{e^h} : W^h \times V^h \longrightarrow \mathbb{R}$, $a_{\beta^h} : W^h \times W^h \longrightarrow \mathbb{R}$ and $a_{M^h} : E^h \times V^h \longrightarrow \mathbb{R}$ by

$$a_{\alpha^h}(u^h, v^h) = \int_{\Omega^h} \alpha^h \nabla u^h \cdot \nabla v^h dx, \tag{69}$$

$$a_{\mu^h}(u^h, v^h) = \int_{\Omega^h} \mu^h \nabla u^h \cdot \nabla v^h dx, \tag{70}$$

$$a_{M^h}(u^h, v^h) = -\int_{\Omega^h} M_e^h u^h \cdot \nabla v^h dx, \tag{71}$$

$$a_{e^h}(u^h, \varphi^h) = \int_{\Omega^h} e^h \nabla u^h \cdot \nabla v^h dx = a_{e^h}(v^h, u^h), \tag{72}$$

$$a_{\beta^h}(\varphi^h, \psi^h) = \int_{\Omega^h} \beta^h \nabla \varphi^h \cdot \nabla \psi^h dx \tag{73}$$

for all $u^h, v^h \in V^h$, $\varphi^h, \psi^h \in W^h$. We note that by assumptions (51)–(54), the above integrals are well defined. Using the definition of norms (1)–(5), we see that the forms a_{α^h}, a_{μ^h}, a_{e^h}, and a_{β^h}, are continuous and that the forms a_{α^h}, a_{μ^h}, and a_{β^h} are symmetric. Furthermore, the form a_{α^h} is V-elliptic, i.e.,

$$a_{\alpha^h}(u^h, v^h) \le \|\alpha^h\|_{L^\infty(\Omega^h)} \|u^h\|_{V^h} \|v^h\|_{V^h}, \ \forall u^h, v^h \in V^h, \tag{74}$$

$$a_\alpha^h(v^h, v^h) \ge \alpha^{h*} \|v^h\|_{V^h}^2 \ \ \forall v^h \in V^h. \tag{75}$$

Now, we can state the variational formulation of our Problem 1.

Problem 2 (Variational Problem)**.** *Determine a displacement field $u^h : [0, T] \longrightarrow V^h$, an electric-potential field $\varphi^h : [0, T] \longrightarrow W^h$, and a temperature field $\theta^h : [0, T] \longrightarrow E^h$, such that*

$$
\begin{aligned}
&a_{\alpha^h}(\dot{u}^h(t), v^h - \dot{u}^h(t)) + a_{\mu^h}(u^h(t), v^h - \dot{u}^h(t)) + a_{e^h}(\varphi(t), v - \dot{u}^h(t)) + a_{M^h}(\theta^h(t), v^h - \dot{u}^h(t)) \\
&\quad + j(\dot{u}^h(t), v^h) - j(\dot{u}^h(t), \dot{u}^h(t)) \geq (f^h, v^h - \dot{u}^h)_{V^h},\ \forall v^h \in V^h,\ t \in [0,T],
\end{aligned} \tag{76}
$$

$$
a_{\beta^h}(\varphi^h, \psi^h) - a_e(u^h, \psi^h) = (q^h, \psi)^h_W,\ \forall \psi^h \in W^h, \tag{77}
$$

$$
\dot{\theta}^h(t) + \widetilde{\mathcal{K}^h}\theta^h(t) = \widetilde{\mathcal{M}^h}\dot{u}^h(t) + P(t),\ in\ E^h, \tag{78}
$$

$$
u^h(0) = u_0^h, \quad \varphi^h(0) = \varphi_0^h, \quad \theta^h(0) = \theta_0^h. \tag{79}
$$

5. Existence and Uniqueness Result

In this section, we use the variational formulation to prove the existence and uniqueness of a weak solution to Problem 1. For that, we make use of some auxiliary problems and lemmas.

Problem 3 (Auxiliary Problem). *Determine a displacement field $u^h : [0,T] \longrightarrow V^h$, such that*

$$
\begin{aligned}
&a^h(u^h(t), v^h - \dot{u}^h(t)) + b^h(\dot{u}^h(t), v^h - \dot{u}^h(t)) + j^h(\dot{u}^h(t), v^h) - j^h(\dot{u}^h(t), \dot{u}^h(t)) \\
&\qquad \geq (F(t), v^h - \dot{u}^h(t))_{V^h},\ \forall v^h \in V^h, \quad t \in [0,T],
\end{aligned} \tag{80}
$$

$$
u^h(0) = u_0^h. \tag{81}
$$

In order to study Auxiliary Problem 3, we assume that

$$
\begin{cases}
a^h : V^h \times V^h \longrightarrow \mathbb{R} \text{ is a bilinear form and there exists } M^h > 0, \text{ such that} \\
|a^h(u^h, v^h)| \leq M^h \|u^h\|_{V^h} \|v^h\|_{V^h},\ \forall u^h, v^h \in V^h;
\end{cases} \tag{82}
$$

$$
\begin{cases}
b^h : V^h \times V^h \longrightarrow \text{ is a bilinear symmetric form verifying:} \\
(a) \text{ There exists } M^{h'} > 0, \text{ such that } |b^h(u^h, v^h)| \leq M^{h'} \|u^h\|_{V^h} \|v^h\|_{V^h},\ \forall u^h, v^h \in X^h, \\
(b) \text{ there exists } m^{h'} > 0, \text{ such that } b^h(v^h, v^h) \geq m^{h'} \|v^h\|^2_{V^h},\ \forall v^h \in V^h;
\end{cases} \tag{83}
$$

$$
\begin{cases}
j^h : V^h \times V^h \longrightarrow \mathbb{R} \text{ satisfies:} \\
(a) \text{ For all } \eta \in V^h,\ j^h(\eta, \cdot) \text{ is convex and l.s.c. on } V^h, \\
(b) \text{ there exists } \alpha^{h'} \geq 0, \text{ such that for all } \eta_1, \eta_2, v_1^h, v_2^h \in V^h \text{ we have} \\
|j^h(\eta_1, v_2^h) - j^h(\eta_1, v_1^h) + j^h(\eta_2, v_1^h) - j^h(\eta_2^h, v_2^h)| \leq \alpha^{h'} \|\eta_1 - \eta_2\|_{V^h} \|v_1^h - v_2^h\|_{V^h};
\end{cases} \tag{84}
$$

$$
u_0^h \in V^h; \tag{85}
$$

$$
F \in W^{1,2}(0,T,V^h). \tag{86}
$$

From this point onward, we denote a generic constant as $c^h > 0$.

Lemma 1. *Assume that (82)–(86) holds. If $m^{h'} > \alpha^{h'}$, then there exists a unique solution $u^h \in W^{1,2}(0,T,V^h)$ to Auxiliary Problem 3.*

Proof. By using (80), for any t_1 and $t_2 \in [0,T]$ we find that

$$
\begin{aligned}
&a^h(u^h(t_1), v^h - \dot{u}^h(t_1)) + b^h(\dot{u}^h(t_1), v^h - \dot{u}^h(t_1)) + j^h(\dot{u}^h(t_1), v^h) - j^h(\dot{u}^h(t_1), \dot{u}^h(t_1)) \\
&\qquad \geq (F(t_1), v^h - \dot{u}^h(t_1))_{V^h},\ \forall v^h \in V^h,
\end{aligned} \tag{87}
$$

$$
\begin{aligned}
&a^h(u^h(t_2), v^h - \dot{u}^h(t_2)) + b^h(\dot{u}^h(t_2), v^h - \dot{u}^h(t_2)) + j^h(\dot{u}(t_2), v) - j^h(\dot{u}^h(t_2), \dot{u}^h(t_2)) \\
&\qquad \geq (F(t_2), v^h - \dot{u}^h(t_2))_{V^h},\ \forall v^h \in V^h.
\end{aligned} \tag{88}
$$

We take $v^h = \dot{u}^h(t_2)$ in (87) and $v^h = \dot{u}^h(t_1)$ in (88) and, adding the two inequalities, we obtain

$$b^h(\dot{u}^h(t_1) - \dot{u}^h(t_2), \dot{u}^h(t_1) - \dot{u}^h(t_2)) + j(\dot{u}^h(t_1), \dot{u}^h(t_2)) - j^h(\dot{u}^h(t_1), \dot{u}^h(t_2)) + j^h(\dot{u}^h(t_2), \dot{u}^h(t_1))$$
$$- j^h(\dot{u}^h(t_2), \dot{u}^h(t_2)) \leq a^h(u^h(t_1) - u^h(t_2), \dot{u}^h(t_2) - \dot{u}^h(t_1)) + (F(t_1) - F(t_2), \dot{u}^h(t_2) - \dot{u}^h(t_1)). \tag{89}$$

We now use assumptions (82)–(84) to obtain

$$\|\dot{u}^h(t_1) - \dot{u}^h(t_2)\|_{V^h} \leq c^h(\|u^h(t_1) - u^h(t_2)\|_{V^h} + \|F(t_1) - F(t_2)\|_{V^h}), \tag{90}$$

where $c^h = \max\left\{\dfrac{M^h}{m^{h'} - \alpha^{h'}}, \dfrac{1}{m^{h'} - \alpha^{h'}}\right\}$. This inequality, combined with the regularity $u^h \in C^1(0, t, V^h)$, shows that $\dot{u}^h : [0, T] \longrightarrow V^h$ is an absolutely continuous function. Furthermore, it should be noted that

$$\|\ddot{u}^h(t)\|_{V^h} \leq c^h(\|\dot{u}^h(t)\|_{V^h} + \|F(t)\|_{V^h}) \text{ a.e., } t \in [0, T].$$

We conclude that $u^h \in W^{2,2}(0, T, V^h)$. $\square$

Using the solution u^h asserted by Lemma 1, we proceed by considering a second auxiliary problem.

Problem 4 (Auxiliary Problem). *Determine a function $\theta^h_\eta : [0, T] \longrightarrow E^h$, such that*

$$\begin{cases} \dot{\theta}^h_\eta(t) + \widetilde{\mathcal{K}^h}\theta^h_\eta(t) = P(t) + \widetilde{\mathcal{M}^h}\dot{u}^h_\eta(t), \\ \theta^h_\eta(0) = \theta^h_0, \quad t \in [0, T]. \end{cases} \tag{91}$$

Lemma 2. *For all $\eta \in C(0, T, V^h)$, there exists a unique solution to Auxiliary Problem 4 with*

$$\theta^h_\eta \in L^2\left(0, T, E^h\right) \cap C\left([0, T], L^2\left(\Omega^h\right)\right) \cap W^{1,2}(0, T, (E^h)'). \tag{92}$$

Moreover, there exists $c^h > 0$, such that, for all $\eta_1, \eta_2 \in C(0, T, V^h)$, we have

$$\|\theta^h_{\eta_1}(t) - \theta^h_{\eta_2}(t)\|_{E^h} \leq c^h \int_0^t \|\dot{u}^h_{\eta_1}(s) - \dot{u}^h_{\eta_2}(s)\|^2_{V^h}\, ds, \quad \forall t \in [0, T]. \tag{93}$$

Proof. The existence and uniqueness result to (91) follows by considering the Gelfand evolution (see, e.g., [30]):

$$E^h \subset F^h \equiv (F^h)' \subset (E^h)',$$

and verifying that the operator $\widetilde{\mathcal{K}^h} : E^h \longrightarrow (E^h)'$ is strongly monotonic and linearly continuous. As inferred from the expression of the operator $\widetilde{\mathcal{M}^h}$, we have

$$\dot{u}^h_\eta(t) \in W^{1,2}(0, T; V^h) \implies \widetilde{\mathcal{M}^h}\dot{u}^h_\eta(t) \in W^{1,2}(0, T, F^h),$$

and

$$P(t) \in W^{1,2}(0, T, E^h) \implies \widetilde{\mathcal{M}^h}\dot{u}^h_\eta(t) + P(t) \in W^{1,2}(0, T, (E^h)').$$

For $\eta_1, \eta_2 \in C(0, T, V^h)$, we have

$$\left(\dot{\theta}^h_{\eta_1}(t) - \dot{\theta}^{\ h}_{\eta_2}(t), \theta^h_{\eta_1}(t) - \theta^h_{\eta_2}(t)\right)_{(E^h)' \times E^h} + \left(\widetilde{\mathcal{K}^h}\theta^h_{\eta_1}(t) - \widetilde{\mathcal{K}^h}\theta^h_{\eta_2}(t), \theta^h_{\eta_1}(t) - \theta^h_{\eta_2}(t)\right)_{E^h} \tag{94}$$
$$= \left(\widetilde{\mathcal{M}^h}\dot{u}^h_{\eta_1}(t) - \widetilde{\mathcal{M}^h}\dot{u}^h_{\eta_2}(t), \theta^h_{\eta_1}(t) - \theta^h_{\eta_2}(t)\right)_{L^2(\Omega^h)'}, \quad t \in [0, T],$$

and, by integrating the last property over $(0,t)$ and utilizing the Lipschitz continuity of $\widetilde{\mathcal{M}^h} : V^h \longrightarrow (E^h)'$, as well as the strong monotonicity of $\widetilde{\mathcal{K}^h}$, we deduce that (93) holds for $t \in [0,T]$. □

We are now ready to prove our main result: the existence of a unique solution to our Problem 1.

Theorem 1. *Assume that (51)–(68) are satisfied. Then, there exists Z_0, which depends on Ω^h, Γ_1^h, Γ_2^h, and Γ_3^h, such that if $L_r < Z_0$ then there exists a unique solution $(u^h, \varphi^h, \theta^h)$ to (VP), satisfying*

$$u^h \in W^{2,2}(0,T,V^h), \tag{95}$$

$$\theta^h \in L^2\left(0,T,E^h\right) \cap C\left([0,T], L^2\left(\Omega^h\right)\right) \cap W^{1,2}(0,T,(E^h)'), \tag{96}$$

$$\varphi^h \in W^{1,2}(0,T,W^h). \tag{97}$$

Proof. From (77) we ascertain that

$$(\beta^h \varphi^h, \psi^h)_{W^h} - (e^h u^h, \psi^h)_{W^h} = (q^h, \psi^h)_{W^h}, \tag{98}$$

while the use of (98) gives

$$\beta^h \varphi^h(t) = e^h u^h(t) + q^h.$$

Hence, we deduce that

$$\varphi^h(t) = \frac{e^h}{\beta^h} u^h(t) + \frac{q^h}{\beta^h}. \tag{99}$$

From (78) and (79), we obtain the solution to

$$\theta^h(t) = \int_0^t \widetilde{\mathcal{M}^h} e^{-\int_s^t \widetilde{\mathcal{K}^h} dx} \dot{u}^h(s) ds + \int_0^t e^{-\int_s^t \widetilde{\mathcal{K}^h} dx} P(s) ds + \theta_0^h e^{-\int_0^t \widetilde{\mathcal{K}^h} ds}. \tag{100}$$

Taking (99) and (100) and substituting them into (76), we obtain

$$\begin{aligned} &a_{\theta^h}(\dot{u}^h(t), v^h - \dot{u}^h(t)) + a_{\mu^h}(u^h(t), v^h - \dot{u}^h(t)) + a_{e^h}\left(\frac{e^h}{\beta^h} u^h(t) + \frac{q^h}{\beta^h}, v^h - \dot{u}^h(t)\right) \\ &\quad + a_{M^h}\left(\int_0^t \widetilde{\mathcal{M}^h} e^{-\int_s^t \widetilde{\mathcal{K}^h} dx} \dot{u}^h(s) ds + \int_0^t e^{-\int_s^t \widetilde{\mathcal{K}^h} dx} P(s) ds + \theta_0^h e^{-\int_0^t \widetilde{\mathcal{K}^h} ds}, v^h - \dot{u}^h(t)\right) \\ &\quad + j^h(\dot{u}^h(t), v^h) - j^h(\dot{u}^h(t), \dot{u}^h(t)) \geq (f^h(t), v^h - \dot{u}^h)_{V^h}, \quad \forall v^h \in V^h,\ t \in [0,T], \\ &\quad u^h(0) = u_0^h. \end{aligned}$$

Next, we define the bilinear forms $a^h : V^h \times V^h \longrightarrow \mathbb{R}$ and $b^h : V^h \times V^h \longrightarrow \mathbb{R}$ as follows:

$$\begin{aligned} a^h(u^h(t), v^h - \dot{u}^h(t)) &= a_{\mu^h}(u^h(t), v^h - \dot{u}^h(t)) + a_{e^h}\left(\frac{e^h}{\beta^h} u^h(t), v^h - \dot{u}^h(t)\right), \\ &\quad + a_{M^h}\left(\int_0^t \widetilde{\mathcal{M}^h} e^{-\int_s^t \widetilde{\mathcal{K}^h} dx} \dot{u}^h(s) ds, v^h - \dot{u}^h(t)\right), \end{aligned} \tag{101}$$

$$b^h(\dot{u}^h(t), v^h - \dot{u}^h(t)) = a_{\theta^h}(\dot{u}^h(t), v^h - \dot{u}^h(t)). \tag{102}$$

We also consider the function $F : [0,T] \longrightarrow V^h$, defined by

$$\begin{aligned} (F(t), v^h - \dot{u}^h)_{V^h} &= (f^h(t), v^h - \dot{u}^h)_{V^h} - a_{e^h}\left(\frac{q^h}{\beta^h}, v^h - \dot{u}^h(t)\right) \\ &\quad - a_{M^h}\left(\int_0^t e^{-\int_s^t \widetilde{\mathcal{K}^h} dx} P(s) ds + \theta_0^h e^{-\int_0^t \widetilde{\mathcal{K}^h} ds}, v^h - \dot{u}^h(t)\right). \end{aligned} \tag{103}$$

From (101)–(103), we obtain the following results:

- the bilinear form a^h and the initial data u_0^h satisfy conditions (82) and (85);
- $f^h \in W^{1,2}(0,T,V^h)$, $q^h \in W^{1,2}(0,T,W^h)$, and $P \in W^{1,2}(0,T,E^h)$, combined with the definition of $F(\cdot)$ in (103), satisfy (86);
- for all $\eta \in V^h$, the functional $j^h(\eta,\cdot) : V^h \longrightarrow \mathbb{R}$ is a continuous seminorm on V^h, thus satisfying the condition (84)-(a) (recall that j^h also fulfills inequality (63), demonstrating that condition (84)-(b) holds with $\alpha^{h'} = c_V L_r$);
- from (74), it is evident that the bilinear form b^h satisfies condition (84) with $m^{h'} = \alpha^{h*}$.

Now, we choose $Z_0 = \dfrac{\alpha^{h*}}{c_V^2}$, which depends on Ω^h, Γ_1^h, Γ_2^h, Γ_3^h, and $\alpha^{h'}$. Then, if $L_r < Z_0$, we have $m^{h'} > \alpha^{h'}$. Therefore, (95) and (96) are a direct consequence of Lemmas 1 and 2. For the proof of (97), let $\varphi^h : [0,T] \longrightarrow W^h$ be the electrical-potential field defined by (99). We have

$$\begin{cases} u^h \in W^{2,2}(0,T,V^h) \\ q^h \in W^{1,2}(0,T,W^h) \end{cases} \Rightarrow \varphi^h \in W^{1,2}(0,T,W^h),$$

which completes the proof. □

6. Conclusions

In this paper, we theoretically investigated a quasi-static-antiplane contact problem with a slip-rate-dependent friction law involving a thermo-electro-visco-elastic body. The slip-rate-dependent friction law was interesting to investigate because the friction level depends on the rate at which the slip changes over time. This implies that variations in the slip rate impact the friction level, with the friction force increasing as the slip-rate speed increases, and decreasing as the slip-rate speed decreases. This is more reasonable because it considers both the quasi-static equilibrium conditions and the localized dynamic interactions that may occur within the contact interface.

We used Green's formula to derive the variational formulation of our problem. Following that, we demonstrated the existence and uniqueness of the weak solution, using various techniques, including the time-dependent variational equation, the variational-evolution equation, the differential equation, and the decoupling of unknowns. The advantages of the developed model include both the quasi-static equilibrium conditions and the localized dynamic interactions that may occur within the contact interface. This approach provides a more comprehensive and realistic representation of the material response under varying thermal and loading conditions. One of the drawbacks we faced was the time consumption and the difficulty of providing practical examples without resorting to numerical analysis and simulation.

We have numerous prospects ahead. In particular, we intend to conduct work on numerical analysis and simulation based on the results obtained for the studied problem.

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References

1. Dosaev, M.; Samsonov, V.; Bekmemetev, V. Comparison between 2D and 3D simulation of contact of two deformable axisymmetric bodies. *Int. J. Nonlinear Sci. Numer. Simul.* **2020**, *21*, 123–133. [CrossRef]
2. Fernandez, F.; Puso, M.A.; Solberg, J.; Tortorelli, D.A. Topology optimization of multiple deformable bodies in contact with large deformations. *Comput. Methods Appl. Mech. Eng.* **2020**, *371*, 113288. [CrossRef]
3. Biber, S.W.; Champneys, A.R.; Szalai, R. Analysis of point-contact models of the bounce of a hard spinning ball on a compliant frictional surface. *IMA J. Appl. Math.* **2023**, *88*, 498–523. [CrossRef]
4. Shaat, M.; Gao, X.-L.; Li, K.; Littlefield, A.G. New analytical model for thermomechanical responses of multi-layered structures with imperfect interfaces. *Acta Mech.* **2023**, *234*, 5779–5818. [CrossRef]
5. Duvaut, G.; Lions, J.-L. *Les Inéquations en Mécanique et en Physique*; Dunod: Paris, France, 1972.
6. Lions, J.-L. *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*; Gauthier-Villars: Paris, France, 1969.
7. Rakici, S.; Kim, J. A stabilized non-ordinary peridynamic model for linear piezoelectricity. *Appl. Math. Model.* **2024**, *125*, 514–538. [CrossRef]
8. Xu, J.; Xu, X. On uniqueness of recovering coefficients from localized Dirichlet-to-Neumann map for piecewise homogeneous piezoelectricity. *SIAM J. Math. Anal.* **2023**, *55*, 571–602. [CrossRef]
9. Benaissa, H.; Essoufi, E.-H.; Fakhar, R. Existence results for unilateral contact problem with friction of thermo-electro-elasticity. *Appl. Math. Mech.* **2015**, *36*, 911–926. [CrossRef]
10. Benaissa, H.; Essoufi, E.-H.; Fakhar, R. Analysis of a Signorini problem with nonlocal friction in thermo-piezoelectricity. *Glas. Mat. Ser. III* **2016**, *51*, 391–411. [CrossRef]
11. Ammar, T.H. Quasistatic contact problem between thermo-electroelastic bodies with long-term memory and adhesion. *Malaya J. Mat.* **2016**, *4*, 211–223. [CrossRef] [PubMed]
12. Sládek, J.; Sládek, V.; Stanak, P. Analysis of thermo-piezoelectricity problems by meshless method. *Acta Mech. Slovaca* **2010**, *14*, 16–27. [CrossRef]
13. Tiersten, H.F. On the nonlinear equations of thermoelectroelasticity. *Internat. J. Engrg. Sci.* **1971**, *9*, 587–604. [CrossRef]
14. Ahmed, A.A.; Boutechebak, S. Analysis of a dynamic thermo-elastic-viscoplastic contact problem. *Electron. J. Qual. Theory Differ. Equ.* **2013**, *2013*, 17. [CrossRef]
15. Merouani, A.; Messelmi, F. Dynamic evolution of damage in elastic-thermo-viscoplastic materials. *Electron. Differ. Equ.* **2010**, *2010*, 1–15.
16. Sofonea, M. Functional Methods in Thermo-Elasto-Visco-Plasticity. Ph.D. Thesis, University of Bucharest, Bucharest, Romania, 1988.
17. Adly, S.; Chau, O. On some dynamic thermal non clamped contact problems. *Math. Program.* **2013**, *139*, 5–26. [CrossRef]
18. Andrews, K.T.; Shillor, M.; Wright, S.; Klarbring, A. A dynamic thermoviscoelastic contact problem with friction and wear. *Internat. J. Engrg. Sci.* **1997**, *35*, 1291–1309. [CrossRef]
19. Dalah, M. Analysis of electro-viscoelastic antiplane contact problem with total slip rate dependent friction. *Electron. J. Differ. Equ.* **2009**, *2009*, 15.
20. Dalah, M.; Sofonea, M. Antiplane frictional contact of electro-viscoelastic cylinders. *Electron. J. Differ. Equ.* **2007**, *2007*, 14.
21. Matei, A.; Motreanu, V.V.; Sofonea, M. A quasistatic antiplane contact problem with slip dependent friction. *Adv. Nonlinear Var. Inequal.* **2001**, *4*, 1–21.
22. Mindlin, R.D. Polarization gradient in elastic dielectrics. *Int. J. Solids Structures* **1968**, *4*, 637–642. [CrossRef]
23. Rachid, C.; Nemira, L. Analysis of a thermo-viscoelastic antiplane contact problem with long-term memory. *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)* **2021**, *67*, 97–111. [CrossRef]
24. Sofonea, M.; Dalah, M.; Ayadi, A. Analysis of an antiplane electro-elastic contact problem. *Adv. Math. Sci. Appl.* **2007**, *17*, 385–400.
25. Sofonea, M.; Matei, A. *Variational Inequalities with Applications: A Study of Antiplane Frictional Contact Problems*; Springer: New York, NY, USA, 2009.
26. Horgan, C.O. Anti-plane shear deformations in linear and nonlinear solid mechanics. *SIAM Rev.* **1995**, *37*, 53–81. [CrossRef]
27. Horgan, C.O.; Miller, K.L. Antiplane shear deformations for homogeneous and inhomogeneous anisotropic linearly elastic solids. *Trans. ASME J. Appl. Mech.* **1994**, *61*, 23–29. [CrossRef]
28. Han, W.; Sofonea, M. *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*; American Mathematical Society: Providence, RI, USA; RIInternational Press: Somerville, MA, USA, 2002.
29. Shillor, M.; Sofonea, M.; Telega, J.J. *Models and Analysis of Quasistatic Contact: Variational Methods*; Springer: Berlin/Heidelberg, Germany, 2004.
30. Zeidler, E. *Nonlinear Functional Analysis and Its Applications*; II/A; Springer: New York, NY, USA, 1990.