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Intuitionistic Fuzzy Modal Multi-Topological Structures and Intuitionistic Fuzzy Multi-Modal Multi-Topological Structures

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Abstract: On the basis of K. Kuratowski's definitions of a topological structure with a closure or interior operator, the concept of a modal topological structure (MTS) with one of these operators was introduced by the author. This new structure was illustrated with examples with intuitionistic fuzzy topological operators from both examples, and for this reason, these structures were named intuitionistic fuzzy MTSs (IFMTSs). In a series of papers, the author introduced some modifications and extensions to the IFMTSs, e.g., intuitionistic fuzzy temporal topological structures, intuitionistic fuzzy level topological structures and others, and intuitionistic fuzzy multi modal topological structures and others. In the present paper, four new examples of intuitionistic fuzzy multi modal topological structures are given. On their base, the concepts of a modal multi-topological structure and of a multi-modal multi-topological structure are introduced and illustrated with examples from the area of the intuitionistic fuzzy sets—intuitionistic fuzzy modal multi-topological structure with a closure or an interior operator; and intuitionistic fuzzy multi-modal multi-topological structure with one of these operators. Two intuitionistic fuzzy topological operators are defined. Their basic properties are studied and they are used in the new structures.

Keywords: intuitionistic fuzzy operation; intuitionistic fuzzy operator; intuitionistic fuzzy set; intuitionistic fuzzy topological structure

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1. Introduction

The first steps in the creation of intuitionistic fuzzy topology were conducted by A. Ban, D. Čoker, M. Demirgi, F.G. Lupiañez, T.K. Mondal, and S.K. Samanta in [1–9]. After these papers, a new series of studies was published, e.g., in [10–38]. All were devoted to giving intuitionistic fuzzy interpretations of objects from ordinary topology.

In the middle of 2022, the author suggested a new direction for the research in the area of intuitionistic fuzzy topology. The aim was to give an intuitionistic fuzzy interpretation of the concept of a topological structure and to find its suitable examples. As a result, as mentioned below, some types of topological structures were generated and their properties were studied.

In [39], K. Kuratowski described topological structures based on the topological operators from closure ($\mathcal{C}$-, cl -) and interior ($\mathcal{I}$-, in-) types (see also, e.g., [6,7,40,41]). Their extensions are the modal topological structures (MTSs) introduced by the author in [42], with the addition of the modal operators: $\Diamond$ ("possibility") from cl-type or $\Box$ ("necessity") from in-type (see, e.g., [43–45]). In [42], the MTSs were illustrated with two examples from the area of intuitionistic fuzzy sets (IFSs; see, e.g., [46])—extensions of fuzzy sets of L. Zadeh [47]—and due to this, they were called "Intuitionistic Fuzzy MTSs" (IFMTSs). In them, the topological and the modal operators have intuitionistic fuzzy forms, discussed in [46].

Later, the IFMTSs were extended in some directions, one of which to intuitionistic fuzzy bimodal topological structures (IF2MTSs, see [48]), and one of which to intuitionistic fuzzy multi-modal topological structures, which, hereafter, will be denoted by "IFmMTSs" (cf. [49], where their abbreviation is "IFMMTS").

Here, following the ideas from [42,49], we will introduce the concept of modal multi topological structures (MmTSs). In them, some different topological operators from *cl*-type and/or *in*-types will exist simultaneously by analogy with the IFmMTSs.

In Section 2, short remarks over IFSs, MTSs, and IFMTSs will be given. In Section 3, two new topological operators from *cl*- and *in*-types will be introduced, and two examples for IFmMTSs will be given. In Section 4, the concept of modal multi topological structures (MmTSs) will be introduced and illustrated with four examples of IFMmTSs. In Section 5, on the basis of all previous definitions, the concept of multi-modal multi-topological structures (mMmTSs) will be given and illustrated with four intuitionistic fuzzy mMmTSs (IFmMmTSs). Some perspectives on the development of the new topological objects will be discussed in the conclusion. There, some possible applications of the new topological structures will be discussed as an object of future research.

2. Preliminaries

First, following [46], we give some necessary definitions about intuitionistic fuzzy sets (IFSs).

With the set E, everywhere below we will denote a fixed universe.

Definition 1. *The set*

$$A^* = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}$$

is called an IFS if the functions $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ define the degrees of membership and of non-membership of an element $x \in E$, respectively, to the set A, and for each $x \in E$:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

Let $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$.

Below, for simplicity, instead of A^*, we will write A.

In [46], different intuitionistic fuzzy (IF) relations, operations, and operators from modal, topological, and other types are defined over IFSs. Here, we will use only the following five. "iff" is an abbreviation of "if and only if".

Definition 2. *For two given IFSs A and B:*

$$\begin{array}{lcl}
A \subseteq B & \textit{iff} & (\forall x \in E)(\mu_A(x) \leq \mu_B(x) \ \&\ \nu_A(x) \geq \nu_B(x)); \\
A = B & \textit{iff} & (\forall x \in E)(\mu_A(x) = \mu_B(x) \ \&\ \nu_A(x) = \nu_B(x)); \\
\neg A & = & \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\}; \\
A \cap B & = & \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\}; \\
A \cup B & = & \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\}.
\end{array}$$

The modifications of the IF modal operators "possibility" (cl-) and "necessity" (in-) are the following operators, introduced in [50] by analogy with the operators $H_{\alpha,\beta}$ and $J_{\alpha,\beta}$ defined in [46].

Definition 3. *For each IFS A:*

$$J^{\#}_{\alpha}(A) = \{\langle x, \mu_A(x), \alpha\nu_A(x)\rangle | x \in E\} = J_{0,\alpha}(A),$$
$$H^{\#}_{\alpha}(A) = \{\langle x, \alpha\mu_A(x), \nu_A(x)\rangle | x \in E\} = H_{\alpha,0}(A).$$

For them, in [50], it is proved that for each IFS A and for each $\alpha \in [0,1]$:

$$H^{\#}_{\alpha}(A) \subseteq A \subseteq J^{\#}_{\alpha}(A),$$

$$H^{\#}_{\alpha}(A) = \neg J^{\#}_{\alpha}\neg(A),$$
$$J^{\#}_{\alpha}(A) = \neg H^{\#}_{\alpha}\neg(A).$$

Following [42], we give some necessary definitions about MTSs and IFMTSs.
Let E be a fixed universe. We define the special sets.

Definition 4. *For each fixed set E:*

$$E^* = \{\langle x, 1, 0\rangle | x \in E\},$$

$$O^* = \{\langle x, 0, 1\rangle | x \in E\},$$

and

$$\mathcal{P}(E^*) = \{A | A \subseteq E^*\},$$

i.e., this is the set of all IFSs over set E.

Let X be a fixed set of subsets. let $\mathcal{O}, \mathcal{Q} : X \to X$ be operators of a cl-type and of an in-type, related to operations $\Delta, \nabla : X \times X \to X$, respectively; let $\circ, \bullet : X \to X$ be modal operators from a cl-type and of an in-type, respectively. Let the following equalities hold:

$$\begin{aligned} A\Delta B &= \neg(\neg A \nabla \neg B), \\ A\nabla B &= \neg(\neg A \Delta \neg B), \\ \mathcal{O}(A) &= \neg(\mathcal{Q}(\neg A)), \\ \mathcal{Q}(A) &= \neg(\mathcal{O}(\neg A)), \\ \circ A &= \neg \bullet \neg A, \\ \bullet A &= \neg \circ \neg A. \end{aligned}$$

Definition 5. *The structures $(cl-cl)$-IFMTS, $(cl-in)$-IFMTS, $(in-cl)$-IFMTS, and $(in-in)$-IFMTS satisfy the following conditions for every two IFSs $A, B \in X$:*

	$cl-cl$-IFMTS		$in-in$-IFMTS
CC1	$\mathcal{O}(A\Delta B) = \mathcal{O}(A)\Delta\mathcal{O}(B)$	II1	$\mathcal{Q}(A\nabla B) = \mathcal{Q}(A)\nabla\mathcal{Q}(B)$
CC2	$A \subseteq \mathcal{O}(A)$	II2	$\mathcal{Q}(A) \subseteq A$
CC3	$\mathcal{O}(O^*) = O^*$	II3	$\mathcal{Q}(E^*) = E^*$
CC4	$\mathcal{O}(\mathcal{O}(A)) = \mathcal{O}(A)$	II4	$\mathcal{Q}(\mathcal{Q}(A)) = \mathcal{Q}(A)$
CC5	$\circ(A\nabla B) = \circ A\nabla \circ B$	II5	$\bullet(A\nabla B) = \bullet A\nabla \bullet B$
CC6	$\circ A \subseteq A$	II6	$\bullet A \subseteq A$
CC7	$\circ E^* = E^*$	II7	$\bullet O^* = O^*$
CC8	$\circ \circ A = \circ A$	II8	$\bullet \bullet A = \bullet A$
CC9	$\circ\, \mathcal{O}(A) = \mathcal{O}(\circ A)$	II9	$\bullet\, \mathcal{Q}(A) = \mathcal{Q}(\bullet A)$

	$cl-in$-IFMTS		$in-cl$-IFMTS
CI1	$\mathcal{O}(A\Delta B)=\mathcal{O}(A)\Delta\mathcal{O}(B)$	IC1	$\mathcal{Q}(A\nabla B)=\mathcal{Q}(A)\nabla\mathcal{Q}(B)$
CI2	$A\subseteq\mathcal{O}(A)$	IC2	$\mathcal{Q}(A)\subseteq A$
CI3	$\mathcal{O}(O^*)=O^*$	IC3	$\mathcal{Q}(E^*)=E^*$
CI4	$\mathcal{O}(\mathcal{O}(A))=\mathcal{O}(A)$	IC4	$\mathcal{Q}(\mathcal{Q}(A))=\mathcal{Q}(A)$
CI5	$\bullet(A\nabla B)=\bullet A\nabla\bullet B$	IC5	$\circ(A\Delta B)=\circ A\Delta\circ B$
CI6	$\bullet A\subseteq A$	IC6	$A\subseteq\circ A$
CI7	$\bullet O^*=O^*$	IC7	$\circ E^*=E^*$
CI8	$\bullet\bullet A=\bullet A$	IC8	$\circ\circ A=\circ A$
CI9	$\bullet\,\mathcal{O}(A)=\mathcal{O}(\bullet A)$	IC9	$\circ\,\mathcal{Q}(A)=\mathcal{Q}(\circ A)$

In [50], it is proved that for each universe E and for each $\alpha\in[0,1]$:

- $\langle\mathcal{P}(E^*),\mathcal{C},\cup,J^{\#}_{\alpha}\rangle$ is a $(cl-cl)$-IFMTS;
- $\langle\mathcal{P}(E^*),\mathcal{I},\cap,H^{\#}_{\alpha}\rangle$ is an $(in-in)$-IFMTS;
- $\langle\mathcal{P}(E^*),\mathcal{C},\cup,H^{\#}_{\alpha}\rangle$ is a $(cl-in)$-IFMTS;
- $\langle\mathcal{P}(E^*),\mathcal{I},\cap,J^{\#}_{\alpha}\rangle$ is an $(in-cl)$-IF2MTS.

3. New Four Examples for Intuitionistic Fuzzy Multi-Modal Topological Structures

First, we define two new IF topological operators that are modifications and extensions of the standard IF topological operators "closure" (*cl*-) and "interior" (*in*-) (see [46]) defined by

$$\begin{aligned}\mathcal{C}(A)&=\{\langle x,\sup_{y\in E}\mu_A(y),\inf_{y\in E}\nu_A(y)\rangle|x\in E\};\\ \mathcal{I}(A)&=\{\langle x,\inf_{y\in E}\mu_A(y),\sup_{y\in E}\nu_A(y)\rangle|x\in E\}.\end{aligned}$$

The new operators have the forms:

$$\begin{aligned}\mathcal{C}_\alpha(A)&=\{\langle x,\sup_{y\in E}\mu_A(y),\alpha\inf_{y\in E}\nu_A(y)\rangle|x\in E\};\\ \mathcal{I}_\alpha(A)&=\{\langle x,\alpha\inf_{y\in E}\mu_A(y),\sup_{y\in E}\nu_A(y)\rangle|x\in E\},\end{aligned}$$

where $\alpha\in[0,1]$.

Obviously, for each IFS A:

$$\mathcal{C}(A)=\mathcal{C}_1(A),$$

$$\mathcal{I}(A)=\mathcal{I}_1(A).$$

For them, we can directly check that for each IFS A and for each $\alpha\in[0,1]$:

$$\mathcal{I}_\alpha(A)\subseteq\mathcal{I}(A)\subseteq A\subseteq\mathcal{C}(A)\subseteq\mathcal{C}_\alpha(A),$$

and therefore operator $\mathcal{C}_\alpha$ is from the *cl*-type and operator $\mathcal{I}_\alpha$ is from the *in*-type (the other properties that characterize these operators as their types will be discussed below). In addition

$$\begin{aligned}\neg\mathcal{C}_\alpha(\neg A) &= \neg\mathcal{C}_\alpha(\{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\})\\ &= \neg\{\langle x, \sup_{y\in E}\nu_A(y), \alpha\inf_{y\in E}\mu_A(y)\rangle | x \in E\}\\ &= \{\langle x, \alpha\inf_{y\in E}\mu_A(y), \sup_{y\in E}\nu_A(y)\rangle | x \in E\}\\ &= \mathcal{I}_\alpha(A)\end{aligned}$$

and

$$\begin{aligned}\neg\mathcal{I}_\alpha(\neg A) &= \neg\mathcal{I}_\alpha(\{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\})\\ &= \neg\{\langle x, \alpha\inf_{y\in E}\nu_A(y), \sup_{y\in E}\nu_A(y)\rangle | x \in E\}\\ &= \{\langle x, \sup_{y\in E}\mu_A(y), \alpha\inf_{y\in E}\nu_A(y)\rangle | x \in E\}\\ &= \mathcal{C}_\alpha(A).\end{aligned}$$

The geometrical interpretations of both new operators are shown in Figure 1 (cf. [46]).

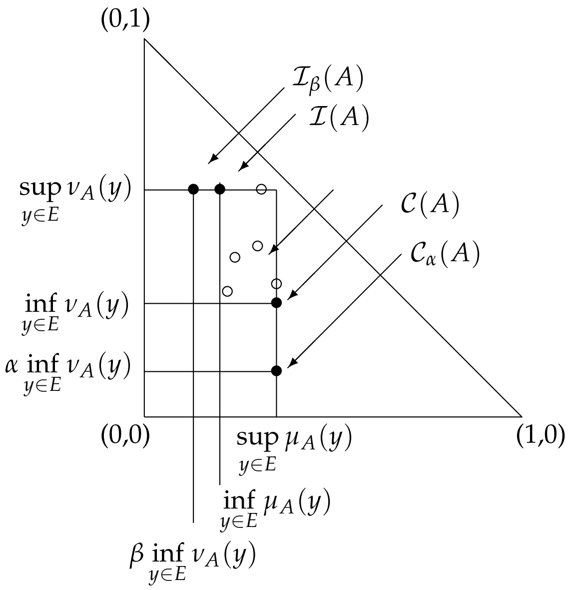

Figure 1. Geometrical interpretations of operators $\mathcal{C}, \mathcal{C}_\alpha, \mathcal{I}, \mathcal{I}_\beta$ represented by points $\bullet$ over IFS A, with elements represented by $\circ$.

Second, we must mention that in [46] the following concept is defined: the IFS is a *proper IFS* iff there exists element $x \in E$ such that $\mu_A(x) + \nu_A(x) < 1$, i.e., $\pi_A(x) > 0$. Let us call this IFS *weak proper IFS*. Now, we can call a given IFS *strong proper IFS* iff, for each, $x \in E, \pi_A(x) > 0$.

Let the following hold true for the strong proper IFS A:

$$\omega = \inf_{y\in E}\pi_A(x) > 0.$$

Let $\varphi \in (1-\omega, 1]$ be an arbitrary number and

$$O^*_\varphi = \{\langle x, 0, \varphi\rangle | x \in E\}.$$

Below, we will change conditions CC3 and CI3 with the weak conditions CC3* and CI3* with the form

$$\mathcal{O}(O^*) = O^*_\varphi.$$

By analogy, let, for $\psi \in (1-\omega, 1]$,

$$E^*_\psi = \{\langle x, \psi, 0\rangle | x \in E\},$$

and let us change conditions II3 and IC3 with the weak conditions II3* and IC3* with the form

$$\mathcal{Q}(E^*) = E^*_\psi.$$

Third, we prove the following theorem.

Theorem 1. *For each universe E:*

(a) $\langle \mathcal{P}(E), \mathcal{C}_\alpha, \cup, J^\#_\beta\rangle$ *is a* $(cl-cl)$*-IFMTS;*

(b) $\langle \mathcal{P}(E), \mathcal{C}_\alpha, \cup, H^\#_\beta\rangle$ *is a* $(cl-in)$*-IFMTS;*

(c) $\langle \mathcal{P}(E), \mathcal{I}_\alpha, \cap, J^\#_\beta\rangle$ *is an* $(in-cl)$*-IFMTS;*

(d) $\langle \mathcal{P}(E), \mathcal{I}_\alpha, \cap, H^\#_\beta\rangle$ *is an* $(in-in)$*-IFMTS.*

Proof. Let the IFSs $A, B \in \mathcal{P}(E^*)$, $\alpha, \beta \in (1-\omega, 1]$.

(a) The check of conditions CC1–CC4 are analogous but different from those in [42], while the checks of conditions CC5–CC8 are the same as in [50] and for this reason we omit them here.

CC1.

$$\begin{aligned}
\mathcal{C}_\alpha(A \cup B) &= \mathcal{C}_\alpha(\{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\}) \\
&= \{\langle x, \sup_{y\in E} \max(\mu_A(y), \mu_B(y)), \alpha \inf_{y \in E} \min(\nu_A(y), \nu_B(y))\rangle | x \in E\} \\
&= \{\langle x, \max(\sup_{y\in E} \mu_A(y), \sup_{y\in E} \mu_B(y)), \min(\alpha \inf_{y\in E} \nu_A(y), \alpha \inf_{y \in E} \nu_B(y))\rangle | x \in E\} \\
&= \{\langle x, \sup_{y\in E} \mu_A(y), \alpha \inf_{y\in E} \nu_A(y)\rangle | x \in E\} \cup \{\langle x, \sup_{y\in E} \mu_B(y), \alpha \inf_{y\in E} \nu_B(y)\rangle | x \in E\} \\
&= \mathcal{C}_\alpha(A) \cup \mathcal{C}_\alpha(B).
\end{aligned}$$

CC2.

$$\begin{aligned}
A &= \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} \\
&\subseteq \{\langle x, \sup_{y\in E} \mu_A(y), \alpha \inf_{y\in E} \nu_A(y)\rangle | x \in E\} \\
&= \mathcal{C}_\alpha(A).
\end{aligned}$$

CC3*.

$$\begin{aligned}
\mathcal{C}_\alpha(O^*) &= \mathcal{C}_\alpha(\{\langle x, 0, 1\rangle | x \in E\}) \\
&= \{\langle x, \sup_{y\in E} 0, \alpha \inf_{y\in E} 1\rangle | x \in E\} \\
&= \{\langle x, 0, \alpha\rangle | x \in E\} \\
&= O^*_\alpha.
\end{aligned}$$

CC4. Having in mind that $\sup\limits_{y\in E}\mu_A(y)$ and $\inf\limits_{y\in E}\nu_A(y)$ are constants, we obtain:

$$\begin{aligned}
\mathcal{C}_\alpha(\mathcal{C}_\alpha(A)) &= \mathcal{C}_\alpha(\{\langle x, \sup_{y\in E}\mu_A(y), \alpha\inf_{y\in E}\nu_A(y)\rangle | x\in E\})\\
&= \{\langle x, \sup_{z\in E}\sup_{y\in E}\mu_A(y), \alpha\inf_{z\in E}\left(\alpha\inf_{y\in E}\nu_A(y)\right)\rangle | x\in E\}\\
&= \{\langle x, \sup_{z\in E}\sup_{y\in E}\mu_A(y), \alpha^2\inf_{y\in E}\nu_A(y)\rangle | x\in E\}\\
&= \mathcal{C}_{\alpha^2}(A).
\end{aligned}$$

CC9.

$$\begin{aligned}
J^\#_\beta(\mathcal{C}_\alpha(A)) &= J^\#_\beta(\{\langle x, \sup_{y\in E}\mu_A(y), \alpha\inf_{y\in E}\nu_A(y)\rangle | x\in E\})\\
&= \{\langle x, \sup_{y\in E}\mu_A(y), \beta(\alpha\inf_{y\in E}\nu_A(y))\rangle | x\in E\}\\
&= \{\langle x, \sup_{y\in E}\mu_A(y), \alpha(\beta\inf_{y\in E}\nu_A(y))\rangle | x\in E\}\\
&= \mathcal{C}_\alpha(\{\langle x, \mu_A(y), \beta\nu_A(y)\rangle | x\in E\})\\
&= \mathcal{C}_\alpha(J^\#_\beta(A)).
\end{aligned}$$

(b) The checks of conditions CI1–CI4 are as above. The checks of conditions CI5–CI8 are the same as in [50], and for this reason we omit them. Therefore, we must check only condition CI9:

$$\begin{aligned}
H^\#_\beta(\mathcal{C}_\alpha(A)) &= H^\#_\beta(\{\langle x, \sup_{y\in E}\mu_A(y), \alpha\inf_{y\in E}\nu_A(y)\rangle | x\in E\})\\
&= \{\langle x, \beta\sup_{y\in E}\mu_A(y), \alpha\inf_{y\in E}\nu_A(y)\rangle | x\in E\}\\
&= \mathcal{C}_\alpha(\{\langle x, \mu_A(y), \beta\nu_A(y)\rangle | x\in E\})\\
&= \mathcal{C}_\alpha(H^\#_\beta(A)).
\end{aligned}$$

The checks of (c) and (d) are similar. □

Fourth, following [49], we will define the concept of an IFmMTS.

The IFmMTS, or more precisely fntuitionistic fuzzy (m,n)-modal topological structure, (IF(m,n)MTS) is an object with the following form:

$$\langle \mathcal{P}(E^*), \mathcal{O}, \Delta, \circ_1, \ldots, \circ_n, *_1, \ldots, *_m\rangle,$$

where $\circ_1, \ldots, \circ_n, *_1, \ldots, *_m$ are IF-modal operators. The components of an IF(m,n)MTS must satisfy the conditions with numbers 1–4 for the topological operator, the conditions with numbers 5–8 for each one of the modal operators, the condition with number 9 (but for the topological operator and for each one of the modal operators), and a new condition (number 10) that has the following form: for each IFS A,

$$*_m A \subset \cdots \subset *_1 A \subset \circ_1 A \subset \cdots \subset \circ_n A, \tag{1}$$

where the relation "$\subset$" denotes a strong inclusion. Its sense is to show that the modal operators are well ordered.

Let the real numbers $\alpha_1, \ldots \alpha_m, \beta_1, \ldots, \beta_n \in [0,1)$ be given, so that

$$\alpha_1 > \alpha_2 > \cdots > \alpha_m, \tag{2}$$

$$\beta_1 > \beta_2 > \cdots > \beta_n. \tag{3}$$

Theorem 2. *For each universe E,* $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, J^{\#}_{\beta_1}, \ldots, J^{\#}_{\beta_n}, H^{\#}_{\alpha_1}, \ldots, H^{\#}_{\alpha_m}\rangle$ *and* $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, J^{\#}_{\beta_1}, \ldots, J^{\#}_{\beta_n}, H^{\#}_{\alpha_1}, \ldots, H^{\#}_{\alpha_m}\rangle$ *are IF(m, n)-MTSs.*

Proof. The checks of the first nine conditions for the *cl*- and for the *in*-types of structures are conducted in Theorem 1 and in [42]. So, we must prove that the $m+n$ IF-modal operators give different values and that they are ordered as (1).

Let $1 \le i < k \le m$ and $1 \le j < l \le n$. Then, from (2) and (3), we check for each IFS A:

$$H^{\#}_{\alpha_k}(A) = \{\langle x, \alpha_k \mu_A(x), \nu_A(x)\rangle | x \in E\} \subset \{\langle x, \alpha_i \mu_A(x), \nu_A(x)\rangle | x \in E\} = H^{\#}_{\alpha_i}(A),$$

$$J^{\#}_{\beta_j}(A) = \{\langle x, \mu_A(x), \beta_j \nu_A(x)\rangle | x \in E\} \subset \{\langle x, \mu_A(x), \beta_l \nu_A(x)\rangle | x \in E\} = J^{\#}_{\beta_l}(A).$$

Hence,

$$H^{\#}_{\alpha_m}(A) \subset \cdots \subset H^{\#}_{\alpha_1}(A)$$

and

$$J^{\#}_{\beta_1}(A) \subset \cdots \subset J^{\#}_{\beta_n}(A).$$

Now, having in mind (1), we must prove that

$$H^{\#}_{\alpha_1}(A) \subset J^{\#}_{\beta_1}(A).$$

Really,

$$H^{\#}_{\alpha_1}(A) = \{\langle x, \alpha_1 \mu_A(x), \nu_A(x)\rangle | x \in E\}$$

(from $\alpha_1 < 1$)

$$\subset A$$

(from $\beta_1 < 1$)

$$\subset \{\langle x, \mu_A(x), \beta_j \nu_A(x)\rangle | x \in E\} = J^{\#}_{\beta_1}(A).$$

Hence, the sequence of inclusions (1) is true, and both structures are IF(m, n)MTSs from *cl*- and *in*-type, respectively. □

4. Intuitionistic Fuzzy Modal Multi-Topological Structures

First, we will introduce the concept of a modal multi-topological structure (MmTS), or more precisely, a modal (u, v)-topological structure (M(u, v)-TSs). It is an object with the following form:

$$\langle \mathcal{P}(E^*), \mathcal{O}_1, \ldots, \mathcal{O}_u, \mathcal{Q}_1, \ldots, \mathcal{Q}_v, \Delta, \circ\rangle,$$

where $\mathcal{O}_1, \ldots, \mathcal{O}_u$ are *cl*-topological operators, $\mathcal{Q}_1, \ldots, \mathcal{Q}_v$ are *in*-topological operators, $u, v \ge 0$ are integers, Δ and ∇ are the operations that generate the *cl*-operators and *in*-operators, respectively, and $\circ$ is a modal operator. The components of an M(u, v)TS must satisfy the conditions with numbers 1–4 for each one of the topological operators from both types, the conditions with numbers 5–8 for the modal operator from both types, the condition with number 9 for the modal operator and for each one of the topological operators, and for each IFS A, the new condition (with number 11) that has the following form:

$$\mathcal{Q}_v(A) \subset \cdots \subset \mathcal{Q}_1(A) \subset \mathcal{O}_1 \subset \cdots \subset \mathcal{O}_u(A). \quad (4)$$

Its sense is to show that the topological operators are well ordered.

Let the real numbers $\gamma_1, \ldots \gamma_v, \delta_1, \ldots, \delta_u \in [0, 1)$ be given, so that

$$\gamma_1 > \gamma_2 > \cdots > \gamma_v, \quad (5)$$

$$\delta_1 > \delta_2 > \cdots > \delta_u. \quad (6)$$

These two sequences will be used below for enumeration of the interior and closure operators, respectively.

Theorem 3. *For each universe E, for every $\gamma_1, \ldots, \gamma_v, \delta_1, \ldots, \delta_u \in [0, 1)$ satisfying conditions (5) and (6), and for each $\alpha \in [0, 1]$ $\langle \mathcal{P}(E^*), \mathcal{C}_{\delta_1}, \ldots, \mathcal{C}_{\delta_u}, \mathcal{I}_{\gamma_1}, \ldots, \mathcal{I}_{\gamma_v}, \cup, \cap, H^{\#}_{\alpha}\rangle$ and $\langle \mathcal{P}(E^*), \mathcal{C}_{\delta_1}, \ldots, \mathcal{C}_{\delta_u}, \mathcal{I}_{\gamma_1}, \ldots, \mathcal{I}_{\gamma_v}, \cup, \cap, J^{\#}_{\alpha}\rangle$ are IFM(u, v)-TSs.*

Proof. The checks of the first eight conditions for the *cl*- and for the *in*-types of structures are conducted in Theorem 1 and in [42] for arbitrary *cl*- and *in*-operators. So, we must prove that the $u + v$ IF-topological operators are different and that they are ordered as in (4).

Let $1 \leq p < r \leq v$ and $1 \leq q < s \leq u$. Then, from (5) and (6), we check for each IFS A:

$$\mathcal{I}_{\gamma_r}(A) = \{\langle x, \gamma_r \inf_{y \in E} \mu_A(y), \sup_{y \in E} \nu_A(y)\rangle | x \in E\}$$

$$\subset \{\langle x, \gamma_p \inf_{y \in E} \mu_A(y), \sup_{y \in E} \nu_A(y)\rangle | x \in E\} = \mathcal{I}_{\gamma_p}(A)$$

and

$$\mathcal{C}_{\delta_q}(A) = \{\langle x, \sup_{y \in E} \mu_A(y), \delta_q \inf_{y \in E} \nu_A(y)\rangle | x \in E\}$$

$$\subset \{\langle x, \sup_{y \in E} \mu_A(y), \delta_s \inf_{y \in E} \nu_A(y)\rangle | x \in E\} = \mathcal{C}_{\delta_s}(A.)$$

Therefore,

$$\mathcal{I}_{\gamma_v}(A) \subset \cdots \subset \mathcal{I}_{\gamma_1}(A)$$

and

$$\mathcal{C}_{\delta_1}(A) \subset \cdots \subset \mathcal{C}_{\delta_u}(A).$$

Now, having in mind (4), we must prove that

$$\mathcal{I}_{\gamma_1}(A) \subset \mathcal{C}_{\delta_1}(A).$$

It is true because

$$\mathcal{I}_{\gamma_1}(A) = \{\langle x, \gamma_1 \inf_{y \in E} \mu_A(y), \sup_{y \in E} \nu_A(y)\rangle | x \in E\}$$

(from $\gamma_1 < 1$)

$$\subset A$$

(from $\delta_1 < 1$)

$$\subset \{\langle x, \sup_{y \in E} \mu_A(y), \delta_1 \inf_{y \in E} \nu_A(y)\rangle | x \in E\} = \mathcal{C}_{\delta_1}(A).$$

Hence, the sequence of inclusions (4) is true and both structures are IFM(m, n)TSs from *cl*- and *in*-type, respectively. □

5. Intuitionistic Fuzzy Multi-Modal Multi-Topological Structures

On the basis of the definitions from the previous two sections, here, first, we will introduce the concept of a multi-modal multi-topological structure, or more precisely, (m, n)-,odal (u, v)-topological structure (m, n)-M(u, v)-TSs). It is the object with the following form:

$$\langle \mathcal{P}(E^*), \mathcal{O}_1, \ldots, \mathcal{O}_u, \mathcal{Q}_1, \ldots, \mathcal{Q}_v, \Delta, \nabla, \circ_1, \ldots, \circ_n, *_1, \ldots, *_m\rangle,$$

where $\mathcal{O}_1, \ldots, \mathcal{O}_u$ are *cl*-topological operators, $\mathcal{Q}_1, \ldots, \mathcal{Q}_v$ are *in*-topological operators, $u, v \geq 0$ are integers, Δ and ∇ are operators generated the *cl*-operator and the *in*-operator, respectively, and $\circ_1, \ldots, \circ_n, *_1, \ldots, *_m$ are IF-modal operators. The components of an (m, n)-M(u, v)TS must satisfy the conditions with numbers 1–4 for each one of the topological operators from both types, the conditions with numbers 5–8 for each one of the modal operators of both types, and for each IFS A, the conditions with numbers 9–11, i.e., the inequalities (1) and (4), must hold.

Theorem 4. *For each universe E, for every $\alpha_1, \dots \alpha_m, \beta_1, \dots, \beta_n \in [0,1)$ satisfying (2) and (3), and for every $\gamma_1, \dots \gamma_v, \delta_1, \dots, \delta_u \in [0,1)$ satisfying (5) and (6), $\langle \mathcal{P}(E^*), \mathcal{C}_{\delta_1}, \dots, \mathcal{C}_{\delta_u}, \mathcal{I}_{\gamma_1}, \dots, \mathcal{I}_{\gamma_v}, \cup, \cap, J^{\#}_{\beta_1}, \dots, J^{\#}_{\beta_n}, H^{\#}_{\alpha_1}, \dots, H^{\#}_{\alpha_m} \rangle$ is an IF(m,n)-M(u,v)-TSs.*

The proof follows from the proofs of Theorems 2 and 3. Also, from these, it follows the validity of the next assertion that is equivalent to Theorem 4.

Theorem 5. *For each universe E, for every $\alpha_1, \dots \alpha_m, \beta_1, \dots, \beta_n \in [0,1)$ satisfying (2) and (3), for every $\gamma_1, \dots \gamma_v, \delta_1, \dots, \delta_u \in [0,1)$ satisfying (5) and (6), and for every $i(1 \le i \le m)$, $j(1 \le j \le n), p(1 \le p \le v), q(1 \le q \le v)$:*

(a) $\langle \mathcal{P}(E), \mathcal{C}_{\alpha_p}, \cup, J^{\#}_{\beta_j} \rangle$ *is a* $(cl - cl)$*-IFMTS;*

(b) $\langle \mathcal{P}(E), \mathcal{C}_{\alpha_p}, \cup, H^{\#}_{\beta_i} \rangle$ *is a* $(cl - in)$*-IFMTS;*

(c) $\langle \mathcal{P}(E), \mathcal{I}_{\alpha_q}, \cap, J^{\#}_{\beta_j} \rangle$ *is an* $(in - cl)$*-IFMTS;*

(d) $\langle \mathcal{P}(E), \mathcal{I}_{\alpha_q}, \cap, H^{\#}_{\beta_i} \rangle$ *is an* $(in - in)$*-IFMTS.*

6. Conclusions

In [42], following [51,52], the idea for maps and atlases generated by different topological structures was joint universe was formulated. These maps and atlases are similar but different from these in differential geometry. Now, we provide additional details about this idea in the IF-case.

When we have a set of *cl*-IFMTSs,

$$\{\langle \mathcal{P}(E^*), \mathcal{C}_\delta, \cup, J^{\#}_{\beta_1} \rangle, \dots \langle \mathcal{P}(E^*), \mathcal{C}_\delta, \cup, J^{\#}_{\beta_n} \rangle, \langle \mathcal{P}(E^*), \mathcal{C}_\delta, \cup, H^{\#}_{\alpha_1} \rangle, \dots \langle \mathcal{P}(E^*), \mathcal{C}_\delta, \cup, H^{\#}_{\alpha_m} \rangle\},$$

and when the modal operators satisfy (1), then we can interpret each one of these structures as a map. Therefore, these structures can be ordered (see Figure 2) and numbered by function $\mathcal{N}$, e.g., as follows:

$$\begin{aligned} \mathcal{N}(H^{\#}_{\alpha_i}) &= i \ (i = 1, \dots, m), \\ \mathcal{N}(J^{\#}_{\beta_j}) &= m + j \ (j = 1, \dots, n). \end{aligned}$$

Hence, the arguments of the modal operators can generate a pagination of the set of maps, i.e., we obtain an *cl*-atlas

$$\langle \mathcal{P}(E^*), \mathcal{C}_\delta, \cup, J^{\#}_{\beta_1}, \dots, J^{\#}_{\beta_n}, H^{\#}_{\alpha_1}, \dots, H^{\#}_{\alpha_m} \rangle$$

over the fixed universe. Analogously, we can construct an *in*-atlas.

In the same way, we can proceed in the case of IFMmTSs (see Figure 3).

Finally, we can perform pagination with pairs of numbers—the number of the topological operator and the number of the modal operator. For the geometrical interpretation, we must assume that the separate pages of the atlas are tiered and ordered, e.g., as is shown in Figure 4.

It will be interesting if this idea is developed in future.

Another direction for near-future research will be related to a representation of graph structures as IF(m,n)-M(u,v)-TSs. In this case, the universe will contain a set of graph vertices. Each vertex and each arc will have its own intuitionistic fuzzy degrees of an existence and a non-existence, and the topological and modal operators will change these degrees in a corresponding way.

The author hopes that in the near future, a new procedure for multi-criteria, multi-person decision-making will be introduced that extends the procedure from [53] and that generates its own IFmMmTS.

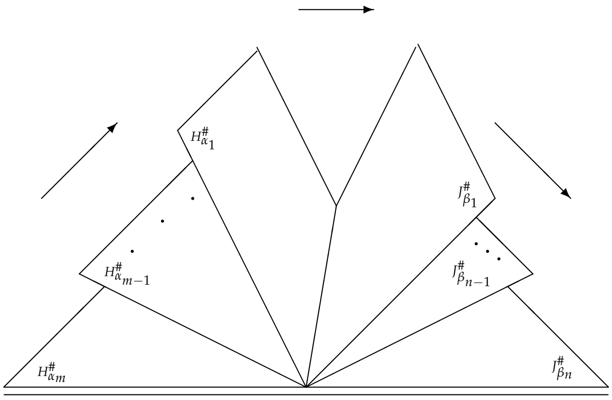

Figure 2. Atlas with maps generated by modal operators $H^{\#}_{\alpha_1}, \ldots, H^{\#}_{\alpha_m}, J^{\#}_{\beta_1}, \ldots, J^{\#}_{\beta_n}$.

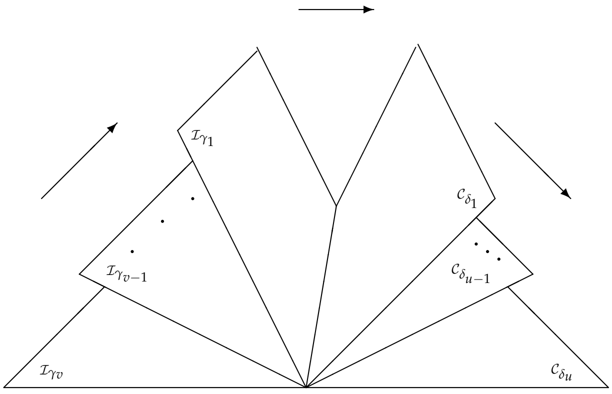

Figure 3. Atlas with maps generated by topological operators $\mathcal{I}_{\gamma_1}, \ldots, \mathcal{I}_{\gamma_v}, \mathcal{C}_{\delta_1}, \ldots, \mathcal{C}_{\delta_u}$.

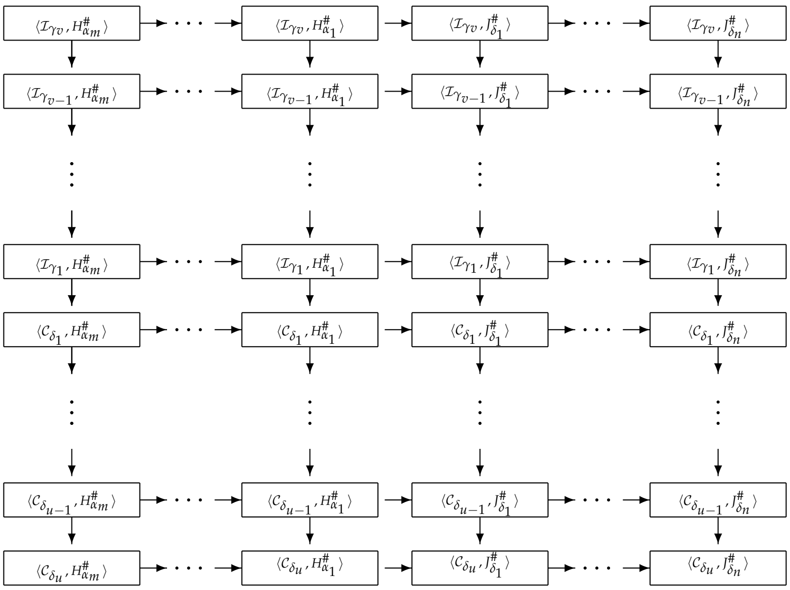

Figure 4. Atlas with maps generated by the topological and modal operators.

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