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A New Class of Bivariate Distributions: Properties, Estimation, and Modeling

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Abstract: The development of bivariate distributions is a challenging area of research, and a novel class of bivariate distributions is proposed in this paper. Certain useful mathematical characteristics of the proposed class of bivariate distributions are explored in general. These properties include the joint and conditional and joint survival function moments and measures of dependence. The parameters of the proposed class of distributions are estimated using the maximum likelihood method of estimation. A multivariate version of the proposed class is also proposed, and some necessary properties are discussed. The multivariate dependence measures are obtained for the proposed multivariate class of distributions. A new bivariate power function distribution is proposed using the power function distribution as a baseline distribution in the proposed bivariate class. Some useful properties of the proposed bivariate distribution are studied. The parameters of the proposed bivariate distribution are estimated using the maximum likelihood method. An extensive simulation study has been conducted to see the performance of the estimation method. The new bivariate power function distribution is used to model some real data.

Keywords: class of distributions; bivariate distributions; dependence measures; reliability analysis; maximum likelihood estimation; power function distribution

MSC: 62E15; 62F10; 62H10; 62H20

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1. Introduction

Probabilistic modeling of certain real-life phenomena is often required in various areas of life. For example, an engineer might be interested in deciding the reliability of a certain component at a large production plant. Any suitable univariate distribution can be used in such situations. It is often the case that the modeling of two phenomena is needed and a bivariate distribution is required. Several standard bivariate distributions are available for modeling two random phenomena, and the most popular of these is the bivariate normal distribution. An excellent review of some bivariate distributions is given by [1].

In certain situations, a standard bivariate distribution cannot be used to model data. In such situations, some non-standard bivariate distributions are required. A method to generate a bivariate distribution from the marginal distributions is introduced by [2] and has the joint distribution function as

$$F_{1,2}(y_1, y_2) = H_1(y_1)H_2(y_2)[1 + \gamma\{1 - H_1(y_1)\}\{1 - H_2(y_2)\}] \;;\; (y_1, y_2) \in \mathbb{R}^2, \quad (1)$$

where $H_1(y_1)$ and $H_2(y_2)$ are marginal cumulative distribution functions (*cdf*s) of Y_1 and Y_2 and γ is the measure of association. The joint distribution function, given in (1), is a class of bivariate distributions, as several bivariate distributions can be generated using different $H_1(y_1)$ and $H_2(y_2)$. Another class of bivariate distributions (*CBD*s) was proposed by [3] on the basis of reciprocals of the univariate *cdf*s. Some more classes of bivariate

distributions have been proposed by many other authors. These classes of distributions use the marginal distribution functions to propose a new joint distribution function with some additional parameters and certain dependence coefficients. The methods of generating bivariate classes of distributions by [2,3] and many others are members of a larger group of generating joint distributions that are known as copulas; see, for example, [1,4].

2. Materials and Methods

The families of probability distributions have recently attracted various authors. A popular family of univariate distributions has been proposed by [5] using the logit of the beta distribution. This family of distributions is known as the beta-generated distributions. The univariate beta-generated family of distributions has been extended to the bivariate case by [6] using a bivariate beta distribution of [7]. The joint distribution function of this bivariate beta-generated family of distributions is

$$F_{Beta-H_1H_2}(y_1,y_2)=\frac{1}{B(\alpha_1,\alpha_2,\beta)}\int_0^{H_2(y_2)}\int_0^{H_1(y_1)}\frac{v_1^{\alpha_1-1}v_2^{\alpha_2-1}(1-v_1)^{\alpha_2+\beta-1}(1-v_2)^{\alpha_1+\beta-1}}{(1-v_1v_2)^{\alpha_1+\alpha_2+\beta}}dv_1dv_2, \quad (2)$$

where $B(\alpha,\beta,\gamma)$ is the extended beta function defined as

$$B(\alpha,\beta,\gamma)=\int_0^1\int_0^1\frac{w_1^{\alpha-1}w_2^{\beta-1}(1-w_1)^{\beta+\gamma-1}(1-w_2)^{\alpha+\gamma-1}}{(1-w_1w_2)^{\alpha+\beta+\gamma}}dw_1dw_2=\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha+\beta+\gamma)},$$

and $\Gamma(w)$ is the complete gamma function. It is shown by [6] that the marginal *cdf*s of (2) are the univariate beta-generated families of distributions. The family (2) is used by [8] to propose a bivariate beta-inverse Weibull distribution.

A general method of obtaining univariate families of distributions is proposed by [9] on the basis of any two baseline distributions. This method has been named the T–X family of distributions. The method by [9] has been extended by [10] to the bivariate case. The joint distribution function of the class of bivariate distributions, proposed by [10], is

$$F_{1,2}(y_1,y_2)=\int_{a_2}^{W_2[H_2(y_2)]}\int_{a_1}^{W_1[H_1(y_1)]}r(v_1,v_2)dv_1dv_2\ ;\ a_1<v_1<b_1\ ;\ a_2<v_2<b_2, \quad (3)$$

where $r(v_1,v_2)$ is the joint density function of some random variables with support on $[a_1,b_1] \times [a_2,b_2]$. Also, $W_1[H_1(y_1)]$ and $W_2[H_2(y_2)]$ are some continuous functions of $H_1(y_1)$ and $H_2(y_2)$ such that $W_1(0)\to a_1$, $W_1(1)\to b_1$, $W_2(0)\to a_2$ and $W_2(1)\to b_2$. A simpler version of (3) is proposed by [11] when the support of $r(v_1,v_2)$ is on $[0, 1] \times [0, 1]$. The family (3), in this case, is

$$F_{1,2}(y_1,y_2)=\int_0^{H_2(y_2)}\int_0^{H_1(y_1)}r(v_1,v_2)dv_1dv_2\ ;\ 0<(v_1,v_2)<1, \quad (4)$$

which obviously holds for $0 < H_1(y_1) < 1$ and $0 < H_2(y_2) < 1$. It is easy to see that the joint distribution function given in (1) can be obtained from (4) using

$$r(v_1,v_2)=1+\gamma(1-2v_1)(1-2v_2)\ ;\ (v_1,v_2)\in[0,1]^2.$$

The family of distributions, (4), is used by [11] to obtain a bivariate family of distributions and is a simpler version of the family of distributions obtained by [12].

This paper deals with a new class of bivariate distributions (*CBD*s) that is obtained using (4). The layout of the paper follows. Section 3 contains new *CBD*s, and certain properties of the proposed class of distributions are given in Section 4. These properties include joint, ratio, conditional moments, reliability analysis, measures of dependence, random number generation, and maximum likelihood estimation. The multivariate extension of the proposed *CBD*s is given in Section 5 with some of its important properties. Section 6 deals with a new bivariate power function (*BPF*) distribution. Section 7 is based on certain properties and parameter estimation of the new *BPF* distribution. Section 8 contains a com-

prehensive simulation study to see the consistency of the maximum likelihood estimates. Some real data applications are given in Section 9 to see the suitability of the proposed bivariate power function distribution. Concluding remarks are given in Section 10.

3. A New Class of Bivariate Distributions

This section deals with new *CBD*s with a motivation that the proposed class will be capable of handling more complex bivariate data. It is clear from (4) that any suitable joint density $r(v_1, v_2)$ over $[0, 1] \times [0, 1]$ can be used to obtain a new bivariate distribution function. We have used

$$r(v_1, v_2) = 1 + \alpha\left(1 - \gamma\phi v_1^{\gamma-1} v_2^{\phi-1}\right) \; ; \; 0 \leq (v_1, v_2) \leq 1 \; , \; \gamma \geq 1, \; \phi \geq 1$$

in (4) to obtain a new class of bivariate distributions. It is hoped that the function $r(v_1, v_2)$ that we have used will provide a new bivariate distribution that will be capable of modeling more complex data. The function that we have used to obtain a new class of bivariate distributions will add two new shape parameters to any baseline distribution. Now, using $r(v_1, v_2) = 1 + \alpha\left(1 - \gamma\phi v_1^{\gamma-1} v_2^{\phi-1}\right)$ in (4), the joint *cdf* of the new *CBD*s is

$$F_{1,2}(y_1, y_2) = \int_0^{H_2(y_2)} \int_0^{H_1(y_1)} \left[1 + \alpha\left(1 - \gamma\phi v_1^{\gamma-1} v_2^{\phi-1}\right)\right] dv_1 dv_2$$

or

$$F_{1,2}(y_1, y_2) = H_1(y_1) H_2(y_2)\left[1 + \alpha - \alpha H_1^{\gamma-1}(y_1) H_2^{\phi-1}(y_2)\right] \; ; \; (y_1, y_2) \in \mathbb{R}^2, \tag{5}$$

where $-1 \leq \alpha \leq 2\min(\gamma^{-1}, \phi^{-1})$ and (γ, ϕ) are the exponentiation parameters. Different choices of these parameters will lead us to a different class of bivariate distributions. Also, different choices of the baseline distributions will provide different new bivariate distributions. It is interesting to note that for $\gamma = \phi = 1$, $F_{1,2}(y_1, y_2) = H_1(y_1) H_2(y_2)$, and hence, Y_1 and Y_2 are independent. The density function corresponding to (5) is

$$f_{1,2}(y_1, y_2) = h_1(y_1) h_2(y_2)\left[1 + \alpha - \alpha\gamma\phi H_1^{\gamma-1}(y_1) H_2^{\phi-1}(y_2)\right] \; ; \; (y_1, y_2) \in \mathbb{R}^2. \tag{6}$$

The marginal *cdf*s of Y_1 and Y_2 for the new *CBD*s, (5), are

$$\left.\begin{array}{rl} & F_1(y_1) = (1 + \alpha) H_1(y_1) - \alpha H_1^{\gamma}(y_1) \; ; \; y_1 \in \mathbb{R} \\ \text{and} & F_2(y_2) = (1 + \alpha) H_2(y_2) - \alpha H_2^{\phi}(y_2) \; ; \; y_2 \in \mathbb{R}. \end{array}\right\} \tag{7}$$

It is easy to see that for $\gamma = \phi = 2$, the marginal *cdf*s, given in (7), are the transmuted distributions proposed by [13]. The marginal density functions of Y_1 and Y_2 are easily obtained from (7) as

$$\left.\begin{array}{rl} & f_1(y_1) = h_1(y_1)\left[(1 + \alpha) - \alpha\gamma H_1^{\gamma-1}(y_1)\right] \; ; \; y_1 \in \mathbb{R} \\ \text{and} & f_2(y_2) = h_2(y_2)\left[(1 + \alpha) - \alpha\phi H_2^{\phi-1}(y_2)\right] \; ; \; y_2 \in \mathbb{R}, \end{array}\right\}$$

which reduce to the transmuted distributions for $\gamma = \phi = 2$. The marginal density functions, given above, are parallel to generalized transmuted distributions proposed by [14].

The conditional density function of Y_1 given $Y_2 = y_2$ for the new *CBD*s is

$$f_{1|2}(y_1|y_2) = \frac{f_{1,2}(y_1, y_2)}{f_2(y_2)} = \frac{h_1(y_1)}{\Delta_2(y_2)}\left[1 + \alpha - \alpha\gamma\phi H_1^{\gamma-1}(y_1) H_2^{\phi-1}(y_2)\right] \; ; \; y_1 \in \mathbb{R}, \tag{8}$$

where $\Delta_2(y_2) = (1+\alpha) - \alpha\phi H_2^{\phi-1}(y_2)$. Again, the conditional density function of Y_2 given $Y_1 = y_1$ is

$$f_{2|1}(y_2|y_1) = \frac{f_{1,2}(y_1,y_2)}{f_1(y_1)} = \frac{h_2(y_2)}{\Delta_1(y_1)}\left[1+\alpha-\alpha\gamma\phi H_1^{\gamma-1}(y_1)H_2^{\phi-1}(y_2)\right] \;;\; y_2 \in \mathbb{R}, \quad (9)$$

where $\Delta_1(y_1) = (1+\alpha) - \alpha\gamma H_1^{\gamma-1}(y_1)$.

We will now discuss some important properties of the new *CBD*s.

4. Useful Properties of the New Class of Bivariate Distributions

In this section, some useful properties of the new *CBD*s are discussed.

4.1. The Joint and Ratio Moments

Certain useful properties of a bivariate distribution can be studied using the joint and ratio moments. The (*r*,*s*)th order joint moment for a bivariate distribution is obtained as

$$\mu_{Y_1,Y_2}^{p,q} = E\left(Y_1^p Y_2^q\right) = \iint_{\mathbb{R}^2} y_1^p y_2^q \, f_{1,2}(y_1,y_2)dy_1dy_2.$$

The product moment for the proposed *CBD*s is derived using (6) in the above equation and is

$$\mu_{Y_1,Y_2}^{p,q} = E\left(Y_1^p Y_2^q\right) = \iint_{\mathbb{R}^2} y_1^p y_2^q h_1(y_1)h_2(y_2)\left[1+\alpha-\alpha\gamma\phi H_1^{\gamma-1}(y_1)H_2^{\phi-1}(y_2)\right]dy_1dy_2.$$

Solving the above integral, the product moment for the new *CBD*s is

$$\mu_{Y_1,Y_2}^{p,q} = (1+\alpha)\mu_{Y_1}^p\mu_{Y_2}^q - \alpha\mu_{Y_1(\gamma:\gamma)}^p\mu_{Y_2(\phi:\phi)}^q, \quad (10)$$

where $\mu_{Y_1}^p$ is the pth raw moment of Y_1, $\mu_{Y_2}^q$ is the qth raw moment of Y_2, $\mu_{Y_1(\gamma:\gamma)}^p$ is the pth raw moment of the γ*–exponentiated* distribution of Y_1, and $\mu_{Y_2(\phi:\phi)}^q$ is the qth raw moment of the φ*–exponentiated* distribution of Y_2. Also, if γ and φ are integers, then $\mu_{Y_1(\gamma:\gamma)}^p$ is the pth raw moment of the largest observation in a sample of size γ from $H_1(y_1)$ and $\mu_{Y_2(\phi:\phi)}^q$ is the qth raw moment of the largest observation in a sample of size φ from $H_2(y_2)$; see [15,16].

The ratio moments are defined in two different ways as

$$\mu_{Y_1,Y_2}^{p/q} = E\left(Y_1^p Y_2^{-q}\right) = \iint_{\mathbb{R}^2} y_1^p y_2^{-q} \, f_{1,2}(y_1,y_2)dy_1dy_2,$$

and

$$\mu_{Y_1,Y_2}^{q/p} = E\left(Y_1^{-p} Y_2^q\right) = \iint_{\mathbb{R}^2} y_1^{-p} y_2^q \, f_{1,2}(y_1,y_2)dy_1dy_2.$$

Using (6) in the above equation, the ratio moments $\mu_{Y_1,Y_2}^{p/q}$ and $\mu_{Y_1,Y_2}^{q/p}$ for the new bivariate family of distributions are

$$\mu_{Y_1,Y_2}^{p/q} = (1+\alpha)\mu_{Y_1}^p\mu_{Y_2}^{-q} - \alpha\mu_{Y_1(\gamma:\gamma)}^p\mu_{Y_2(\phi:\phi)}^{-q} \;\;\&\;\; \mu_{Y_1,Y_2}^{q/p} = (1+\alpha)\mu_{Y_1}^{-p}\mu_{Y_2}^q - \alpha\mu_{Y_1(\gamma:\gamma)}^{-p}\mu_{Y_2(\phi:\phi)}^q, \quad (11)$$

where $\mu_{Y_1}^p$ is the pth rae moment of Y_1, $\mu_{Y_2}^{-q}$ is qth inverse moment of Y_2, $\mu_{Y_1(\gamma:\gamma)}^{-p}$ is the pth inverse moment of the γ*–exponentiated* distribution of Y_1, and $\mu_{Y_2(\phi:\phi)}^{-q}$ is the qth inverse moment of the φ*–exponentiated* distribution of Y_2. Also, if γ and φ are integers, then $\mu_{Y_1(\gamma:\gamma)}^{-p}$ and $\mu_{Y_2(\phi:\phi)}^{-q}$ are the inverse moments of the largest observation in a sample of size γ and φ from $H_1(y_1)$ and $H_2(y_2)$, respectively.

4.2. The Joint Survival and Hazard Rate Functions

The joint survival function is a useful function to see the joint reliability of two components. The bivariate reliability function (e.g., [17]) is defined as

$$R_{1,2}(y_1, y_2) = 1 - [F_1(y_1) + F_2(y_2) - F_{1,2}(y_1, y_2)]. \tag{12}$$

Using the marginal and joint *cdf*s for the new *CBD*s from (5) and (7), the bivariate reliability function is obtained as

$$\begin{aligned} R_{1,2}(y_1, y_2) = 1 - \quad & \Big[(1+\alpha)H_1(y_1) - \alpha H_1^{\gamma}(y_1) + (1+\alpha)H_2(y_2) - \alpha H_2^{\phi}(y_2) \\ & - \Big\{(1+\alpha)H_1(y_1)H_2(y_2) - \alpha H_1^{\gamma}(y_1)H_2^{\phi}(y_2)\Big\}\Big]. \end{aligned}$$

Rearranging the above expression, the bivariate reliability function for the new bivariate family of distributions is

$$R_{1,2}(y_1, y_2) = (1+\alpha)R_1(y_1)R_2(y_2) - \alpha\left[1 - H_1^{\gamma}(y_1)\right]\left[1 - H_2^{\phi}(y_2)\right] \;;\; y_1, y_2 \in \mathbb{R}^2, \tag{13}$$

where $R_1(y_1) = 1 - H_1(y_1)$ and $R_2(y_2) = 1 - H_2(y_2)$ are the marginal reliability functions of Y_1 and Y_2, respectively.

The joint hazard rate function (e.g., [18]) is defined as

$$\eta_{1,2}(y_1, y_2) = \frac{f_{1,2}(y_1, y_2)}{R_{1,2}(y_1, y_2)}.$$

The joint hazard rate function for the new *CBD*s is

$$\eta_{1,2}(y_1, y_2) = \frac{h_1(y_1)h_2(y_2)\left[1 + \alpha - \alpha\gamma\phi H_1^{\gamma-1}(y_1)H_2^{\phi-1}(y_2)\right]}{(1+\alpha)R_1(y_1)R_2(y_2) - \alpha\left[1 - H_1^{\gamma}(y_1)\right]\left[1 - H_2^{\phi}(y_2)\right]}. \tag{14}$$

The joint hazard rate function can be computed for any baseline distribution.

4.3. Hazard Rate and Mean Residual Life Vectors

The joint hazard rate function, given in (14), provides an instantaneous failure of two components. It is sometimes required to separately compute the hazard rate of both variables. One way of performing this is to obtain the hazard rate functions for both variables using their marginal distributions. In this method, the dependency of the variables is completely ignored. A better way is to obtain the hazard rate vector; see, for example [19], is as follows:

$$\boldsymbol{\eta}(y_1, y_2) = \left[-\tfrac{\partial}{\partial y_1}\ln R_{1,2}(y_1, y_2) \quad -\tfrac{\partial}{\partial y_2}\ln R_{1,2}(y_1, y_2)\right]^{/}. \tag{15}$$

Now, for the new family of distributions, we have

$$R_{1,2}(y_1, y_2) = (1+\alpha)R_1(y_1)R_2(y_2) - \alpha\left[1 - H_1^{\gamma}(y_1)\right]\left[1 - H_2^{\phi}(y_2)\right] \;;\; y_1, y_2 \in \mathbb{R}^2.$$

Now

$$-\frac{\partial}{\partial y_1}\ln R_{1,2}(y_1, y_2) = \frac{-1}{R_{1,2}(y_1, y_2)}\left[h_1(y_1)\left\{(1+\alpha)R_2(y_2) - \alpha\gamma H_1^{\gamma-1}(y_1)\left[1 - H_2^{\phi}(y_2)\right]\right\}\right] \tag{16}$$

and

$$-\frac{\partial}{\partial y_2}\ln R_{1,2}(y_1, y_2) = \frac{-1}{R_{1,2}(y_1, y_2)}\left[h_2(y_2)\left\{(1+\alpha)R_1(y_1) - \alpha\phi H_2^{\phi-1}(y_2)\left[1 - H_1^{\gamma}(y_1)\right]\right\}\right] \tag{17}$$

The hazard rate vector can be written using (16) and (17) in (15).

The mean residual life vector is another useful measure in bivariate reliability theory. The mean residual life vector is defined as

$$\mathbf{m}(y_1, y_2) = \begin{bmatrix} m_1(y_1, y_2) & m_2(y_1, y_2) \end{bmatrix}', \tag{18}$$

where

$$m_1(y_1, y_2) = E(Y_1 - y_1 | Y_1 > y_1, Y_2 > y_2) = \frac{1}{R_{1,2}(y_1, y_2)} \int_{t_1}^{\infty} R_{1,2}(y_1, y_2) dy_1 = \frac{I_1}{R_{1,2}(y_1, y_2)} \tag{19}$$

and

$$m_2(y_1, y_2) = E(Y_2 - y_2 | Y_1 > y_1, Y_2 > y_2) = \frac{1}{R_{1,2}(y_1, y_2)} \int_{y_2}^{\infty} R_{1,2}(y_1, y_2) dy_2 = \frac{I_2}{R_{1,2}(y_1, y_2)}. \tag{20}$$

Now

$$\begin{aligned} I_1 &= \textstyle\int_{y_1}^{\infty} R_{1,2}(y_1, y_2) dy_1 \\ &= \textstyle\int_{y_1}^{\infty} \Big[(1+\alpha)\{1 - H_1(y_1)\}\{1 - H_2(y_2)\} - \alpha\{1 - H_1^{\gamma}(y_1)\}\Big\{1 - H_2^{\phi}(y_2)\Big\}\Big] dy_1. \end{aligned}$$

Making the transformation $H_1(y_1) = u_1$, we have

$$I_1 = \frac{1}{h_1(y_1)} \int_{H_1(y_1)}^{1} \Big[(1+\alpha)(1-u_1)\{1 - H_2(y_2)\} - \alpha\big(1 - u_1^{\gamma}\big)\Big\{1 - H_2^{\phi}(y_2)\Big\}\Big] du_1.$$

Solving the integral, we have

$$I_1 = \frac{R_1(y_1)}{h_1(y_1)} \left[\frac{1}{2}(1+\alpha) R_1(y_1) R_2(y_2) - \alpha\Big\{1 - H_2^{\phi}(y_2)\Big\}\left\{1 - \frac{1 - H_1^{\gamma+1}(y_1)}{(\gamma+1) R_1(y_1)}, \right\}\right]$$

or

$$I_1 = \frac{R_1(y_1)}{h_1(y_1)} \left[\frac{1}{2}(1+\alpha) R_1(y_1) R_2(y_2) - \frac{\alpha}{\gamma+1}\Big\{1 - H_2^{\phi}(y_2)\Big\}\big\{\gamma - \big[1 - H_1^{\gamma}(y_1)\big]\big\}\right]. \tag{21}$$

Similarly

$$I_2 = \frac{R_2(y_2)}{h_2(y_2)} \left[\frac{1}{2}(1+\alpha) R_1(y_1) R_2(y_2) - \alpha\{1 - H_1^{\gamma}(y_1)\}\left\{1 - \frac{1 - H_2^{\phi+1}(y_2)}{(\phi+1) R_2(y_2)}\right\}\right],$$

or

$$I_2 = \frac{R_2(y_2)}{h_2(y_2)} \left[\frac{1}{2}(1+\alpha) R_1(y_1) R_2(y_2) - \frac{\alpha}{\phi+1}\{1 - H_1^{\gamma}(y_1)\}\Big\{\phi - \Big[1 - H_2^{\phi}(y_2)\Big]\Big\}\right]. \tag{22}$$

Using (21) and (22) in (19) and (20), we have the entries for the mean residual life vector. The hazard rate vector, (15), and the mean residual life vector, (18), can be obtained for any baseline distribution.

4.4. Moments of the Conditional Distributions

The conditional distributions provide the basis to obtain the moments of one random variable under certain conditions on the other and are known as the conditional moments. The pth conditional moment of Y_1 given $Y_2 = y_2$ is defined as

$$\mu_{Y_1|y_2}^{p} = E\Big(Y_1^{p}\Big|y_2\Big) = \int_{-\infty}^{\infty} y_1^{p} f_{1|2}(y_1|y_2) dy_1$$

Also, the qth conditional moment of Y_2 given $Y_1 = y_1$ is defined as

$$\mu^q_{Y_2|y_1} = E\left(Y_2^q \middle| y_1\right) = \int_{-\infty}^{\infty} y_2^q f_{2|1}(y_2|y_1)dy_2.$$

Now, using the conditional density function of Y_1 given $Y_2 = y_2$ from (8), the pth conditional moment of Y_1 given $Y_2 = y_2$ is

$$\mu^p_{Y_1|y_2} = \frac{1}{\Delta_2(y_2)}\int_{-\infty}^{\infty} y_1^p h_1(y_1)\left[1+\alpha-\alpha\gamma\phi H_1^{\gamma-1}(y_1)H_2^{\phi-1}(y_2)\right]dy_1.$$

Solving the above integral, the pth conditional moment of Y_1 given $Y_2 = y_2$ is

$$\mu^p_{Y_1|y_2} = \frac{1}{\Delta_2(y_2)}\left[(1+\alpha)\mu^p_{Y_1} - \alpha\phi H_2^{\phi-1}(y_2)\mu^p_{Y_1(\gamma:\gamma)}\right], \tag{23}$$

where $\Delta_2(y_2)$ and $\mu^p_{Y_1(\gamma:\gamma)}$ are defined earlier. Similarly, using the conditional density function of Y_2 given $Y_1 = y_1$, from (9), the qth conditional moment of Y_2 given $Y_1 = y_1$ is

$$\mu^q_{Y_2|y_1} = \frac{1}{\Delta_1(y_1)}\left[(1+\alpha)\mu^q_{Y_2} - \alpha\gamma H_1^{\gamma-1}(y_1)\mu^q_{Y_2(\phi:\phi)}\right],$$

where $\Delta_1(y_1)$ and $\mu^q_{Y_2(\phi:\phi)}$. are defined earlier. The conditional moments can be computed for any baseline distribution.

4.5. The Measures of Dependence

A measure of dependence is a useful measure to see the association between two variables. Various dependence measures are available in the literature; see, for example, [1,4,20], among others. In the following, some popular dependence measures are obtained for the new *CBD*s.

4.5.1. The Tau Coefficient

The τ coefficient is a popular dependence measure for two random variables. It is defined as

$$\tau = 4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{1,2}(y_1,y_2)f_{1,2}(y_1,y_2)dy_1dy_2 - 1$$

Using the distribution and density functions from (5) and (6), in the above equation, the τ coefficient for the new *CBD*s is

$$\begin{aligned}\tau = 4\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} &\left[H_1(y_1)H_2(y_2)\left\{1+\alpha-\alpha H_1^{\gamma-1}(y_1)H_2^{\phi-1}(y_2)\right\}\right]\\ &\times\left[h_1(y_1)h_2(y_2)\left\{1+\alpha-\alpha\gamma\phi H_1^{\gamma-1}(y_1)H_2^{\phi-1}(y_2)\right\}\right]dy_1dy_2 - 1.\end{aligned}$$

Using the transformation $w_1 = H_1(y_1)$ and $w_2 = H_2(y_2)$, the above integral can be written as

$$\tau = 4\int_0^1\int_0^1 \left[w_1w_2\left\{(1+\alpha)-\alpha w_1^{\gamma-1}w_2^{\phi-1}\right\}\right]\left[(1+\alpha)-\alpha\gamma\phi w_1^{\gamma-1}w_2^{\phi-1}\right]dw_1dw_2 - 1.$$

Solving the above integral, the τ coefficient for the new *CBD*s is

$$\tau = -\frac{2(\gamma-1)(\phi-1)\alpha(1+\alpha)}{(\gamma+1)(\phi+1)}. \tag{24}$$

It can be seen that the τ coefficient is free from the baseline distribution. We can also see that if $\alpha < 0$, then τ is positive, and if $\alpha > 0$, then τ is negative. The plots of the tau coefficient for different values of the parameters are given in Figure 1 below.

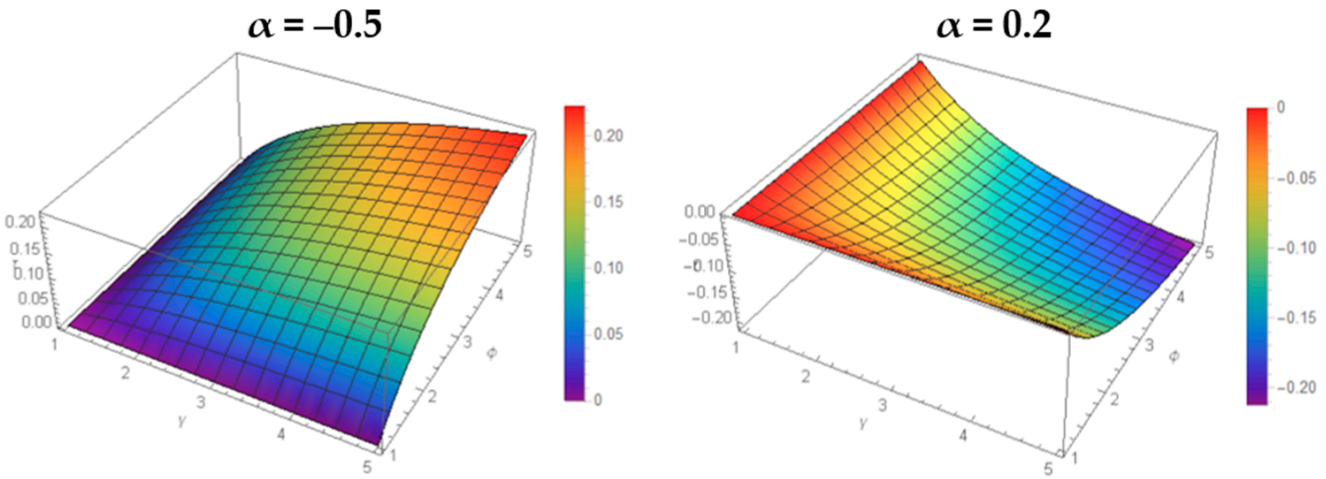


Figure 1. Kendall's τ coefficient for the new class of bivariate distributions.

The above plot indicates that when α is negative, Kendall's τ has an increasing trend with an increase in the values of γ and φ, and it has a decreasing trend with an increase in γ and φ when α is positive.

4.5.2. The Rho Coefficient

The ρ coefficient is another popular measure of dependence for two random variables. The measure is computed as

$$\rho = 12\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{1,2}(y_1, y_2) f_1(y_1) f_2(y_2) dy_1 dy_2 - 3.$$

Using the distribution and density functions from (5) and (6), in the above equation, Spearman's ρ for the new *CBD*s is

$$\begin{aligned}\rho = 12\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}&\left[H_1(y_1)H_2(y_2)\left\{(1+\alpha) - \alpha H_1^{\gamma-1}(y_1)H_2^{\phi-1}(y_2)\right\}\right]\\ &\times\left[h_1(y_1)h_2(y_2)\left\{(1+\alpha) - \alpha\gamma H_1^{\gamma-1}(y_1)\right\}\left\{(1+\alpha) - \alpha\phi H_2^{\phi-1}(y_2)\right\}\right]dy_1 dy_2 - 3.\end{aligned}$$

Using the transformation $w_1 = H_1(y_1)$ and $w_2 = H_2(y_2)$, the above integral can be written as

$$\rho = 12\int_0^1\int_0^1 \left[w_1 w_2\left\{(1+\alpha) - \alpha w_1^{\gamma-1} w_2^{\phi-1}\right\}\right]\left[\left\{(1+\alpha) - \alpha\gamma w_1^{\gamma-1}\right\}\left\{(1+\alpha) - \alpha\phi w_2^{\phi-1}\right\}\right]dw_1 dw_2 - 3.$$

Solving the above integral, the ρ coefficient for the new *CBD*s is

$$\rho = -\frac{3(\gamma-1)(\phi-1)\alpha(1+\alpha)}{(\gamma+1)(\phi+1)} = \frac{3}{2}\tau. \tag{25}$$

It can be seen that Spearman's ρ is also free from the baseline distribution. It can be seen that if $\alpha < 0$, then $\rho > \tau$, and if $\alpha > 0$, then $\rho < \tau$. Since Spearman's ρ is a multiple of Kendall's τ, its plots will show the same behavior as given in Figure 1.

4.5.3. The Local Dependence Measure

The local dependence measure for *CBD*s is defined by [20] as

$$\gamma(y_1, y_2) = \frac{\partial^2}{\partial y_1 \partial y_2} \ln f_{1,2}(y_1, y_2)$$

Now, for the new *CBD*s, we have

$$\ln f_{1,2}(y_1, y_2) = \ln h_1(y_1) + \ln h_2(y_2) + \ln\left[1 + \alpha - \alpha\gamma\phi H_1^{\gamma-1}(y_1) H_2^{\phi-1}(y_2)\right],$$

and the local dependence measure is

$$\gamma(y_1, y_2) = -\frac{\alpha\gamma\phi(\gamma-1)(\phi-1)(1+\alpha) h_1(y_1) h_2(y_2) H_1^{\gamma}(y_1) H_2^{\phi}(y_2)}{\left[H_1(y_1) H_2(y_2)\left\{(1+\alpha) - \alpha\gamma\phi H_1^{\gamma-1}(y_1) H_2^{\phi-1}(y_2)\right\}\right]^2}. \tag{26}$$

It is easy to see that the local dependence measure is based on the underlying distributions.

4.6. Stress- and Strength-Type Reliability

The stress- and strength-type reliabilities are often needed in industrial engineering. If a positive random variable Y_1 represents the strength and a positive random variable Y_2 represents stress and their joint distribution is $f_{1,2}(y_1,y_2)$, then the stress–strength reliability is computed as

$$R = P(Y_2 < Y_1) = \int_0^{\infty} \int_0^{y_1} f_{1,2}(y_1, y_2) dy_2 dy_1.$$

Now, suppose that the joint distribution of random variables Y_1 and Y_2 is as given in (6); then, the stress–strength reliability is

$$R = \int_0^{\infty} \int_0^{y_1} h_1(y_1) h_2(y_2) \left[1 + \alpha - \alpha\gamma\phi H_1^{\gamma-1}(y_1) H_2^{\phi-1}(y_2)\right] dy_2 dy_1.$$

Making the transformation $u_2 = H_2(y_2)$, we have

$$R = \int_0^{\infty} \int_0^{H_2(y_1)} h_1(y_1) \left[1 + \alpha - \alpha\gamma\phi H_1^{\gamma-1}(y_1) u_2^{\phi-1}\right] du_2 dy_1,$$

or

$$R = \int_0^{\infty} h_1(y_1) H_2(y_1) \left[(1+\alpha) - \alpha\gamma H_1^{\gamma-1}(y_1) H_2^{\phi-1}(y_1)\right] dy_1. \tag{27}$$

The reliability coefficient, (27), can be computed for given baseline distributions. Also, if Y_1 and Y_2 have the same distributions, that is $H_1(y_1) = H_2(y_2) = H(y)$, then the reliability coefficient is given as

$$R = P(Y_2 < Y_1) = \int_0^{\infty} h(y) H(y) \left[(1+\alpha) - \alpha\gamma H^{\gamma+\phi-2}(y)\right] dy.$$

Making the transformation $u = H(y)$, we have

$$R = P(Y_2 < Y_1) = \int_0^1 u\left(1 + \alpha - \alpha\gamma u^{\gamma+\phi-2}\right) du = \frac{1}{2}\left(1 + \alpha - \frac{2\alpha\gamma}{\gamma+\phi}\right). \tag{28}$$

We can see that when Y_1 and Y_2 have the same distribution, then the reliability coefficient does not depend on that underlying distribution. It is also interesting to see that if either $\alpha = 0$ or $\gamma = \varphi$, then the reliability coefficient will be $R = 1/2$, as it should be.

4.7. Random Sampling

The conditional distribution approach is a useful method to draw a random sample from a bivariate distribution. In the following, the same is discussed for the new *CBD*s. In this technique, a random observation is first drawn from the marginal distribution of Y_1, and then the same is used to draw a random observation from the conditional distribution of Y_2 given $Y_1 = y_1$. Now, a random observation is drawn from the marginal distribution of Y_1 by solving $F_1(y_1) = u_1$ or by solving

$$(1+\alpha)H_1(y_1) - \alpha H_1^{\gamma}(y_1) = u_1, \tag{29}$$

for y_1, where u_1 is a uniform random number. The above equation is solved for a specific value of γ and for given values of the parameters of $H_1(y_1)$. A random observation y_2 is generated using the conditional distribution of Y_2 given that $Y_1 = y_1$, that is a random observation and y_2 is obtained by solving $F_{1|2}(y_2 \,|\, y_1) = u_2$ for y_2, where u_2 is another uniform random number.

Now, the conditional *cdf* of Y_2 given $Y_1 = y_1$ is

$$F_{2|1}(y_2|y_1) = \int_{-\infty}^{y_2} f_{Y_2|y_1}(u_2|y_1)du_2 = \int_{-\infty}^{y_2} \frac{g_2(u_2)}{\Delta_1(y_1)}\Big[(1+\alpha) - \alpha\gamma\phi H_1^{\gamma-1}(y_1)H_2^{\phi-1}(u)\Big]du_2.$$

Solving the above integral, the conditional *cdf* of Y_2 given $Y_1 = y_1$ is

$$F_{2|1}(y_2|y_1) = \frac{1}{\Delta_1(y_1)}\Big[(1+\alpha)H_2(y_2) - \alpha\gamma H_1^{\gamma-1}(y_1)H_2^{\phi}(y_2)\Big] \;\; ; \;\; y_2 \in \mathbb{R}.$$

So, a random observation y_2 is generated by solving

$$\frac{1}{\Delta_1(y_1)}\Big[(1+\alpha)H_2(y_2) - \alpha\gamma H_1^{\gamma-1}(y_1)H_2^{\phi}(y_2)\Big] = u_2 \tag{30}$$

for y_2. Equations (29) and (30) are essentially non–linear equations. The pair (y_1, y_2) will constitute a bivariate random observation from the new bivariate family of distributions.

4.8. Parameter Estimation

The estimates of the parameters are required to apply any distribution for modeling some real data. In the following, the maximum likelihood estimation of the parameters of the new *CBD*s is discussed, assuming that all the parameters of the baseline distributions are known.

Suppose a sample of n bivariate observations is available from the new *CBD*s. The likelihood function is

$$L(\alpha,\gamma,\phi;\mathbf{y}_1,\mathbf{y}_2) = \Big[\prod_{j=1}^{n} h_1(y_{1j})\Big]\Big[\prod_{j=1}^{n} h_2(y_{2j})\Big]\prod_{j=1}^{n}\Big[(1+\alpha) - \alpha\gamma\phi H_1^{\gamma-1}(y_{1j})H_2^{\phi-1}(y_{2j})\Big].$$

The log-likelihood function is

$$\ell(\alpha,\gamma,\phi;\mathbf{y}_1,\mathbf{y}_2) = \sum_{j=1}^{n}\ln h_1(y_{1j}) + \sum_{j=1}^{n}\ln h_2(y_{2j}) + \sum_{j=1}^{n}\ln\Big[(1+\alpha) - \alpha\gamma\phi H_1^{\gamma-1}(y_{1j})H_2^{\phi-1}(y_{2j})\Big].$$

The derivatives of the log-likelihood function with respect to the unknown parameters are

$$\frac{\partial}{\partial\alpha}\ell(\alpha,\gamma,\phi;\mathbf{y}_1,\mathbf{y}_2)=\sum_{j=1}^{n}\frac{1-\gamma\phi H_1^{\gamma-1}(y_{1j})H_2^{\phi-1}(y_{2j})}{1+\alpha-\alpha\gamma\phi H_1^{\gamma-1}(y_{1j})H_2^{\phi-1}(y_{2j})}, \tag{31}$$

$$\frac{\partial}{\partial\gamma}\ell(\alpha,\gamma,\phi;\mathbf{y}_1,\mathbf{y}_2)=-\sum_{j=1}^{n}\frac{\alpha\phi H_1^{\gamma-1}(y_{1j})H_2^{\phi-1}(y_{2j})\left[1+\gamma\ln H_1(y_{1j})\right]}{1+\alpha-\alpha\gamma\phi H_1^{\gamma-1}(y_{1j})H_2^{\phi-1}(y_{2j})}, \tag{32}$$

and

$$\frac{\partial}{\partial\phi}\ell(\alpha,\gamma,\phi;\mathbf{y}_1,\mathbf{y}_2)=-\sum_{j=1}^{n}\frac{\alpha\gamma H_1^{\gamma-1}(y_{1j})H_2^{\phi-1}(y_{2j})\left[1+\phi\ln H_2(y_{2j})\right]}{1+\alpha-\alpha\gamma\phi H_1^{\gamma-1}(y_{1j})H_2^{\phi-1}(y_{2j})}. \tag{33}$$

The maximum likelihood estimates α, γ, and ϕ can be obtained by equating the derivatives in (31)–(33) to zero and simultaneously solving the resulting equations. If all the parameters of the baseline distribution are known, then, asymptotically, the maximum likelihood estimates $\hat{\alpha}$, $\hat{\gamma}$, and $\hat{\phi}$ have a multivariate normal distribution. Specifically,

$$\begin{bmatrix}\hat{\alpha} & \hat{\gamma} & \hat{\phi}\end{bmatrix}' \sim N_3\Big(\boldsymbol{\mu}=\begin{bmatrix}\alpha & \gamma & \phi\end{bmatrix}' \;;\; \boldsymbol{\Sigma}=(\sigma_{ij})\Big),$$

where

$$\boldsymbol{\Sigma}^{-1}=-E\begin{bmatrix}\partial^2\ell/\partial\alpha^2 & \partial^2\ell/\partial\alpha\partial\gamma & \partial^2\ell/\partial\alpha\partial\phi\\ & \partial^2\ell/\partial\gamma^2 & \partial^2\ell/\partial\gamma\partial\phi\\ & & \partial^2\ell/\partial\phi^2\end{bmatrix}$$

is the Fisher information matrix. The entries of this matrix are given in Appendix A.

5. The Multivariate Extension

In this section, a multivariate extension of the new class of distributions, (5), is obtained. For this, we first see that a multivariate extension of (4) is immediately given as

$$F_{\mathbf{y}}(y_1,\ldots,y_p)=\int_0^{H_p(y_p)}\cdots\int_0^{H_1(y_1)} r(v_1,\ldots,v_p)dv_1\cdots dv_p \;;\; 0<v_i<1. \tag{34}$$

Now, using

$$r(v_1,\ldots,v_p)=1+\alpha\Big(1-\textstyle\prod_{i=1}^{p}\gamma_i v_i^{a_i-1}\Big)$$

in (34), the *cdf* of a new class of multivariate distributions is obtained as

$$F_{\mathbf{y}}(y_1,\ldots,y_p)=\int_0^{H_p(y_p)}\cdots\int_0^{H_1(y_1)}\Big[1+\alpha\Big(1-\textstyle\prod_{i=1}^{p}\gamma_i v_i^{a_i-1}\Big)\Big]dv_1\cdots dv_p$$

or

$$F_{\mathbf{y}}(y_1,\ldots,y_p)=\Big[\textstyle\prod_{i=1}^{p}H_i(y_i)\Big]\Big[(1+\alpha)-\alpha\prod_{i=1}^{p}H_i^{\gamma_i-1}(y_i)\Big] \;;\; \mathbf{y}\in\mathbb{R}^p, \tag{35}$$

where $-1\le\alpha\le 2\min\Big(\gamma_i^{-1}\Big)$. The density function corresponding to (35) is

$$f_{\mathbf{y}}(\mathbf{y})=\Big[\textstyle\prod_{i=1}^{p}h_i(y_i)\Big]\Big[(1+\alpha)-\alpha\prod_{i=1}^{p}\gamma_i H_i^{\gamma_i-1}(y_i)\Big] \;;\; \mathbf{y}\in\mathbb{R}^p \;,\; -1\le\alpha\le 2\min\Big(\gamma_i^{-1}\Big). \tag{36}$$

It is easy to see that the marginal density of Y_i for the joint density function in (36) is

$$f_{Y_i}(y_i)=h(y_i)\Big[(1+\alpha)-\alpha\gamma_i H_i^{\gamma_i-1}(y_i)\Big] \;;\; y_i\in\mathbb{R} \;;\; -1\le\alpha\le 2\gamma_i^{-1}$$

and the joint density function for any pair (Y_i, Y_m) is the same as given in (6). The conditional density function of a specific Y_i given the information of all other variables is

$$f_{Y_i|\mathbf{y}_{(i)}}\left(y_i\middle|\mathbf{y}_{(i)}\right) = \frac{h_i(y_i)}{\Delta_i\left(\mathbf{y}_{(i)}\right)}\left[(1+\alpha) - \alpha\prod_{i=1}^{p}\gamma_i H_i^{\gamma_i-1}(y_i)\right], \tag{37}$$

where $\mathbf{y}_{(i)} = \left(Y_1,\ldots,Y_{(i-1)},Y_{(i+1)},\ldots,Y_p\right)$ and $\Delta_i\left(\mathbf{y}_{(i)}\right) = \left[(1+\alpha) - \alpha\prod_{h\neq i=1}^{p}\gamma_h H_h^{\gamma_h-1}(y_h)\right]$. The joint conditional distribution of any pair, say (Y_i, Y_m), given the information of all other variables, can be easily obtained as

$$f_{Y_i,Y_m|\mathbf{y}_{(i,m)}}\left(y_i,y_m\middle|\mathbf{y}_{(i,m)}\right) = \frac{h_i(y_i)h_m(y_m)}{\Delta_{i,m}\left(\mathbf{y}_{(i,m)}\right)}\left[(1+\alpha) - \alpha\prod_{i=1}^{p}\gamma_i H_i^{\gamma_i-1}(y_i)\right] \ , \ (y_i,y_m)\in\mathbb{R}^2, \tag{38}$$

where $\mathbf{y}_{(i,m)}$ is the random vector $\mathbf{y}$ without Y_i and Y_m and

$$\Delta_{i,m}\left(\mathbf{y}_{(i,m)}\right) = \left[(1+\alpha) - \alpha\prod_{h\neq i,m=1}^{p}\gamma_h H_h^{\gamma_h-1}(y_h)\right].$$

The conditional distributions, (37) and (38), can be used to obtain the conditional means, conditional variances, and conditional covariances.

5.1. Multivariate Dependence Measures

The multivariate dependence measures are the extension of the bivariate dependence measures for a set of random vectors. The multivariate dependence measures for the new class of multivariate distributions are discussed as follows.

5.1.1. Multivariate Tau Coefficient

The multivariate extension of Kendall's coefficient, τ_p, is given by [21] as

$$\tau_p = \frac{2^p}{2^{p-1}-1}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}F_{\mathbf{y}}(\mathbf{y})f_{\mathbf{y}}(\mathbf{y})dy_1\ldots dy_p - \frac{1}{2^{p-1}-1}.$$

Now, using (35) and (36) in the above equation, the τ_p coefficient for the new class of multivariate distributions is

$$\begin{aligned}\tau_p = \frac{2^p}{2^{p-1}-1}&\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\prod_{i=1}^{p}H_i(y_i)\left\{(1+\alpha) - \alpha\prod_{i=1}^{p}H_i^{\gamma_i-1}(y_i)\right\}\right]\\ &\times\left[\prod_{i=1}^{p}h_i(y_i)\left\{(1+\alpha) - \alpha\prod_{i=1}^{p}\gamma_i H_i^{\gamma_i-1}(y_i)\right\}\right]dy_1\ldots dy_p - \frac{1}{2^{p-1}-1}.\end{aligned}$$

Making the transformation $w_i = H_i(y_i)$, we have

$$\begin{aligned}\tau_p = \frac{2^p}{2^{p-1}-1}&\int_0^1\cdots\int_0^1\left[\prod_{i=1}^{p}w_i\left\{(1+\alpha) - \alpha\prod_{i=1}^{p}w_i^{\gamma_i-1}\right\}\right]\\ &\times\left[\left\{(1+\alpha) - \alpha\prod_{i=1}^{p}\gamma_i w_i^{\gamma_i-1}\right\}\right]dw_1\cdots dw_p - \frac{1}{2^{p-1}-1}\end{aligned}$$

or

$$\begin{aligned}\tau_p = \frac{2^p}{2^{p-1}-1}&\int_0^1\cdots\int_0^1\left[(1+\alpha)^2\prod_{i=1}^{p}w_i - \alpha(1+\alpha)\prod_{i=1}^{p}\gamma_i w_i^{\gamma_i}\right.\\ &\left.-\alpha(1+\alpha)\prod_{i=1}^{p}w_i^{\gamma_i} + \alpha^2\prod_{i=1}^{p}\gamma_i w_i^{2\gamma_i-1}\right]dw_1\cdots dw_p - \frac{1}{2^{p-1}-1}.\end{aligned}$$

Solving the above multiple integrals and simplifying τ_p for the new class of multivariate distributions results in

$$\tau_p = \frac{-\alpha(1+\alpha)}{2^{p-1}-1}\left[2^p\left(\prod_{i=1}^{p}\gamma_i+1\right)/\prod_{i=1}^{p}(\gamma_i+1)+2\right]. \tag{39}$$

It is easy to see that for p = 2, (39) reduces to (24), as it should.

5.1.2. Multivariate Rho Coefficient

The multivariate extension of Spearman's coefficient, ρ_p, is defined by [22] as

$$\rho_p = \frac{2^p(p+1)}{2^p-(p+1)}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}F_{\mathbf{y}}(\mathbf{y})\left[\prod_{i=1}^{p}f_i(y_i)\right]dy_1\ldots dy_p-\frac{(p+1)}{2^p-(p+1)}.$$

Now, using the joint density of a new multivariate family of distributions from (36) and the marginal density functions of each of Y_i, Spearman's ρ_p is

$$\rho_p = \frac{2^p(p+1)}{2^p-(p+1)}\quad\begin{aligned}&\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\prod_{i=1}^{p}H_i(y_i)\left\{(1+\alpha)-\alpha\prod_{i=1}^{p}H_i^{\gamma_i-1}(y_i)\right\}\right]\\&\times\left[\prod_{i=1}^{p}\left\{h_i(y_i)\left[(1+\alpha)-\alpha\gamma_iH_i^{\gamma_i-1}(y_i)\right]\right\}\right]dy_1\ldots dy_p-\frac{(p+1)}{2^p-(p+1)}.\end{aligned}$$

Making the transformation $w_i = H_i(y_i)$, we have

$$\begin{aligned}\rho_p = \frac{2^p(p+1)}{2^p-(p+1)}&\int_0^1\cdots\int_0^1\left[\prod_{i=1}^{p}w_i\left\{(1+\alpha)-\alpha\prod_{i=1}^{p}w_i^{\gamma_i-1}\right\}\right]\\&\times\left[\prod_{i=1}^{p}\left\{(1+\alpha)-\alpha\gamma_iw_i^{\gamma_i-1}\right\}\right]dw_1\ldots dw_p-\frac{(p+1)}{2^p-(p+1)}\end{aligned}$$

or

$$\begin{aligned}\rho_p = \frac{2^p(p+1)}{2^p-(p+1)}&\int_0^1\cdots\int_0^1\left[(1+\alpha)\prod_{i=1}^{p}w_i\left\{(1+\alpha)-\alpha\gamma_iw_i^{\gamma_i-1}\right\}\right]dw_1\cdots dw_p\\-\frac{2^p(p+1)}{2^p-(p+1)}&\int_0^1\cdots\int_0^1\left[\alpha\prod_{i=1}^{p}w_i^{\gamma_i}\left\{(1+\alpha)-\alpha\gamma_iw_i^{\gamma_i-1}\right\}\right]dw_1\cdots dw_p-\frac{(p+1)}{2^p-(p+1)}.\end{aligned}$$

Solving the above integrals and simplifying the multivariate Spearman's ρ_p coefficient for the new multivariate family of distributions results in

$$\rho_p = \frac{(p+1)}{2^p-(p+1)}\left[(1+\alpha)\prod_{i=1}^{p}\left\{1-\frac{\alpha(\gamma_i-1)}{\gamma_i+1}\right\}-\alpha\prod_{i=1}^{p}\left\{\frac{2-\alpha(\gamma_i-1)}{\gamma_i+1}\right\}-1\right]. \tag{40}$$

We can see that for p = 2, (40) reduces to (25), as it should.

5.2. Parameter Estimation for the Multivariate Extension

In the following, we have discussed the maximum likelihood estimation of the parameters of the new multivariate family of distributions. Suppose n vector observations are available from the new class of multivariate distributions, assuming that all the parameters of the baseline distributions are known. The log-likelihood function is then

$$\ell(\alpha,\boldsymbol{\gamma};\mathbf{Y}) = \sum_{j=1}^{n}\sum_{i=1}^{p}\ln h_i(y_{ij})+\sum_{j=1}^{n}\ln\left[(1+\alpha)-\alpha\prod_{i=1}^{p}\gamma_iH_i^{\gamma_i-1}(y_{ij})\right].$$

where $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_p)$ and $\mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n)$. The derivatives of the log-likelihood function with respect to the unknown parameters are

$$(1\text{–p})\ \frac{\partial}{\partial \gamma_i}\ell(\alpha, \gamma_i; \mathbf{Y}) = -\alpha \sum_{j=1}^{n} \frac{\left\{\gamma_i^{-1} + \ln H_i(y_{ij})\right\} \prod_{i=1}^{p} \gamma_i H_i^{\gamma_i - 1}(y_{ij})}{(1+\alpha) - \alpha \prod_{i=1}^{p} \gamma_i H_i^{\gamma_i - 1}(y_{ij})}, \tag{41}$$

and

$$\frac{\partial}{\partial \alpha}\ell(\alpha, \gamma_i; \mathbf{Y}) = \sum_{j=1}^{n} \frac{1 - \prod_{i=1}^{p} \gamma_i H_i^{\gamma_i - 1}(y_{ij})}{(1+\alpha) - \alpha \prod_{i=1}^{p} \gamma_i H_i^{\gamma_i - 1}(y_{ij})}. \tag{42}$$

Equating the above derivatives to zero and numerically solving the resulting $(p + 1)$ equations, we can obtain the maximum likelihood estimators of γ_i, $i = 1, 2, \ldots, p$, and α.

We will now discuss a member of the new bivariate family of distributions when the baseline distributions are the power function distributions.

6. A Bivariate Power Function Distribution

The power function distribution is a simple yet powerful probability distribution. The density and distribution function of the power function distribution are given as

$$h(y) = \frac{\beta y^{\beta - 1}}{k^{\beta}} \ \&\ H(y) = \left(\frac{y}{k}\right)^{\beta} \ ;\ y < k \ ;\ k, \beta > 0$$

and are written as $PF(k, \beta)$. Several authors have studied this distribution in different contexts. The moments of generalized order statistics for the distribution are obtained by [23], whereas [24] has given a characterization of the distribution based on lower record values. The distribution has been used by [25] to propose a general transmuted power function distribution using a general transmutation technique by [26]. Some other notable references of the work on the distribution are [27–31], among others.

In the following, we have proposed a bivariate power function distribution by using the new bivariate family of distributions. For this, suppose that the random variables Y_1 and Y_2 have $PF(k_1, \beta_1)$ and $PF(k_2, \beta_2)$ distributions, respectively. Using these baseline distributions in (6), the joint *cdf* of the new bivariate power function (*BPF*) distribution is obtained as

$$F_{1,2}(y_1, y_2) = (1+\alpha)\left(\frac{y_1}{k_1}\right)^{\beta_1}\left(\frac{y_2}{k_2}\right)^{\beta_2} - \alpha\left(\frac{y_1}{k_1}\right)^{\gamma\beta_1}\left(\frac{y_2}{k_2}\right)^{\phi\beta_2} \ ;\ \begin{matrix} 0 < y_1 < k_1 \\ 0 < y_2 < k_2, \end{matrix} \tag{43}$$

where $(\gamma, \phi) \geq 1$ and $-1 \leq \alpha \leq 2\min(\gamma^{-1}, \phi^{-1})$. The density function of the new *BPF* distribution is immediately written from (35) as

$$f_{1,2}(y_1, y_2) = \left(\frac{\beta_1 y_1^{\beta_1 - 1}}{k_1^{\beta_1}}\right)\left(\frac{\beta_2 y_2^{\beta_2 - 1}}{k_2^{\beta_2}}\right)\left[(1+\alpha) - \alpha\gamma\phi\left(\frac{y_1}{k_1}\right)^{(\gamma-1)\beta_1}\left(\frac{y_2}{k_2}\right)^{(\phi-1)\beta_2}\right] \ ;\ \begin{matrix} 0 < y_1 < k_1 \\ 0 < y_2 < k_2. \end{matrix} \tag{44}$$

The distribution given in (44) will be written as $BPF(\alpha, \gamma, \varphi, \beta_1, \beta_2, k_1, k_2)$. It is interesting to note that if $\gamma = \varphi = 1$ or if $\alpha = 0$, then the variables Y_1 and Y_2 in $BPF(\alpha, \gamma, \varphi, \beta_1, \beta_2, k_1, k_2)$ have independent $PF(k_1, \beta_1)$ and $PF(k_2, \beta_2)$ distributions. The plot of the joint density function of the bivariate power function distribution for $k_1 = k_2 = 2$ and $\gamma = \varphi = 3.0$ and for various of α, β_1, and β_2 is given in Figure 2 below.

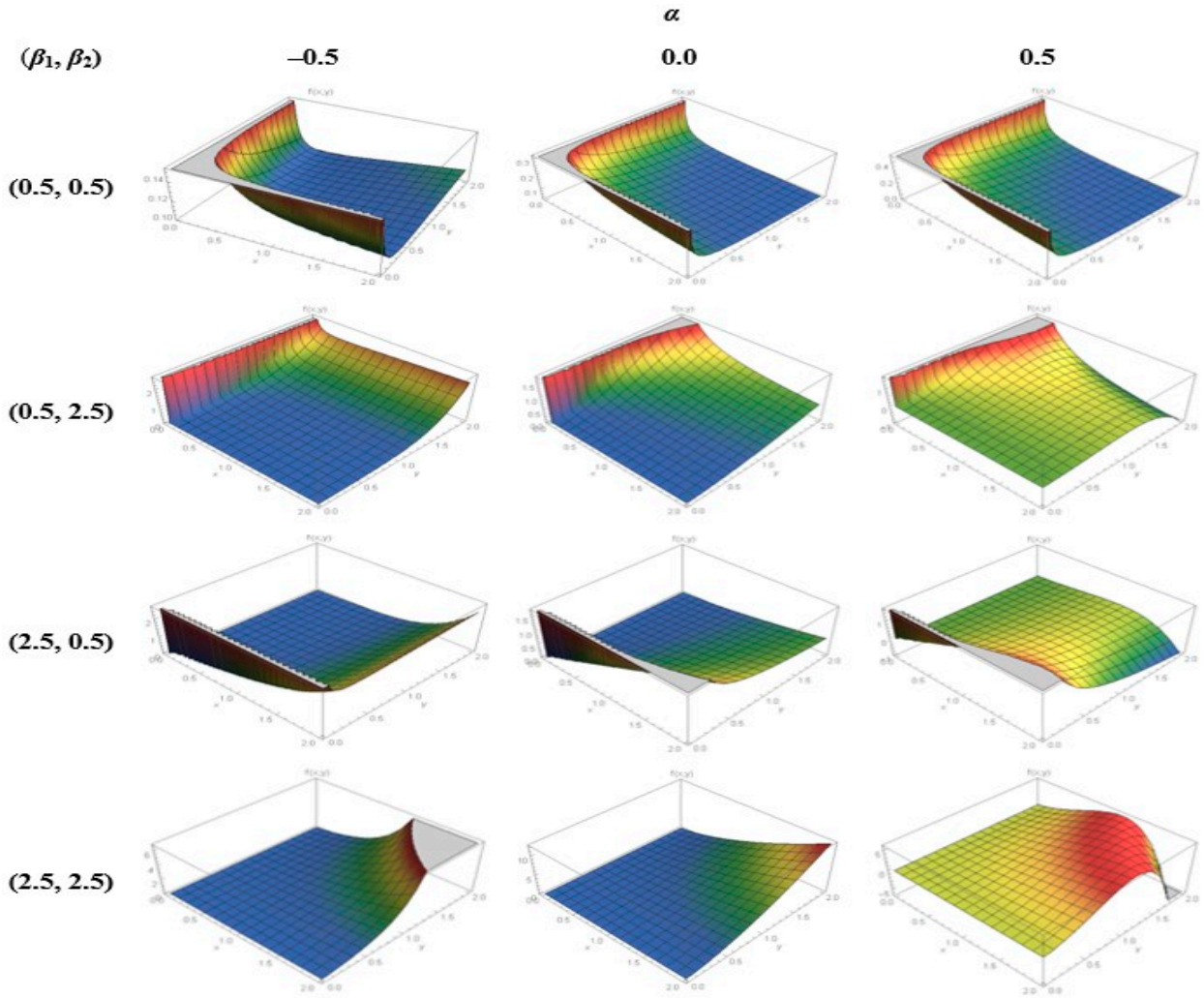


Figure 2. Joint density function of the bivariate power function distribution.

The plot shows that the density function has low ordinate when any one of β_1 or β_2 is less than 1 and has high ordinate when both β_1 or β_2 are greater than 1. Also, when β_1, β_2, and α are greater than 1, then the ordinate first increases and then decreases.

The marginal density functions of Y_1 and Y_2 for (44) are

$$f_1(y_1) = \frac{\beta_1 y_1^{\beta_1-1}}{k_1^{\beta_1}}\left[(1+\alpha) - \alpha\gamma(y_1/k_1)^{(\gamma-1)\beta_1}\right] \;\; ; \;\; 0 < y_1 < k_1$$

and

$$f_2(y_2) = \frac{\beta_2 y_2^{\beta_2-1}}{k_2^{\beta_2}}\left[(1+\alpha) - \alpha\phi(y_2/k_2)^{(\phi-1)k_2}\right] \;\; ; \;\; 0 < y_2 < k_2.$$

The marginal distributions, given above, reduce to the transmuted power function distribution given by [27] for $\gamma = 2$ or $\varphi = 2$.

The conditional density function of Y_1 given $Y_2 = y_2$ for the new *BPF* distribution is

$$f_{1|2}(y_1|y_2) = \frac{1}{\Delta_2(y_2)}\left(\frac{\beta_1 y_1^{\beta_1-1}}{k_1^{\beta_1}}\right)\left[(1+\alpha) - \alpha\gamma\phi(y_1/k_1)^{(\gamma-1)\beta_1}(y_2/k_2)^{(\phi-1)k_2}\right] \;\; ; \;\; 0 < y_1 < k_1, \tag{45}$$

where $\Delta_2(y_2) = (1+\alpha) - \alpha\phi(y_2/k_2)^{(\phi-1)\beta_2}$. Also, the conditional distribution of Y_2 given $Y_1 = y_1$ is

$$f_{2|1}(y_2|y_1) = \frac{1}{\Delta_1(y_1)}\left(\frac{\beta_2 y_2^{\beta_2-1}}{k_2^{\beta_2}}\right)\left[(1+\alpha) - \alpha\gamma\phi(y_1/k_1)^{(\gamma-1)\beta_1}(y_2/k_2)^{(\phi-1)k_2}\right] \; ; \; 0 < y_2 < k_2, \tag{46}$$

where $\Delta_1(y_1) = (1+\alpha) - \alpha\gamma(y_1/k_1)^{(\gamma-1)\beta_1}$.

7. Some Properties of the Proposed Bivariate Power Function Distribution

In this section, some important properties of the bivariate power function distribution are discussed alongside the estimation of the model parameters.

7.1. The Joint and Ratio Moments

The product and ratio moments for the new bivariate family of distributions are given in (10) and (11), respectively. The product and ratio moments for the new *BPF* distribution are obtained using the moments of power function and exponentiated power function distributions. Now, if a random variable Y has a power function distribution with parameters k and β, then we have

$$\mu_Y^p = \frac{\beta k^p}{\beta + p} \; , \; \mu_Y^{-p} = \frac{\beta k^{-p}}{\beta - p} \; , \; \mu_{Y(a:a)}^p = \frac{a\beta k^p}{a\beta + p} \text{ and } \mu_{Y(a:a)}^{-p} = \frac{a\beta k^{-p}}{a\beta - p} \tag{47}$$

where $\mu_{Y(a:a)}^p$ and $\mu_{Y(a:a)}^{-p}$ are, respectively, the pth raw and pth inverse moments of an exponentiated power function distribution with exponentiation parameter a. Now, using (47) in (10), the (p,q)th joint moment for the new bivariate power function distribution is

$$\mu_{Y_1,Y_2}^{p,q} = \beta_1\beta_2 k_1^p k_2^q\left[\frac{1+\alpha}{(\beta_1+p)(\beta_2+q)} - \frac{\alpha\gamma\phi}{(\gamma\beta_1+p)(\phi\beta_2+q)}\right]. \tag{48}$$

The product moments can be used to compute the means and variances of each of the random variables. The product moments can also be used to compute the correlation coefficient between the two variables for the *BPF* distribution. We have plotted the Pearson correlation coefficient between the two variables for the *BPF* distribution for different values of the parameters. The plots are given in Figure 3 below.

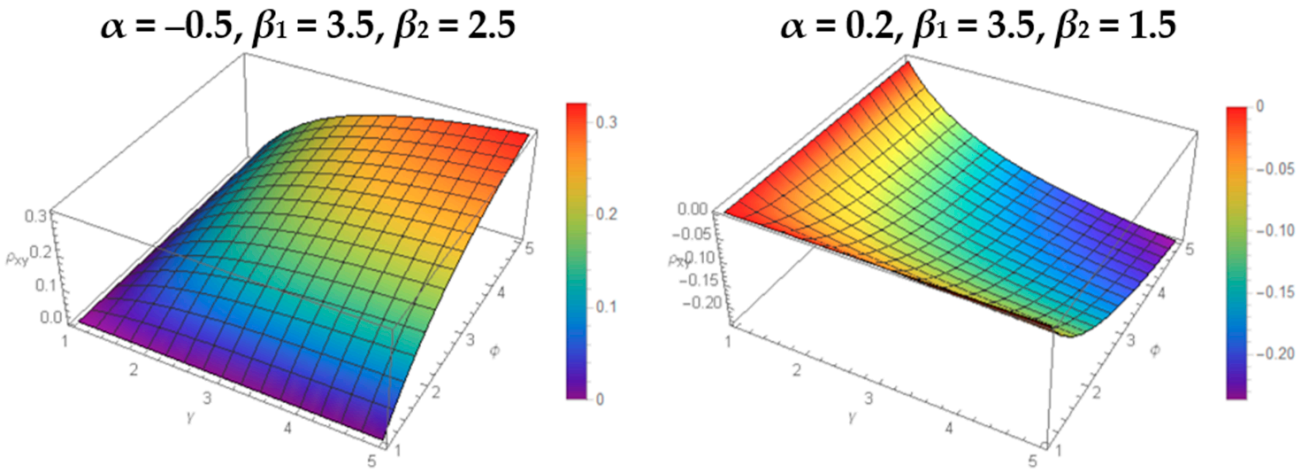


Figure 3. Plots of the correlation coefficient for the bivariate power function distribution.

In the above figure, we can see that the behavior of the correlation coefficient for the *BPF* distribution is similar to the behavior of Kendall's τ coefficient for the new *CBD*s.

The ratio moments $\mu^{p/q}_{Y_1,Y_2}$ and $\mu^{q/p}_{Y_1,Y_2}$ are easily obtained using (47) in (11) and are given as

$$\mu^{p/q}_{Y_1,Y_2} = \beta_1\beta_2 k_1^p k_2^{-q}\left[\frac{1+\alpha}{(\beta_1+p)(\beta_2-q)} - \frac{\alpha\gamma\phi}{(\gamma\beta_1+p)(\phi\beta_2-q)}\right] \;;\; \beta_2 > q$$

and

$$\mu^{q/p}_{Y_1,Y_2} = \beta_1\beta_2 k_1^{-p} k_2^{q}\left[\frac{1+\alpha}{(\beta_1-p)(\beta_2+q)} - \frac{\alpha\gamma\phi}{(\gamma\beta_1-p)(\phi\beta_2+q)}\right] \;;\; \beta_1 > p.$$

The correlation coefficient can be computed using (48).

7.2. Moments of the Conditional Distributions

The conditional moments for the new *BPF* distribution can be readily obtained using the general expressions for the conditional moments for the proposed *CBD*s. The *p*th conditional moment of Y_1 given $Y_2 = y_2$ for the new *CBD*s is given in (23) as

$$\mu^p_{Y_1|y_2} = \frac{1}{\Delta_2(y_2)}\left[(1+\alpha)\mu^p_{Y_1} - \alpha\phi H_2^{\phi-1}(y_2)\mu^p_{Y_1(\gamma:\gamma)}\right].$$

Now, for the *BPF* distribution, we have $\Delta_2(y_2) = (1+\alpha) - \alpha\phi(y_2/k_2)^{(\phi-1)\beta_2}$, and hence the *p*th conditional moment of Y_1 given $Y_2 = y_2$ is

$$\mu^p_{Y_1|y_2} = \frac{\beta_1 k_1^p}{\Delta_2(y_2)}\left[\frac{1+\alpha}{\beta_1+p} - \left(\frac{y_2}{k_2}\right)^{(\phi-1)\beta_2}\frac{\alpha\gamma\phi}{\gamma\beta_1+p}\right]. \tag{49}$$

Similarly, the *q*th conditional moment of Y_2 given $Y_1 = y_1$ is obtained as

$$\mu^q_{Y_2|y_1} = \frac{\beta_2 k_2^q}{\Delta_1(y_1)}\left[\frac{1+\alpha}{\beta_2+q} - \left(\frac{y_1}{k_1}\right)^{(\gamma-1)\beta_1}\frac{\alpha\gamma\phi}{\phi\beta_2+q}\right], \tag{50}$$

where $\Delta_1(y_1) = (1+\alpha) - \alpha\gamma(y_1/k_1)^{(\gamma-1)\beta_1}$,. The conditional means and variances can be obtained from (49) and (50).

7.3. Random Sample Generation

The bivariate random sample can be generated from the *BPF* distribution using the conditional distribution approach. For this, a random y_1 observation is generated from the marginal distribution of Y_1 by solving

$$(1+\alpha)(y_1/k_1)^{\beta_1} - \alpha(y_1/k_1)^{\gamma\beta_1} = u_1$$

for y_1 and specific values of the parameters. Once the random y_1 observation is obtained, the random y_2 observation is obtained using the conditional distribution function of Y_2 given $Y_1 = y_1$, as given in (30). Now, for the *BPF* distribution, the conditional distribution function of Y_2 given $Y_1 = y_1$ is

$$F_{2|1}(y_2|y_1) = \Delta_1^{-1}(y_1)\left[(1+\alpha)(y_2/k_2)^{\beta_2} - \alpha\gamma(y_1/k_1)^{(\gamma-1)\beta_1}(y_2/k_2)^{\phi\beta_2}\right],$$

where $\Delta_1(y_1)$ is defined earlier. Hence, a random y_2 observation is obtained by solving

$$\Delta_1^{-1}(y_1)\left[(1+\alpha)(y_2/k_2)^{\beta_2} - \alpha\gamma(y_1/k_1)^{(\gamma-1)\beta_1}(y_2/k_2)^{\phi\beta_2}\right] = u_2$$

for y_2.

7.4. Bivariate Survival and Hazard Rate Functions

The bivariate survival or reliability function for the new *CBD*s is given in (13) as

$$R_{1,2}(y_1, y_2) = (1+\alpha)R_1(y_1)R_2(y_2) - \alpha\left[1 - H_1^{\gamma}(y_1)\right]\left[1 - H_2^{\phi}(y_2)\right] \;;\; y_1, y_2 \in \mathbb{R}^2.$$

The bivariate reliability function for the new *BPF* distribution is immediately written as

$$R_{1,2}(y_1, y_2) = (1+\alpha)\left[1 - (y_1/k_1)^{\beta_1}\right]\left[1 - (y_2/k_2)^{\beta_2}\right] - \alpha\left[1 - (y_1/k_1)^{\gamma\beta_1}\right]\left[1 - (y_2/k_2)^{\phi\beta_2}\right].$$

Also, the bivariate hazard rate function for the new bivariate family of distributions is given in (14) as

$$\eta_{1,2}(y_1, y_2) = \frac{h_1(y_1)h_2(y_2)\left[1 + \alpha - \alpha\gamma\phi H_1^{\gamma-1}(y_1)H_2^{\phi-1}(y_2)\right]}{(1+\alpha)R_1(y_1)R_2(y_2) - \alpha\left[1 - H_1^{\gamma}(y_1)\right]\left[1 - H_2^{\phi}(y_2)\right]}.$$

The bivariate hazard function for the new *BPF* distribution is readily written as

$$h_{1,2}(y_1, y_2) = \frac{\left(\beta_1 y_1^{\beta_1-1}/k_1^{\beta_1}\right)\left(\beta_2 y_2^{\beta_2-1}/k_2^{\beta_2}\right)\left[1 + \alpha - \alpha\gamma\phi(y_1/k_1)^{(\gamma-1)\beta_1}(y_2/k_2)^{(\phi-1)\beta_2}\right]}{(1+\alpha)\left[1 - (y_1/k_1)^{\beta_1}\right]\left[1 - (y_2/k_2)^{\beta_2}\right] - \alpha\left[1 - (y_1/k_1)^{\gamma\beta_1}\right]\left[1 - (y_2/k_2)^{\phi\beta_2}\right]}. \quad (51)$$

The plot of the hazard rate function for $k_1 = k_2 = 2$ and $\gamma = \varphi = 3.0$ and for various values of α, β_1, and β_2 are given in Figure 4 below.

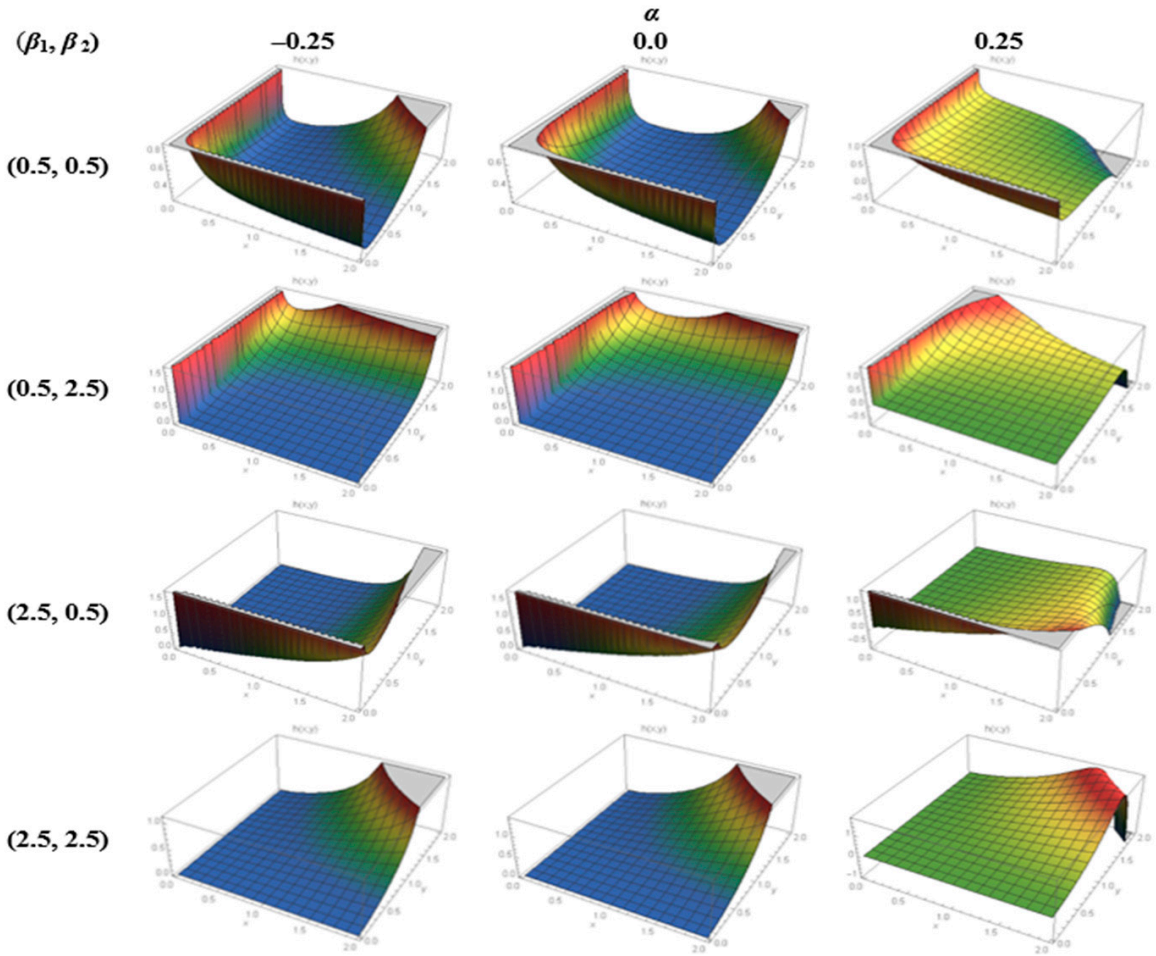


Figure 4. Joint hazard rate function of the bivariate power function distribution.

The hazard rate function plots indicate that the new *BPF* distribution has both increasing and decreasing trends.

7.5. Stress and Strength Reliability

Stress- and strength-based reliability is often used in engineering problems. Stress- and strength-based reliabilities for the new *CBD*s are given in (27) as

$$R = P(Y_2 < Y_1) = \int_0^{\infty} h_1(y_1)H_2(y_1)\left[(1+\alpha) - \alpha\gamma H_1^{\gamma-1}(y_1)H_2^{\phi-1}(y_1)\right]dy_1.$$

Now, if the random vector (Y_1,Y_2) has a joint *BPF* distribution, then the measure is given as

$$R = P(Y_2 < Y_1) = \int_0^{k_1} \left(\frac{\beta_1 y_1^{\beta_1-1}}{k_1^{\beta_1}}\right)\left(\frac{y_1}{k_2}\right)^{\beta_2}\left[(1+\alpha) - \alpha\gamma\left(\frac{y_1}{k_1}\right)^{(\gamma-1)\beta_1}\left(\frac{y_1}{k_2}\right)^{(\phi-1)\beta_2}\right]dy_1$$

or

$$R = P(Y_2 < Y_1) = \int_0^{k_1} \left(\frac{\beta_1 y_1^{\beta_1+\beta_2-1}}{k_1^{\beta_1}k_2^{\beta_2}}\right)\left[(1+\alpha) - \frac{\alpha\gamma y_1^{\gamma\beta_1+\phi\beta_2-\beta_1-\beta_2}}{k_1^{(\gamma-1)\beta_1}k_2^{(\phi-1)\beta_2}}\right]dy_1.$$

Solving the integral, the stress–strength reliability coefficient, R, for the new *BPF* distribution is

$$R = \frac{(1+\alpha)\beta_1 k_1^{\beta_2}}{k_2^{\beta_2}(\beta_1+\beta_2)} - \frac{\alpha\beta_1\gamma k_1^{\phi\beta_2}}{k_2^{\phi\beta_2}(\gamma\beta_1+\phi\beta_2)}. \tag{52}$$

It is easy to see that if Y_1 and Y_2 have the same distributions, that is if $\beta_1 = \beta_2$ and $k_1 = k_2$, then the reliability coefficient given in (52) reduces to the reliability coefficient given in (28). The plots of the reliability coefficient for different values of the parameters are given in Figure 5 below.

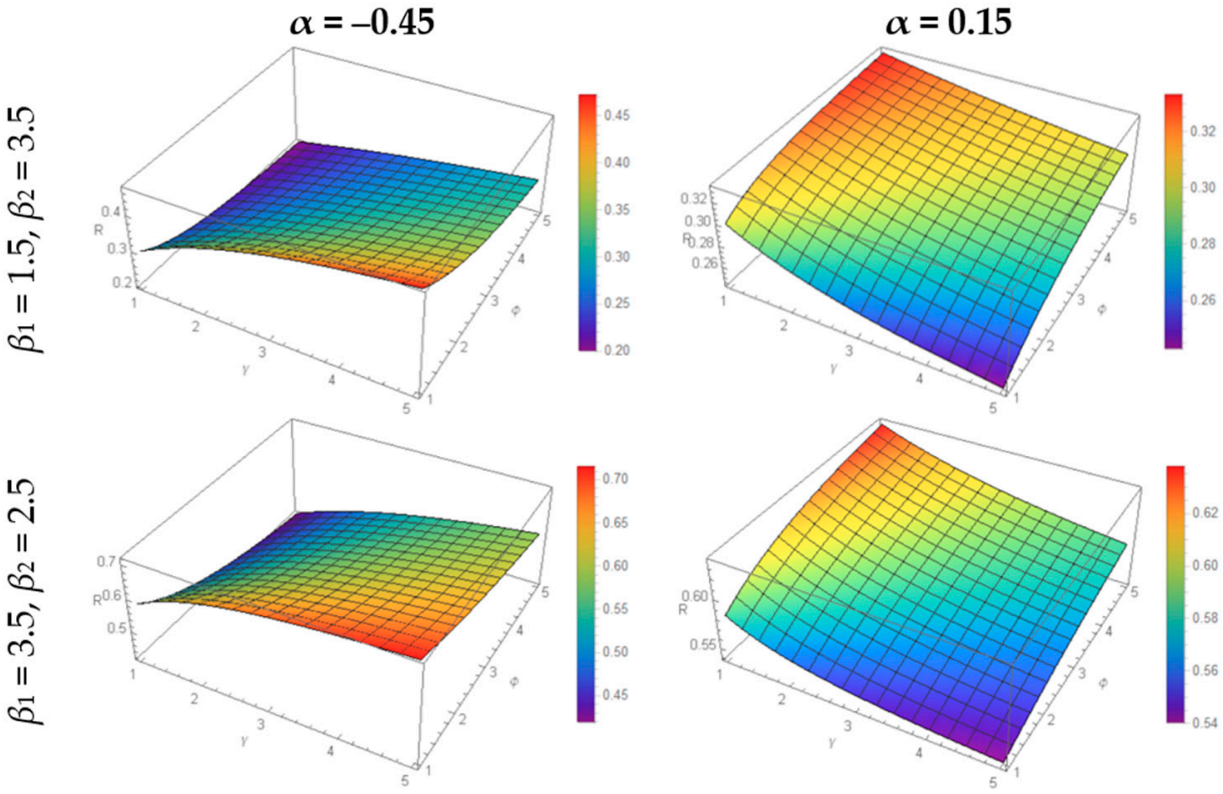


Figure 5. The reliability coefficient for the bivariate power function distribution.

The plots of the reliability coefficient given above indicate that the reliability coefficient has an increasing trend when $\alpha < 0$ and has a decreasing trend when $\alpha > 0$. We can also see in the above plots that the reliability coefficient remains less than 0.5 if $\beta_1 < \beta_2$, and it remains above 0.5 if $\beta_1 > \beta_2$.

7.6. The Local Dependence Measure

The local dependence measure for the new *CBD*s is given in (26) as

$$\gamma(y_1, y_2) = -\frac{\alpha\gamma\phi(\gamma-1)(\phi-1)(1+\alpha)h_1(y_1)h_2(y_2)H_1^{\gamma}(y_1)H_2^{\phi}(y_2)}{\left[H_1(y_1)H_2(y_2)\left\{(1+\alpha)-\alpha\gamma\phi H_1^{\gamma-1}(y_1)H_2^{\phi-1}(y_2)\right\}\right]^2}.$$

Now, using the marginal density and distribution functions of the power function distribution in the above equation, the local dependence measure for the *BPF* distribution is

$$\gamma(y_1, y_2) = -\frac{\alpha\gamma\phi(\gamma-1)(\phi-1)(1+\alpha)\beta_1\beta_2 y_1^{(\gamma+1)\beta_1-1} y_2^{(\phi+1)\beta_2-1}}{k_1^{(\gamma+1)\beta_1}k_2^{(\phi+1)\beta_2}\Delta_{1,2}^2(y_1, y_2)},$$

where $\Delta_{1,2}(y_1, y_2) = (y_1/k_1)^{\gamma\beta_1}(y_2/k_2)^{\phi\beta_2}\left[(1+\alpha)-\alpha\gamma\phi(y_1/k_1)^{(\gamma-1)\beta_1}(y_2/k_2)^{(\phi-1)\beta_2}\right]$. It is easy to see that $\gamma(y_1, y_2) > 0$ if α is negative and $\gamma(y_1, y_2) < 0$ if α is negative. We have also seen in Section 4.5 that Kendall's τ and Spearman's ρ are free from the choice of the baseline distribution.

7.7. Parameter Estimation

In this section, we have discussed the maximum likelihood estimation for the parameters of the new *BPF* distribution. Let n bivariate observations be available from (44). The likelihood function is

$$L(\theta; \mathbf{y}_1, \mathbf{y}_2) = \prod_{j=1}^{n}\left(\frac{\beta_1 y_{1j}^{\beta_1-1}}{k_1^{\beta_1}}\right)\prod_{j=1}^{n}\left(\frac{\beta_2 y_{2j}^{\beta_2-1}}{k_2^{\beta_2}}\right)\prod_{j=1}^{n}\left[(1+\alpha)-\alpha\gamma\phi\left(\frac{y_{1j}}{k_1}\right)^{(\gamma-1)\beta_1}\left(\frac{y_{2j}}{k_2}\right)^{(\phi-1)\beta_2}\right],$$

where $\theta = (\alpha,\gamma,\varphi,\beta_1,\beta_2,k_1,k_2)$. The log-likelihood function is

$$\begin{aligned}\ell(\theta; \mathbf{y}_1, \mathbf{y}_2) &= n\ln\beta_1 - n\beta_1\ln k_1 + (\beta_1-1)\sum_{j=1}^{n}\ln y_{1j} + n\ln\beta_2 - n\beta_2\ln k_2 \\ &+(\beta_2-1)\sum_{j=1}^{n}\ln y_{2j} + \sum_{j=1}^{n}\ln\left[(1+\alpha)-\alpha\gamma\phi\left(y_{1j}/k_1\right)^{(\gamma-1)\beta_1}\left(y_{2j}/k_2\right)^{(\phi-1)\beta_2}\right].\end{aligned} \tag{53}$$

Since k_1 and k_2 appear in the domain of the distribution, their maximum likelihood estimates are immediately obtained as $\hat{k}_1 = \max(y_{1j})$ and $\hat{k}_2 = \max(y_{2j})$. The derivatives of (53) with respect to the unknown parameters are

$$\frac{\partial}{\partial\alpha}\ell(\theta; \mathbf{y}_1, \mathbf{y}_2) = \sum_{j=1}^{n}\frac{1-\gamma\phi\left(y_{1j}/k_1\right)^{(\gamma-1)\beta_1}\left(y_{2j}/k_2\right)^{(\phi-1)\beta_2}}{(1+\alpha)-\alpha\gamma\phi\left(y_{1j}/k_1\right)^{(\gamma-1)\beta_1}\left(y_{2j}/k_2\right)^{(\phi-1)\beta_2}}, \tag{54}$$

$$\frac{\partial}{\partial\gamma}\ell(\theta; \mathbf{y}_1, \mathbf{y}_2) = -\sum_{j=1}^{n}\frac{\alpha\phi\left(y_{1j}/k_1\right)^{(\gamma-1)\beta_1}\left(y_{2j}/k_2\right)^{(\phi-1)\beta_2}\left[1+\beta_1\gamma\ln\left(y_{1j}/k_1\right)^{\beta_1}\right]}{(1+\alpha)-\alpha\gamma\phi\left(y_{1j}/k_1\right)^{(\gamma-1)\beta_1}\left(y_{2j}/k_2\right)^{(\phi-1)\beta_2}}, \tag{55}$$

$$\frac{\partial}{\partial\phi}\ell(\theta; \mathbf{y}_1, \mathbf{y}_2) = -\sum_{j=1}^{n}\frac{\alpha\gamma\left(y_{1j}/k_1\right)^{(\gamma-1)\beta_1}\left(y_{2j}/k_2\right)^{(\phi-1)\beta_2}\left[1+\beta_2\phi\ln\left(y_{2j}/k_2\right)^{\beta_2}\right]}{(1+\alpha)-\alpha\gamma\phi\left(y_{1j}/k_1\right)^{(\gamma-1)\beta_1}\left(y_{2j}/k_2\right)^{(\phi-1)\beta_2}}, \tag{56}$$

$$\frac{\partial}{\partial\beta_1}\ell(\boldsymbol{\theta};\mathbf{y}_1,\mathbf{y}_2) = n\left(\beta_1^{-1}-\ln k_1\right)+\sum_{j=1}^{n}\ln y_{1j} - \sum_{j=1}^{n}\frac{\alpha\gamma\phi(\gamma-1)\left(y_{1j}/k_1\right)^{(\gamma-1)\beta_1}\left(y_{2j}/k_2\right)^{(\phi-1)\beta_2}\ln\left(y_{1j}/k_1\right)}{(1+\alpha)-\alpha\gamma\phi\left(y_{1j}/k_1\right)^{(\gamma-1)\beta_1}\left(y_{2j}/k_2\right)^{(\phi-1)\beta_2}}, \tag{57}$$

and

$$\frac{\partial}{\partial\beta_2}\ell(\boldsymbol{\theta};\mathbf{y}_1,\mathbf{y}_2) = n\left(\beta_2^{-1}-\ln k_2\right)+\sum_{j=1}^{n}\ln y_{2j} - \sum_{j=1}^{n}\frac{\alpha\gamma\phi(\phi-1)\left(y_{1j}/k_1\right)^{(\gamma-1)\beta_1}\left(y_{2j}/k_2\right)^{(\phi-1)\beta_2}\ln\left(y_{2j}/k_2\right)}{(1+\alpha)-\alpha\gamma\phi\left(y_{1j}/k_1\right)^{(\gamma-1)\beta_1}\left(y_{2j}/k_2\right)^{(\phi-1)\beta_2}}. \tag{58}$$

The maximum likelihood estimates of α, γ, φ, β_1, and β_2 are obtained by equating the derivatives in (54)–(58) to zero and simultaneously solving the resulting equations. The asymptotic property of the maximum likelihood estimate states that the maximum likelihood estimate is asymptotically normal, with the mean equal to the corresponding parameter and the variance given as the inverse of the Fisher information. Now, for the proposed bivariate power function distribution, the maximum likelihood estimates $\boldsymbol{\Theta} = (\hat{\alpha},\hat{\gamma},\hat{\phi},\hat{\beta}_1,\hat{\beta}_2)$ have a multivariate normal distribution, that is

$$\begin{bmatrix}\hat{\alpha}\\ \hat{\gamma}\\ \hat{\phi}\\ \hat{\beta}_1\\ \hat{\beta}_2\end{bmatrix} \sim N\left(\boldsymbol{\mu}=\begin{bmatrix}\alpha\\ \gamma\\ \phi\\ \beta_1\\ \beta_2\end{bmatrix};\ \boldsymbol{\Sigma}=\begin{bmatrix}V_{11} & V_{12} & V_{13} & V_{14} & V_{14}\\ & V_{22} & V_{23} & V_{24} & V_{25}\\ & & V_{33} & V_{34} & V_{35}\\ & & & V_{44} & V_{45}\\ & & & & V_{55}\end{bmatrix}\right),$$

where

$$\mathbf{V}^{-1} = -E\begin{bmatrix}\partial^2\ell/\partial\alpha^2 & \partial^2\ell/\partial\alpha\partial\gamma & \partial^2\ell/\partial\alpha\partial\phi & \partial^2\ell/\partial\alpha\partial\beta_1 & \partial^2\ell/\partial\alpha\partial\beta_2\\ & \partial^2\ell/\partial\gamma^2 & \partial^2\ell/\partial\gamma\partial\phi & \partial^2\ell/\partial\gamma\partial\beta_1 & \partial^2\ell/\partial\gamma\partial\beta_2\\ & & \partial^2\ell/\partial\phi^2 & \partial^2\ell/\partial\phi\partial\beta_1 & \partial^2\ell/\partial\phi\partial\beta_2\\ & & & \partial^2\ell/\partial\beta_1^2 & \partial^2\ell/\partial\beta_1\partial\beta_2\\ & & & & \partial^2\ell/\partial\beta_2^2\end{bmatrix}$$

is the Fisher information matrix. The entries of this matrix are given in Appendix B.

8. Simulation Study

In this section, we have given the results of the simulation study to see the consistency of the maximum likelihood estimates. The simulation study has been performed as per the following algorithm:

1. Generate a bivariate random sample of size n = 20, 50, 100, 200, 500, and 1000 from the *BPF* distribution using some pre-specified values of the parameters, as discussed in Sub-Section 7.3.
2. Use the sample in step 1 to compute the maximum likelihood estimates of the parameters.
3. Fix the number of simulations, say, N, and repeat steps 1 and 2 for N times.
4. Compute the estimated value of a specific parameter, alongside its mean square error, using

$$\hat{\theta}_h = N^{-1}\sum_{j=1}^{N}\hat{\theta}_{hj} \ \ and \ \ MSE\left(\hat{\theta}_h\right) = (N-1)^{-1}\sum_{j=1}^{N}\left(\hat{\theta}_{hj}-\hat{\theta}_h\right)^2$$

where $\hat{\theta}_h$ is the hth element of $\boldsymbol{\theta} = (\alpha,\gamma,\phi,\beta_1,\beta_2)$ and is the estimated value of the hth parameter obtained from the jth sample.

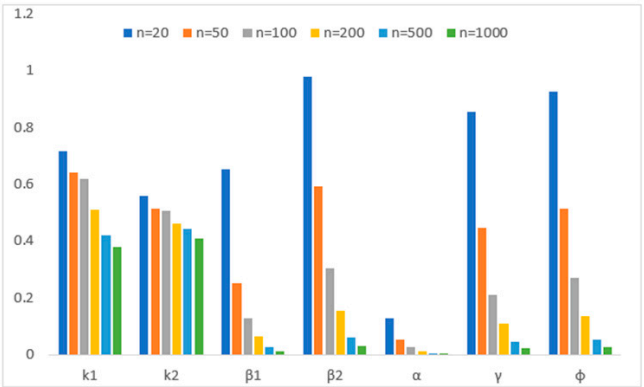
It is to be noted that the maximum likelihood estimate of k_1 is the maximum value of Y_1, and the maximum likelihood estimate of k_2 is the maximum value of Y_2 in the generated sample.

The results of the simulation study, using N = 10000 simulations, are given in Table 1 below.

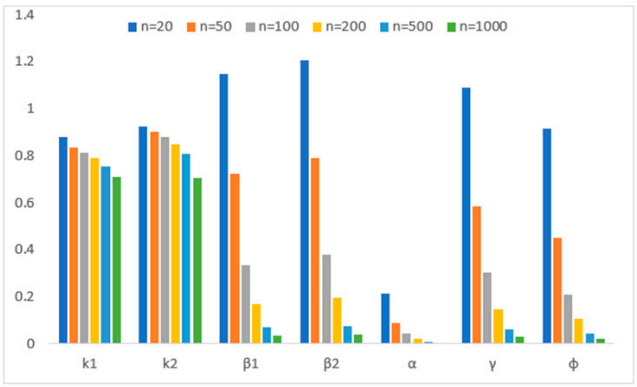
Table 1. Results of the simulation study for the bivariate power function distribution.

n	$k_1 = 3.5$	$k_2 = 2.5$	$\beta_1 = 1.5$	$\beta_2 = 3.5$	$\alpha = 0.3$	$\gamma = 2.5$	$\phi = 3.0$
20	3.217	2.148	1.503	3.502	0.329	2.501	2.997
	(0.715)	(0.558)	(0.653)	(0.977)	(0.129)	(0.854)	(0.927)
50	3.253	2.263	1.494	3.499	0.333	2.504	3.001
	(0.642)	(0.513)	(0.250)	(0.594)	(0.052)	(0.448)	(0.512)
100	3.339	2.376	1.499	3.503	0.321	2.503	2.999
	(0.618)	(0.507)	(0.127)	(0.304)	(0.026)	(0.212)	(0.271)
200	3.397	2.461	1.499	3.502	0.276	2.499	3.002
	(0.509)	(0.461)	(0.064)	(0.156)	(0.013)	(0.11)	(0.135)
500	3.454	2.487	1.497	3.498	0.327	2.498	3.002
	(0.421)	(0.442)	(0.027)	(0.061)	(0.006)	(0.046)	(0.054)
1000	3.489	2.488	1.495	3.498	0.332	2.501	2.998
	(0.379)	(0.407)	(0.013)	(0.032)	(0.003)	(0.022)	(0.028)
n	$k_1 = 4.0$	$k_2 = 4.5$	$\beta_1 = 2.5$	$\beta_2 = 4.5$	$\alpha = -0.5$	$\gamma = 3.5$	$\phi = 2.5$
20	3.653	4.479	4.002	4.421	-0.502	3.485	2.497
	(0.882)	(0.926)	(1.147)	(1.208)	(0.212)	(1.091)	(0.917)
50	3.719	4.438	4.000	4.486	-0.506	3.506	2.498
	(0.836)	(0.904)	(0.725)	(0.791)	(0.087)	(0.584)	(0.449)
100	3.908	4.508	3.999	4.500	-0.513	3.498	2.499
	(0.814)	(0.881)	(0.335)	(0.381)	(0.045)	(0.303)	(0.209)
200	3.934	4.503	3.998	4.501	-0.494	3.502	2.503
	(0.792)	(0.849)	(0.170)	(0.194)	(0.022)	(0.148)	(0.105)
500	4.052	4.967	3.998	4.508	-0.509	3.495	2.499
	(0.753)	(0.807)	(0.069)	(0.076)	(0.009)	(0.063)	(0.045)
1000	3.936	4.985	4.001	4.502	-0.508	3.499	2.504
	(0.709)	(0.706)	(0.036)	(0.039)	(0.005)	(0.031)	(0.023)

The results indicate that the maximum estimates are consistent, as the estimated values are close to the true parameter values. Also, the mean square error of the estimates decreases with an increase in the sample size. The mean square errors for the estimates of various parameters are plotted against different sample sizes. The plots are given in Figure 6 below.



(**a**) $k_1 = 3.5$, $k_2 = 2.5$, $\beta_1 = 1.5$, $\beta_2 = 3.5$, $\alpha = 0.3$, $\gamma = 2.5$, $\phi = 3.0$



(**b**) $k_1 = 4.0$, $k_2 = 4.5$, $\beta_1 = 2.5$, $\beta_2 = 4.5$, $\alpha = -0.5$, $\gamma = 3.5$, $\phi = 2.5$

Figure 6. Mean square errors for different estimates.

The above figure indicates that the mean square error of all estimates decreases with the increase in the sample size. This shows the consistency of the maximum likelihood estimates.

9. Real Data Applications

In this section, we have given some real data applications to see the suitability of the proposed *BPF* distribution. We have used four data sets for the real data application. The first three data sets have been taken from [32] and the fourth data set has been taken from [33]. The first data are about the percentage of the population between a specific age group in US counties and the second data are about the company *IPOs*. The description of variables is in Table 2 below.

Table 2. Description of the Data Sets.

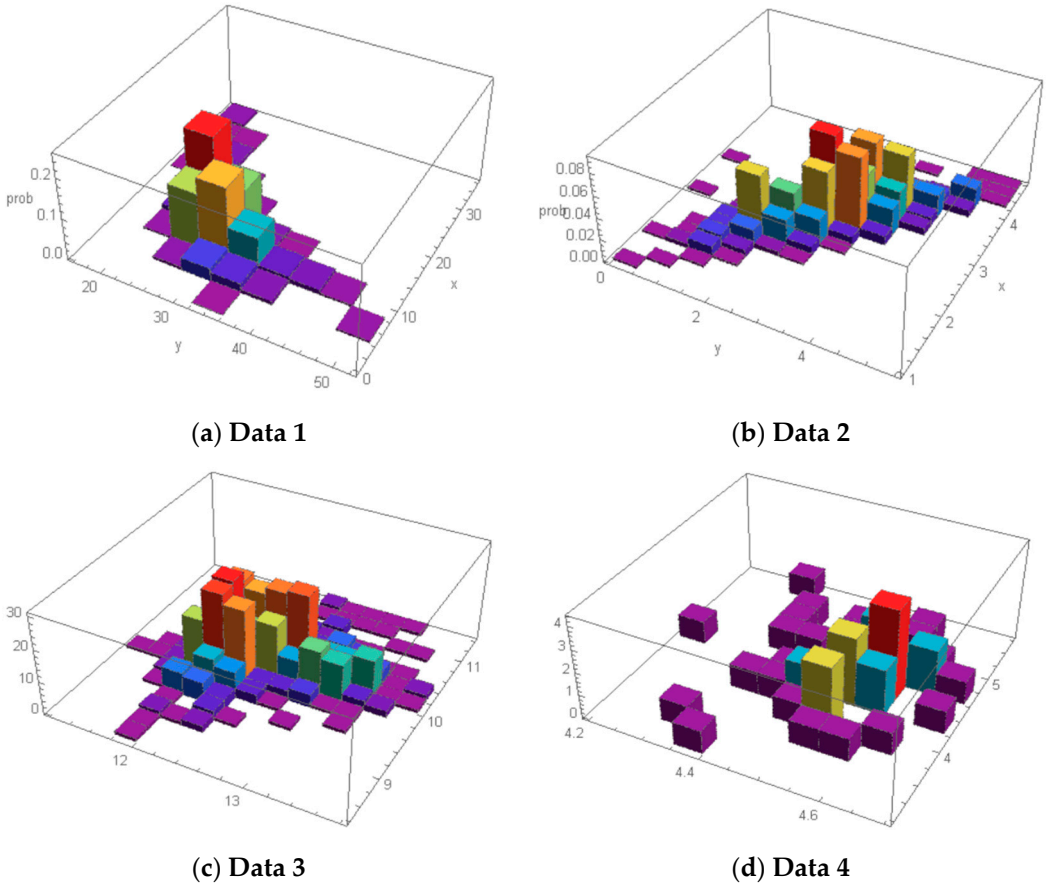
Data	*n*	Variables
1	440	Y_1: Percent of the population aged 18–34 Y_2: Percent of the population aged above 65
2	482	Y_1: Log of the face value of the company in millions (USD) Y_2: Log of shares offered in hundreds of thousands
3	522	Y_1: Log of the sale price of a house Y_2: Log of the lot size in square feet
4	46	Y_1: Log of fasting plasma glucose Y_2: Log of insulin resistance

Summary measures of the two data sets are given in Table 3 below.

Table 3. Summary measures of the two data sets.

Data		Min	Q_1	Median	Mean	Q_3	Max	Var.	Skew.	Corr.
1	Y_1	16.40	26.20	28.10	28.57	30.02	49.70	17.57	1.22	–0.616
	Y_2	3.00	9.88	11.75	12.17	13.62	33.80	15.94	1.92	
2	Y_1	0.18	2.33	2.97	2.89	3.48	5.46	0.82	–0.21	0.892
	Y_2	1.10	2.56	3.00	2.94	3.30	4.70	0.31	0.07	
3	Y_1	11.34	12.10	12.35	12.43	12.72	13.73	0.19	0.60	0.294
	Y_2	8.43	9.75	10.01	10.01	10.20	11.37	0.17	0.19	
4	Y_1	4.25	4.45	4.50	4.50	4.58	4.66	0.01	–0.55	0.046
	Y_2	3.37	4.12	4.52	4.47	4.81	5.46	0.24	–0.23	

The summary measures show that in the first data set, the variables are negatively correlated, whereas in the other data sets, the variables are positively correlated. Also, in the fourth data set, the variables have a very low correlation coefficient. The summary measures also indicate that both variables are positively skewed in the first and third data sets; in the second data set, one variable is positively skewed and the other variable is negatively skewed. In the fourth data set, both of the variables are negatively skewed. The bivariate histograms of both data sets are given in Figure 7 below.



(a) Data 1 **(b) Data 2**

(c) Data 3 **(d) Data 4**

Figure 7. Bivariate histograms of the data sets.

We have fitted the following bivariate power function distributions on these two data sets alongside the proposed bivariate power function distribution.

- Gumbel's Bivariate Power Function (*GBPF*) [2] is

$$f_{1,2(G)}(y_1,y_2)=\left(\frac{\beta_1 y_1^{\beta_1-1}}{k_1^{\beta_1}}\right)\left(\frac{\beta_2 y_2^{\beta_2-1}}{k_2^{\beta_2}}\right)\left[1+\alpha\left\{1-2\left(\frac{y_1}{k_1}\right)^{\beta_1}\right\}\left\{1-2\left(\frac{y_2}{k_2}\right)^{\beta_2}\right\}\right].$$

- Huang and Kotz's Bivariate Power Function (*HKBPF*) [34] is

$$f_{1,2(HK)}(y_1,y_2)=\left(\frac{\beta_1 y_1^{\beta_1-1}}{k_1^{\beta_1}}\right)\left(\frac{\beta_2 y_2^{\beta_2-1}}{k_2^{\beta_2}}\right)\left[1+\alpha\left\{1-\gamma\left(\frac{y_1}{k_1}\right)^{(\gamma-1)\beta_1}\right\}\left\{1-\gamma\left(\frac{y_2}{k_2}\right)^{(\gamma-1)\beta_2}\right\}\right].$$

- Cambanis's Bivariate Power Function (*CBPF*) [12] is

$$\begin{aligned}f_{1,2(C)}(y_1,y_2)=&\left(\frac{\beta_1 y_1^{\beta_1-1}}{k_1^{\beta_1}}\right)\left(\frac{\beta_2 y_2^{\beta_2-1}}{k_2^{\beta_2}}\right)\left[1+\alpha\left\{1-2\left(\frac{y_1}{k_1}\right)^{\beta_1}\right\}+\gamma\left\{1-2\left(\frac{y_2}{k_2}\right)^{\beta_2}\right\}\right.\\&\left.+\phi\left\{1-2\left(\frac{y_1}{k_1}\right)^{\beta_1}\right\}\left\{1-2\left(\frac{y_2}{k_2}\right)^{\beta_2}\right\}\right].\end{aligned}$$

It is to be noted that, for each of the above distributions, $0<y_1<k_1$ and $0<y_2<k_2$. The maximum likelihood estimates of the parameters of different distributions for the data sets are given in Table 4 below. It is to be noted that the maximum likelihood estimates of k_1 and k_2 for each distribution are the maximum values of the respective variables in each data set. This indicates that the maximum likelihood estimators of k_1 and k_2 for each distribution in

data sets 1 are $\hat{k}_1 = 49.70$ and $\hat{k}_2 = 33.80$, respectively. The maximum likelihood estimators of k_1 and k_2 for the second data set can be similarly written from Table 3.

Table 4. Maximum likelihood estimates of parameters.

Distribution	Parameter	Data 1	Data 2	Data 3	Data 4
Proposed Bivariate Power Function Distribution (*NBPF*)	β_1	1.880 (0.325)	2.274 (0.425)	4.252 (0.721)	4.954 (0.802)
	β_2	1.008 (0.415)	1.935 (0.468)	2.106 (0.213)	0.379 (0.198)
	α	0.354 (0.086)	–0.154 (0.027)	–0.985 (0.013)	–0.852 (0.184)
	γ	5.172 (0.751)	2.312 (0.941)	2.355 (0.413)	2.466 (0.405)
	ϕ	4.521 (1.152)	1.587 (0.612)	3.702 (0.419)	12.392 (4.262)
Gumbel Bivariate Power Function Distribution (*GBPF*)	β_1	2.459 (1.218)	1.535 (0.446)	11.892 (3.485)	29.250 (5.412)
	β_2	1.545 (0.487)	2.182 (0.748)	9.235 (3.221)	5.029 (0.742)
	α	1.000 (0.358)	0.481 (0.227)	0.962 (0.284)	0.488 (0.548)
Huang and Kotz Bivariate Power Function Distribution (*HKBPF*)	β_1	2.145 (0.718)	1.577 (0.627)	3.199 (0.134)	5.128 (0.759)
	β_2	1.136 (0.584)	2.207 (0.881)	3.615 (0.143)	2.103 (0.269)
	α	0.653 (0.617)	0.351 (0.174)	0.812 (0.262	0.985 (0.209)
	γ	2.452 (0.923)	3.915 (1.376)	2.221 (1.547)	0.954 (0.311)
Cambanis Bivariate Power Function Distribution (*CBPF*)	β_1	2.570 (0.868)	2.107 (0.944)	4.264 (0.485)	5.027 (1.019)
	β_2	0.471 (0.163)	1.026 (0.627)	3.173 (1.085)	2.444 (0.841)
	α	0.992 (0.367)	0.991 (0.334)	–0.885 (0.281)	–0.919 (0.347)
	γ	–0.992 (0.291)	–1.000 (0.391)	–0.945 (0.457)	–0.919 (3.363)
	ϕ	–1.000 (0.882)	–0.991 (0.667)	0.999 (0.854)	0.999 (5.166)

We have computed two famous goodness of fit measures to see the suitability of the fitted distributions. These include the Akaike information criteria (*AIC*) and Bayesian information criteria (*BIC*). These goodness of fit measures are given in Table 5 below.

Table 5. The fit measures for different distributions.

Data	Measure	Distribution			
		NBPFD	GBPFD	HKBPFD	CBPFD
1	Log-like	–3079.09	–3184.20	–3167.17	–3095.84
	AIC	6172.18	6378.4	6346.34	6205.68
	BIC	6200.787	6398.834	6370.861	6234.287
	KS *p*-value	0.4635	0.0285	0.0596	0.2541
2	Log-like	–1323.26	–1407.23	–1352.08	–1334.10
	AIC	2660.52	2824.46	2716.16	2682.2
	BIC	2689.766	2845.35	2741.228	2711.446
	KS *p*-value	0.5084	0.0684	0.0847	0.2867
3	Log-like	–1086.56	–1286.45	–1242.53	–1147.52
	AIC	2183.12	2578.90	2493.06	2305.04
	BIC	2204.40	2591.67	2510.09	2326.33
	KS *p*-value	0.6628	0.0035	0.0085	0.3251
4	Log-like.	–16.14	–33.67	–24.88	–21.11
	AIC	42.28	73.34	57.76	52.21
	BIC	51.42	78.83	65.07	61.36
	KS *p*-value	0.5548	0.0658	0.0837	0.2395

The above table indicates that the proposed bivariate power function distribution is the best fit for all the data sets, as it has the smallest values of *AIC* and *BIC* of all of the four data sets. Table 4 also contains the *p*-values of a bivariate Kolmogorov–Smirnov test to test the goodness of fit of the various distributions; see, for example, [35]. The *p*-values clearly indicate that the proposed bivariate power function distribution is the best fit for all four of the data sets. The Cambanis bivariate power function distribution is the second-best distribution.

The plots of the fitted marginal density functions for the four data sets are given in Figure 8 below.

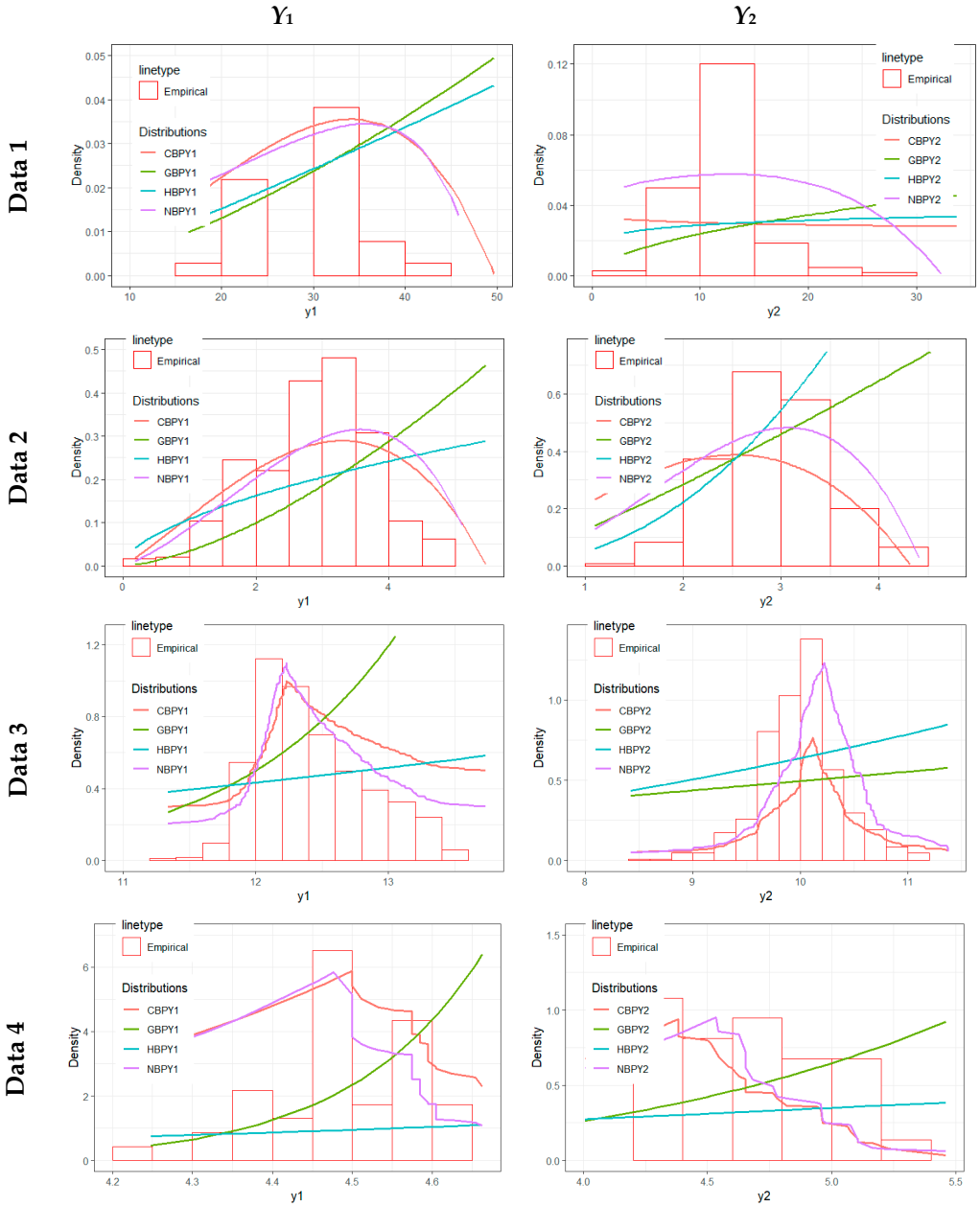


Figure 8. Fitted marginal density functions for four data sets.

The above graphs indicate that the proposed bivariate power function distribution is the best fit for the data among all the competing distributions. The marginals of the bivariate power function distribution obtained using the Cambanis [12] family of distributions closely follow the marginals of the proposed bivariate power function distribution.

10. Conclusions and Further Suggestions

In this paper, a new *CBD* is proposed. The proposed class of distributions provides the power-transmuted distributions as marginals. Certain desirable properties of the proposed class of distributions are studied. These include joint, ratio, and conditional moments. We have seen that these moments depend on the moments of the exponentiated baseline distribution. We have also obtained the dependence measures for the proposed family of distributions alongside the maximum likelihood estimation of the parameters. The proposed family of distributions has been extended to the multivariate case, and some useful properties of the multivariate family of distributions are also studied. We have used the power function distribution as a baseline distribution in the proposed family of distributions to obtain a new bivariate power function (*BPF*) distribution. Some properties of the new *BPF* distribution are studied. We have obtained the maximum likelihood estimates of the parameters of the new *BPF* distribution. A simulation study has been conducted to see the consistency of the maximum likelihood estimates, and it has been observed that the maximum likelihood estimates are consistent. We have used four data sets to see the suitability of the new *BPF* distribution alongside the *BPF* distributions obtained using some other families of distributions. We have found that our proposed *BPF* distribution performs better than the competing *BPF* distributions. We hope that the proposed bivariate family of distributions can be used to propose some new bivariate distributions for modeling real bivariate data. The proposed multivariate family of distributions can also be used to propose some new multivariate distributions using different baseline distributions.

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Appendix A. Entries of the Fisher Information Matrix for the New Class of Bivariate Distributions

The entries of the Fisher information matrix for the new class of distributions are given below when all the parameters of the baseline distribution are known. In these entries, we have

$$\eta_{1,2}(y_{1j}, y_{2j}) = H_1^{\gamma-1}(y_{1j}) H_2^{\gamma-1}(y_{2j}) \text{ and } \Phi_{1,2}(y_{1j}, y_{2j}) = (1+\alpha) - \alpha\gamma\phi\eta_{1,2}(y_{1j}, y_{2j})$$

$$\frac{\partial^2 \ell}{\partial \alpha^2} = -\sum_{j=1}^{n} \frac{\left[1 - \gamma\phi\eta_{1,2}(y_{1j}, y_{2j})\right]^2}{\Phi_{1,2}^2(y_{1j}, y_{2j})}$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \gamma} = \sum_{j=1}^{n} \frac{\{1-\gamma\phi\eta_{1,2}(y_{1j},y_{2j})\}[\alpha\phi\eta_{1,2}(y_{1j},y_{2j})\{1+\gamma \ln H_1(y_{1j})\}]}{\Phi_{1,2}^2(y_{1j},y_{2j})} - \sum_{j=1}^{n} \frac{\phi\eta_{1,2}(y_{1j},y_{2j})[1+\gamma \ln H_1(y_{1j})]}{\Phi_{1,2}(y_{1j},y_{2j})}$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \phi} = \sum_{j=1}^{n} \frac{\{1-\gamma\phi\eta_{1,2}(y_{1j},y_{2j})\}[\alpha\gamma\eta_{1,2}(y_{1j},y_{2j})\{1+\beta_2\phi \ln H_1(y_{1j})\}]}{\Phi_{1,2}^2(y_{1j},y_{2j})} - \sum_{j=1}^{n} \frac{\gamma\eta_{1,2}(y_{1j},y_{2j})\{1+\beta_2\phi \ln H_2(y_{2j})\}}{\Phi_{1,2}(y_{1j},y_{2j})}$$

$$\frac{\partial^2 \ell}{\partial \gamma^2} = -\sum_{j=1}^{n} \frac{[\alpha\phi\eta_{1,2}(y_{1j},y_{2j})\{1+\gamma \ln H_1(y_{1j})\}]^2}{\Phi_{1,2}^2(y_{1j},y_{2j})} - \sum_{j=1}^{n} \frac{\alpha\phi\eta_{1,2}(y_{1j},y_{2j}) \ln H_1(y_{1j})[2+\gamma \ln H_1(y_{1j})]}{\Phi_{1,2}(y_{1j},y_{2j})}$$

$$\frac{\partial^2 \ell}{\partial \gamma \partial \phi} = -\sum_{j=1}^{n} \frac{[\alpha\phi\eta_{1,2}(y_{1j},y_{2j})\{1+\gamma \ln H_1(y_{1j})\}][\alpha\gamma\eta_{1,2}(y_{1j},y_{2j})\{1+\phi \ln H_2(y_{2j})\}]}{\Phi_{1,2}^2(y_{1j},y_{2j})} - \sum_{j=1}^{n} \frac{\alpha\eta_{1,2}(y_{1j},y_{2j})[1+\gamma \ln H_1(y_{1j})+\phi \ln H_2(y_{2j})+\phi\gamma \ln H_1(y_{1j}) \ln H_2(y_{2j})]}{\Phi_{1,2}(y_{1j},y_{2j})}$$

$$\frac{\partial^2 \ell}{\partial \phi^2} = -\sum_{j=1}^{n} \frac{[\alpha\gamma\eta_{1,2}(y_{1j},y_{2j})\{1+\phi \ln H_2(y_{2j})\}]^2}{\Delta_{1,2}^2(y_{1j},y_{2j})} - \sum_{j=1}^{n} \frac{\alpha\gamma\eta_{1,2}(y_{1j},y_{2j}) \ln H_2(y_{2j})[2+\phi \ln H_2(y_{2j})]}{\Phi_{1,2}(y_{1j},y_{2j})}.$$

Appendix B. Entries of the Fisher Information Matrix for Bivariate Power Function Distribution

The entries of the Fisher information matrix are given below. In these entries, we have

$$\varphi_{1,2}(y_{1j},y_{2j}) = (y_{1j}/k_1)^{(\gamma-1)\beta_1}(y_{2j}/k_2)^{(\phi-1)\beta_2} \text{ and } \Delta_{1,2}(y_{1j},y_{2j}) = (1+\alpha) - \alpha\gamma\phi\varphi_{1,2}(y_{1j},y_{2j})$$

$$\frac{\partial^2 \ell}{\partial \alpha^2} = -\sum_{j=1}^{n} \frac{[1-\gamma\phi\varphi_{1,2}(y_{1j},y_{2j})]^2}{\Delta_{1,2}^2(y_{1j},y_{2j})}$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \gamma} = \sum_{j=1}^{n} \frac{\{1-\gamma\phi\varphi_{1,2}(y_{1j},y_{2j})\}[\alpha\phi\varphi_{1,2}(y_{1j},y_{2j})\{1+\beta_1\gamma \ln(y_{1j}/k_1)\}]}{\Delta_{1,2}^2(y_{1j},y_{2j})} - \sum_{j=1}^{n} \frac{\phi\varphi_{1,2}(y_{1j},y_{2j})[1+\beta_1\gamma \ln(y_{1j}/k_1)]}{\Delta_{1,2}(y_{1j},y_{2j})}$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \phi} = \sum_{j=1}^{n} \frac{\{1-\gamma\phi\varphi_{1,2}(y_{1j},y_{2j})\}[\alpha\gamma\varphi_{1,2}(y_{1j},y_{2j})\{1+\beta_2\phi \ln(y_{2j}/k_2)\}]}{\Delta_{1,2}^2(y_{1j},y_{2j})} - \sum_{j=1}^{n} \frac{\gamma\varphi_{1,2}(y_{1j},y_{2j})\{1+\beta_2\phi \ln(y_{2j}/k_2)\}}{\Delta_{1,2}(y_{1j},y_{2j})}$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \beta_1} = \sum_{j=1}^{n} \frac{\alpha\phi\gamma(\gamma-1)\ln(y_{1j}/k_1)\varphi_{1,2}(y_{1j},y_{2j})\left[1-\gamma\phi\varphi_{1,2}(y_{1j},y_{2j})\right]}{\Delta_{1,2}^2(y_{1j},y_{2j})} - \sum_{j=1}^{n} \frac{\phi\gamma(\gamma-1)\ln(y_{1j}/k_1)\varphi_{1,2}(y_{1j},y_{2j})}{\Delta_{1,2}(y_{1j},y_{2j})}$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \beta_2} = \sum_{j=1}^{n} \frac{\alpha\phi\gamma(\phi-1)\ln(y_{2j}/k_2)\varphi_{1,2}(y_{1j},y_{2j})\left[1-\gamma\phi\varphi_{1,2}(y_{1j},y_{2j})\right]}{\Delta_{1,2}^2(y_{1j},y_{2j})} - \sum_{j=1}^{n} \frac{\phi\gamma(\phi-1)\ln(y_{2j}/k_2)\varphi_{1,2}(y_{1j},y_{2j})}{\Delta_{1,2}(y_{1j},y_{2j})}$$

$$\frac{\partial^2 \ell}{\partial \gamma^2} = -\sum_{j=1}^{n} \frac{\left[\alpha\phi\varphi_{1,2}(y_{1j},y_{2j})\left\{1+\beta_1\gamma\ln(y_{1j}/k_1)\right\}\right]^2}{\Delta_{1,2}^2(y_{1j},y_{2j})} - \sum_{j=1}^{n} \frac{\alpha\beta_1\phi\varphi_{1,2}(y_{1j},y_{2j})\ln(y_{1j}/k_1)\left[2+\beta_1\gamma\ln(y_{1j}/k_1)\right]}{\Delta_{1,2}(y_{1j},y_{2j})}$$

$$\frac{\partial^2 \ell}{\partial \gamma \partial \phi} = -\sum_{j=1}^{n} \frac{\left[\alpha\phi\varphi_{1,2}(y_{1j},y_{2j})\left\{1+\beta_1\gamma\ln(y_{1j}/k_1)\right\}\right]\left[\alpha\gamma\varphi_{1,2}(y_{1j},y_{2j})\left\{1+\beta_2\phi\ln(y_{2j}/k_2)\right\}\right]}{\Delta_{1,2}^2(y_{1j},y_{2j})} - \sum_{j=1}^{n} \frac{\alpha\varphi_{1,2}(y_{1j},y_{2j})\left[1+\beta_1\gamma\ln(y_{1j}/k_1)+\beta_2\phi\ln(y_{2j}/k_2)+\beta_1\beta_2\phi\gamma\ln(y_{1j}/k_1)\ln(y_{2j}/k_2)\right]}{\Delta_{1,2}(y_{1j},y_{2j})}$$

$$\frac{\partial^2 \ell}{\partial \gamma \partial \beta_1} = -\sum_{j=1}^{n} \frac{\alpha^2\gamma\phi^2(\gamma-1)\varphi_{1,2}^2(y_{1j},y_{2j})\ln(y_{1j}/k_1)\left[1+\beta_1\gamma\ln(y_{1j}/k_1)\right]}{\Delta_{1,2}^2(y_{1j},y_{2j})} - \sum_{j=1}^{n} \frac{\alpha\phi\varphi_{1,2}(y_{1j},y_{2j})\ln(y_{1j}/k_1)\left[(2\gamma-1)+\beta_1\gamma(\gamma-1)\ln(y_{1j}/k_1)\right]}{\Delta_{1,2}(y_{1j},y_{2j})}$$

$$\frac{\partial^2 \ell}{\partial \gamma \partial \beta_2} = -\sum_{j=1}^{n} \frac{\alpha^2\gamma\phi^2(\phi-1)\varphi_{1,2}^2(y_{1j},y_{2j})\ln(y_{2j}/k_2)\left[1+\beta_1\gamma\ln(y_{1j}/k_1)\right]}{\Delta_{1,2}^2(y_{1j},y_{2j})} - \sum_{j=1}^{n} \frac{\alpha\phi(\phi-1)\varphi_{1,2}(y_{1j},y_{2j})\ln(y_{2j}/k_2)\left[1+\beta_1\gamma(\gamma-1)\ln(y_{1j}/k_1)\right]}{\Delta_{1,2}(y_{1j},y_{2j})}$$

$$\frac{\partial^2 \ell}{\partial \phi^2} = -\sum_{j=1}^{n} \frac{\left[\alpha\gamma\varphi_{1,2}(y_{1j},y_{2j})\left\{1+\beta_2\phi\ln(y_{2j}/k_2)\right\}\right]^2}{\Delta_{1,2}^2(y_{1j},y_{2j})} - \sum_{j=1}^{n} \frac{\alpha\gamma\beta_2\varphi_{1,2}(y_{1j},y_{2j})\ln(y_{2j}/k_2)\left[2+\beta_2\phi\ln(y_{2j}/k_2)\right]}{\Delta_{1,2}(y_{1j},y_{2j})}$$

$$\frac{\partial^2 \ell}{\partial \phi \partial \beta_1} = -\sum_{j=1}^{n} \frac{\alpha^2\gamma^2(\gamma-1)\phi\varphi_{1,2}^2(y_{1j},y_{2j})\ln(y_{1j}/k_1)\left[1+\beta_2\phi\ln(y_{2j}/k_2)\right]}{\Delta_{1,2}^2(y_{1j},y_{2j})} - \sum_{j=1}^{n} \frac{\alpha\gamma(\gamma-1)\varphi_{1,2}(y_{1j},y_{2j})\ln(y_{1j}/k_1)\left[1+\beta_2\phi\ln(y_{2j}/k_2)\right]}{\Delta_{1,2}(y_{1j},y_{2j})}$$

$$\frac{\partial^2 \ell}{\partial \phi \partial \beta_2} = \sum_{j=1}^{n} \frac{\alpha^2\gamma^2\phi(\phi-1)\varphi_{1,2}^2(y_{1j},y_{2j})\ln(y_{2j}/k_2)\left[1+\beta_2\phi\ln(y_{2j}/k_2)\right]}{\Delta_{1,2}^2(y_{1j},y_{2j})} - \sum_{j=1}^{n} \frac{\alpha\gamma\varphi_{1,2}(y_{1j},y_{2j})\ln(y_{2j}/k_2)\left[(2\phi-1)+\beta_2\phi(\phi-1)\ln(y_{2j}/k_2)\right]}{\Delta_{1,2}(y_{1j},y_{2j})}$$

$$\frac{\partial^2 \ell}{\partial \beta_1^2} = -\frac{n}{\beta_1^2} - \sum_{j=1}^{n} \frac{\alpha^2\gamma^2\phi^2(\gamma-1)^2\left[\varphi_{1,2}(y_{1j},y_{2j})\ln(y_{1j}/k_1)\right]^2}{\Delta_{1,2}^2(y_{1j},y_{2j})} - \sum_{j=1}^{n} \frac{\alpha\gamma\phi(\gamma-1)^2\varphi_{1,2}(y_{1j},y_{2j})\left[\ln(y_{1j}/k_1)\right]^2}{\Delta_{1,2}(y_{1j},y_{2j})}$$

$$\frac{\partial^2 \ell}{\partial \beta_1 \partial \beta_2} = -\sum_{j=1}^{n} \frac{\alpha^2\gamma^2\phi^2(\gamma-1)(\phi-1)\varphi_{1,2}(y_{1j},y_{2j})\ln(y_{1j}/k_1)\ln(y_{2j}/k_2)}{\Delta_{1,2}^2(y_{1j},y_{2j})} - \sum_{j=1}^{n} \frac{\alpha\gamma\phi(\gamma-1)(\phi-1)\varphi_{1,2}(y_{1j},y_{2j})\ln(y_{1j}/k_1)\ln(y_{2j}/k_2)}{\Delta_{1,2}(y_{1j},y_{2j})}$$

$$\frac{\partial^2 \ell}{\partial \beta_2^2} = -\frac{n}{\beta_2^2} - \sum_{j=1}^{n} \frac{\alpha^2\gamma^2\phi^2(\phi-1)^2\left[\varphi_{1,2}(y_{1j},y_{2j})\ln(y_{2j}/k_2)\right]^2}{\Delta_{1,2}^2(y_{1j},y_{2j})} - \sum_{j=1}^{n} \frac{\alpha\gamma\phi(\phi-1)^2\varphi_{1,2}(y_{1j},y_{2j})\left[\ln(y_{2j}/k_2)\right]^2}{\Delta_{1,2}(y_{1j},y_{2j})}.$$

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