

Article

Existence of Solutions for Generalized Nonlinear Fourth-Order Differential Equations

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Abstract: In this article, we studied the existence of solutions for a more general form of nonlinear fourth-order differential equations by using a new generalization of the Arzelà–Ascoli theorem and Schauder fixed theorem under easier and general conditions. Moreover, we provided some sufficient conditions on the nonlinear function that allowed us to deduce the nonexistence results. Finally, we outlined an example to illustrate our main results.

Keywords: generalization of Arzelà–Ascoli theorem; fixed point theorem; fourth-order differential equations

MSC: 34K04; 34A34; 47H10; 34C25



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1. Introduction

Many phenomena we experience require mathematical modeling for effective control and monitoring. One of the most significant types of models is the nonlinear differential equations, either with or without delays. Numerous authors presented second- and third-order nonlinear differential equations (see [1–11]).

Additionally, various studies explored second- and third-order nonlinear differential equations with delays as models for topics in physics, chemistry, and biology (see, for example [12–15]).

To our knowledge, there are a relatively few papers concerning the study of fourth-order differential equations, and in recent years, significant attention has been devoted to solving fourth-order differential equations, which have applications across various fields of pure and applied sciences. (see [16–18]).

In our work, we considered a fourth-order nonlinear differential equation

$$\begin{cases} x^{(4)}(t) + ax^{(3)}(t) + bx^{(2)}(t) + cx^{(1)}(t) + dx(t) = f(t, x(t), x'(t), x''(t), x'''(t)), t \in [0, T], \\ x^{(i)}(0) = x^{(i)}(T), i = 0, \dots, 3, \end{cases} \quad (1)$$

which extends the study of the following nonlinear third-order differential equation conducted by Benhiouana et al. [19]

$$x^{(3)}(t) + p(t)x^{(2)}(t) + q(t)x^{(1)}(t) + r(t)x(t) = f(t, x(t), x'(t), x''(t)), t \in \mathbb{R}.$$

It is important to point out the following general form of the fourth-order nonlinear differential equation

$$\begin{aligned} x^{(4)}(t) + p(t)x^{(3)}(t) + q(t)x^{(2)}(t) + r(t)x^{(1)}(t) + \varphi(t)x(t) \\ = g(t, x(t), x'(t), x''(t), x'''(t)), t \in [0, T], \end{aligned}$$

can be seen as a particular form of Equation (1) where the nonlinear function can be defined as follows:

$$\begin{aligned} f(t, x(t), x'(t), x''(t), x'''(t)) = g(t, x(t), x'(t), x''(t), x'''(t)) \\ - \left[(p(t) - a)x^{(3)}(t) + (q(t) - b)x^{(2)}(t) + (r(t) - c)x^{(1)}(t) + (\varphi(t) - d)x(t) \right]. \end{aligned}$$

There are some very recent results on the fourth-order problems. For example, several studies [20–27] presented particular forms of Equation (1). More precisely, Conti et al. [23] addressed the following particular form of Equation (1):

$$x^{(4)}(t) - cx^{(2)}(t) = f(t, x(t)), t \in [0, T],$$

while the following equation was analyzed in [24]:

$$\begin{cases} x^{(4)}(t) = f(t, x(t), -x^{(2)}(t)), 0 < t < 1. \\ x(0) = x(1) = x^{(2)}(0) = x^{(2)}(1). \end{cases}$$

On the other hand, Xin et al. in [26] provided Green's function and studied the existence of positive solution of (1) in the case of the nonlinear function $f(t, x(t), x'(t), x''(t)) = h(t)$.

In this work, we used the Green's function given in [9,26] to generalize the existence results from [26] to a more general form of the nonlinear term f .

Ahmad and Arshad [27] studied the existence and the numerical solution of Equation (1) for $a = b = c = d = 0$ under some boundary condition by using a novel fixed-point approach based on Green's function.

Dimitrov and Jonnalagadda [28] studied the existence and nonexistence of the solution for the following nonlinear differential equations of the fourth order:

$$\begin{cases} x^{(4)}(t) = f(t, x(t)), t \in [0, 1]. \\ x(0) = x^{(1)}(0) = x^{(2)}(0) = 0, x^{(3)}(\eta) = \alpha x^{(1)}(1), 0 \leq \alpha < 2, \frac{1}{2} \leq \eta < 1. \end{cases}$$

This equation describes the stationary deflection states of an elastic beam of a given length 1 (see [28]).

A novel aspect of the equation we studied is the presence of derivatives x' , x'' , x''' in the second nonlinear term. To investigate the existence of solutions, we employed Schauder fixed-point theorem and the generalization of Arzelà–Ascoli theorem in [29] by using more general and straightforward conditions on the data functions. On the other hand, we provided some sufficient conditions on the nonlinear function to prove that there is no nontrivial solution of Equation (1).

2. Preliminaries and Lemmas

In this section, we provided the relevant notations and definition.

Let E_1, E_2 be two finite-dimensional Banach spaces equipped with the norms $\|\cdot\|_1, \|\cdot\|_2$ respectively, and let $F \subset E_1$ be a compact subset. We denote the vector space of all functions from F to E_2 with the continuous third derivative by $C^3(F, E_2)$. This space is equipped with the norm $\|\varphi\| = \sum_{i=0}^3 \|\varphi^{(i)}\|_\infty$ such that $\|\varphi^{(i)}\|_\infty = \sup_{x \in F} \{\|\varphi^{(i)}\|_2\}$ such that $\varphi^{(i)}$ is i -th derivative of the function φ .

For our purposes, we need the following definition in $C^3(F, E_2)$.

Definition 1 ([29]). The family $G \subset C^3(F, E_2)$ is said to be equicontinuous if, for every $\epsilon > 0$, there exists $\delta > 0$ such that $\|\varphi^{(i)}(t) - \varphi^{(i)}(s)\|_2 < \epsilon$ for all $i \in \{0, 1, 2, 3\}$, $\varphi \in G$, and for all $t, s \in F$, satisfying $\|t - s\|_1 < \delta$.

The family $G \subset C^3(F, E_2)$ is said to be equibounded if there exists a positive constant L such that $\|\varphi(t)\|_2 \leq L$ for all $\varphi \in G$ and $t \in F$.

The following theorem gives a new generalization of Arzelà–Ascoli theorem in the space $C^3(F, E_2)$.

Theorem 1 ([29]). Let G be a subset of the space $C^3(F, E_2)$. Then, G is relatively compact if and only if F is equicontinuous and equibounded.

It is clear that $C^i([0, T], \mathbb{R})$ ($i \in \{0, 1, 2, 3\}$) is a Banach space quipped with the following norm:

$$\|\varphi\| = \sum_{i=0}^3 \sup_{t \in [0, T]} |\varphi^{(i)}(t)|.$$

We studied Equation (1) under the following conditions:

(H_1) $f \in C(\mathbb{R}^5, \mathbb{R})$.

(H_2) There exist $k_i > 0$, $i \in \{1, \dots, 4\}$ and $\phi \in C([0, T], \mathbb{R}^+)$ such that

$$|f(t, u_1, u_2, u_3, u_4)| \leq \phi(t) + \sum_{i=1}^4 k_i |u_i|, \quad \forall (t, u_1, u_2, u_3, u_4) \in [0, T]^5.$$

(H_3) $a = -2\rho$, $b = 3\rho^3$, $c = -2\rho^3$, $d = \rho^3$, where ρ is a positive constant such that $\rho < \frac{2\pi}{\sqrt{3}T}$.

Therefore, by using [26], Equation (1) can be transformed to the following system of two second-order differential equations;

$$\begin{cases} y''(t) - \rho y'(t) + \rho^2 y(t) = f(t, y(t), y'(t), y''(t), y'''(t)) \\ y^i(0) = y^i(T), i = 0, 1, \end{cases} \quad (2)$$

$$\begin{cases} x''(t) - \rho x'(t) + \rho^2 x(t) = y(t) \\ x^i(0) = x^i(T), i = 0, 1. \end{cases} \quad (3)$$

In what follows, we used the following lemma.

Lemma 1 ([9]). The boundary problem (2) is equivalent to the following integral equation

$$y(t) = \int_0^T G(t, s) f(s, y(s), y'(s), y''(s), y'''(s)) ds, \quad (4)$$

where

$$G(t, s) = \begin{cases} \frac{2e^{\frac{\rho}{2}(t-s)} \left[\sin(\frac{\sqrt{3}}{2}\rho(T-t+s)) + e^{-\frac{\rho T}{2}} \sin(\frac{\sqrt{3}}{2}\rho(t-s)) \right]}{\sqrt{3}\rho \left(e^{\frac{\rho T}{2}} + e^{-\frac{\rho T}{2}} - 2\cos(\frac{\sqrt{3}}{2}\rho T) \right)}, & \text{if } 0 \leq s \leq t \leq T, \\ \frac{2e^{\frac{\rho}{2}(\omega+t-s)} \left[\sin(\frac{\sqrt{3}}{2}\rho(s-t)) + e^{-\frac{\rho T}{2}} \sin(\frac{\sqrt{3}}{2}\rho(T-s+t)) \right]}{\sqrt{3}\rho \left(e^{\frac{\rho T}{2}} + e^{-\frac{\rho T}{2}} - 2\cos(\frac{\sqrt{3}}{2}\rho T) \right)}, & \text{if } 0 \leq t \leq s \leq T. \end{cases}$$

Moreover, if $\rho < \frac{2\pi}{\sqrt{3}T}$, we have the following estimation:

$$0 \leq G(t, s) \leq M = \frac{2}{\sqrt{3} \sin(\frac{\sqrt{3}\rho T}{2})},$$

then by using the above lemma, the solution of (3) is given by the following:

$$x(t) = \int_0^T G(t, s)y(s)ds. \quad (5)$$

It is easy to prove the following lemma.

Lemma 2. The first, second, and third derivatives, with respect to the first variable of the function G , are given by the following formulas:

$$\frac{\partial G}{\partial t}(t, s) = \begin{cases} \frac{\rho}{2} G(t, s) + \frac{e^{\frac{\rho}{2}(t-s)} \left[-\cos(\frac{\sqrt{3}}{2}\rho(T-t+s)) + e^{-\frac{\rho T}{2}} \cos(\frac{\sqrt{3}}{2}\rho(t-s)) \right]}{e^{\frac{\rho T}{2}} + e^{-\frac{\rho T}{2}} - 2\cos(\frac{\sqrt{3}}{2}\rho T)}, & \text{if } 0 \leq s \leq t \leq T, \\ \frac{\rho}{2} G(t, s) + \frac{e^{\frac{\rho}{2}(T+t-s)} \left[-\cos(\frac{\sqrt{3}}{2}\rho(s-t)) + e^{-\frac{\rho T}{2}} \cos(\frac{\sqrt{3}}{2}\rho(T-s+t)) \right]}{e^{\frac{\rho T}{2}} + e^{-\frac{\rho T}{2}} - 2\cos(\frac{\sqrt{3}}{2}\rho T)}, & \text{if } 0 \leq t \leq s \leq T. \end{cases}$$

$$\frac{\partial^2 G}{\partial t^2}(t, s) = \begin{cases} -\frac{\rho^2}{2} G(t, s) + \rho \frac{e^{\frac{\rho}{2}(t-s)} \left[-\cos(\frac{\sqrt{3}}{2}\rho(T-t+s)) + e^{-\frac{\rho T}{2}} \cos(\frac{\sqrt{3}}{2}\rho(t-s)) \right]}{e^{\frac{\rho T}{2}} + e^{-\frac{\rho T}{2}} - 2\cos(\frac{\sqrt{3}}{2}\rho T)}, & \text{if } 0 \leq s \leq t \leq T, \\ -\frac{\rho^2}{2} G(t, s) + \rho \frac{e^{\frac{\rho}{2}(T+t-s)} \left[-\cos(\frac{\sqrt{3}}{2}\rho(s-t)) + e^{-\frac{\rho T}{2}} \cos(\frac{\sqrt{3}}{2}\rho(T-s+t)) \right]}{e^{\frac{\rho T}{2}} + e^{-\frac{\rho T}{2}} - 2\cos(\frac{\sqrt{3}}{2}\rho T)}, & \text{if } 0 \leq t \leq s \leq T. \end{cases}$$

and

$$\frac{\partial^3 G}{\partial t^3}(t, s) = \rho^3 G(t, s).$$

Now, from (4) and (5), the Equation (1) can be written by using the following integral equation:

$$\begin{aligned} x(t) &= \int_0^T G(t, \tau)y(\tau)d\tau \\ &= \int_0^T \int_0^T G(t, \tau)G(\tau, s)f(s, x(s), x'(s), x''(s), x'''(s))dsd\tau \\ &= \int_0^T \left(\int_0^T G(t, \tau)G(\tau, s)d\tau \right) f(s, x(s), x'(s), x''(s), x'''(s))ds, \end{aligned}$$

which implies that

$$x(t) = \int_0^T g(t, s)f(s, x(s), x'(s), x''(s), x'''(s))ds, \quad (6)$$

where the kernel

$$g(t, s) = \int_0^T G(t, \tau)G(\tau, s)d\tau.$$

In the sequence, we need the following lemma.

Lemma 3. If $\rho < \frac{2\pi}{\sqrt{3}T}$, then the function g , and its first, second, and third derivatives, with respect to the first variable, can be bounded as follows:

$$\begin{aligned} |g(t, s)| &\leq M^2 T, \\ \left| \frac{\partial g}{\partial t}(t, s) \right| &= \left| \int_0^T \frac{\partial G}{\partial t}(t, \tau) G(\tau, s) d\tau \right| \leq MT \left(\frac{\rho}{2} M + \frac{e^{\rho T} + e^{\frac{\rho T}{2}}}{e^{\frac{\rho T}{2}} + e^{-\frac{\rho T}{3}} - 2 \cos(\frac{\sqrt{3}}{2} \rho T)} \right) \\ \left| \frac{\partial^2 g}{\partial t^2}(t, s) \right| &= \left| \int_0^T \frac{\partial^2 G}{\partial t^2}(t, \tau) G(\tau, s) d\tau \right| \leq MT \rho \left(\frac{\rho}{2} M + \frac{e^{\rho T} + e^{\frac{\rho T}{2}}}{e^{\frac{\rho T}{2}} + e^{-\frac{\rho T}{3}} - 2 \cos(\frac{\sqrt{3}}{2} \rho T)} \right), \end{aligned}$$

and

$$\left| \frac{\partial^3 g}{\partial t^3}(t, s) \right| = \left| \int_0^T \frac{\partial^3 G(t, \tau)}{\partial t^3} G(\tau, s) d\tau \right| \leq M^2 T \rho^3.$$

Proof. It is obvious, so we omit it. $\square$

We end this section by stating Schauder's fixed-point theorem.

Theorem 2 ([30]). Let F be a nonempty bounded, closed, and convex subset of a Banach space E , and A is a continuous operator from F into itself. If $A(F)$ is relatively compact, then A has a fixed point.

3. Main Results

It is easy to check, under the above assumptions and Lemma 1, that x is a solution of (1) in $C^4([0, T], \mathbb{R})$ if and only if x is the solution of the integral Equation (6) in $C^3([0, T], \mathbb{R})$. Under the hypothesis $(H_1 - H_4)$ and the above lemmas, we will use Schauder's fixed-point theorem to prove the following existence result.

Theorem 3. If the assumptions $(H_1 - H_4)$ hold, and under the following condition

$$k_5 \theta < 1, \quad (7)$$

such that $k_5 = \max\{k_1, k_2, k_3, k_4\}$ and $\theta = \sum_{i=0}^3 \theta_i$, where

$$\begin{cases} \theta_0 &= M^2 T, \\ \theta_1 &= MT \left(\frac{\rho}{2} M + \frac{e^{\rho T} + e^{\frac{\rho T}{2}}}{e^{\frac{\rho T}{2}} + e^{-\frac{\rho T}{3}} - 2 \cos(\frac{\sqrt{3}}{2} \rho T)} \right), \\ \theta_2 &= MT \rho \left(\frac{\rho}{2} M + \frac{e^{\rho T} + e^{\frac{\rho T}{2}}}{e^{\frac{\rho T}{2}} + e^{-\frac{\rho T}{3}} - 2 \cos(\frac{\sqrt{3}}{2} \rho T)} \right), \\ \theta_3 &= M^2 T \rho^3, \end{cases}$$

then, the integral Equation (6) has a solution in $C^3([0, T], \mathbb{R})$.

Proof. Solving Equation (6) is equivalent to finding a fixed point of the operator A defined by the following formula:

$$Ax(t) = \int_0^T g(t, s) \left[f(s, x(s), x'(s), x''(s), x'''(s)) \right] ds.$$

It is clear that the operator A is well defined from $C^3([0, T], \mathbb{R})$ into itself. Moreover, for $i = 0, \dots, 3$,

$$(Ax)^i(t) = \int_0^T \frac{\partial^i g}{\partial t^i}(t, s) \left[f(s, x(s), x'(s), x''(s), x'''(s)) \right] ds.$$

The rest of the proof is divided into three steps.

Step 1. There exists $\alpha > 0$ such that A transforms $C = \{x \in C^3([0, T], \mathbb{R}), \|x\| \leq \alpha\}$ into itself. It is easy to see that C is nonempty, bounded, convex, and closed.

Moreover, for all $t \in [0, T]$ and $x \in C$, we have

$$\begin{aligned} |Ax(t)| &= \left| \int_0^T g(t, s) f(s, x(s), x'(s), x''(s), x'''(s)) ds \right| \\ &\leq \int_0^T |g(t, s)| \left(|\phi(s)| + \sum_{i=0}^3 k_{i+1} |x^{(i)}(s)| \right) ds \\ &\leq M^2 T^2 (\|\phi\|_\infty + \sum_{i=0}^3 k_{i+1} \|x^{(i)}\|_\infty) \\ &\leq \theta_0 (\|\phi\|_\infty + \sum_{i=0}^3 k_{i+1} \|x^{(i)}\|_\infty) \\ &\leq \theta_0 (\lambda + k_5 \|x\|), \end{aligned} \tag{8}$$

such that $\lambda = \|\phi\|_\infty$.

$$\begin{aligned} |(Ax)'(t)| &= \left| \int_0^T \frac{\partial g}{\partial t}(t, s) f(s, x(s), x'(s), x''(s), x'''(s)) ds \right| \\ &\leq \int_0^T \left| \frac{\partial g}{\partial t}(t, s) \right| \left(\|\phi\|_\infty + \sum_{i=0}^3 k_{i+1} |x^{(i)}(s)| \right) ds \\ &\leq \theta_1 \left(\|\phi\|_\infty + \sum_{i=0}^3 k_{i+1} \|x^{(i)}\|_\infty \right) \\ &\leq \theta_1 (\lambda + k_5 \|x\|). \end{aligned} \tag{9}$$

$$\begin{aligned} |(Ax)''(t)| &= \left| \int_0^T \frac{\partial^2 g}{\partial t^2}(t, s) f(s, x(s), x'(s), x''(s), x'''(s)) ds \right| \\ &\leq \int_0^T \left| \frac{\partial^2 g}{\partial t^2}(t, s) \right| \left(|\phi(s)| + \sum_{i=0}^3 k_{i+1} |x^{(i)}(s)| \right) ds \\ &\leq \left(\|\phi\|_\infty + \sum_{i=0}^3 k_{i+1} \|x^{(i)}\|_\infty \right) \int_0^T \left| \frac{\partial^2 g}{\partial t^2}(t, s) \right| ds \\ &\leq \theta_2 \times (\lambda + k_5 \|x\|). \end{aligned} \tag{10}$$

$$\begin{aligned}
|(Ax)'''(t)| &= \left| \int_0^T \frac{\partial^3 g}{\partial t^3}(t, s) f(s, x(s), x'(s), x''(s), x'''(s)) ds \right| \\
&\leq \int_0^T \left| \frac{\partial^3 g}{\partial t^3}(t, s) \right| \left(|\phi(s)| + \sum_{i=0}^3 k_{i+1} |x^{(i)}(s)| \right) ds \\
&\leq \theta_3 \times (\lambda + k_5 \|x\|).
\end{aligned} \tag{11}$$

Hence, using (8)–(11), we obtain

$$\|A(x)\| \leq \theta \times (\lambda + k_5 \|x\|),$$

where $\theta = \sum_{i=0}^3 \theta_i$.

We deduced that A transforms C into itself if

$$\|A(x)\| \leq \theta \times (\lambda + k_5 \|x\|) \leq \theta \times (\lambda + k_5 \alpha) \leq \alpha,$$

which implies, under condition (7), that

$$0 < \frac{\lambda \theta}{1 - k_5 \theta} \leq \alpha,$$

then, A transforms C into itself for

$$\alpha = \frac{\lambda \theta}{1 - k_5 \theta}.$$

Step 2. The operator A is continuous. Let $(x_n)_n \in C$ be a convergence sequence to $x \in C$, which implies that $(x_n^{(i)})$ converges to $x^{(i)}$ ($i = 0, \dots, 3$) in $C([0, T], [-\alpha, \alpha])$. Since f is uniformly continuous on the compact set $[0, T] \times [-\alpha, \alpha]^4$, then the sequence $(f(s, x_n(s), x_n'(s), x_n''(s), x_n'''(s)))$ converges to $f(s, x(s), x'(s), x''(s), x'''(s))$ in $C([0, T], \mathbb{R})$. It follows that, for all $i = 0, \dots, 3$,

$$\begin{aligned}
\left\| (Ax_n)^{(i)} - (Ax)^{(i)} \right\|_{\infty} &\leq \theta_i \left(\|f(s, x_n(s), x_n'(s), x_n''(s), x_n'''(s))\right. \\
&\quad \left. - f(s, x(s), x'(s), x''(s), x'''(s))\|_{\infty} \right),
\end{aligned}$$

which implies that (Ax_n) converges to Ax and A is continuous.

Step 3. $A(C)$ is relatively compact, it is clear that $A(C)$ is equibounded.

Now, to prove that $A(C)$ is equicontinuous, let t_1 and t_2 in $[0, T]$.

Let $H(s) = f(s, x(s), x'(s), x''(s), x'''(s))$, by using the assumption (H_2) , we obtain

$$\|H\|_{\infty} \leq \lambda + k_5 \alpha.$$

It follows that for all $i = 0, \dots, 3$,

$$\begin{aligned}
|(Ax)^{(i)}(t_1) - (Ax)^{(i)}(t_2)| &= \left| \int_0^T \left(\frac{\partial^i g}{\partial t^i}(t_1, s) - \frac{\partial^i g}{\partial t^i}(t_2, s) \right) H(s) ds \right| \\
&\leq \|H\| \left| \int_0^T \left(\frac{\partial^i g}{\partial t^i}(t_1, s) - \frac{\partial^i g}{\partial t^i}(t_2, s) \right) ds \right| \\
&\leq (\lambda + k_5 \alpha) \int_0^T \left| \frac{\partial^i g}{\partial t^i}(t_1, s) - \frac{\partial^i g}{\partial t^i}(t_2, s) \right| ds.
\end{aligned} \tag{12}$$

Now, let $\varepsilon > 0$, since the functions $\frac{\partial^i g}{\partial t^i}(t, s), i = 0, \dots, 3$ are uniformly continuous on the compact set $[0, T]^2$, then there exists $\delta > 0$ such that; if $|t_2 - t_1| \leq \delta$, we have for all $s \in [0, T]$ and $i = 0, \dots, 3$

$$\left| \frac{\partial^i g}{\partial t^i}(t_1, s) - \frac{\partial^i g}{\partial t^i}(t_2, s) \right| \leq \frac{\varepsilon}{T(\lambda + k_5 \alpha)}, \forall s \in [0, T],$$

then, for $i = 0, \dots, 3$, we deduce that

$$|Ax^{(i)}(t_2) - Ax^{(i)}(t_1)| \leq \varepsilon.$$

which implies that the set $A(C)$ is equicontinuous. Hence, using Theorem 1, $A(C)$ is relatively compact. The proof of Theorem 3 then follows from Schauder's fixed-point theorem. $\square$

In the following result, we provided some sufficient conditions on the data functions to ensure that Equation (6) has no nontrivial solution in $C^3([0, T], \mathbb{R})$.

Theorem 4. *If the hypotheses $(H_1 - H_4)$ hold such that $\phi = 0$, and under the following condition:*

$$k_5 \theta < 1,$$

then the integral Equation (6) has no nontrivial solution in $C^3([0, T], \mathbb{R})$.

Proof. Suppose that there exists $x \in C^3([0, T], \mathbb{R})$ such that $x \neq 0$ and $x = Ax$. From (8)–(11), we obtain that for all $t \in [0, T]$.

$$|(Ax)^{(i)}(t)| \leq \theta_i k_5 \|x\|, i = 0, \dots, 3,$$

hence,

$$\|x\| = \|Ax\| \leq \theta k_5 \|x\|,$$

it follows that

$$\theta k_5 \geq 1,$$

which is a contradiction. $\square$

4. Application

Consider the following fourth-order differential equation:

$$\begin{cases} x^{(4)}(t) + ax^{(3)}(t) + bx^{(2)}(t) + cx^{(1)}(t) + dx(t) = \cos(6t) + \ln \left(1 + \left(kx(t) + k \sum_{i=0}^3 x^{(i)}(s) \right)^2 \right), t \in [0, 1], \\ x^{(i)}(0) = x^{(i)}(1), i = 0, \dots, 3, \end{cases} \tag{13}$$

such that $\rho = \frac{\sqrt{3}\pi}{2}$. Hence, by using the notations of Theorem 3, we have $a = -2\rho$, $b = 3\rho^3$, $c = -2\rho^3$, $d = \rho^3$, $T = 1$, $\phi(t) = \cos(6t)$, and

$$f(t, u_1, u_2, u_3, u_4) = \cos(6t) + \ln(1 + (ku_1 + ku_2 + ku_3 + ku_4 + ku_5)^2).$$

Carrying out straightforward computations, it is easy to obtain the following:

$$\begin{cases} \rho < \frac{2\pi}{\sqrt{3T}}, \\ M = \frac{2\sqrt{2}}{\sqrt{3}}, \\ \theta_0 = \frac{8}{3}, \\ \theta_1 = \frac{2\sqrt{2}}{\sqrt{3}} \left(\frac{\pi}{\sqrt{3}} + \frac{e^{\frac{\sqrt{3}\pi}{2}} + e^{\frac{\sqrt{3}\pi}{4}}}{e^{\frac{\sqrt{3}\pi}{4}} + e^{\frac{-\sqrt{3}\pi}{6}} + \sqrt{2}} \right), \\ \theta_2 = \sqrt{2}\pi \left(\frac{\pi}{\sqrt{3}} + \frac{e^{\frac{\sqrt{3}\pi}{2}} + e^{\frac{\sqrt{3}\pi}{4}}}{e^{\frac{\sqrt{3}\pi}{4}} + e^{\frac{-\sqrt{3}\pi}{6}} + \sqrt{2}} \right), \\ \theta_3 = \sqrt{3}\pi^3. \end{cases}$$

Therefore, the inequality in Theorem 3 takes the following form:

$$|k| \left(\frac{8}{3} + \sqrt{3}\pi^3 + \left(\frac{2\sqrt{2}}{\sqrt{3}} + \sqrt{2}\pi \right) \left(\frac{\pi}{\sqrt{3}} + \frac{e^{\frac{\sqrt{3}\pi}{2}} + e^{\frac{\sqrt{3}\pi}{4}}}{e^{\frac{\sqrt{3}\pi}{4}} + e^{\frac{-\sqrt{3}\pi}{6}} + \sqrt{2}} \right) \right) < 1. \quad (14)$$

Then, with Theorem 3, we concluded, from the inequality (14), that the third-order nonlinear differential Equation (13) has a solution $u \in C^4(I, \mathbb{R})$ if

$$|k| < 1.20 \times 10^{-2}.$$

5. Conclusions

This paper studied a general form of a fourth-order nonlinear differential equation, where the derivatives x' , x'' , and x''' are included in the nonlinear function. We investigated the existence of solutions under relatively simple conditions by applying a new generalization of the Arzelà–Ascoli theorem along with Schauder’s fixed-point theorem. Moreover, we proved, under some sufficient conditions, the nonexistence of nontrivial solutions. The key benefit of using the new generalization of the Arzelà–Ascoli theorem, as presented in [29], was that it enabled us to address the existence of solutions for fourth-order differential equations where x' , x'' , and x''' are part of the nonlinear function, while they are not addressed in many studies on fourth-order differential equations (e.g., [21–25]). Notably, this general form of the fourth-order nonlinear differential Equation (1) encompasses several important differential equations found in existing literature as special cases.

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