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Two Families of Bi-Univalent Functions Associating the (p, q) -Derivative with Generalized Bivariate Fibonacci Polynomials

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Abstract: Making use of generalized bivariate Fibonacci polynomials, we propose two families of regular functions of the type $\phi(\zeta) = \zeta + \sum_{j=2}^{\infty} d_j \zeta^j$, which are bi-univalent in the disc $\{\zeta \in \mathbb{C} : |\zeta| < 1\}$ involving the (p, q) -derivative operator. We find estimates on the coefficients $|d_2|$, $|d_3|$ and the of Fekete–Szegő functional for members of these families. Relevant connections to the existing results and new consequences of the main result are presented.

Keywords: bi-univalent functions; (p, q) -derivative operator; subordination; Horodam polynomials; Fekete–Szegő functional

MSC: 30C45; 33C45; 11B39



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1. Preliminaries

The q -analysis is a generalization of the ordinary analysis that does not employ limit notation. Jackson presented the use and application of the q -calculus in [1]. The q -analogue of the derivative and integral operator were first defined in [2], along with some of their applications. The multifaceted uses of the q -derivative operator make it extremely important in the study of geometric function theory. Initially, Ismail et al. proposed the concept of q -extension of the class of q -starlike functions in [3]. Later, a number of mathematicians studied q -calculus in the context of geometric functions theory: the Ruscheweyh differential operator's q -analogue was first presented in [4]; some applications for multivalent functions were examined in [5,6]; q -starlike functions associated with the generalized conic domain, by applying the convolution concept was investigated in [7]; an operator associated with q -hypergeometric function was investigated in [8], and so forth. Numerous authors have recently released a number of articles that examine a class of q -starlike functions from different perspectives (see [9–14]).

An extension of the q -calculus to the (p, q) -calculus, was taken into consideration by the researchers. The (p, q) -number, was first examined around the same time (1991) and subsequently on its own by [15–18]. Fibonacci oscillators were studied with the presentation of the (p, q) -number in [15]. The investigation of the (p, q) -number in [16] allows for the construction of a (p, q) -Harmonic oscillator. The (p, q) -number was investigated in [17] as a means of combining different types of q -oscillator algebras and in [18], the (p, q) -number was examined in order to determine the (p, q) -Stirling numbers.

Building on the aforementioned publications, since 1991, a large number of scientists have investigated (p, q) -calculus in a range of research domains. The results in [19] provided a syntax for embedding the q -series into a (p, q) -series. They also investigated (p, q) -hypergeometric series and found some results corresponding to (p, q) -extensions of the known q -identities. The q -identities are extended correspondingly to yield the (p, q) -series (see, e.g., [20]). We provide some basic definitions of the (p, q) -calculus concepts used in this paper. The (p, q) -bracket number is given by $[j]_{p,q} = p^{j-1} + p^{j-2}q + p^{j-3}q^2 + \dots + pq^{j-2} + q^{j-1} = \frac{p^j - q^j}{p - q}$ ($p \neq q$), which is an extension of q -number (see [2]), that is $[j]_q = \frac{1 - q^j}{1 - q}$ ($q \neq 1$). Note that $[j]_{p,q}$ is symmetric and if $p = 1$, then $[j]_{p,q} = [j]_q$.

Let $\{\zeta \in \mathbb{C} : |\zeta| < 1\} = \mathfrak{D}$ be the open unit disk, and $\mathbb{C}$ be the complex plane. Let $\mathbb{N}$ represent the natural number set and $\mathbb{R}$ be the real number set.

Definition 1 ([21]). Let φ be a function defined on $\mathbb{C}$ and $0 < q < p \leq 1$. Then, the (p, q) -derivative of φ is defined by

$$D_{p,q}\varphi(\zeta) = \frac{\varphi(p\zeta) - \varphi(q\zeta)}{(p - q)\zeta} \quad (\zeta \neq 0),$$

and $D_{p,q}\varphi(0) = \varphi'(0)$, provided $\varphi'(0)$ exists.

We note that $D_{p,q}\zeta^j = [j]_{p,q}\zeta^{j-1}$, $D_{p,q}\ln(\zeta) = \frac{\ln(p/q)}{(p-q)\zeta}$ and $[j]_{p,q} \rightarrow j$, if $p = 1$ and $q \rightarrow 1^-$. Therefore, $D_{p,q}\varphi(\zeta) \rightarrow \varphi'(\zeta)$ as $p = 1$ and $q \rightarrow 1^-$. For any constants δ and κ , it is obvious that $D_{p,q}(\delta\varphi_1(\zeta) + \kappa\varphi_2(\zeta)) = \delta D_{p,q}\varphi_1(\zeta) + \kappa D_{p,q}\varphi_2(\zeta)$. The product rules and quotient rules are satisfied by the (p, q) -derivative (see [22]). The exponential functions are used to define the (p, q) -analogues of many functions, including sine, cosine, and tangent, in the same way as their well-known Euler expressions. Duran et al. [23] examined the (p, q) -derivatives of these functions.

The set of functions ϕ , which are regular in $\mathfrak{D}$ and have the following form, is represented by $\mathcal{A}$:

$$\phi(\zeta) = \zeta + \sum_{j=2}^{\infty} d_j \zeta^j, \quad (\zeta \in \mathfrak{D}), \quad (1)$$

with $\phi(0) = 0 = \phi'(0) - 1$, and we denote a sub-set of $\mathcal{A}$ containing univalent functions in $\mathfrak{D}$ by $\mathcal{S}$. If $\phi \in \mathcal{A}$ is of the form (1), then

$$D_{p,q}\phi(\zeta) = 1 + \sum_{j=2}^{\infty} [j]_{p,q} d_j \zeta^{j-1}, \quad (\zeta \in \mathfrak{D}). \quad (2)$$

The Koebe theorem (see [24]) states that the inverse of each function ϕ in $\mathcal{S}$ is given by

$$\phi^{-1}(\omega) = \psi(\omega) = \omega - d_2\omega^2 + (2d_2^2 - d_3)\omega^3 - (5d_2^3 - 5d_2d_3 + d_4)\omega^4 + \dots \quad (3)$$

satisfying $\zeta = \phi^{-1}(\phi(\zeta))$ and $\omega = \phi(\phi^{-1}(\omega))$, $|\omega| < r_0(\phi)$, $r_0(\phi) \geq 1/4$, $\zeta, \omega \in \mathfrak{D}$. In $\mathfrak{D}$, a member ϕ of $\mathcal{A}$ given by (1) is called bi-univalent if $\phi \in \mathcal{S}$ and $\phi^{-1} \in \mathcal{S}$. The set of such functions in $\mathfrak{D}$ is represented by σ . $\frac{1}{2} \log\left(\frac{1+\zeta}{1-\zeta}\right)$, $-\log(1-\zeta)$, and $\frac{\zeta}{1-\zeta^2}$ are some of the functions in the σ family. Nevertheless, despite being in $\mathcal{S}$, $\zeta - \frac{\zeta^2}{2}$, $\frac{\zeta}{1-\zeta^2}$, and the Koebe function does not belong to σ . For a concise analysis and to discover some of the remarkable characteristics of the family σ , see [25–28] and the citation provided in these papers. Srivastava et al. [29] introduced several subclasses of the family σ , which are similar to the well-known subclasses of the family $\mathcal{S}$. As follow-ups, numerous authors have since examined a number of different subfamilies of σ (see, for instance, [30–34]).

Several subclasses of the class σ were studied using the (p, q) -calculus. In [35], the (p, q) -derivative operator and the subordination principle were used to introduce the new

generalized classes of (p, q) -starlike and (p, q) -convex functions. The Fekete–Szegő inequalities are also examined and the (p, q) -Bernardi integral operator for analytic functions is defined. The (p, q) -Bernardi integral operator was used to obtain some applications of the main results. Several studies have also introduced and investigated novel subclasses of the class σ related to the (p, q) -differential operator (see [36–41]).

Let $k(x, y)$ and $l(x, y)$ be polynomials with real coefficients. For, $j \geq 2$, the generalized bivariate Fibonacci polynomials (GBFP) are defined by the recurrence relation:

$$\mathcal{F}_j(x, y) = k(x, y)\mathcal{F}_{j-1}(x, y) + l(x, y)\mathcal{F}_{j-2}(x, y), \quad (4)$$

where $\mathcal{F}_0(x, y) = 0$, $\mathcal{F}_1(x, y) = 1$ and $k^2(x, y) + 4l(x, y) > 0$. The generating function of GBFP is (see [42])

$$\mathcal{F}(x, y, \zeta) = \sum_{j=0}^{\infty} \mathcal{F}_j(x, y)\zeta^j = \frac{\zeta}{1 - k(x, y)\zeta - l(x, y)\zeta^2}. \quad (5)$$

For specific selections of $k(x, y)$ and $l(x, y)$, GBFP leads to various known polynomials (see [43]). Readers with an interest in GBFP can find a brief history and extensive information in [44] and its references. Interesting findings regarding coefficient estimates and Fekete–Szegő functional have been presented in [45–47] for members of certain subclasses of σ associated with GBFP.

For brevity, we write hereafter that $k(x, y) = k$ and $l(x, y) = l$. $\mathcal{F}_2(x, y) = k$ and $\mathcal{F}_3(x, y) = k^2 + l$ are evident from (4).

Remark 1. By specializing k and l , many polynomial sequences can be inferred from GBFP. They are as follows: (i) the bivariate Fibonacci polynomials are obtained if $k = x$ and $l = y$; (ii) the Pell polynomials are achieved if $k = 2x$ and $l = 1$; (iii) we derive the Jacobsthal polynomials if $k = 1$ and $l = x$; and (iv) we arrive at the Fermat polynomials, if $k = 3x$ and $l = -2$, and so forth.

For $\Omega_1, \Omega_2 \in \mathcal{A}$ holomorphic in $\mathfrak{D}$, Ω_1 is subordinate to Ω_2 , if there is $h(\zeta)$ a Schwarz function that is holomorphic in $\mathfrak{D}$ with $h(0) = 0$ and $|h(\zeta)| < 1$, such that $\Omega_1(\zeta) = \Omega_2(h(\zeta))$. The notation $\Omega_1 \prec \Omega_2$ or $\Omega_1(\zeta) \prec \Omega_2(\zeta)$ ($\zeta \in \mathfrak{D}$) denotes this subordination. Specifically, when $\Omega_2 \in \mathcal{S}$, we have

$$\Omega_1(\zeta) \prec \Omega_2(\zeta) \Leftrightarrow \Omega_1(0) = \Omega_2(0) \text{ and } \Omega_1(\mathfrak{D}) \subset \Omega_2(\mathfrak{D}).$$

We present two new families of σ subordinate to GBFP $\mathcal{F}_j(x, y)$ as in (4), with the generating function $\mathcal{F}(x, y, z)$ as in (5). These families are motivated by the aforementioned trends on coefficient-related problems and the Fekete–Szegő functional [48] on certain subclasses of σ linked to GBFP. $\mathcal{F}(x, y, z)$ is as in (5), $\phi^{-1}(\omega) = \psi(\omega)$, an inverse function as in (3), $x \in \mathbb{R}$, $\zeta \in \mathfrak{D}$, and $\omega \in \mathfrak{D}$ are assumed throughout this paper, unless otherwise mentioned.

Definition 2. A function ϕ given by (1), which is said to be in the set $\mathfrak{Y}_{\sigma, p, q}^{\lambda, \eta}(\gamma, F)$, $\gamma \geq 1, 0 < \eta \leq 1$ and $\lambda \geq 1$, if

$$\frac{1}{2} \left\{ \frac{\gamma [D_{p, q}(\zeta D_{p, q} \phi(\zeta))]^\lambda + (1 - \gamma)}{D_{p, q} \phi(\zeta)} + \left(\frac{\gamma [D_{p, q}(\zeta D_{p, q} \phi(\zeta))]^\lambda + (1 - \gamma)}{D_{p, q} \phi(\zeta)} \right)^{\frac{1}{\eta}} \right\} \prec F(\zeta) \\ = \frac{\mathcal{F}(x, y, \zeta)}{\zeta},$$

and

$$\frac{1}{2} \left\{ \frac{\gamma [D_{p,q}(\omega D_{p,q}\psi(\omega))]^\lambda + (1-\gamma)}{D_{p,q}\psi(\omega)} + \left(\frac{\gamma [D_{p,q}(\omega D_{p,q}\psi(\omega))]^\lambda + (1-\gamma)}{D_{p,q}\psi(\omega)} \right)^{\frac{1}{\eta}} \right\} \prec F(\omega) \\ = \frac{\mathcal{F}(\varkappa, y, \omega)}{\omega},$$

where

$$F(\zeta) = \frac{1}{1 - k\zeta - l\zeta^2}, k \neq 0, \text{ and } k^2 + 4l > 0. \quad (6)$$

For specific choices of p, q, γ and λ , the family $\mathfrak{Y}_{\sigma,p,q}^{\lambda,\eta}(\gamma, F)$ includes many new and existing subfamilies of σ . This is shown below:

1. let $\lambda = 1$. Then $\mathfrak{H}_{\sigma,p,q}^\eta(\gamma, F) \equiv \mathfrak{Y}_{\sigma,p,q}^{1,\eta}(\gamma, F), \gamma \geq 1, 0 < \eta \leq 1$ is the collection of members ϕ of σ that satisfy

$$\frac{1}{2} \left\{ \frac{\gamma D_{p,q}(\zeta D_{p,q}\phi(\zeta)) + (1-\gamma)}{D_{p,q}\phi(\zeta)} + \left(\frac{\gamma D_{p,q}(\zeta D_{p,q}\phi(\zeta)) + (1-\gamma)}{D_{p,q}\phi(\zeta)} \right)^{\frac{1}{\eta}} \right\} \prec F(\zeta) \\ = \frac{\mathcal{F}(\varkappa, y, \zeta)}{\zeta},$$

and

$$\frac{1}{2} \left\{ \frac{\gamma D_{p,q}(\omega D_{p,q}\psi(\omega)) + (1-\gamma)}{D_{p,q}\psi(\omega)} + \left(\frac{\gamma D_{p,q}(\omega D_{p,q}\psi(\omega)) + (1-\gamma)}{D_{p,q}\psi(\omega)} \right)^{\frac{1}{\delta}} \right\} \prec F(\omega) \\ = \frac{\mathcal{F}(\varkappa, y, \omega)}{\omega},$$

where $F(\zeta)$ is as mentioned in (6).

2. Assume that $\gamma = 1$. Then $\mathfrak{J}_{\sigma,p,q}^{\lambda,\eta}(F) \equiv \mathfrak{Y}_{\sigma,p,q}^{\lambda,\eta}(1, F), 0 < \eta \leq 1$ and $\lambda \geq 1$, is the collection of elements ϕ of σ that satisfy

$$\frac{1}{2} \left\{ \frac{[D_{p,q}(\zeta D_{p,q}\phi(\zeta))]^\lambda}{D_{p,q}\phi(\zeta)} + \left(\frac{[D_{p,q}(\zeta D_{p,q}\phi(\zeta))]^\lambda}{D_{p,q}\phi(\zeta)} \right)^{\frac{1}{\eta}} \right\} \prec F(\zeta) = \frac{\mathcal{F}(\varkappa, y, \zeta)}{\zeta},$$

and

$$\frac{1}{2} \left\{ \frac{[D_{p,q}(\omega D_{p,q}\psi(\omega))]^\lambda}{D_{p,q}\psi(\omega)} + \left(\frac{[D_{p,q}(\omega D_{p,q}\psi(\omega))]^\lambda}{D_{p,q}\psi(\omega)} \right)^{\frac{1}{\eta}} \right\} \prec F(\omega) = \frac{\mathcal{F}(\varkappa, y, \omega)}{\omega},$$

where $F(\zeta)$ is as mentioned in (6).

3. If $p = 1$ and $q \rightarrow 1^-$ in the class $\mathfrak{Y}_{\sigma,p,q}^{\lambda,\eta}(\gamma, F)$, then we obtain a subset $\mathfrak{Y}_{\sigma}^{\lambda,\eta}(\gamma, F)$ ($\lambda \geq 1, 0 < \eta \leq 1, \gamma \geq 1$) which is the collection of members ϕ of σ that satisfy

$$\frac{1}{2} \left\{ \frac{\gamma [(\zeta \phi'(\zeta))']^\lambda + (1-\gamma)}{\phi'(\zeta)} + \left(\frac{\gamma [(\zeta \phi'(\zeta))']^\lambda + (1-\gamma)}{\phi'(\zeta)} \right)^{\frac{1}{\eta}} \right\} \prec F(\zeta) = \frac{\mathcal{F}(\varkappa, y, \zeta)}{\zeta},$$

and

$$\frac{1}{2} \left\{ \frac{\gamma[(\omega\psi'(\omega))']^\lambda + (1-\gamma)}{\psi'(\omega)} + \left(\frac{\gamma[(\omega\psi'(\omega))']^\lambda + (1-\gamma)}{\psi'(\omega)} \right)^{\frac{1}{\eta}} \right\} \prec F(\omega) = \frac{\mathcal{F}(\varkappa, y, \omega)}{\omega},$$

where $F(\zeta)$ is as mentioned in (6).

4. Let $\eta = 1$. Then $\mathfrak{Z}_{\sigma,p,q}^\lambda(\gamma, F) \equiv \mathfrak{Y}_{\sigma,p,q}^{\lambda,1}(\gamma, F)$, $\gamma \geq 1$ and $\lambda \geq 1$ is the set of elements ϕ of σ that satisfy

$$\left\{ \frac{\gamma[D_{p,q}(\zeta D_{p,q}\phi(\zeta))]^\lambda + (1-\gamma)}{D_{p,q}\phi(\zeta)} \right\} \prec F(\zeta) = \frac{\mathcal{F}(\varkappa, y, \zeta)}{\zeta},$$

and

$$\left\{ \frac{\gamma[D_{p,q}(\omega D_{p,q}\psi(\omega))]^\lambda + (1-\gamma)}{D_{p,q}\psi(\omega)} \right\} \prec F(\omega) = \frac{\mathcal{F}(\varkappa, y, \omega)}{\omega},$$

where $F(\zeta)$ is as mentioned in (6).

Remark 2. (i). $\mathfrak{H}_{\sigma,p,q}^\eta(1, F) \equiv \mathfrak{I}_{\sigma,p,q}^{1,\eta}(F)$, (ii). $\mathfrak{H}_{\sigma,p,q}^1(\gamma, F) \equiv \mathfrak{Z}_{\sigma,p,q}^1(\gamma, F)$, and (iii). $\mathfrak{Z}_{\sigma,p,q}^\lambda(1, F) \equiv \mathfrak{I}_{\sigma,p,q}^{\lambda,1}(F)$.

Definition 3. A function ϕ given by (1) is said to a member of the set $\mathfrak{T}_{\sigma,p,q}^{\lambda,\eta}(\nu, F)$, $0 \leq \nu \leq 1$, $0 < \eta \leq 1$ and $\lambda \geq 1$, if

$$\frac{1}{2} \left\{ \frac{\zeta(D_{p,q}\phi(\zeta))^\lambda}{\nu\phi(\zeta) + (1-\nu)\zeta} + \left(\frac{\zeta(D_{p,q}\phi(\zeta))^\lambda}{\nu\phi(\zeta) + (1-\nu)\zeta} \right)^{\frac{1}{\eta}} \right\} \prec F(\zeta) = \frac{\mathcal{F}(\varkappa, y, \zeta)}{\zeta},$$

and

$$\frac{1}{2} \left\{ \frac{\omega(D_{p,q}\psi(\omega))^\lambda}{\nu\psi(\omega) + (1-\nu)\omega} + \left(\frac{\omega(D_{p,q}\psi(\omega))^\lambda}{\nu\psi(\omega) + (1-\nu)\omega} \right)^{\frac{1}{\eta}} \right\} \prec F(\omega) = \frac{\mathcal{F}(\varkappa, y, \omega)}{\omega},$$

where $F(\zeta)$ is as mentioned in (6).

The family $\mathfrak{T}_{\sigma,p,q}^{\lambda,\eta}(\nu, F)$ contains numerous new and existing subfamilies of σ for specific choices of ν and λ , as shown below:

1. Let $\nu = 0$. Then, $\mathfrak{C}_{\sigma,p,q}^{\lambda,\eta}(F) \equiv \mathfrak{T}_{\sigma,p,q}^{\lambda,\eta}(0, F)$, $0 < \eta \leq 1$ and $\lambda \geq 1$, is the collection of elements ϕ of σ that satisfy

$$\frac{1}{2} \left((D_{p,q}\phi(\zeta))^\lambda + (D_{p,q}\phi(\zeta))^{\frac{\lambda}{\eta}} \right) \prec F(\zeta) = \frac{\mathcal{F}(\varkappa, y, \zeta)}{\zeta},$$

and

$$\frac{1}{2} \left((D_{p,q}\psi(\omega))^\lambda + (D_{p,q}\psi(\omega))^{\frac{\lambda}{\eta}} \right) \prec F(\omega) = \frac{\mathcal{F}(\varkappa, y, \omega)}{\omega},$$

where $F(\zeta)$ is as mentioned in (6).

2. Let $\nu = 1$. Then, $\mathfrak{D}_{\sigma,p,q}^{\lambda,\eta}(F) \equiv \mathfrak{T}_{\sigma,p,q}^{\lambda,\eta}(1, F)$, $0 < \eta \leq 1$ and $\lambda \geq 1$, is the family of members ϕ of σ that satisfy

$$\frac{1}{2} \left\{ \frac{\zeta(D_{p,q}\phi(\zeta))^\lambda}{\phi(\zeta)} + \left(\frac{\zeta(D_{p,q}\phi(\zeta))^\lambda}{\phi(\zeta)} \right)^{\frac{1}{\eta}} \right\} \prec F(\zeta) = \frac{\mathcal{F}(\varkappa, y, \zeta)}{\zeta},$$

and

$$\frac{1}{2} \left\{ \frac{\omega(D_{p,q}\psi(\omega))^\lambda}{f(\omega)} + \left(\frac{\omega(D_{p,q}\psi(\omega))^\lambda}{f(\omega)} \right)^{\frac{1}{\eta}} \right\} \prec F(\omega) = \frac{\mathcal{F}(\varkappa, y, \omega)}{\omega},$$

where $F(\zeta)$ is as mentioned in (6).

3. If $p = 1$ and $q \rightarrow 1^-$ in the class $\mathfrak{T}_{\sigma,p,q}^{\lambda,\eta}(\nu, F)$, then we get a subclass $\Gamma_{\sigma}^{\lambda,\eta}(\nu, F)$ ($\lambda \geq 1$, $0 < \eta \leq 1$, $0 \leq \nu \leq 1$) of functions $\phi \in \sigma$ satisfying

$$\frac{1}{2} \left\{ \frac{\zeta(\phi'(\zeta))^\lambda}{\nu\phi(\zeta) + (1-\nu)\zeta} + \left(\frac{\zeta(\phi'(\zeta))^\lambda}{\nu\phi(\zeta) + (1-\nu)\zeta} \right)^{\frac{1}{\eta}} \right\} \prec F(\zeta) = \frac{\mathcal{F}(\varkappa, y, \zeta)}{\zeta},$$

and

$$\frac{1}{2} \left\{ \frac{\omega(\psi'(\omega))^\lambda}{\nu\psi(\omega) + (1-\nu)\omega} + \left(\frac{\omega(\psi'(\omega))^\lambda}{\nu\psi(\omega) + (1-\nu)\omega} \right)^{\frac{1}{\eta}} \right\} \prec F(\omega) = \frac{\mathcal{F}(\varkappa, y, \omega)}{\omega},$$

where $F(\zeta)$ is as mentioned in (6).

4. Suppose $\eta = 1$. Then, $\mathfrak{D}_{\sigma,p,q}^{\lambda}(\nu, F) \equiv \mathfrak{T}_{\sigma,p,q}^{\lambda,1}(\nu, F)$, $0 \leq \nu \leq 1$ and $\lambda \geq 1$, is the set of functions $\phi \in \sigma$ that satisfy

$$\frac{\zeta(D_{p,q}\phi(\zeta))^\lambda}{\nu\phi(\zeta) + (1-\nu)\zeta} \prec F(\zeta) = \frac{\mathcal{F}(\varkappa, y, \zeta)}{\zeta}, \text{ and } \frac{\omega(D_{p,q}\psi(\omega))^\lambda}{\nu\psi(\omega) + (1-\nu)\omega} \prec F(\omega) = \frac{\mathcal{F}(\varkappa, y, \omega)}{\omega},$$

where $F(\zeta)$ is as mentioned in (6).

Remark 3. (i) $\mathfrak{D}_{\sigma,p,q}^{\lambda,1}(F) \equiv \mathfrak{D}_{\sigma,p,q}^{\lambda}(1, F)$, and (ii) $\Gamma_{\sigma}^{\lambda,1}(\nu, F) \equiv \mathfrak{D}_{\sigma,p=1,q \rightarrow 1^-}^{\lambda}(\nu, F)$.

Estimates for bounds on $|d_2|$, $|d_3|$, and $|d_3 - \mu d_2^2|$, $\mu \in \mathbb{R}$ for functions in the class $\mathfrak{Y}_{\sigma,p,q}^{\lambda,\eta}(\gamma, F)$ are found in Section 2. The upper bounds for $|d_2|$, $|d_3|$, and $|d_3 - \mu d_2^2|$, $\mu \in \mathbb{R}$ for functions in the class $\mathfrak{T}_{\sigma,p,q}^{\lambda,\eta}(\nu, F)$ are derived in Section 3. Along with pertinent connections to the published research, interesting results are also presented.

2. Results for the Class $\mathfrak{Y}_{\sigma,p,q}^{\lambda,\eta}(\gamma, F)$

First, for $\phi \in \mathfrak{Y}_{\sigma,p,q}^{\lambda,\eta}(\nu, F)$, the class as defined in Definition 2, we find the coefficient estimates.

Theorem 1. Let $\gamma \geq 1$, $0 < \eta \leq 1$, and $\lambda \geq 1$. If $\phi \in \mathfrak{Y}_{\sigma,p,q}^{\lambda,\eta}(\gamma, F)$, then

$$|d_2| \leq \tag{7}$$

$$\frac{2\eta|k|\sqrt{|k|}}{\sqrt{|(2\eta(\eta+1)(P[3]_{p,q} + Q[2]_{p,q}^2) + (1-\eta)R^2[2]_{p,q}^2k^2 - (\eta+1)^2R^2[2]_{p,q}^2(k^2+l))|}},$$

$$|d_3| \leq \frac{4\eta^2k^2}{(\eta+1)^2R^2[2]_{p,q}^2} + \frac{2\eta|k|}{(\eta+1)P[3]_{p,q}}, \tag{8}$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{2\eta|k|}{(\eta+1)P[3]_{p,q}}; & |1-\mu| \leq J \\ \frac{4\eta^2|k|^3|1-\mu|}{|(2\eta(\eta+1)(P[3]_{p,q} + Q[2]_{p,q}^2) + (1-\eta)R^2[2]_{p,q}^2k^2 - (\eta+1)^2R^2[2]_{p,q}^2(k^2+l))|}; & |1-\mu| \geq J, \end{cases} \tag{9}$$

where

$$J = \left| \frac{(2\eta(\eta+1)(P[3]_{p,q} + Q[2]_{p,q}^2) + (1-\eta)R^2[2]_{p,q}^2)k^2 - (\eta+1)^2R^2[2]^2(k^2+l)}{2\eta(1+\eta)P[3]_{p,q}k^2} \right|, \quad (10)$$

$$P = \gamma\lambda[3]_{p,q} - 1, \quad (11)$$

$$Q = 1 - \gamma\lambda[2]_{p,q} + \frac{\gamma\lambda(\lambda-1)}{2}[2]_{p,q}^2, \quad (12)$$

and

$$R = \gamma\lambda[2]_{p,q} - 1. \quad (13)$$

Proof. Let $\phi \in \mathfrak{H}_{\sigma,p,q}^{\lambda,\eta}(\gamma, \varkappa)$. Then, due to Definition 2, we obtain

$$\frac{1}{2} \left\{ \frac{\gamma[D_{p,q}(\zeta D_{p,q}\phi(\zeta))]^\lambda + (1-\gamma)}{D_{p,q}\phi(\zeta)} + \left(\frac{\gamma[D_{p,q}(\zeta D_{p,q}\phi(\zeta))]^\lambda + (1-\gamma)}{D_{p,q}\phi(\zeta)} \right)^{\frac{1}{\eta}} \right\} = F(\mathfrak{m}(\zeta)) \quad (14)$$

and

$$\frac{1}{2} \left\{ \frac{\gamma[D_{p,q}(\omega D_{p,q}\psi(\omega))]^\lambda + (1-\gamma)}{D_{p,q}\psi(\omega)} + \left(\frac{\gamma[D_{p,q}(\omega D_{p,q}\psi(\omega))]^\lambda + (1-\gamma)}{D_{p,q}\psi(\omega)} \right)^{\frac{1}{\eta}} \right\} = F(\mathfrak{n}(\omega)) \quad (15)$$

where

$$\mathfrak{m}(\zeta) = m_1\zeta + m_2\zeta^2 + m_3\zeta^3 + \dots \quad \text{and} \quad \mathfrak{n}(\omega) = n_1\omega + n_2\omega^2 + n_3\omega^3 + \dots, \quad (16)$$

are some holomorphic functions with $|\mathfrak{m}(\zeta)| < 1$, $|\mathfrak{n}(\omega)| < 1$, $\zeta, \omega \in \mathfrak{D}$ and $\mathfrak{m}(0) = 0 = \mathfrak{n}(0)$. It is known that

$$|\mathfrak{m}_i| \leq 1, \text{ and } |\mathfrak{n}_i| \leq 1, (i \in \mathbb{N}). \quad (17)$$

From (14)–(16), it follows that

$$\frac{1}{2} \left\{ \frac{\gamma[D_{p,q}(\zeta D_{p,q}\phi(\zeta))]^\lambda + (1-\gamma)}{D_{p,q}\phi(\zeta)} + \left(\frac{\gamma[D_{p,q}(\zeta D_{p,q}\phi(\zeta))]^\lambda + (1-\gamma)}{D_{p,q}\phi(\zeta)} \right)^{\frac{1}{\eta}} \right\} = 1 + \mathcal{F}_2(\varkappa, y)\mathfrak{m}(\zeta) + \mathcal{F}_3(\varkappa, y)\mathfrak{m}^2(\zeta) + \dots, \quad (18)$$

and

$$\frac{1}{2} \left\{ \frac{\gamma[D_{p,q}(\omega D_{p,q}\psi(\omega))]^\lambda + (1-\gamma)}{D_{p,q}\psi(\omega)} + \left(\frac{\gamma[D_{p,q}(\omega D_{p,q}\psi(\omega))]^\lambda + (1-\gamma)}{D_{p,q}\psi(\omega)} \right)^{\frac{1}{\eta}} \right\} = 1 + \mathcal{F}_2(\varkappa, y)\mathfrak{n}(\omega) + \mathcal{F}_3(\varkappa, y)\mathfrak{n}^2(\omega) + \dots. \quad (19)$$

In light of (4), Equations (18) and (19) can be expressed as follows.

$$\frac{1}{2} \left\{ \frac{\gamma[D_{p,q}(\zeta D_{p,q}\phi(\zeta))]^\lambda + (1-\gamma)}{D_{p,q}\phi(\zeta)} + \left(\frac{\gamma[D_{p,q}(\zeta D_{p,q}\phi(\zeta))]^\lambda + (1-\gamma)}{D_{p,q}\phi(\zeta)} \right)^{\frac{1}{\eta}} \right\} = 1 + \mathcal{F}_2(\varkappa, y)\mathfrak{m}_1\zeta + [\mathcal{F}_2(\varkappa, y)\mathfrak{m}_2 + \mathcal{F}_3(\varkappa, y)\mathfrak{m}_1^2]\zeta^2 + \dots, \quad (20)$$

and

$$\frac{1}{2} \left\{ \frac{\gamma [D_{p,q}(\omega D_{p,q}\psi(\omega))]^\lambda + (1-\gamma)}{D_{p,q}\psi(\omega)} + \left(\frac{\gamma [D_{p,q}(\omega D_{p,q}\psi(\omega))]^\lambda + (1-\gamma)}{D_{p,q}\psi(\omega)} \right)^{\frac{1}{\eta}} \right\} = \quad (21)$$

$$1 + \mathcal{F}_2(\varkappa, y) \mathfrak{n}_1 \omega + [\mathcal{F}_2(\varkappa, y) \mathfrak{n}_2 + \mathcal{F}_3(\varkappa, y) \mathfrak{n}_1^2] \mathfrak{s}^2 + \dots$$

Comparing (20) and (21), we have

$$\frac{(\eta+1)R[2]_{p,q}}{2\eta} d_2 = \mathcal{F}_2(\varkappa, y) \mathfrak{m}_1, \quad (22)$$

$$\left(\frac{\eta+1}{2\eta} \right) (P[3]_{p,q} d_3 + Q[2]_{p,q}^2 d_2^2) + \left(\frac{1-\eta}{4\eta^2} \right) R^2[2]_{p,q}^2 d_2^2 = \mathcal{F}_2(\varkappa, y) \mathfrak{m}_2 + \mathcal{F}_3(\varkappa, y) \mathfrak{m}_1^2, \quad (23)$$

$$-\frac{(\eta+1)R[2]_{p,q}}{2\eta} d_2 = \mathcal{F}_2(\varkappa, y) \mathfrak{n}_1, \quad (24)$$

and

$$\left(\frac{\eta+1}{2\eta} \right) (P[3]_{p,q} (2d_2^2 - d_3) + Q[2]_{p,q}^2 d_2^2) + \left(\frac{1-\eta}{4\eta^2} \right) R^2[2]_{p,q}^2 d_2^2 = \mathcal{F}_2(\varkappa, y) \mathfrak{n}_2 + \mathcal{F}_3(\varkappa, y) \mathfrak{n}_1^2, \quad (25)$$

where P , Q , and R are as mentioned in (11), (12) and (13), respectively. From (22) and (24), we easily obtain

$$\mathfrak{m}_1 = -\mathfrak{n}_1 \quad (26)$$

and also

$$\frac{(\eta+1)^2 R^2[2]_{p,q}^2}{2\eta^2} d_2^2 = (\mathfrak{m}_1^2 + \mathfrak{n}_1^2) \mathcal{F}_2^2(\varkappa, y). \quad (27)$$

By adding (23) and (25), we obtain

$$\left[\left(\frac{\eta+1}{\eta} \right) (P[3]_{p,q} + Q[2]_{p,q}^2) + \left(\frac{1-\eta}{2\eta^2} \right) R^2[2]_{p,q}^2 \right] d_2^2 = \mathcal{F}_2(\varkappa, y) (\mathfrak{m}_2 + \mathfrak{n}_2) + \mathcal{F}_3(\varkappa, y) (\mathfrak{m}_1^2 + \mathfrak{n}_1^2). \quad (28)$$

The value of $\mathfrak{m}_1^2 + \mathfrak{n}_1^2$ from (27) is substituted in (28), yielding

$$d_2^2 = \frac{2\eta^2 \mathcal{F}_2^3(\varkappa, y) (\mathfrak{m}_2 + \mathfrak{n}_2)}{(2\eta(\eta+1)(P[3]_{p,q} + Q[2]_{p,q}^2) + (1-\eta)R^2[2]_{p,q}^2) \mathcal{F}_2^2(\varkappa, y) - (\eta+1)^2 R^2[2]_{p,q}^2 \mathcal{F}_3(\varkappa, y)}. \quad (29)$$

Applying (17) to the coefficients $\mathfrak{m}_2$ and $\mathfrak{n}_2$ yields (7).

We deduct (25) from (23) to get the bound on $|d_3|$:

$$d_3 = d_2^2 + \frac{\mathcal{F}_2(\varkappa, y) (\mathfrak{m}_2 - \mathfrak{n}_2)}{\left(\frac{\eta+1}{\eta} \right) P[3]_{p,q}}. \quad (30)$$

Then in view of (26) and (27), (30) becomes

$$d_3 = \frac{2\eta^2 \mathcal{F}_2^2(\varkappa, y) (\mathfrak{m}_1^2 + \mathfrak{n}_1^2)}{(\eta+1)^2 R^2[2]_{p,q}^2} + \frac{\eta \mathcal{F}_2(\varkappa, y) (\mathfrak{m}_2 - \mathfrak{n}_2)}{(\eta+1) P[3]_{p,q}},$$

and applying (17) for the coefficients $\mathfrak{m}_1$, $\mathfrak{m}_2$, $\mathfrak{n}_1$ and $\mathfrak{n}_2$ we obtain (8).

From (29) and (30), for $\mu \in \mathbb{R}$, we obtain

$$|d_3 - \mu d_2^2| = |\mathcal{F}_2(\varkappa, y)| \left| \left(\mathfrak{B}(\mu) + \frac{\eta}{(\eta+1) P[3]_{p,q}} \right) \mathfrak{m}_2 + \left(\mathfrak{B}(\mu) - \frac{\eta}{(\eta+1) P[3]_{p,q}} \right) \mathfrak{n}_2 \right|,$$

where

$$\mathfrak{B}(\mu) = \frac{2\eta^2(1-\mu)\mathcal{F}_2^2(\varkappa, y)}{(2\eta(\eta+1)(P[3]_{p,q} + Q[2]_{p,q}^2) + (1-\eta)R^2[2]_{p,q}^2)\mathcal{F}_2^2(\varkappa, y) - (\eta+1)^2R^2[2]_{p,q}^2\mathcal{F}_3(\varkappa, y)}.$$

Clearly

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{2\eta|\mathcal{F}_2(\varkappa, y)|}{(\eta+1)P[3]_{p,q}} & ; 0 \leq |\mathfrak{B}(\mu)| \leq \frac{\eta}{(\eta+1)P[3]_{p,q}} \\ 2|\mathcal{F}_2(\varkappa, y)||\mathfrak{B}(\mu)| & ; |\mathfrak{B}(\mu)| \geq \frac{\eta}{(\eta+1)P[3]_{p,q}}, \end{cases}$$

which results in (9) and J as in (10). $\square$

Corollary 1. Let $\lambda = 1$ in the above theorem. Then, for any function $\phi \in \mathfrak{H}_{\sigma, p, q}^\eta(\gamma, F)$ the upper bounds of $|d_2|$, $|d_3|$, and $|d_3 - \mu d_2^2|$, $\mu \in \mathbb{R}$, are given by (7), (8), and (9), respectively, with $P = P_1 = \gamma[3]_{p,q} - 1$, $Q = Q_1 = 1 - \gamma[2]_{p,q}$ and $R = R_1 = -Q_1$ in (11), (12), and (13), respectively. P_1, Q_1 , and R_1 are to be used in place of P, Q , and R for J in (10).

Corollary 2. Let $\gamma = 1$ in the above theorem. Then, for any function $\phi \in \mathfrak{I}_{\sigma, p, q}^{\lambda, \eta}(\varkappa)$ the upper bounds of $|d_2|$, $|d_3|$, and $|d_3 - \mu d_2^2|$, $\mu \in \mathbb{R}$, are given by (7), (8), and (9), respectively, where $P = P_2 = \lambda[3]_{p,q} - 1$, $Q = Q_2 = 1 - \lambda[2]_{p,q} + \frac{\lambda(\lambda-1)}{2}[2]_{p,q}^2$, and $R = R_2 = \lambda[2]_{p,q} - 1$. P_2, Q_2 , and R_2 are to be used in place of P, Q , and R for J in (10).

From Theorem 1, taking $p = 1$ and $q \rightarrow 1^-$, we obtain

Corollary 3. Let $\gamma \geq 1$, $0 < \eta \leq 1$, and $\lambda \geq 1$. If $\phi \in \mathcal{Y}_{\sigma}^{\lambda, \eta}(\gamma, F)$, then

$$|d_2| \leq \frac{\eta|k|\sqrt{2|k|}}{\sqrt{|(\eta(\eta+1)(8\gamma\lambda^2 - 7\gamma\lambda + 1) + 2(1-\eta)(2\gamma\lambda - 1)^2)k^2 - 2(\eta+1)^2(2\gamma\lambda - 1)^2(k^2 + l)|}},$$

$$|d_3| \leq \left(\frac{\eta k}{(\eta+1)(2\gamma\lambda - 1)} \right)^2 + \frac{2\eta|k|}{3(\eta+1)(3\gamma\lambda - 1)},$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{2\eta|k|}{3(\eta+1)(3\gamma\lambda - 1)}; & |1 - \mu| \leq J_1 \\ \frac{2\eta^2|k|^3|1-\mu|}{|(\eta(\eta+1)(8\gamma\lambda^2 - 7\gamma\lambda + 1) + 2(1-\eta)(2\gamma\lambda - 1)^2)k^2 - 2(\eta+1)^2(2\gamma\lambda - 1)^2(k^2 + l)|}; & |1 - \mu| \geq J_1, \end{cases}$$

where

$$J_1 = \left| \frac{(\eta(\eta+1)(8\gamma\lambda^2 - 7\gamma\lambda + 1) + 2(1-\eta)(2\gamma\lambda - 1)^2)k^2 - 2(\eta+1)^2(2\gamma\lambda - 1)^2(k^2 + l)}{\eta(\eta+1)(3\gamma\lambda - 1)k^2} \right|,$$

Here are a few examples of the above corollary's special cases.

Example 1. Letting $\eta = 1$ in the class $\mathcal{Y}_{\sigma}^{\lambda, \eta}(\gamma, F)$, we obtain a subclass $\mathcal{X}_{\sigma}^{\lambda}(\gamma, F) \equiv \mathcal{Y}_{\sigma}^{\lambda, 1}(\gamma, F)$ of functions $\phi \in \sigma$ satisfying

$$\frac{\gamma[(\zeta\phi'(\zeta))']^{\lambda} + (1-\gamma)}{\phi'(\zeta)} \prec F(\zeta) = \frac{\mathcal{F}(\varkappa, y, \zeta)}{\zeta}$$

and

$$\frac{\gamma[(\omega\psi'(\omega))']^{\lambda} + (1-\gamma)}{\psi'(\omega)} \prec F(\omega) = \frac{\mathcal{F}(\varkappa, y, \omega)}{\omega},$$

where $F(\zeta)$ is as mentioned in (6).

The class $\chi_{\sigma}^{\lambda}(\gamma, F)$ is not an empty set, which is an important point to note. Consider the following example, where $\gamma = \lambda = 1, k = 1$ and $l = 0$. The function $u(\zeta) = \zeta + \frac{\zeta^2}{6} + \frac{\zeta^3}{27} + \dots = -3 \log(1 - \frac{\zeta}{3}) \in \mathcal{S}$ is in the class $\chi_{\sigma}^1(1, F)$. Because, its inverse $v(\omega) = \omega - \frac{\omega^2}{6} + \frac{\omega^3}{54} - \dots = \frac{3(e^{\frac{\omega}{3}} - 1)}{e^{\frac{\omega}{3}}} \in \mathcal{S}$. So, we obtain that $u(\zeta) \in \sigma$ (see [46]).

Corollary 4. Let $\gamma \geq 1$, and $\lambda \geq 1$. If $\phi \in \chi_{\sigma}^{\lambda}(\gamma, F)$, then

$$|d_2| \leq \frac{|k|\sqrt{|k|}}{\sqrt{|(8\gamma\lambda^2 - 7\gamma\lambda + 1)k^2 - 4(2\gamma\lambda - 1)^2(k^2 + l)|}},$$

$$|d_3| \leq \left(\frac{k}{2(2\gamma\lambda - 1)}\right)^2 + \frac{|k|}{3(3\gamma\lambda - 1)},$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{|k|}{3(3\gamma\lambda - 1)}; & |1 - \mu| \leq J_2 \\ \frac{2|k|^3|1 - \mu|}{|(8\gamma\lambda^2 - 7\gamma\lambda + 1)k^2 - 4(2\gamma\lambda - 1)^2(k^2 + l)|}; & |1 - \mu| \geq J_2, \end{cases}$$

where

$$J_2 = \left| \frac{(8\gamma\lambda^2 - 7\gamma\lambda + 1)k^2 - 4(2\gamma\lambda - 1)^2(k^2 + l)}{(3\gamma\lambda - 1)k^2} \right|.$$

Example 2. Letting $\gamma = 1$ in the class $\chi_{\sigma}^{\lambda}(\gamma, F)$, we obtain a subclass $\chi_{\sigma}^{\lambda}(1, F)$ of functions $\phi \in \sigma$ satisfying

$$\frac{[(\zeta\phi'(\zeta))']^{\lambda}}{\phi'(\zeta)} \prec F(\zeta) = \frac{\mathcal{F}(\varkappa, y, \zeta)}{\zeta}, \text{ and } \frac{[(\omega\psi'(\omega))']^{\lambda}}{\psi'(\omega)} \prec F(\omega) = \frac{\mathcal{F}(\varkappa, y, \omega)}{\omega},$$

where $F(\zeta)$ is as mentioned in (6).

Corollary 5. Let $\lambda \geq 1$. If $\phi \in \chi_{\sigma}^{\lambda}(1, F)$ then

$$|d_2| \leq \frac{|k|\sqrt{|k|}}{\sqrt{|(8\lambda^2 - 7\lambda + 1)k^2 - 4(2\lambda - 1)^2(k^2 + l)|}}, \quad |d_3| \leq \left(\frac{k}{2(2\lambda - 1)}\right)^2 + \frac{|k|}{3(3\lambda - 1)},$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{|k|}{3(3\lambda - 1)}; & |1 - \mu| \leq J_3 \\ \frac{2|k|^3|1 - \mu|}{|(8\lambda^2 - 7\lambda + 1)k^2 - 4(2\lambda - 1)^2(k^2 + l)|}; & |1 - \mu| \geq J_3, \end{cases}$$

where

$$J_3 = \left| \frac{(8\lambda^2 - 7\lambda + 1)k^2 - 4(2\lambda - 1)^2(k^2 + l)}{(3\gamma\lambda - 1)k^2} \right|.$$

Remark 4. $\lambda = 1$ in Corollary 5 yields Corollaries 3 and 7 of Yilmaz and Aktas [43].

Taking $\eta = 1$ in the above theorem, we obtain

Corollary 6. Let $\gamma \geq 1$, and $\lambda \geq 1$. If $\phi \in \mathfrak{Z}_{\sigma, p, q}^{\lambda}(\gamma, \varkappa)$, then

$$|d_2| \leq \frac{|k|\sqrt{|k|}}{\sqrt{|(P[3]_{p, q} + Q[2]_{p, q}^2)k^2 - R^2[2]_{p, q}^2(k^2 + l)|}}, \quad |d_3| \leq \frac{k^2}{R^2[2]_{p, q}^2} + \frac{|k|}{P[3]_{p, q}},$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{|k|}{P[3]_{p,q}}; & |1 - \mu| \leq J_4 \\ \frac{|k|^3 |1 - \mu|}{|(P[3]_{p,q} + Q[2]_{p,q}^2)k^2 - R^2[2]_{p,q}^2(k^2 + l)|}; & |1 - \mu| \geq J_4, \end{cases}$$

where

$$J_4 = \left| \frac{(P[3]_{p,q} + Q[2]_{p,q}^2)k^2 - R^2[2]_{p,q}^2(k^2 + l)}{P[3]_{p,q}k^2} \right|,$$

and P , Q , and R are as detailed in (11), (12), and (13), respectively.

Remark 5. $3_{\sigma,p=1,q \rightarrow 1^-}^\lambda(\gamma, F) \equiv \chi_\sigma^\lambda(\gamma, F)$

3. Results for the Class $\mathfrak{T}_{\sigma,p,q}^{\lambda,\eta}(\nu, \varkappa)$

First, for the class $\phi \in \mathfrak{T}_{\sigma,p,q}^{\lambda,\eta}(\nu, \varkappa)$, as defined in Definition 3, we determine the coefficient estimates.

Theorem 2. Let $0 \leq \nu \leq 1$, $0 < \eta \leq 1$, and $\lambda \geq 1$. If $\phi \in \mathfrak{T}_{\sigma,p,q}^{\lambda,\eta}(\nu, \varkappa)$, then

$$|d_2| \leq \frac{2\eta|k|\sqrt{|k|}}{\sqrt{|(2\eta(\eta+1)(L+N) + (1-\eta)M^2)k^2 - (\eta+1)^2B^2(k^2+l)|}}}, \quad (31)$$

$$|d_3| \leq \frac{4\eta^2k^2}{(\eta+1)^2M^2} + \frac{2\eta|k|}{(\eta+1)L}, \quad (32)$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{2\eta|k|}{(\eta+1)L} & ; |1 - \mu| \leq \mathbb{Q} \\ \frac{4\eta^2|k|^3 |1 - \mu|}{|(2\eta(\eta+1)(L+N) + (1-\eta)M^2)k^2 - (\eta+1)^2B^2(k^2+l)|} & ; |1 - \mu| \geq \mathbb{Q}, \end{cases} \quad (33)$$

where

$$\mathbb{Q} = \left| \frac{(2\eta(\eta+1)(L+N) + (1-\eta)M^2)k^2 - (\eta+1)^2M^2(k^2+l)}{2\eta(1+\eta)Lk^2} \right|, \quad (34)$$

$$L = \lambda[3]_{p,q} - \nu, \quad (35)$$

$$N = \frac{\lambda(\lambda-1)[2]_{p,q}^2}{2} - \nu\lambda[2]_{p,q} + \nu^2, \quad (36)$$

and

$$M = \lambda[2]_{p,q} - \nu. \quad (37)$$

Proof. Let $\phi \in \mathfrak{T}_{\sigma,p,q}^{\lambda,\eta}(\nu, \varkappa)$. Following that, due to Definition 3, we obtain

$$\frac{1}{2} \left\{ \frac{\zeta(D_{p,q}\phi(\zeta))^\lambda}{\nu\phi(\zeta) + (1-\nu)\zeta} + \left(\frac{\zeta(D_{p,q}\phi(\zeta))^\lambda}{\nu\phi(\zeta) + (1-\nu)\zeta} \right)^{\frac{1}{\eta}} \right\} = F(\mathfrak{m}(\zeta)), \quad (38)$$

and

$$\frac{1}{2} \left\{ \frac{\omega(D_{p,q}\psi(\omega))^\lambda}{\nu\psi(\omega) + (1-\nu)\omega} + \left(\frac{\omega(D_{p,q}\psi(\omega))^\lambda}{\nu\psi(\omega) + (1-\nu)\omega} \right)^{\frac{1}{\eta}} \right\} = F(\mathfrak{n}(\omega)), \quad (39)$$

where $m(\zeta)$ and $n(\omega)$ are analytic functions as given in (16), with $|m(\zeta)| < 1, |n(\omega)| < 1$, $\zeta, \omega \in \mathfrak{D}$, and it is known that

$$|m_i| \leq 1, \text{ and } |n_i| \leq 1, (i \in \mathbb{N}). \quad (40)$$

It follows from (38), (39) and (16) that

$$\frac{1}{2} \left\{ \frac{\zeta(D_{p,q}\phi(\zeta))^\tau}{v\phi(\zeta) + (1-v)\zeta} + \left(\frac{\zeta(D_{p,q}\phi(\zeta))^\lambda}{v\phi(\zeta) + (1-v)\zeta} \right)^{\frac{1}{\eta}} \right\} = 1 + \mathcal{F}_2(\varkappa, y)m(\zeta) + \mathcal{F}_3(\varkappa, y)m^2(\zeta) + \dots, \quad (41)$$

and

$$\frac{1}{2} \left\{ \frac{\omega(D_{p,q}\psi(\omega))^\lambda}{v\psi(\omega) + (1-v)\omega} + \left(\frac{w(D_{p,q}\psi(\omega))^\lambda}{v\psi(\omega) + (1-v)\omega} \right)^{\frac{1}{\eta}} \right\} = 1 + \mathcal{F}_2(\varkappa, y)n(\omega) + \mathcal{F}_3(\varkappa, y)n^2(\omega) + \dots. \quad (42)$$

In light of (4), Equations (41) and (42) can be expressed as follows.

$$\frac{1}{2} \left\{ \frac{\zeta(D_{p,q}\phi(\zeta))^\lambda}{v\phi(\zeta) + (1-v)\zeta} + \left(\frac{\zeta(D_{p,q}\phi(\zeta))^\lambda}{v\phi(\zeta) + (1-v)\zeta} \right)^{\frac{1}{\eta}} \right\} = 1 + \mathcal{F}_2(\varkappa, y)m_1\zeta + [\mathcal{F}_2(\varkappa, y)m_2 + \mathcal{F}_3(\varkappa, y)m_1^2]\zeta^2 + \dots, \quad (43)$$

and

$$\frac{1}{2} \left\{ \frac{\omega(D_{p,q}\psi(\omega))^\lambda}{v\psi(\omega) + (1-v)\omega} + \left(\frac{w(D_{p,q}\psi(\omega))^\lambda}{v\psi(\omega) + (1-v)\omega} \right)^{\frac{1}{\eta}} \right\} = 1 + \mathcal{F}_2(\varkappa, y)n_1\omega + [\mathcal{F}_2(\varkappa, y)n_2 + \mathcal{F}_3(\varkappa, y)n_1^2]\omega^2 + \dots. \quad (44)$$

Comparing (43) and (44), we have

$$\frac{(\eta+1)M}{2\eta} d_2 = \mathcal{F}_2(\varkappa, y)m_1, \quad (45)$$

$$\left(\frac{\eta+1}{2\eta} \right) (Ld_3 + Nd_2^2) + \left(\frac{1-\eta}{4\eta^2} \right) M^2 d_2^2 = \mathcal{F}_2(\varkappa, y)m_2 + \mathcal{F}_3(\varkappa, y)m_1^2, \quad (46)$$

$$-\frac{(\eta+1)M}{2\eta} d_2 = \mathcal{F}_2(\varkappa, y)n_1, \quad (47)$$

and

$$\left(\frac{\eta+1}{2\eta} \right) (L(2d_2^2 - d_3) + Nd_2^2) + \left(\frac{1-\eta}{4\eta^2} \right) M^2 d_2^2 = \mathcal{F}_2(\varkappa, y)n_2 + \mathcal{F}_3(\varkappa, y)n_1^2, \quad (48)$$

where L, N , and M are as in (35), (36) and (37), respectively. From (45) and (47), we easily obtain

$$m_1 = -n_1 \quad (49)$$

and also

$$\frac{(\eta+1)^2 M^2}{2\eta^2} d_2^2 = (m_1^2 + n_1^2)(\mathcal{H}_2(\varkappa))^2. \quad (50)$$

The bound on $|d_2|$, is obtained by adding (46) and (48).

$$\left(\left(\frac{\eta+1}{\eta} \right) (L+N) + \left(\frac{1-\eta}{2\eta^2} \right) M^2 \right) d_2^2 = \mathcal{F}_2(\varkappa, y)(m_2 + n_2) + \mathcal{F}_3(\varkappa, y)(m_1^2 + n_1^2). \quad (51)$$

Substituting the value of $m_1^2 + n_1^2$ from (50) in (51), we obtain

$$d_2^2 = \frac{2\eta^2 \mathcal{F}_2^3(\mathcal{x}, y)(\mathcal{m}_2 + \mathcal{n}_2)}{(2\eta(\eta + 1)(L + N) + (1 - \eta)M^2)\mathcal{F}_2^2(\mathcal{x}, y) - (\eta + 1)^2 M^2 \mathcal{F}_3(\mathcal{x}, y)}. \quad (52)$$

Applying (40) for the coefficients m_2 and n_2 , we obtain (31).

To obtain the bound on $|d_3|$, we deduct (48) from (46):

$$d_3 = d_2^2 + \frac{\mathcal{F}_2(\mathcal{x}, y)(\mathcal{m}_2 - \mathcal{n}_2)}{\left(\frac{\eta+1}{\eta}\right)L}. \quad (53)$$

Then, in view of (49) and (50), (53) becomes

$$d_3 = \frac{2\eta^2 \mathcal{F}_2^2(\mathcal{x}, y)(\mathcal{m}_1^2 + \mathcal{n}_1^2)}{(\eta + 1)^2 M^2} + \frac{\eta \mathcal{F}_2(\mathcal{x}, y)(\mathcal{m}_2 - \mathcal{n}_2)}{(\eta + 1)L},$$

and applying (40) for the coefficients m_1, m_2, n_1 and n_2 we obtain (32).

From (52) and (53), for $\mu \in \mathbb{R}$, we obtain

$$|d_3 - \mu d_2^2| = |\mathcal{F}_2(\mathcal{x}, y)| \left| \left(\mathfrak{B}_1(\mu, \mathcal{x}) + \frac{\eta}{(\eta + 1)L} \right) \mathcal{m}_2 + \left(\mathfrak{B}_1(\mu, \mathcal{x}) - \frac{\eta}{(\delta + 1)L} \right) \mathcal{n}_2 \right|,$$

where

$$\mathfrak{B}_1(\mu, \mathcal{x}) = \frac{2\eta^2(1 - \mu)\mathcal{F}_2^2(\mathcal{x}, y)}{(2\eta(\eta + 1)(L + N) + (1 - \eta)M^2)\mathcal{F}_2^2(\mathcal{x}, y) - (\eta + 1)^2 M^2 \mathcal{F}_3(\mathcal{x}, y)}.$$

Clearly

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{2\eta|\mathcal{F}_2(\mathcal{x}, y)|}{(\eta+1)L} & ; 0 \leq |\mathfrak{B}_1(\mu, \mathcal{x})| \leq \frac{\eta}{(\eta+1)L} \\ 2|\mathcal{F}_2(\mathcal{x}, y)||\mathfrak{B}_1(\mu, \mathcal{x})| & ; |\mathfrak{B}_1(\mu, \mathcal{x})| \geq \frac{\eta}{(\eta+1)L}, \end{cases}$$

from which we conclude (33) with $\mathbb{Q}$ as in (34). $\square$

Corollary 7. Let $v = 0$ in the above theorem. Then, for any function $\phi \in \mathfrak{C}_{\sigma, p, q}^{\lambda, \eta}(F)$ the upper bounds of $|d_2|$, $|d_3|$, and $|d_3 - \mu d_2^2|$, $\mu \in \mathbb{R}$, are given by (31), (32) and (33), respectively, with $L = L_1 = \lambda[3]_{p, q}$, $N = N_1 = \frac{\lambda(\lambda-1)}{2}[2]_{p, q}^2$ and $M = M_1 = \lambda[2]_{p, q}$. L_1, N_1 , and M_1 are to be used in place of L, N , and M for $\mathbb{Q}$ in (34).

Corollary 8. Let $v = 1$ in the above theorem. Then, for any function $\phi \in \mathfrak{D}_{\sigma, p, q}^{\lambda, \eta}(F)$ the upper bounds of $|d_2|$, $|d_3|$, and $|d_3 - \mu d_2^2|$, $\mu \in \mathbb{R}$, are given by (31), (32) and (33), respectively, with $L = L_2 = \lambda[3]_{p, q} - 1$, $N = N_2 = \frac{\lambda(\lambda-1)}{2}[2]_{p, q}^2 - \lambda[2]_{p, q} + 1$ and $M = M_2 = \lambda[2]_{p, q} - 1$. L_2, N_2 , and M_2 are to be used in place of L, N , and M for $\mathbb{Q}$ in (34).

From Theorem 2, taking $p = 1$ and $q \rightarrow 1^-$, we obtain

Corollary 9. Let $0 \leq v \leq 1, 0 < \eta \leq 1$, and $\lambda \geq 1$. If $\phi \in \Gamma_{\sigma}^{\lambda, \eta}(v, F)$ then

$$|d_2| \leq \frac{2\eta|k|\sqrt{|k|}}{\sqrt{|(2\eta(\eta + 1)((2\lambda + 1)(\lambda - v) + v^2) + (1 - \eta)(2\lambda - v)^2 k^2 - (\eta + 1)^2 (2\lambda - v)^2 (k^2 + I)|}},$$

$$|d_3| \leq \left(\frac{2\eta k}{(\eta + 1)(2\lambda - v)} \right)^2 + \frac{2\eta|k|}{(\eta + 1)(3\lambda - v)},$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{2\eta|k|}{(\eta+1)(3\lambda-\nu)}; & |1-\mu| \leq \mathbb{Q}_1 \\ \frac{4\eta^2|k|^3|1-\mu|}{|(2\eta(\eta+1)((2\lambda+1)(\lambda-\nu)+\nu^2)+(1-\eta)(2\lambda-\nu)^2k^2-(\eta+1)^2(2\lambda-\nu)^2(k^2+I))|}; & |1-\mu| \geq \mathbb{Q}_1, \end{cases}$$

where

$$\mathbb{Q}_1 = \left| \frac{(2\eta(\eta+1)((2\lambda+1)(\lambda-\nu)+\nu^2)+(1-\eta)(2\lambda-\nu)^2k^2-(\eta+1)^2(2\lambda-\nu)^2(k^2+I))}{2\eta(\eta+1)(3\lambda-\nu)k^2} \right|.$$

Here are a few examples of the above corollary's special cases.

Example 3. Letting $\eta = 1$ in the class $\Gamma_{\sigma}^{\lambda,\eta}(\nu, F)$, we obtain a subclass $\Psi_{\sigma}^{\lambda}(\nu, F) \equiv \Gamma_{\sigma}^{\lambda,1}(\nu, F)$ of functions $\phi \in \sigma$ satisfying

$$\frac{\zeta(\phi'(\zeta))^{\lambda}}{\nu\phi(\zeta) + (1-\nu)\zeta} \prec F(\zeta) = \frac{\mathcal{F}(\varkappa, y, \zeta)}{\zeta},$$

and

$$\frac{\omega(\psi'(\omega))^{\lambda}}{\nu\psi(\omega) + (1-\nu)\omega} \prec F(\omega) = \frac{\mathcal{F}(\varkappa, y, \omega)}{\omega},$$

where $F(\zeta)$ is as mentioned in (6), $0 \leq \nu \leq 1$, and $\lambda \geq 1$.

Corollary 10. Let $0 \leq \nu \leq 1$, and $\lambda \geq 1$. If $\phi \in \Psi_{\sigma}^{\lambda}(\nu, F)$ then

$$|d_2| \leq \frac{|k|\sqrt{|k|}}{\sqrt{|((2\lambda+1)(\lambda-\nu)+\nu^2)k^2-(2\lambda-\nu)^2(k^2+I)|}},$$

$$|d_3| \leq \left(\frac{k}{2\lambda-\nu}\right)^2 + \frac{|k|}{3\lambda-\nu},$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{|k|}{3\lambda-\nu}; & |1-\mu| \leq \mathbb{Q}_2 \\ \frac{|k|^3|1-\mu|}{|((2\lambda+1)(\lambda-\nu)+\nu^2)k^2-(2\lambda-\nu)^2(k^2+I)|}; & |1-\mu| \geq \mathbb{Q}_2, \end{cases}$$

where

$$\mathbb{Q}_2 = \left| \frac{((2\lambda+1)(\lambda-\nu)+\nu^2)k^2-(2\lambda-\nu)^2(k^2+I)}{(3\lambda-\nu)k^2} \right|.$$

Example 4. Letting $\nu = 1$, then $\Psi_{\sigma}^{\lambda}(1, F)$ is the subclass of functions $\phi \in \sigma$ satisfying

$$\frac{\zeta(\phi'(\zeta))^{\lambda}}{\phi(\zeta)} \prec F(\zeta) = \frac{\mathcal{F}(\varkappa, y, \zeta)}{\zeta}, \text{ and } \frac{\omega(\psi'(\omega))^{\lambda}}{\psi(\omega)} \prec F(\omega) = \frac{\mathcal{F}(\varkappa, y, \omega)}{\omega},$$

where $F(\zeta)$ is as mentioned in (6).

Corollary 11. Let $\lambda \geq 1$. If $\phi \in \Psi_{\sigma}^{\lambda}(1, F)$ then

$$|d_2| \leq \frac{|k|\sqrt{|k|}}{\sqrt{|\lambda(2\lambda-1)k^2-(2\lambda-1)^2(k^2+I)|}}, \quad |d_3| \leq \left(\frac{k}{2\lambda-1}\right)^2 + \frac{|k|}{3\lambda-1},$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{|k|}{3\lambda-1}; & |1-\mu| \leq \left| \frac{\lambda(2\lambda-1)k^2 - (2\lambda-1)^2(k^2+l)}{(3\lambda-1)k^2} \right| \\ \frac{|k|^3|1-\mu|}{|\lambda(2\lambda-1)k^2 - (2\lambda-1)^2(k^2+l)|}; & |1-\mu| \geq \left| \frac{\lambda(2\lambda-1)k^2 - (2\lambda-1)^2(k^2+l)}{(3\lambda-1)k^2} \right|. \end{cases}$$

Remark 6. $\lambda = 1$ in Corollary 11 yields Corollaries 2 and 6 of Yilmaz and Aktas [43].

Example 5. Letting $\nu = 0$, then $\Psi_\sigma^\lambda(0, F)$ is the subclass of functions $\phi \in \sigma$ satisfying

$$(\phi'(\zeta))^\lambda \prec F(\zeta) = \frac{\mathcal{F}(\varkappa, y, \zeta)}{\zeta}, \text{ and } (\psi'(\omega))^\lambda \prec F(\omega) = \frac{\mathcal{F}(\varkappa, y, \omega)}{\omega},$$

where $F(\zeta)$ is as mentioned in (6).

The class $\Psi_\sigma^\lambda(0, F)$ is not an empty set, which is an important point to note. Consider the following example, where $\lambda = 1, k = 1$ and $l = 0$. The function $u(\zeta) = \frac{1}{2} \log\left(\frac{1+\zeta}{1-\zeta}\right) \in \mathcal{S}$ is in the class $\Psi_\sigma^\lambda(0, F)$. Because, its inverse $v(\omega) = \frac{e^{2\omega}-1}{e^{2\omega}+1} \in \mathcal{S}$. So, we get that $u(\zeta) \in \sigma$ (see [46]).

Corollary 12. Let $\lambda \geq 1$. If $\phi \in \Psi_\sigma^\lambda(0, F)$, then

$$|d_2| \leq \frac{|k|\sqrt{|k|}}{\sqrt{|(2\lambda+1)\lambda k^2 - 4\lambda^2(k^2+l)|}}, \quad |d_3| \leq \left(\frac{k}{2\lambda}\right)^2 + \frac{|k|}{3\lambda},$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{|k|}{3\lambda}; & |1-\mu| \leq \left| \frac{(2\lambda+1)\lambda k^2 - 4\lambda^2(k^2+l)}{3\lambda k^2} \right| \\ \frac{|k|^3|1-\mu|}{|(2\lambda+1)\lambda k^2 - 4\lambda^2(k^2+l)|}; & |1-\mu| \geq \left| \frac{(2\lambda+1)\lambda k^2 - 4\lambda^2(k^2+l)}{3\lambda k^2} \right|. \end{cases}$$

Corollary 13. Let $0 \leq \nu \leq 1$, and $\lambda \geq 1$. If $\phi \in \mathfrak{D}_{\sigma,p,q}^\lambda(\nu, F)$, then

$$|d_2| \leq \frac{|k|\sqrt{|k|}}{\sqrt{|(L+N)k^2 - M^2(k^2+l)|}}, \quad |d_3| \leq \frac{k^2}{M^2} + \frac{|k|}{L},$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{|k|}{L} & ; |1-\mu| \leq \left| \frac{(L+N)k^2 - M^2(k^2+l)}{Lk^2} \right| \\ \frac{|k|^3|1-\mu|}{|(L+N)k^2 - M^2(k^2+l)|} & ; |1-\mu| \geq \left| \frac{(L+N)k^2 - M^2(k^2+l)}{Lk^2} \right|, \end{cases}$$

where L, N , and M are as mentioned in (31), (32), and (33), respectively.

Remark 7. $\mathfrak{D}_{\sigma,p=1,q \rightarrow 1^-}^\lambda(\nu, F) \equiv \Psi_\sigma^\lambda(\nu, F)$.

4. Conclusions

The current work defines two subfamilies of σ associated with GBFP and derives upper bounds of $|d_2|, |d_3|$ and the Fekete–Szegő functional $|d_3 - \mu d_2^2|, \mu \in \mathbb{R}$ for functions in these subfamilies. We have been able to draw attention to several implications by altering the parameters in Theorem 1 and Theorem 2. Relevant connections to the current research are also discovered. The subfamilies this paper studies may encourage researchers to

concentrate on the (p, q) -derivative operator. Future studies might look into extending obtained results to fractional derivatives, higher-order Hankel determinants, or Toeplitz determinants. The findings presented here give these advancements a strong foundation and emphasize the importance of geometric factors in the study of analytic function theory.

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References

1. Jackson, F.H. q -difference equations. *Am. J. Math.* **1910**, *32*, 305–314. [\[CrossRef\]](#)
2. Jackson, F.H. On q -functions and a certain difference operator. *Earth Environ. Sci. Trans. R. Soc. Edinb.* **1908**, *46*, 253–281. [\[CrossRef\]](#)
3. Ismail, M.E.H.; Merkes, E.; Styer, D. A generalization of starlike functions. *Complex Var. Theory Appl.* **1990**, *14*, 77–84. [\[CrossRef\]](#)
4. Kanas, S.; Raducanu, D. Some class of analytic functions related to conic domains. *Math. Slovaca* **2014**, *64*, 1183–1196. [\[CrossRef\]](#)
5. Arif, M.; Barkub, O.; Srivastava, H.M.; Abdullah, S.; Khan, S.A. Some Janowski type harmonic q -starlike functions associated with symmetrical points. *Mathematics* **2020**, *8*, 629. [\[CrossRef\]](#)
6. Arif, M.; Srivastava, H.M.; Uma, S. Some applications of a q -analogue of the Ruscheweyh type operator for multivalent functions. *Rev. Real Acad. Cienc. Exactas Fis. Natur. Ser. A Mat. (RACSAM)* **2019**, *113*, 1211–1221. [\[CrossRef\]](#)
7. Zhang, X.; Khan, S.; Hussain, S.; Tang, H.; Shareef, Z. New subclass of q -starlike functions associated with generalized conic domain. *AIMS Math.* **2020**, *5*, 4830–4848. [\[CrossRef\]](#)
8. Mohammed, A.; Darus, M. A generalized operator involving the q -hypergeometric function. *Mat. Vesnik* **2013**, *65*, 454–465.
9. Ahmad, Q.Z.; Khan, N.; Raza, M. Certain q -difference operators and their applications to the subclass of meromorphic q -starlike functions. *Filomat* **2019**, *33*, 3385–3397. [\[CrossRef\]](#)
10. Frasin, B.A.; Darus, M. Subclass of analytic functions defined by q -derivative operator associated with Pascal distribution series. *AIMS Math.* **2021**, *6*, 5008–5019. [\[CrossRef\]](#)
11. Srivastava, H.M.; Khan, B.; Khan, N.; Ahmad, Q.Z. Coefficient inequalities for q -starlike functions associated with the Janowski functions. *Hokkaido Math. J.* **2019**, *48*, 407–425. [\[CrossRef\]](#)
12. Srivastava, H.M.; Khan, N.; Darus, M.; Khan, S.; Ahmad, Q.Z.; Hussain, S. Fekete-Szegő type problems and their applications for a subclass of q -starlike functions with respect to symmetrical points. *Mathematics* **2020**, *8*, 842. [\[CrossRef\]](#)
13. El-Deeb, S.M.; Bulboacă, T.; El-Matary, B.M. Maclaurin coefficient estimates of bi-univalent functions connected with the q -derivative. *Mathematics* **2020**, *8*, 418. [\[CrossRef\]](#)
14. Srivastava, H.M. Operators of basic (or q -) calculus and fractional q -calculus and their applications in geometric function theory of complex analysis. *Iran J. Sci. Technol. Trans. A Sci.* **2020**, *44*, 327–344. [\[CrossRef\]](#)
15. Arik, M.; Demircan, E.; Turgut, T.; Ekinci, L.; Mungan, M. Fibonacci oscillators. *Z. Phys. C Part. Fields* **1992**, *55*, 89–95. [\[CrossRef\]](#)
16. Brodimas, G.; Jannussis, A.; Mignani, R. Two-parameter Quantum Groups. *Dipartimento di Fisica Universit di Roma “La Sapienza” I.N.F.N.-Sezione di Roma* **1991**, 1–16, preprint.
17. Chakrabarti, R.; Jagannathan, R.A. (p, q) -oscillator realization of two-parameter quantum algebras. *J. Phys. A Math. Gen.* **1991**, *24*, L711–L718. [\[CrossRef\]](#)
18. Wachs, M.; White, D. (p, q) -Stirling numbers and set partition statistics. *J. Combin. Theory Ser. A* **1991**, *56*, 27–46. [\[CrossRef\]](#)
19. Jagannathan, R.; Rao, K.S. Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series. In *Proceeding of the International Conference on Number Theory and Mathematical Physics*, Srinivasa Ramanujan Centre, Kumbakonam, India, 20–21 December 2005.
20. Araci, S.; Duran, U.; Acikgoz, M.; Srivastava, H.M. A certain (p, q) -derivative operator and associated divided differences. *J. Inequal. Appl.* **2016**, *301*, 2016. [\[CrossRef\]](#)
21. Tuncer, A.; Ali, A.; Syed Abdul, M. On Kantorovich modification of (p, q) -Baskakov operators. *J. Inequal. Appl.* **2016**, *2016*, 98.
22. Sadjang, P.N. On the fundamental theorem of (p, q) -calculus and some (p, q) -Taylor formulas. *Results Math.* **2018**, *73*, 39. [\[CrossRef\]](#)
23. Duran, U.; Acikgoz, M.; Aracta, S. Study on some new results arising from (p, q) -calculus. *TWMS J. Pure Appl. Math.* **2020**, *11*, 57–71.
24. Duren, P.L. *Univalent Functions, Grundlehren der Mathematischen Wissenschaften*; Band 259; Springer-Verlag: New York, NY, USA, 1983.

25. Gehring, F.W. Aspects of contemporary complex analysis. In *Proceedings of an Instructional Conference Organized by the London Mathematical Society at the University of Durham, July 1979*; Brannan, D.A., Clunie, J.G., Eds.; Academic Press: Cambridge, MA, USA, 1979.
26. Brannan, D.A.; Taha, T.S. On some classes of bi-univalent functions. *Studia Univ. Babes-Bolyai Math.* **1986**, *31*, 70–77.
27. Lewin, M. On a coefficient problem for bi-univalent functions. *Proc. Am. Math. Soc.* **1967**, *18*, 63–68. [[CrossRef](#)]
28. Tan, D.L. Coefficient estimates for bi-univalent functions. *Chin. Ann. Math. Ser. A.* **1984**, *5*, 559–568.
29. Srivastava, H.M.; Mishra, A.K.; Gochhayat, P. Certain subclasses of analytic and bi-univalent functions. *Appl. Math. Lett.* **2010**, *23*, 1188–1192. [[CrossRef](#)]
30. Deniz, E. Certain subclasses of bi-univalent functions satisfying subordinate conditions. *J. Class. Anal.* **2013**, *2*, 49–60. [[CrossRef](#)]
31. Frasin, B.A. Coefficient bounds for certain classes of bi-univalent functions. *Hacet. J. Math. stat.* **2014**, *43*, 383–389. [[CrossRef](#)]
32. Frasin, B.A.; Aouf, M.K. New subclasses of bi-univalent functions. *Appl. Math.* **2011**, *24*, 1569–1573. [[CrossRef](#)]
33. Srivastava, H.M.; Gaboury, S.; Ghanim, F. Coefficients estimate for some general subclasses of analytic and bi-univalent functions. *Afr. Mat.* **2017**, *28*, 693–706. [[CrossRef](#)]
34. Tang, H.; Deng, G.; Li, S. Coefficient estimates for new subclasses of Ma-Minda bi-univalent functions. *J. Ineq. Appl.* **2013**, *2013*, 317. [[CrossRef](#)]
35. Srivastava, H.M.; Raza, N.; AbuJarad, E.S.A.; Srivastava, G.; AbuJarad, M.H. Fekete-Szegő inequality for classes of (p, q) -Starlike and (p, q) -convex functions. *Rev. La Real Acad. Cienc. Exactas Fisicasy Nat. Ser. A Mat.* **2019**, *113*, 3563–3584. [[CrossRef](#)]
36. Altinkaya, Ş.; Yalçın, S. Certain classes of bi-univalent functions of complex order associated with quasi-subordination involving (p, q) -derivative operator. *Kragujevac J. Math.* **2020**, *44*, 639–649. [[CrossRef](#)]
37. Altinkaya, Ş.; Yalçın, S. Lucas polynomials and applications to an unified class of bi-univalent functions equipped with (p, q) -derivative operators. *TWMS J. Pure. Appl. Math.* **2020**, *11*, 100–108.
38. Cotîrlă, L.-I. New classes of analytic and bi-univalent functions. *AIMS Math.* **2021**, *6*, 10642–10651. [[CrossRef](#)]
39. Vijayalakshmi, S.P.; Sudharsan, T.V.; Bulboacă, T. Symmetric Toeplitz determinants for classes defined by post quantum operators subordinated to the limaçon function. *Stud. Univ. Babes-Bolyai Math.* **2024**, *69*, 299–316.
40. Almutairi, N.S.; Sahen, A.; Catas, A.; Darwish, H. On the Fekete Szegő problem for certain classes of (γ, δ) -starlike and (γ, δ) -convex functions related to quasi-subordinations. *Symmetry* **2024**, *16*, 1043. [[CrossRef](#)]
41. Motamednezhad, A.; Salehian, S. New subclass of bi-univalent functions by $(p; q)$ -derivative operator. *Honam Math. J.* **2019**, *41*, 381–390.
42. Kocer, E.G.; Tuncel, S. Bazilevic Fibonacci and Lucas like polynomials. *Gazi Univ. J. Sci.* **2016**, *29*, 109–113.
43. Yilmaz, N.; Aktas, I. On some new subclasses of bi-univalent functions defined by generalized bivariate Fibonacci polynomial. *Afr. Mat.* **2022**, *33*, 3–12.
44. Panwar, Y.K.; Singh, M. Generalized bivariate Fibonacci-like polynomials. *Int. J. Pure Math.* **2014**, *1*, 8–13.
45. Aktaş, I.; Hamarat, D. Generalized bivariate Fibonacci polynomial and two new subclasses of bi-univalent functions. *Asian-Eur. J. Math.* **2023**, *16*, 2350147. [[CrossRef](#)]
46. Öztürk, R.; Aktaş, I. Coefficient estimate and Fekete-Szegő problems for certain new subclasses of Bi-univalent functions defined by generalized bivariate Fibonacci polynomial. *Sahand Commun. Math. Anal.* **2024**, *21*, 35–53.
47. Frasin, B.A.; Swamy, S.R.; Amourah, A.; Salah, J.; Ranjitha, H.M. A family of bi-univalent functions defined by (p, q) -derivative operator subordinate to a generalized bivariate Fibonacci polynomials. *Eur. J. Pure Appl. Math.* **2024**, *17*, 3801–3814. [[CrossRef](#)]
48. Fekete, M.; Szegő, G. Eine Bemerkung Über Ungerade Schlichte Funktionen. *J. Lond. Math. Soc.* **1933**, *89*, 85–89. [[CrossRef](#)]

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