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First Eigenvalues of Some Operators Under the Backward Ricci Flow on Bianchi Classes

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Abstract: Let $\lambda(t)$ be the first eigenvalue of the operator $-\Delta + aR^b$ on locally three-dimensional homogeneous manifolds along the backward Ricci flow, where a, b are real constants and R is the scalar curvature. In this paper, we study the properties of $\lambda(t)$ on Bianchi classes. We begin by deriving an evolution equation for the quantity $\lambda(t)$ on three-dimensional homogeneous manifolds in the context of the backward Ricci flow. Utilizing this equation, we subsequently establish a monotonic quantity that is contingent upon $\lambda(t)$. Additionally, we present both upper and lower bounds for $\lambda(t)$ within the framework of Bianchi classes.

Keywords: Ricci flow; p-Laplacian operator; eigenvalue

MSC: 53E20; 53C21; 53C40

1. Introduction

Over the last few years, the study of the eigenvalues of geometric operators has become more important in mathematics and physics. Understanding the behavior and nature of the eigenvalues of the operators reveals useful information about the topology and geometry of the underlying Riemannian manifolds.

Suppose that (M, g) is a Riemannian manifold, $-\Delta$ is the Laplace–Beltrami operator, and R is the scalar curvature correponding to the metric g. Richard Hamilton first defined and introduced the concept of Ricci flow (see [1]). Perlman was able to prove Poincaré's conjecture by using Ricci flow. In [2], Perelman showed that the first non-zero eigenvalue of the operator $-4\Delta + R$ was non-decreasing under Ricci flow. Perelman's results were built on by many authors. Under the Ricci flow, Cao [3] extended Perelman's results by considering evolving the first eigenvalue of $-\Delta + \frac{1}{2}R$ on Riemannian manifolds with $R \geq 0$, and in [4], he investigated the monotonicity of the first eigenvalue of $-\Delta + cR$, $c \geq \frac{1}{4}$ on closed Riemannian manifolds. Li [5] removed Cao's curvature assumption for the first eigenvalue of $-\Delta + \frac{1}{2}R$ and came to a similar conclusion. In [6], Fang and Yang studied the first eigenvalues of $-\Delta_\phi + \frac{R}{2}$ under the Yamabe flow, where $-\Delta_\phi$ is the Witten–Laplacian. In [7], the first author investigated the first eigenvalue of $-\Delta_\phi + cS$ for $c > \frac{1}{4}$ under the rescaled List extended Ricci flow, where $S = R - \alpha|\nabla\phi|^2$, $\alpha > 0$ is a constant, and $\phi(t) : M \to \mathbb{R}$ is a smooth function such that the couple $(g(t), \phi(t))$ evolves under rescaled List extended Ricci flow as $\frac{\partial g}{\partial t} = -2Ric + 2\alpha\nabla \otimes \nabla\phi + \frac{2r}{n}g$ and $\frac{\partial \phi}{\partial t} = \Delta\phi$ and $r(t)$ is a smooth function only dependent on the t. Yang and Zhang [8] studied the first eigenvalue of $-\Delta + cR^a$ along the Ricci flow, where a, c are real constants. It is important to recognize that the Laplacian expressed as $\nabla^*\nabla + \frac{1}{4}R$ corresponds to the familiar Dirac Laplacian, where the equation $D = \nabla^*\nabla + \frac{1}{4}R$ defines the Dirac operator D where ∇^* is an adjoint of ∇. For further details, please refer to sources [9] (p. 352), [10], [11] (p. 55), and [12].

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Recently, the authors studied the eigenvalue of $-\Delta_\phi + cS^a$ along the rescaled List extended Ricci flow [13]. Hou, in [14], studied the first eigenvalue of $-\Delta$ under the backward Ricci flow. Then, Korouki and Razavi [15] obtained similar results for $(-\Delta - R)$ along the Ricci flow. In [16], Hou and Yang investigated the eigenvalues of $-\Delta + aR$, $a > 0$ on locally homogeneous three-dimensional manifolds along the backward Ricci flow.

In this paper, we consider the first eigenvalues of the operators $-\Delta + aR^b$ on locally three-dimensional homogeneous manifolds along the backward Ricci flow. Then, on Bianchi classes, we obtain some monotonic quantities and bounds for the first eigenvalues of the operators $-\Delta + aR^b$ along the backward Ricci flow. If $a = 0$, the findings presented in paper [14] will be derived. In the case where $a = -1$ and $b = 1$, the outcomes from article [15] will be achieved. Furthermore, when $b = 1$, the results from article [16] will also be obtained. Consequently, our findings represent a novel contribution and serve as a generalization of the results from the aforementioned studies.

2. Preliminaries

There are nine classes of three-dimensional locally homogeneous closed manifolds, which are divided into two families [17]. The first family includes of $H(3)$, $SO(3) \times \mathbb{R}^1$, and $H(2) \times \mathbb{R}^1$; the second family consists $\mathbb{R}^3$, $SL(2,\mathbb{R})$, $SU(2)$, $E(1,1)$, $E(2)$, and *Heisenberg*. The second family is called Bianchi classes. Milnor [18] obtain a basis $\{X_i\}_{i=1}^3$ on Bianchi classes such that the initial metric and the Ricci tensors can be diagonalized with respect to this basis, and this property is maintained by the Ricci flow. Suppose that $\{\theta\}_{i=1}^3$ is the frame dual to $\{X_i\}_{i=1}^3$. Hence, we consider the following metric:

$$g(t) = \mathcal{J}(t)(\theta_1)^2 + \mathcal{K}(t)(\theta_2)^2 + \mathcal{L}(t)(\theta_3)^2.$$

Consider M to be a closed n-dimensional manifold with the Riemannian metric g_0. The Ricci flow [19,20] is defined as follows:

$$\frac{d}{dt}g(t) + 2Ric(g(t)) = 0, \quad g(0) = g_0. \tag{1}$$

Here, Ric indicates the Ricci curvature tensor of $g(t)$. Let $g(t)$ be a solution of (1) and $d\mu$ be the volume form of $g(t)$. For $r = \frac{\int_M R dv}{\int_M dv}$, we assume

$$\gamma(t) = \frac{1}{1 - \frac{2}{n}\int_0^t r(\tau)d\tau}$$

and $\bar{t} = \int_0^t \gamma(\tau)d\tau$; then, $\bar{g}(\bar{t}) - \gamma(t)g(t) = 0$ solves the normalized Ricci flow

$$\frac{\partial \bar{g}_{ij}}{\partial t} = -2(\bar{R}_{ij} - \frac{r}{n}\bar{g}_{ij}). \tag{2}$$

Under the normalized Ricci flow, the volume of $(M^n, g(t))$ is a constant. Suppose that $g(t), t \in (-T_b, T_c)$ is the maximal solution of (2). We consider the backward Ricci flow equation

$$\frac{d}{dt}g(t) + 2Ric(g(t)) + \frac{2r}{n}g = 0, \quad g(0) = g_0, \tag{3}$$

which on locally homogeneous closed three-dimensional manifolds becomes

$$\frac{d}{dt}g(t) + 2Ric(g(t)) + \frac{2R}{3}g = 0, \quad g(0) = g_0. \tag{4}$$

See [21–24] for more detail of the backward Ricci flow.

Suppose that $\mathcal{I} \in [0, +\infty]$ is the maximal existence time for (4).

For real constants a and b, we say that λ is an eigenvalue of the operator $-\Delta + aR^b$ if there is a function u defined on M that satisfies the equation

$$-\Delta u + aR^b u = \lambda u. \tag{5}$$

In this scenario, u is called the eigenfunction corresponding to the eigenvalue λ. Now, we give an evolution formula for $\lambda(t)$ on three-homogeneous manifold along the backward Ricci flow.

Proposition 1. *Suppose that the backward Ricci flow (4) on a locally homogeneous three-dimensional manifold has the solution $(M, g(t))$ and $\lambda(t)$ is the first eigenvalue of $-\Delta + aR^b$. Then,*

$$\begin{aligned}\frac{d}{dt}\lambda(t) &= -\frac{2R}{3}\lambda(t) - 2\int_M R_{ij}u_{ij}u\,d\mu + 2abR^{b-1}|Ric|^2 \\ &\quad + \frac{2}{3}a(1-b)R^{b+1}.\end{aligned} \tag{6}$$

Proof. Along the backward Ricci flow, we obtain

$$\frac{\partial}{\partial t}\Delta u = \Delta u_t + 2R_{ij}u_{ij} - \frac{2}{3}R\Delta u.$$

Therefore, we find

$$\begin{aligned}\frac{d}{dt}\lambda(t) &= \frac{d}{dt}\left(\int_M \left(-\Delta u + aR^b u\right)u\,d\mu\right) \\ &= \int_M \left(-2R_{ij}u_{ij} + \frac{2}{3}R\Delta u + abR^{b-1}\frac{\partial R}{\partial t}u\right)u\,d\mu \\ &\quad + \int_M \left(-\Delta u_t + aR^b u_t\right)u\,d\mu + \int_M \left(-\Delta u + aR^b u\right)\frac{d}{dt}(u\,d\mu).\end{aligned} \tag{7}$$

Integration by parts implies that

$$\int_M \left(-\Delta u_t + aR^b u_t\right)u\,d\mu = \int_M \left(-\Delta u + aR^b u\right)u_t\,d\mu.$$

From the normalization condition $\int_M u^2 d\mu = 1$, we conclude that

$$0 = \frac{d}{dt}\left(\int_M u^2\,d\mu\right) = \int_M u\left(u_t\,d\mu + (u\,d\mu)_t\right).$$

Hence, we infer

$$\begin{aligned}&\int_M \left(-\Delta u_t + aR^b u_t\right)u\,d\mu + \int_M \left(-\Delta_\phi u + aR^b u\right)\frac{d}{dt}(u\,d\mu) \\ &= \int_M \left(-\Delta u + aR^b u\right)\left(u_t\,d\mu + (u\,d\mu)_t\right) = \lambda(t_0)\int_M u\left(u_t\,d\mu + (u\,d\mu)_t\right) = 0.\end{aligned} \tag{8}$$

Plugging (8) into (7), we have

$$\frac{d}{dt}\lambda(t) = \int_M \left(-2R_{ij}u_{ij} + \frac{2}{3}R\Delta u + abR^{b-1}\frac{\partial R}{\partial t}u\right)u\,d\mu. \tag{9}$$

Along the backward Ricci flow, we arrive at

$$\frac{\partial R}{\partial t} = \Delta R + 2|Ric|^2 - \frac{2}{3}R^2. \tag{10}$$

Applying (10) in the evolution formula (9), we deduce

$$\begin{aligned}\frac{d}{dt}\lambda(t) &= \int_M \left[-2R_{ij}u_{ij}u + \frac{2R}{3}u\Delta u\right] d\mu \\ &\quad + ab\int_M R^{b-1}\left[\Delta R + 2|Ric|^2 - \frac{2}{3}R^2\right]u^2\, d\mu. \end{aligned} \tag{11}$$

On locally homogeneous three-dimensional manifolds, the scalar curvature R is a constant; then, $\Delta R = 0$. Also, $|Ric|^2$ is a constant; hence, we obtain (6). □

3. Bounds for $\lambda(t)$ on Bianchi Classes

Now, we are going to estimate the eigenvalue in Bianchi classes.

Case 1: $\mathbb{R}^3$

On $\mathbb{R}^3$, we have $g(t) = g_0$; thus, $\lambda(t)$ becomes a constant.

Case 2: Heisenberg H^3

For a given the metric $g_0 = \mathcal{J}_0(\theta_1)^2 + \mathcal{K}_0(\theta_2)^2 + \mathcal{L}_0(\theta_3)^2$ on Heisenberg, we have the frame $\{X_i\}_{i=1}^3$ as follows:

$$[X_2, X_3] = 2X_1, \quad [X_3, X_1] = 0, \quad [X_1, X_2] = 0,$$

also under the normalization $\mathcal{J}_0\mathcal{K}_0\mathcal{L}_0 = 4$, we have

$$\begin{aligned} &R_{11} = \frac{1}{2}\mathcal{J}^3, \quad R_{22} = -\frac{1}{2}\mathcal{J}^2\mathcal{K}, \quad R_{33} = -\frac{1}{2}\mathcal{J}^2\mathcal{L}, \\ &R = -\frac{1}{2}\mathcal{J}^2, \quad |Ric|^2 = \frac{3}{4}\mathcal{J}^4. \end{aligned}$$

Therefore, the backward Ricci flow equations become

$$\begin{cases} \frac{d\mathcal{J}}{dt} = \frac{4}{3}\mathcal{J}^3, & \mathcal{J}(0) = \mathcal{J}_0, \\ \frac{d\mathcal{K}}{dt} = -\frac{2}{3}\mathcal{J}^2\mathcal{K}, & \mathcal{K}(0) = \mathcal{K}_0, \\ \frac{d\mathcal{L}}{dt} = -\frac{2}{3}\mathcal{J}^2\mathcal{L}, & \mathcal{L}(0) = \mathcal{L}_0, \end{cases}$$

and this system has a solution as follows:

$$\mathcal{J} = \mathcal{J}_0\left(1 - \frac{8}{3}\mathcal{J}_0^2 t\right)^{-\frac{1}{2}}, \quad \mathcal{K} = \mathcal{K}_0\left(1 - \frac{8}{3}\mathcal{J}_0^2 t\right)^{\frac{1}{4}}, \quad \mathcal{L} = \mathcal{L}_0\left(1 - \frac{8}{3}\mathcal{J}_0^2 t\right)^{\frac{1}{4}}.$$

Theorem 1. *Suppose that $\lambda(t)$ is the first eigenvalue of $-\Delta + aR^b$, $a \geq 0$ on $\left(M = H^3, g_0\right)$. Let $\mathcal{I}$ be the maximal existence time for (4). Then, for any $b \geq 0$ such that $(-1)^b$ is a real constant and $b \neq \frac{1}{2}$, the quantity*

$$\lambda(t)\left(1 - \frac{8}{3}\mathcal{J}_0^2 t\right)^{-\frac{1}{4}} - a(-\frac{1}{2})^b(\frac{5}{3} + \frac{5}{6}b)\frac{3\mathcal{J}_0^{2b}}{2(4b+1)}\left((1 - \frac{8}{3}\mathcal{J}_0^2 t)^{-b-\frac{1}{4}} - 1\right)$$

is non-increasing along flow (4) and

$$\lambda(t) \leq \left(1 - \frac{8}{3}\mathcal{J}_0^2 t\right)^{\frac{1}{4}}\left[\lambda(0) - a(-\frac{1}{2})^b(\frac{5}{3} + \frac{5}{6}b)\frac{3\mathcal{J}_0^{2b}}{2(4b+1)}\left((1 - \frac{8}{3}\mathcal{J}_0^2 t)^{-b-\frac{1}{4}} - 1\right)\right]. \tag{12}$$

Also, the quantity

$$\lambda(t)\left(1 - \frac{8}{3}\mathcal{J}_0^2 t\right)^{\frac{1}{2}} - a(-\frac{1}{2})^b(-\frac{1}{3} + \frac{5}{6}b)\frac{3\mathcal{J}_0^{2b}}{4(-2b+1)}\left((1 - \frac{8}{3}\mathcal{J}_0^2 t)^{-b+\frac{1}{2}} - 1\right)$$

is non-decreasing along flow (4) and

$$\lambda(t) \geq \left(1-\frac{8}{3}\mathcal{J}_0^2 t\right)^{-\frac{1}{2}}\left[\lambda(0)-a(-\frac{1}{2})^b(-\frac{1}{3}+\frac{5}{6}b)\frac{3\mathcal{J}_0^{2b}}{4(-2b+1)}\left((1-\frac{8}{3}\mathcal{J}_0^2 t)^{-b+\frac{1}{2}}-1\right)\right] \tag{13}$$

on $[0,\mathcal{I})$.

Proof. Proposition 1 implies that

$$\begin{aligned}\frac{d}{dt}\lambda(t) &\leq \frac{1}{3}\mathcal{J}^2\lambda+\mathcal{J}^2\int_M u\Delta u d\mu+\frac{3}{2}ab(-\frac{1}{2}\mathcal{J}^2)^{b-1}\mathcal{J}^4+\frac{2}{3}(-\frac{1}{2}\mathcal{J}^2)^{b+1}a(1-b)\\ &= -\frac{2}{3}\mathcal{J}^2\lambda+a(-\frac{1}{2}\mathcal{J}^2)^b\mathcal{J}^2+\frac{3}{2}ab(-\frac{1}{2}\mathcal{J}^2)^{b-1}\mathcal{J}^4+\frac{2}{3}(-\frac{1}{2}\mathcal{J}^2)^{b+1}a(1-b)\\ &= -\frac{2}{3}\mathcal{J}^2\lambda+a(-\frac{1}{2})^b\mathcal{J}^{2b+2}(\frac{5}{3}+\frac{5}{6}b).\end{aligned} \tag{14}$$

Then,

$$\frac{d}{dt}\left(\lambda(t)e^{\frac{2}{3}\int_0^t\mathcal{J}^2(s)ds}-\int_0^t a(-\frac{1}{2})^b\mathcal{J}^{2b+2}(s)(\frac{5}{3}+\frac{5}{6}b)e^{\frac{2}{3}\int_0^s\mathcal{J}^2(\tau)d\tau}ds\right)\leq 0.$$

Since $\mathcal{J}=\mathcal{J}_0(1-\frac{8}{3}\mathcal{J}_0^2 t)^{-\frac{1}{2}}$, then we have

$$\frac{2}{3}\int_0^t\mathcal{J}^2(s)ds=\ln(1-\frac{8}{3}\mathcal{J}_0^2 t)^{-\frac{1}{4}}$$

and

$$\begin{aligned}\int_0^t\mathcal{J}^{2b+2}(s)e^{\frac{2}{3}\int_0^s\mathcal{J}^2(\tau)d\tau}ds &= \mathcal{J}_0^{2b+2}\int_0^s(1-\frac{8}{3}\mathcal{J}_0^2 s)^{-b-\frac{5}{4}}ds\\ &= \frac{3\mathcal{J}_0^{2b}}{2(4b+1)}\left((1-\frac{8}{3}\mathcal{J}_0^2 t)^{-b-\frac{1}{4}}-1\right).\end{aligned}$$

Therefore, the quantity

$$\lambda(t)\left(1-\frac{8}{3}\mathcal{J}_0^2 t\right)^{-\frac{1}{4}}-a(-\frac{1}{2})^b(\frac{5}{3}+\frac{5}{6}b)\frac{3\mathcal{J}_0^{2b}}{2(4b+1)}\left((1-\frac{8}{3}\mathcal{J}_0^2 t)^{-b-\frac{1}{4}}-1\right)$$

is non-increasing along flow (4), and it follows from the integration that

$$\lambda(t)\leq\left(1-\frac{8}{3}\mathcal{J}_0^2 t\right)^{\frac{1}{4}}\left[\lambda(0)-a(-\frac{1}{2})^b(\frac{5}{3}+\frac{5}{6}b)\frac{3\mathcal{J}_0^{2b}}{2(4b+1)}\left((1-\frac{8}{3}\mathcal{J}_0^2 t)^{-b-\frac{1}{4}}-1\right)\right]. \tag{15}$$

In a similar way, we also have

$$\begin{aligned}\frac{d}{dt}\lambda(t) &\geq \frac{1}{3}\mathcal{J}^2\lambda-\mathcal{J}^2\int_M u\Delta u d\mu+\frac{3}{2}ab(-\frac{1}{2}\mathcal{J}^2)^{b-1}\mathcal{J}^4+\frac{2}{3}(-\frac{1}{2}\mathcal{J}^2)^{b+1}a(1-b)\\ &= \frac{4}{3}\mathcal{J}^2\lambda+a(-\frac{1}{2})^b\mathcal{J}^{2b+2}(-\frac{1}{3}+\frac{5}{6}b).\end{aligned} \tag{16}$$

Hence,

$$\frac{d}{dt}\left(\lambda(t)e^{-\frac{4}{3}\int_0^t\mathcal{J}^2(s)ds}-\int_0^t a(-\frac{1}{2})^b\mathcal{J}^{2b+2}(s)(-\frac{1}{3}+\frac{5}{6}b)e^{-\frac{4}{3}\int_0^s\mathcal{J}^2(\tau)d\tau}ds\right)\geq 0.$$

Therefore, the quantity

$$\lambda(t)\left(1-\frac{8}{3}\mathcal{J}_0^2 t\right)^{\frac{1}{2}}-a(-\frac{1}{2})^b(-\frac{1}{3}+\frac{5}{6}b)\frac{3\mathcal{J}_0^{2b}}{4(-2b+1)}\left((1-\frac{8}{3}\mathcal{J}_0^2 t)^{-b+\frac{1}{2}}-1\right)$$

is non-decreasing along flow (4), and from the integration, we obtain

$$\lambda(t) \geq \left(1-\frac{8}{3}\mathcal{J}_0^2 t\right)^{-\frac{1}{2}}\left[\lambda(0)-a(-\frac{1}{2})^b(-\frac{1}{3}+\frac{5}{6}b)\frac{3\mathcal{J}_0^{2b}}{4(-2b+1)}\left((1-\frac{8}{3}\mathcal{J}_0^2 t)^{-b+\frac{1}{2}}-1\right)\right]. \quad (17)$$

This completes the proof of our theorem. □

Case 3: E(2)

E(2) represents the collection of isometries within the Euclidean plane.

The manifold E(2) is an Einstein manifold. For a given metric g_0, there is a Milnor frame $\{X_i\}_{i=0}^3$ such that

$$[X_1, X_2] = 0, \quad [X_3, X_1] = 2X_2 \quad [X_2, X_3] = 2X_1.$$

In this case, under the normalization $\mathcal{J}_0\mathcal{K}_0\mathcal{L}_0 = 4$, we have

$$R_{11} = \frac{1}{2}\mathcal{J}\left(\mathcal{J}^2-\mathcal{K}^2\right), \quad R_{22} = \frac{1}{2}\mathcal{K}\left(\mathcal{K}^2-\mathcal{J}^2\right), \quad R_{33} = -\frac{1}{2}\mathcal{L}(\mathcal{J}-\mathcal{K})^2,$$
$$R = -\frac{1}{2}(\mathcal{J}-\mathcal{K})^2, \quad |Ric|^2 = \frac{1}{4}[2(\mathcal{J}^2-\mathcal{K}^2)^2+(\mathcal{J}-\mathcal{K})^4].$$

Cao and Saloff-Coste [22] proved the following:

- if $\mathcal{J}_0 = \mathcal{K}_0$, then there holds $\mathcal{J} = \mathcal{K}, \mathcal{I} = \infty$, and $g(t) = g_0$;
- If $\mathcal{J}_0 > \mathcal{K}_0$, then $\mathcal{J} > \mathcal{K}, \mathcal{I} < \infty$.

$$\mathcal{J}(t) \sim \frac{\sqrt{6}}{4}(\mathcal{I}-t)^{-\frac{1}{2}}, \quad \mathcal{K}(t) \sim \eta_1(\mathcal{I}-t)^{\frac{1}{4}}, \quad \mathcal{L}(t) \sim \eta_2(\mathcal{I}-t)^{\frac{1}{4}}$$

as $t \to \mathcal{I}$, where η_1 and η_2 are two positive constants.

Theorem 2. *Suppose that $\lambda(t)$ is a first eigenvalue of $-\Delta + aR^b$, $a \geq 0$ on a three-dimensional homogeneous Riemannian manifold $(M = E(2), g_0)$. Let $1 \leq b$ be an odd number. Then, the following is true:*

(i) If $\mathcal{J}_0 = \mathcal{K}_0$ then $\lambda(t) = \lambda(0)$;

(ii) If $\mathcal{J}_0 > \mathcal{K}_0$ and $\mathcal{L}_0 > \mathcal{K}_0$, then the quantity

$$\lambda(t)e^{\frac{1}{3}\int_0^t(\mathcal{J}^2(s)-\mathcal{K}^2(s))ds} - 3ab(\frac{1}{2})^b\int_0^t(\mathcal{J}^2(s)-\mathcal{K}^2(s))^{b+1}e^{\frac{1}{3}\int_0^s(\mathcal{J}^2(r)-\mathcal{K}^2(r))dr}ds$$

is non-increasing along flow (4) and the quantity

$$\lambda(t)e^{\frac{1}{3}\int_0^t(\mathcal{J}(s)-\mathcal{K}(s))^2ds} + \frac{1}{3}a(2+b)(\frac{1}{2})^b\int_0^t(\mathcal{J}(s)-\mathcal{K}(s))^{2b+2}e^{\frac{1}{3}\int_0^s(\mathcal{J}(r)-\mathcal{K}(r))^2dr}ds$$

is non-decreasing under (4). Also,

$$\lambda(\tau)\left(\frac{\mathcal{I}-t}{\mathcal{I}-\tau}\right)^{c_1} - \frac{c_2}{b+c_1}(\mathcal{I}-t)^{c_1}\left((\mathcal{I}-t)^{-b-c_1}-(\mathcal{I}-\tau)^{-b-c_1}\right).$$
$$\leq \lambda(t) \leq \lambda(\tau)\left(\frac{\mathcal{I}-t}{\mathcal{I}-\tau}\right)^{c_1} + \frac{c_3}{b+c_1}(\mathcal{I}-t)^{c_1}\left((\mathcal{I}-t)^{-b-c_1}-(\mathcal{I}-\tau)^{-b-c_1}\right)$$

for $t \geq \tau$ and some positive constants c_1, c_2 and c_3.

Proof. For the case $\mathcal{J}_0 = \mathcal{K}_0$, we have $g(t) = g_0$, and consequently $\lambda(t) = \lambda(0)$. For $\mathcal{J}_0 > \mathcal{K}_0$, $1 \leq b$ is an odd number, and $a > 0$; then, from [21], we obtain $\mathcal{J} > \mathcal{K}$ and form Proposition 1, so we have

$$\begin{aligned}\frac{d}{dt}\lambda(t) \leq & \frac{2}{3}(\mathcal{J}-\mathcal{K})^2\lambda(t) + (\mathcal{J}^2-\mathcal{K}^2)\int_M u\Delta u d\mu + \frac{2}{3}a(1-b)(-\frac{1}{2})^{b-1}(\mathcal{J}-\mathcal{K})^{2b+2} \\ & + \frac{1}{2}ab(-\frac{1}{2})^{b-1}(\mathcal{J}-\mathcal{K})^{2b-2}[2(\mathcal{J}^2-\mathcal{K}^2)^2 + (\mathcal{J}-\mathcal{K})^4] \\ & = \left(\frac{2}{3}(\mathcal{J}-\mathcal{K})^2 - (\mathcal{J}^2-\mathcal{K}^2)\right)\lambda(t) + a(\mathcal{J}^2-\mathcal{K}^2)(-\frac{1}{2})^b(\mathcal{J}-\mathcal{K})^{2b} \\ & + \frac{2}{3}a(1-b)(-\frac{1}{2})^{b-1}(\mathcal{J}-\mathcal{K})^{2b+2} \\ & + \frac{1}{2}ab(-\frac{1}{2})^{b-1}(\mathcal{J}-\mathcal{K})^{2b-2}[2(\mathcal{J}^2-\mathcal{K}^2)^2 + (\mathcal{J}-\mathcal{K})^4].\end{aligned}$$

Therefore, we obtain

$$\frac{d}{dt}\lambda(t) \leq -\frac{1}{3}(\mathcal{J}^2-\mathcal{K}^2)\lambda(t) + 3ab(\frac{1}{2})^b(\mathcal{J}^2-\mathcal{K}^2)^{b+1}.$$

Then,

$$\frac{d}{dt}\left(\lambda(t)e^{\frac{1}{3}\int_0^t(\mathcal{J}^2(s)-\mathcal{K}^2(s))ds} - 3ab(\frac{1}{2})^b\int_0^t(\mathcal{J}^2(s)-\mathcal{K}^2(s))^{b+1}e^{\frac{1}{3}\int_0^s(\mathcal{J}^2(r)-\mathcal{K}^2(r))dr}ds\right) \leq 0$$

and it yields the quantity

$$\lambda(t)e^{\frac{1}{3}\int_0^t(\mathcal{J}^2(s)-\mathcal{K}^2(s))ds} - 3ab(\frac{1}{2})^b\int_0^t(\mathcal{J}^2(s)-\mathcal{K}^2(s))^{b+1}e^{\frac{1}{3}\int_0^s(\mathcal{J}^2(r)-\mathcal{K}^2(r))dr}ds$$

which is non-increasing along flow (4). Also, we conclude

$$\frac{d}{dt}\left(\lambda(t)e^{c_1\int_\tau^t(\mathcal{I}-s)^{-1}ds} - c_3\int_\tau^t(\mathcal{I}-s)^{-b-1}e^{c_1\int_\tau^s(\mathcal{I}-r)^{-1}dr}ds\right) \leq 0.$$

Then,

$$\lambda(t) \leq \lambda(\tau)\left(\frac{\mathcal{I}-t}{\mathcal{I}-\tau}\right)^{c_1} + \frac{c_3}{b+c_1}(\mathcal{I}-t)^{c_1}\left((\mathcal{I}-t)^{-b-c_1} - (\mathcal{I}-\tau)^{-b-c_1}\right).$$

In a similar way, we obtain

$$\frac{d}{dt}\lambda(t) \geq -\frac{1}{3}(\mathcal{J}-\mathcal{K})^2\lambda(t) - a(\frac{1}{2})^b(\frac{2}{3}+\frac{b}{3})(\mathcal{J}-\mathcal{K})^{2b+2},$$

which implies that

$$\frac{d}{dt}\left(\lambda(t)e^{\frac{1}{3}\int_0^t(\mathcal{J}(s)-\mathcal{K}(s))^2ds} + \frac{1}{3}a(2+b)(\frac{1}{2})^b\int_0^t(\mathcal{J}(s)-\mathcal{K}(s))^{2b+2}e^{\frac{1}{3}\int_0^s(\mathcal{J}(r)-\mathcal{K}(r))^2dr}ds\right) \geq 0$$

and it yields the quantity

$$\lambda(t)e^{\frac{1}{3}\int_0^t(\mathcal{J}(s)-\mathcal{K}(s))^2ds} + \frac{1}{3}a(2+b)(\frac{1}{2})^b\int_0^t(\mathcal{J}(s)-\mathcal{K}(s))^{2b+2}e^{\frac{1}{3}\int_0^s(\mathcal{J}(r)-\mathcal{K}(r))^2dr}ds$$

which is non-decreasing along (4). We also infer

$$\frac{d}{dt}\left(\lambda(t)e^{c_1\int_\tau^t(\mathcal{I}-s)^{-1}ds} + c_2\int_\tau^t(\mathcal{I}-s)^{-b-1}e^{c_1\int_\tau^s(\mathcal{I}-r)^{-1}dr}ds\right) \geq 0.$$

Then,

$$\lambda(t) \geq \lambda(\tau)\left(\frac{\mathcal{I}-t}{\mathcal{I}-\tau}\right)^{c_1} - \frac{c_2}{b+c_1}(\mathcal{I}-t)^{c_1}\left((\mathcal{I}-t)^{-b-c_1} - (\mathcal{I}-\tau)^{-b-c_1}\right).$$

□

Case 4: E(1,1)

$E(1,1)$ represents the collection of isometries within a two-dimensional plane characterized by a flat Lorentzian metric. In this context, there is an absence of an Einstein metric. Instead, all elements within this group exhibit asymptotic behavior akin to cigar degeneracies. Given a metric g_0 on $E(1,1)$, we choose a Milnor frame $\{X_i\}_{i=0}^3$ such that

$$[X_3, X_1] = 0, \quad [X_2, X_3] = 2X_1, \quad [X_1, X_2] = -2X_3. \quad .$$

Also, under the normalization $\mathcal{J}_0\mathcal{K}_0\mathcal{L}_0 = 4$, we have

$$R_{11} = \frac{1}{2}\mathcal{J}\left(\mathcal{J}^2 - \mathcal{L}^2\right), \quad R_{22} = -\frac{1}{2}\mathcal{K}(\mathcal{J}+\mathcal{L})^2, \quad R_{33} = \frac{1}{2}\mathcal{L}\left(\mathcal{L}^2 - \mathcal{J}^2\right),$$
$$R = -\frac{1}{2}(\mathcal{J}+\mathcal{L})^2, \quad |Ric|^2 = \frac{1}{4}[(2(\mathcal{J}^2-\mathcal{L}^2)^2 + (\mathcal{J}+\mathcal{L})^2].$$

Under the condition $\mathcal{J}_0 \geq \mathcal{L}_0$, Cao and Saloff-Coste [22] proved the following:

(i) if $\mathcal{J}_0 = \mathcal{L}_0$, then we have $\mathcal{J}(t) = \mathcal{L}(t)$, $\mathcal{I} = \frac{3}{32}\mathcal{K}_0$, and

$$\mathcal{J}(t) \sim \frac{\sqrt{6}}{4}(\mathcal{I}-t)^{-\frac{1}{2}}, \quad \mathcal{K}(t) \sim \frac{32}{3}(\mathcal{I}-t), \quad t \in [0, \mathcal{I}).$$

(ii) If $\mathcal{J}_0 > \mathcal{L}_0$, then $\mathcal{J} > \mathcal{L}$, $\mathcal{I} < \infty$, and

$$\mathcal{J}(t) \sim \frac{\sqrt{6}}{4}(\mathcal{I}-t)^{-\frac{1}{2}}, \quad \mathcal{K}(t) \sim \eta_1(\mathcal{I}-t)^{\frac{1}{4}}, \quad \mathcal{L}(t) \sim \eta_2(\mathcal{I}-t)^{\frac{1}{4}}$$

as t goes to $\mathcal{I}$, where η_1 and η_2 are two positive constants and $\mathcal{I}$ is the maximal existence time.

Theorem 3. *Suppose that $\lambda(t)$ is the first eigenvalue of $-\Delta + aR^b$, $a \geq 0$ on $(M = E(1,1), g_0)$. Let $1 \leq b$ be a odd number. Then, the quantity*

$$\lambda(t)e^{\frac{2}{3}\int_0^t(\mathcal{J}+\mathcal{L})^2(s)ds} - \frac{1}{3}a(\frac{1}{2})^b(2+b)\int_0^s(\mathcal{J}+\mathcal{L})^{2b+2}(s)e^{\frac{2}{3}\int_0^t(\mathcal{J}+\mathcal{L})^2(r)dr}ds$$

is non-increasing along flow (4), and the quantity

$$\lambda(t)e^{\frac{2}{3}\int_0^t(\mathcal{J}^2(s)-\mathcal{L}^2(s))ds} - \frac{1}{3}a(\frac{1}{2})^b(2+b)\int_0^s(\mathcal{J}+\mathcal{L})^{2b+2}(s)e^{\frac{2}{3}\int_0^t(\mathcal{J}^2(r)-\mathcal{L}^2(r))dr}ds$$

is non-decreasing along flow (4). If $\mathcal{J}_0 \geq \mathcal{L}_0$, then

$$\begin{aligned}&\lambda(\tau)\left(\frac{\mathcal{I}-t}{\mathcal{I}-\tau}\right)^{c_3} + \frac{c_2}{b+c_3}(\mathcal{I}-t)^{c_3}\left((\mathcal{I}-t)^{-b-c_3} - (\mathcal{I}-\tau)^{-b-c_3}\right)\\ &\leq \lambda(t) \leq \lambda(\tau)\left(\frac{\mathcal{I}-t}{\mathcal{I}-\tau}\right)^{c_1} + \frac{c_2}{b+c_1}(\mathcal{I}-t)^{c_1}\left((\mathcal{I}-t)^{-b-c_1} - (\mathcal{I}-\tau)^{-b-c_1}\right)\end{aligned} \tag{18}$$

for $t \geq \tau$ and some positive constants c_1 and c_3.

Proof.

$$\begin{aligned}\frac{d}{dt}\lambda(t) \leq & \frac{1}{3}(\mathcal{J}+\mathcal{L})^2\lambda(t) + (\mathcal{J}+\mathcal{L})^2\int_M u\Delta u\, d\mu \\ & + \frac{1}{2}ab\left(-\frac{1}{2}(\mathcal{J}+\mathcal{L})^2\right)^{b-1}\left[(2(\mathcal{J}^2-\mathcal{L}^2)^2 + (\mathcal{J}+\mathcal{L})^2\right] \\ & + \frac{2}{3}a(1-b)\left(-\frac{1}{2}(\mathcal{J}+\mathcal{L})^2\right)^{b+1} \\ & \leq -\frac{2}{3}(\mathcal{J}+\mathcal{L})^2\lambda(t) + \frac{1}{3}a(\frac{1}{2})^b(2+b)(\mathcal{J}+\mathcal{L})^{2b+2}.\end{aligned}$$

Thus,

$$\frac{d}{dt}\left(\lambda(t)e^{\frac{2}{3}\int_0^t(\mathcal{J}+\mathcal{L})^2(s)ds} - \frac{1}{3}a(\frac{1}{2})^b(2+b)\int_0^s(\mathcal{J}+\mathcal{L})^{2b+2}(s)e^{\frac{2}{3}\int_0^t(\mathcal{J}+\mathcal{L})^2(r)dr}ds\right) \leq 0,$$

which yields the quantity

$$\lambda(t)e^{\frac{2}{3}\int_0^t(\mathcal{J}+\mathcal{L})^2(s)ds} - \frac{1}{3}a(\frac{1}{2})^b(2+b)\int_0^s(\mathcal{J}+\mathcal{L})^{2b+2}(s)e^{\frac{2}{3}\int_0^t(\mathcal{J}+\mathcal{L})^2(r)dr}ds$$

which is non-increasing along flow (4). Similarly, we have

$$\begin{aligned}\frac{d}{dt}\lambda(t) \geq & -\frac{2}{3}(\mathcal{J}^2-\mathcal{L}^2)\lambda(t) + a(\mathcal{J}^2-\mathcal{L}^2)\left(-\frac{1}{2}(\mathcal{J}+\mathcal{L})^2\right)^b + \frac{2}{3}a(1-b)\left(-\frac{1}{2}(\mathcal{J}+\mathcal{L})^2\right)^{b+1} \\ & \geq -\frac{2}{3}(\mathcal{J}^2-\mathcal{L}^2)\lambda(t)\lambda(t) - \frac{1}{3}a(\frac{1}{2})^b(2+b)(\mathcal{J}+\mathcal{L})^{2b+2}.\end{aligned}$$

Thus,

$$\frac{d}{dt}\left(\lambda(t)e^{\frac{2}{3}\int_0^t(\mathcal{J}^2(s)-\mathcal{L}^2(s))ds} - \frac{1}{3}a(\frac{1}{2})^b(2+b)\int_0^s(\mathcal{J}+\mathcal{L})^{2b+2}(s)e^{\frac{2}{3}\int_0^t(\mathcal{J}^2(r)-\mathcal{L}^2(r))dr}ds,\right) \geq 0$$

which yields the quantity

$$\lambda(t)e^{\frac{2}{3}\int_0^t(\mathcal{J}^2(s)-\mathcal{L}^2(s))ds} - \frac{1}{3}a(\frac{1}{2})^b(2+b)\int_0^s(\mathcal{J}+\mathcal{L})^{2b+2}(s)e^{\frac{2}{3}\int_0^t(\mathcal{J}^2(r)-\mathcal{L}^2(r))dr}ds$$

which is non-decreasing along flow (4). If $\mathcal{J}_0 \leq \mathcal{L}_0$, then $\frac{2}{3}(\mathcal{J}+\mathcal{L})^2 \sim c_1(\mathcal{I}-t)^{-1}$ and $\frac{2}{3}(\mathcal{J}^2-\mathcal{L}^2) \sim c_3(\mathcal{I}-t)^{-1}$ for some positive constant c_1. Hence,

$$\lambda(t) \leq \lambda(\tau)\left(\frac{\mathcal{I}-t}{\mathcal{I}-\tau}\right)^{c_1} + \frac{c_2}{b+c_1}(\mathcal{I}-t)^{c_1}\left((\mathcal{I}-t)^{-b-c_1} - (\mathcal{I}-\tau)^{-b-c_1}\right)$$

and

$$\lambda(t) \geq \lambda(\tau)\left(\frac{\mathcal{I}-t}{\mathcal{I}-\tau}\right)^{c_3} + \frac{c_2}{b+c_3}(\mathcal{I}-t)^{c_3}\left((\mathcal{I}-t)^{-b-c_3} - (\mathcal{I}-\tau)^{-b-c_3}\right)$$

which implies that (18) holds. □

Case 5: SU(2)

A Manifold $SU(2)$ is an Einstein manifold. Given a metric g_0 on $SU(2)$, we choose a Milnor frame $\{X_i\}_{i=0}^3$ such that

$$[X_1, X_2] = 2X_3, \quad [X_3, X_1] = 2X_2, \quad [X_2, X_3] = 2X_1.$$

In this case, under the normalization $\mathcal{J}_0\mathcal{K}_0\mathcal{L}_0 = 4$, we have

$$R_{11} = \frac{1}{2}\mathcal{J}[\mathcal{J}^2 - (\mathcal{K} - \mathcal{L})^2], \quad R_{22} = \frac{1}{2}\mathcal{K}[\mathcal{K}^2 - (\mathcal{J} - \mathcal{L})^2],$$
$$R_{33} = \frac{1}{2}\mathcal{L}[\mathcal{L}^2 - (\mathcal{J} - \mathcal{K})^2].$$

Then,

$$R = \frac{1}{2}\left[\mathcal{J}^2 - (\mathcal{K} - \mathcal{L})^2\right] + \frac{1}{2}\left[\mathcal{K}^2 - (\mathcal{J} - \mathcal{L})^2\right] + \frac{1}{2}\left[\mathcal{L}^2 - (\mathcal{J} - \mathcal{K})^2\right],$$

and

$$|Ric|^2 = \frac{1}{4}\Big(\left[\mathcal{J}^2 - (\mathcal{K} - \mathcal{L})^2\right]^2 + \left[\mathcal{K}^2 - (\mathcal{J} - \mathcal{L})^2\right]^2 + \left[\mathcal{L}^2 - (\mathcal{J} - \mathcal{K})^2\right]^2\Big).$$

From [22], we have:

Theorem 4. *Suppose that $g(t)$, $t \in [0, \mathcal{I})$ is the solution of (4) and $\mathcal{L}_0 \leq \mathcal{K}_0 \leq \mathcal{J}_0$. Then, the following is true:*

(i) *If $\mathcal{L}_0 = \mathcal{K}_0 = \mathcal{J}_0$, then $\mathcal{I} = \infty$ and $g(t) = g_0$, $t \in [0, \infty)$;*
(ii) *If $\mathcal{L}_0 < \mathcal{J}_0 = \mathcal{K}_0$, then $\mathcal{I} = \infty$, $\mathcal{J} = \mathcal{K} > \mathcal{L}$ and $\mathcal{J} \sim \frac{8}{3}t$, $\mathcal{L} \sim \frac{9}{16}t^{-2}$ as $t \in \infty$;*
(iii) *If $\mathcal{L}_0 \leq \mathcal{K}_0 < \mathcal{J}_0$, then $\mathcal{I} < \infty$, $\mathcal{J} > \mathcal{K} \geq \mathcal{L}$ and there are two positive constants η_1, η_2, such that*

$$\mathcal{J}(t) \sim \frac{\sqrt{6}}{4}(\mathcal{I} - t)^{-\frac{1}{2}}, \quad \mathcal{K}(t) \sim \eta_1(\mathcal{I} - t)^{\frac{1}{4}}, \quad \mathcal{L}(t) \sim \eta_2(\mathcal{I} - t)^{\frac{1}{4}}$$

as t tends to $\mathcal{I}$.

Theorem 5. *Suppose that $\lambda(t)$ is a first eigenvalue of $-\Delta + aR^b$, $a \geq 0$ on $(M = SU(2), g_0)$. Let $1 \leq b$ be an odd number. The following is then true:*

(i) *If $\mathcal{J}_0 = \mathcal{K}_0 = \mathcal{L}_0$, then $\lambda(t) = \lambda(0)$.*
(ii) *If $\mathcal{J}_0 = \mathcal{K}_0 > \mathcal{L}_0$, then the quantity*

$$\lambda(t) - \frac{9}{4}ab2^b \int_0^t \mathcal{J}^{2b+2}(s)ds$$

is non-increasing along (4) and the quantity

$$\lambda(t)e^{-\frac{2}{3}\int_0^t \mathcal{J}^2(s)ds} + \frac{2}{3}a(-2 + 4b)2^b \int_0^t \mathcal{J}^{2b+2}(s)e^{-\frac{2}{3}\int_0^s \mathcal{J}^2(r)dr}ds$$

is non-decreasing under (4). Also,

$$\lambda(t) \leq \lambda(\tau) + \frac{6ab2^b}{2b+3}\left(t^{2b+3} - \tau^{2b+3}\right)$$

for $\tau \leq t$, so τ is a fixed time.
(iii) *If $\mathcal{J}_0 > \mathcal{K}_0 \geq \mathcal{L}_0$, then the quantity*

$$\lambda(t)e^{\frac{2}{3}\int_0^t \mathcal{J}^2(s)ds} - a2^b(1 + \frac{3}{4}b)\int_0^t \mathcal{J}^{2b+2}(s)e^{\frac{2}{3}\int_0^s \mathcal{J}^2(r)dr}ds$$

is non-increasing along (4) and the quantity

$$\lambda(t)e^{-\frac{2}{3}\int_0^t \mathcal{J}^2(s)ds} - \frac{2}{3}a(1 - b)(\frac{9}{2})^{b+1}\int_0^t \mathcal{J}^{2b+2}(s)e^{-\frac{2}{3}\int_0^s \mathcal{J}^2(r)dr}ds$$

is non-decreasing under (4). Also,

$$\lambda(\tau)\left(\frac{\mathcal{I}-t}{\mathcal{I}-\tau}\right)^{c_1}+\frac{c_2}{b+c_1}(\mathcal{I}-t)^{c_1}\left((\mathcal{I}-t)^{-b-c_1}-(\mathcal{I}-\tau)^{-b-c_1}\right)$$
$$\leq \lambda(t)\leq \lambda(\tau)\left(\frac{\mathcal{I}-t}{\mathcal{I}-\tau}\right)^{c_1}+\frac{c_3}{b+c_1}(\mathcal{I}-t)^{c_1}\left((\mathcal{I}-t)^{-b-c_1}-(\mathcal{I}-\tau)^{-b-c_1}\right).$$

Proof. (i) If $\mathcal{J}_0=\mathcal{K}_0=\mathcal{L}_0$, then we obtain $g(t)=g_0,\ t\in[0,\infty)$ and it follows that $\lambda(t)=\lambda(0)$.

(ii) If $\mathcal{J}_0=\mathcal{K}_0>\mathcal{L}_0$, then we have $\mathcal{J}=\mathcal{K}>\mathcal{L}$,

$$\begin{aligned}
-2R_{11}&=-2R_{22}=\mathcal{J}\left[\mathcal{L}^2-2\mathcal{J}\mathcal{L}\right]\leq-\mathcal{J}\mathcal{L}^2,\\
-R_{33}&=-\mathcal{L}^3,\\
-\frac{2}{3}R&=\frac{1}{3}(\mathcal{L}^2-4\mathcal{J}\mathcal{L})\leq-\mathcal{L}^2,\\
2abR^{b-1}|Ric|^2&=\frac{ab}{2}\left(-\frac{1}{2}\mathcal{L}^2+2\mathcal{J}\mathcal{L}\right)^{b-1}\left[2(-\mathcal{L}^2+2\mathcal{J}\mathcal{L})^2+\mathcal{L}^4\right]\\
&\leq\frac{9}{4}ab2^b\mathcal{J}^{2b+2},
\end{aligned}$$

and

$$\frac{2}{3}a(1-b)R^{b+1}=\frac{2}{3}a(1-b)\left(-\frac{1}{2}\mathcal{L}^2+2\mathcal{J}\mathcal{L}\right)^{b+1}\leq 0.$$

Therefore, based on Proposition 1, we can write

$$\begin{aligned}
\frac{d}{dt}\lambda(t)&\leq-\mathcal{L}^2\lambda(t)-\mathcal{L}^2\int_M u\Delta u d\mu+\frac{9}{4}ab2^b\mathcal{J}^{2b+2}\\
&=-a\mathcal{L}^2\left(-\frac{1}{2}\mathcal{L}^2+2\mathcal{J}\mathcal{L}\right)^{b}+\frac{9}{4}ab2^b\mathcal{J}^{2b+2}\\
&\leq\frac{9}{4}ab2^b\mathcal{J}^{2b+2}
\end{aligned}$$

which implies that the quantity

$$\lambda(t)-\frac{9}{4}ab2^b\int_0^t\mathcal{J}^{2b+2}(s)ds$$

is non-increasing along (4). Also, we infer

$$\lambda(t)\leq\lambda(\tau)+\frac{6ab2^b}{2b+3}\left(t^{2b+3}-\tau^{2b+3}\right)$$

for $t\geq\tau$. In a similar way for $b\geq1$, we conclude

$$\begin{aligned}
&-2R_{11}=-2R_{22}=\mathcal{J}\left[\mathcal{L}^2-2\mathcal{J}\mathcal{L}\right]\geq-2\mathcal{J}^3,\\
&-R_{33}=-\mathcal{L}^3\geq-2\mathcal{J}^2\mathcal{L},\\
&-\frac{2}{3}R=\frac{1}{3}(\mathcal{L}^2-4\mathcal{J}\mathcal{L})\geq-\frac{4}{3}\mathcal{J}^2,\\
&2abR^{b-1}|Ric|^2\geq0,\\
&\frac{2}{3}a(1-b)R^{b+1}\geq\frac{2}{3}a(1-b)\left(2\mathcal{J}^2\right)^{b+1}.
\end{aligned}$$

It follows from the above inequalities and Proposition 1 that

$$\begin{aligned}\frac{d}{dt}\lambda(t) &\geq -\frac{4}{3}\mathcal{J}^2\lambda(t) - 2\mathcal{J}^2\int_M u\Delta u d\mu + \frac{4}{3}a(1-b)2^b\mathcal{J}^{2b+2}\\ &= \frac{2}{3}\mathcal{J}^2\lambda(t) - 2a\mathcal{J}^2(-\frac{1}{2}\mathcal{L}^2 + 2\mathcal{J}\mathcal{L})^b + \frac{4}{3}a(1-b)2^b\mathcal{J}^{2b+2}\\ &\geq \frac{2}{3}\mathcal{J}^2\lambda(t) - 2a\mathcal{J}^2(2\mathcal{J}^2)^b + \frac{4}{3}a(1-b)2^b\mathcal{J}^{2b+2}\\ &= \frac{2}{3}\mathcal{J}^2\lambda(t) - \frac{1}{3}a2^b(-2+4b)\mathcal{J}^{2b+2}.\end{aligned}$$

It is easy to check that the quantity

$$\lambda(t)e^{-\frac{2}{3}\int_0^t \mathcal{J}^2(s)ds} + \frac{2}{3}a(-2+4b)2^b\int_0^t \mathcal{J}^{2b+2}(s)e^{-\frac{2}{3}\int_0^s \mathcal{J}^2(r)dr}ds$$

is non-decreasing along flow (4).

(iii) If $\mathcal{J}_0 > \mathcal{K}_0 \geq \mathcal{L}_0$ then $\mathcal{J} > \mathcal{K} \geq \mathcal{L}$,

$$\begin{aligned}&-2R_{11} \leq \mathcal{J}^3, \quad -2R_{22} \leq \mathcal{J}^2\mathcal{K}, \quad -2R_{33} \leq \mathcal{J}^2\mathcal{L},\\ &-\frac{2}{3}R = \frac{1}{3}\Big(\mathcal{J}^2 + \mathcal{K}^2 + \mathcal{L}^2 - 2\mathcal{J}\mathcal{K} - 2\mathcal{A}\mathcal{L} - 2\mathcal{K}\mathcal{L}\Big) \leq \frac{1}{3}\mathcal{J}^2,\\ &2abR^{b-1}|Ric|^2 = ab(\frac{1}{2})^b\Big(-\mathcal{J}^2 - \mathcal{K}^2 - \mathcal{L}^2 + 2\mathcal{J}\mathcal{K} + 2\mathcal{J}\mathcal{L} + 2\mathcal{K}\mathcal{L}\Big)\\ &\qquad\qquad \times\Big(\big[\mathcal{J}^2 - (\mathcal{K}-\mathcal{L})^2\big]^2 + \big[\mathcal{K}^2 - (\mathcal{J}-\mathcal{L})^2\big]^2 + \big[\mathcal{L}^2 - (\mathcal{J}-\mathcal{K})^2\big]^2\Big)\\ &\qquad\qquad \leq ab(\frac{1}{2})^b(4\mathcal{J}^2)^{b-1}(3\mathcal{J}^4) = \frac{3}{4}ab2^b\mathcal{J}^{2b+2},\\ &\frac{2}{3}a(1-b)R^{b+1} \leq 0.\end{aligned}$$

According to Proposition 1 and the above inequalities, we find

$$\begin{aligned}\frac{d}{dt}\lambda(t) &\leq \frac{1}{3}\mathcal{J}^2\lambda(t) + \mathcal{J}^2\int_M u\Delta u d\mu + \frac{3}{4}ab2^b\mathcal{J}^{2b+2}\\ &= -\frac{2}{3}\mathcal{J}^2\lambda(t) + a2^b\mathcal{J}^{2b+2} + \frac{3}{4}ab2^b\mathcal{J}^{2b+2}\\ &= -\frac{2}{3}\mathcal{J}^2\lambda(t) + a2^b(1+\frac{3}{4}b)\mathcal{J}^{2b+2},\end{aligned}$$

which leads to the quantity

$$\lambda(t)e^{\frac{2}{3}\int_0^t \mathcal{J}^2(s)ds} - a2^b(1+\frac{3}{4}b)\int_0^t \mathcal{J}^{2b+2}(s)e^{\frac{2}{3}\int_0^s \mathcal{J}^2(r)dr}ds$$

which is non-increasing along flow (4) and

$$\lambda(t) \leq \lambda(\tau)\Big(\frac{\mathcal{I}-t}{\mathcal{I}-\tau}\Big)^{c_1} + \frac{c_3}{b+c_1}(\mathcal{I}-t)^{c_1}\Big((\mathcal{I}-t)^{-b-c_1} - (\mathcal{I}-\tau)^{-b-c_1}\Big).$$

Similarly, we have

$$
\begin{aligned}
&-2R_{11} = -\mathcal{J}\left[\mathcal{J}^2 - (\mathcal{K}-\mathcal{L})^2\right] \geq \mathcal{J}\left[-\mathcal{J}^2 + \mathcal{L}^2 - \mathcal{K}^2\right] \geq -2\mathcal{J}^3,\\
&-2R_{22} \leq -2\mathcal{J}^2\mathcal{K}, \quad -2R_{33} \leq -2\mathcal{J}^2\mathcal{L}, \quad -\frac{2}{3}R \geq -\frac{4}{3}\mathcal{J}^2,\\
&2abR^{b-1}|Ric|^2 \geq 0,\\
&\frac{2}{3}a(1-b)R^{b+1} \geq \frac{2}{3}a(1-b)(\frac{9}{2})^{b+1}\mathcal{J}^{2b+2},
\end{aligned}
$$

then

$$
\begin{aligned}
\frac{d}{dt}\lambda(t) &\geq -\frac{4}{3}\mathcal{J}^2\lambda(t) - 2\mathcal{J}^2\int_M u\Delta u d\mu + \frac{2}{3}a(1-b)(\frac{9}{2})^{b+1}\mathcal{J}^{2b+2}\\
&\geq \frac{2}{3}\mathcal{J}^2\lambda(t) + \frac{2}{3}a(1-b)(\frac{9}{2})^{b+1}\mathcal{J}^{2b+2}.
\end{aligned}
$$

We deduce that

$$
\lambda(t)e^{-\frac{2}{3}\int_0^t \mathcal{J}^2(s)ds} - \frac{2}{3}a(1-b)(\frac{9}{2})^{b+1}\int_0^t \mathcal{J}^{2b+2}(s)e^{-\frac{2}{3}\int_0^s \mathcal{J}^2(r)dr}ds
$$

is non-decreasing under (4) and

$$
\lambda(t) \geq \lambda(\tau)\left(\frac{\mathcal{I}-t}{\mathcal{I}-\tau}\right)^{c_1} + \frac{c_2}{b+c_1}(\mathcal{I}-t)^{c_1}\left((\mathcal{I}-t)^{-b-c_1} - (\mathcal{I}-\tau)^{-b-c_1}\right).
$$

This completes the proof of our theorem. □

Case 6: $SL(2,\mathbb{R})$

Using the Milnor frame $\{X_i\}_{i=0}^3$ on $SL(2,\mathbb{R})$, we obtain

$$
[X_1, X_2] = 2X_3, \quad [X_3, X_1] = 2X_2, \quad [X_2, X_3] = -2X_1,
$$

and in this case, under the normalization $\mathcal{J}_0\mathcal{K}_0\mathcal{L}_0 = 4$, we have

$$
\begin{aligned}
&R_{11} = \frac{1}{2}\mathcal{J}[\mathcal{J}^2 - (\mathcal{K}-\mathcal{L})^2], \quad R_{22} = \frac{1}{2}\mathcal{K}[\mathcal{K}^2 - (\mathcal{J}+\mathcal{L})^2]\\
&R_{33} = \frac{1}{2}\mathcal{L}[\mathcal{L}^2 - (\mathcal{J}+\mathcal{K})^2].
\end{aligned}
$$

Then,

$$
R = \frac{1}{2}\left[\mathcal{J}^2 - (\mathcal{K}-\mathcal{L})^2\right] + \frac{1}{2}\left[\mathcal{K}^2 - (\mathcal{J}+\mathcal{L})^2\right] + \frac{1}{2}\left[\mathcal{L}^2 - (\mathcal{J}+\mathcal{K})^2\right],
$$

and

$$
|Ric|^2 = \frac{1}{4}\Big(\left[\mathcal{J}^2 - (\mathcal{K}-\mathcal{L})^2\right]^2 + \left[\mathcal{K}^2 - (\mathcal{J}+\mathcal{L})^2\right]^2 + \left[\mathcal{L}^2 - (\mathcal{J}+\mathcal{K})^2\right]^2\Big).
$$

Under the condition $\mathcal{K}_0 \geq \mathcal{L}_0$, from [22], we have

(i) if $\mathcal{J}_0 \geq \mathcal{K}_0 \geq \mathcal{L}_0$, then $\mathcal{J} \geq \mathcal{K} \geq \mathcal{L}$ and

$$
\mathcal{J}(t) \sim \frac{\sqrt{6}}{4}(\mathcal{I}-t)^{-\frac{1}{2}}, \quad \mathcal{K}(t) \sim \eta_1(\mathcal{I}-t)^{\frac{1}{4}}, \quad \mathcal{L}(t) \sim \eta_2(\mathcal{I}-t)^{\frac{1}{4}}
$$

as t goes to $\mathcal{I}$, where η_1 and η_2 are two positive constants.

(ii) If $\mathcal{J}_0 \leq \mathcal{K}_0 - \mathcal{L}_0$, then $\mathcal{J} \leq \mathcal{K} - \mathcal{L}$ and

$$\mathcal{J}(t) \sim \eta_1(\mathcal{I}-t)^{\frac{1}{4}}, \quad \mathcal{K}(t) \sim \frac{\sqrt{6}}{4}(\mathcal{I}-t)^{-\frac{1}{2}}, \quad \mathcal{L}(t) \sim \eta_2(\mathcal{I}-t)^{\frac{1}{4}}$$

as t goes to $\mathcal{I}$, where η_1 and η_2 are two positive constants.

(iii) If $\mathcal{J} < \mathcal{K} < \mathcal{J} + \mathcal{L}$ for all time $t \in [0, \mathcal{I})$, then we obtain

$$\mathcal{J}(t) \sim \frac{\sqrt{6}}{4}(\mathcal{I}-t)^{-\frac{1}{2}}, \quad \mathcal{K}(t) \sim \frac{\sqrt{6}}{4}(\mathcal{I}-t)^{-\frac{1}{2}}, \quad \mathcal{L}(t) \sim \eta_2(\mathcal{I}-t)$$

as t goes to $\mathcal{I}$, where η_1 is a positive constant.

Theorem 6. *Suppose that $\lambda(t)$ is the first eigenvalue of $-\Delta + aR^b$, $a \geq 0$ on $(M = SL(2,\mathbb{R}), g_0)$. Let $1 \geq b$ be an even number.*

If $\mathcal{J}_0 \geq \mathcal{K}_0 \geq \mathcal{L}_0$, then the quantity

$$\lambda(t)e^{\frac{11}{3}\int_0^t \mathcal{J}^2(s)ds} - \frac{1}{3}a(\frac{1}{2})^b(11+b)\int_0^t \mathcal{J}^{2b+2}(s)e^{\frac{11}{3}\int_0^t \mathcal{J}^2(r)dr}ds$$

is non-increasing along flow (4) and the quantity

$$\lambda(t)e^{-\int_0^t \mathcal{J}^2(s)ds} - a(\frac{1}{2})^b(1+9b)\int_0^t \mathcal{J}^{2b+2}(s)e^{-\int_0^t \mathcal{J}^2(r)dr}ds$$

is non-decreasing along (4). Also,

$$\begin{aligned}
&\lambda(\tau)\left(\frac{\mathcal{I}-\tau}{\mathcal{I}-t}\right)^{\frac{3}{8}} + a(\frac{1}{2})^b(1+9b)\frac{1}{b+\frac{3}{8}}(\mathcal{I}-t)^{-\frac{3}{8}}\left[(\mathcal{I}-t)^{-b+\frac{3}{8}} - (\mathcal{I}-\tau)^{-b+\frac{3}{8}}\right] \\
&\leq \lambda(t) \leq \lambda(\tau)\left(\frac{\mathcal{I}-t}{\mathcal{I}-\tau}\right)^{c_1} + \frac{c_2}{b+c_1}(\mathcal{I}-t)^{c_1}\Big((\mathcal{I}-t)^{-b-c_1} - (\mathcal{I}-\tau)^{-b-c_1}\Big),
\end{aligned}$$

for $t \geq \tau$ and some positive constants c_1 and c_2, so τ is a fixed time.

Proof. If $\mathcal{J}_0 \geq \mathcal{K}_0 \geq \mathcal{L}_0$, then $\mathcal{J} \geq \mathcal{K} \geq \mathcal{L}$, $R_{11} > 0, R_{22} < 0, R_{33} < 0$,

$$\begin{aligned}
&-2R_{11} < 0, \\
&-2R_{22} = -\mathcal{K}\left[\mathcal{K}^2 - (\mathcal{J}+\mathcal{L})^2\right] \leq 4\mathcal{K}\mathcal{J}^2, \\
&-2R_{33} = -\mathcal{L}\left[\mathcal{L}^2 - (\mathcal{J}+\mathcal{K})^2\right] \leq 4\mathcal{L}\mathcal{J}^2, \\
&-\frac{2}{3}R = \frac{1}{3}(\mathcal{J}-\mathcal{K}-\mathcal{L})^2 \leq \frac{1}{3}\mathcal{J}^2, \\
&2abR^{b-1}|Ric|^2 = -2ab(\frac{1}{2})^{b-1}(\mathcal{J}-\mathcal{K}-\mathcal{L})^{2b-2}|Ric|^2 \leq 0,
\end{aligned}$$

and

$$\frac{2}{3}a(1-b)R^{b+1} = \frac{2}{3}a(b-1)(\frac{1}{2})^{b+1}(\mathcal{J}-\mathcal{K}-\mathcal{L})^{2b+2} \leq \frac{2}{3}a(b-1)(\frac{1}{2})^{b+1}\mathcal{J}^{2b+2}.$$

Hence, Proposition 1 yields

$$\begin{aligned}\frac{d}{dt}\lambda(t) &\leq \frac{1}{3}\mathcal{J}^2\lambda(t) + 4\mathcal{J}^2\int_M u\Delta u d\mu + \frac{2}{3}a(b-1)(\frac{1}{2})^{b+1}\mathcal{J}^{2b+2}\\ &= -\frac{11}{3}\mathcal{J}^2\lambda(t) + 4a\mathcal{J}^2(\frac{1}{2})^b(\mathcal{J}-\mathcal{K}-\mathcal{L})^b + \frac{2}{3}a(b-1)(\frac{1}{2})^{b+1}\mathcal{J}^{2b+2}\\ &\leq -\frac{11}{3}\mathcal{J}^2\lambda(t) + \frac{1}{3}a(\frac{1}{2})^b(11+b)\mathcal{J}^{2b+2}.\end{aligned}$$

Then, the quantity

$$\lambda(t)e^{\frac{11}{3}\int_0^t \mathcal{J}^2(s)ds} - \frac{1}{3}a(\frac{1}{2})^b(11+b)\int_0^t \mathcal{J}^{2b+2}(s)e^{\frac{11}{3}\int_0^t \mathcal{J}^2(r)dr}ds$$

is non-increasing along flow (4) and

$$\lambda(t) \leq \lambda(\tau)\left(\frac{\mathcal{I}-t}{\mathcal{I}-\tau}\right)^{c_1} + \frac{c_2}{b+c_1}(\mathcal{I}-t)^{c_1}\Big((\mathcal{I}-t)^{-b-c_1} - (\mathcal{I}-\tau)^{-b-c_1}\Big).$$

Also,

$$\begin{aligned}&-2R_{11} = \mathcal{J}\Big[-\mathcal{J}^2 + (\mathcal{K}-\mathcal{L})^2\Big] \geq -\mathcal{J}^3,\\ &-2R_{22} \geq -\mathcal{K}\mathcal{J}^2,\\ &-2R_{33} \geq -\mathcal{L}\mathcal{J}^2,\\ &-\frac{2}{3}R = \frac{1}{3}(\mathcal{J}-\mathcal{K}-\mathcal{L})^2 \geq 0,\\ &2abR^{b-1}|Ric|^2 = -2ab(\frac{1}{2})^{b-1}(\mathcal{J}-\mathcal{K}-\mathcal{L})^{2b-2}|Ric|^2 \geq -\frac{9ab}{2}(\frac{1}{2})^{b-1}\mathcal{J}^{2b+2},\\ &\frac{2}{3}a(1-b)R^{b+1} \geq 0.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\frac{d}{dt}\lambda(t) &\geq -\mathcal{J}^2\int_M u\Delta u d\mu - \frac{9ab}{2}(\frac{1}{2})^{b-1}\mathcal{J}^{2b+2}\\ &\geq \mathcal{J}^2\lambda(t) - a(\frac{1}{2})^b(1+9b)\mathcal{J}^{2b+2}.\end{aligned}$$

This yields the quantity

$$\lambda(t)e^{-\int_0^t \mathcal{J}^2(s)ds} - a(\frac{1}{2})^b(1+9b)\int_0^t \mathcal{J}^{2b+2}(s)e^{-\int_0^t \mathcal{J}^2(r)dr}ds$$

is non-decreasing under (4) and

$$\lambda(t) \geq \lambda(\tau)\left(\frac{\mathcal{I}-\tau}{\mathcal{I}-t}\right)^{\frac{3}{8}} + a(\frac{1}{2})^b(1+9b)\frac{1}{b+\frac{3}{8}}(\mathcal{I}-t)^{-\frac{3}{8}}\Big[(\mathcal{I}-t)^{-b+\frac{3}{8}} - (\mathcal{I}-\tau)^{-b+\frac{3}{8}}\Big].$$

□

4. Conclusions

In this paper, we study the first eigenvalues $\lambda(t)$ of the geometric perators $-\Delta + aR^b$ on locally three-dimensional homogeneous manifolds along the backward Ricci flow, where a, b are constants. First, we obtain an evolution formula for $\lambda(t)$ on three-homogeneous manifold along the backward Ricci flow. Then, on Bianchi classes, we obtain some monotonic quantities and bounds for $\lambda(t)$, which is constant on $\mathbb{R}^3$.

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