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Embedding the Different Families of Fuzzy Sets into Banach Spaces by Using Cauchy Sequences

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Abstract: The family $\mathcal{F}(U)$ of all fuzzy sets in a normed space cannot be a vector space. This deficiency affects its application to practical problems. The topics of functional analysis and nonlinear analysis in mathematics are based on vector spaces. Since these two topics have been well developed such that their tools can be used to solve practical economics and engineering problems, lacking a vector structure for the family $\mathcal{F}(U)$ diminishes its applications to these kinds of practical problems when fuzzy uncertainty has been detected in a real environment. Embedding the whole family $\mathcal{F}(U)$ into a Banach space is still not possible. However, it is possible to embed some interesting and important subfamilies of $\mathcal{F}(U)$ into some suitable Banach spaces. This paper presents the concrete and detailed structures of these kinds of Banach spaces such that their mathematical structures can penetrate the core of practical economics and engineering problems in fuzzy environments. The important issue of uniqueness for these Banach spaces is also addressed via the concept of isometry.

Keywords: Banach space; Cauchy sequence; embedding theorem; Hausdorff metric; normed space

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1. Introduction

Let $\tilde{A}$ be a fuzzy set in a universal set U with a membership function denoted by $\xi_{\tilde{A}}$. For $\alpha \in (0,1]$, the α-level set of $\tilde{A}$ is denoted and defined by

$$\tilde{A}_\alpha = \{x \in U : \xi_{\tilde{A}}(x) \geq \alpha\}. \tag{1}$$

The support of a fuzzy set $\tilde{A}$ within a universal set U is the crisp set defined by

$$\tilde{A}_{0+} = \{x \in U : \xi_{\tilde{A}}(x) > 0\}.$$

The definition of a 0-level set is an important issue in fuzzy set theory. When the universal set U is endowed with a topology τ, the 0-level set $\tilde{A}_0$ can be defined as the closure of the support of $\tilde{A}$, i.e.,

$$\tilde{A}_0 = \mathrm{cl}(\tilde{A}_{0+}).$$

When U is not endowed with a topological structure, the intuitive way to define the 0-level set is to follow equality (1) for $\alpha = 0$. In this case, the 0-level set of $\tilde{A}$ is the whole universal set U. This kind of 0-level set does not seem that useful. Therefore, we always endow the universal set U with a topological structure such that the 0-level set can be seriously considered.

Let $(U, \|\cdot\|)$ be a normed space. We denote by $\mathcal{F}(U)$ the family of all fuzzy sets in U. Given any $\tilde{A}, \tilde{B} \in \mathcal{F}(U)$, the addition $\tilde{A} \oplus \tilde{B}$ of fuzzy sets is well known. Any given real number $\lambda \in \mathbb{R}$ can also be treated as a fuzzy number $\tilde{1}_{\{\lambda\}}$, which is sometimes called a crisp number with value λ. More precisely, the membership function of $\tilde{1}_{\{\lambda\}}$ only takes

a value of membership degree 1 at λ and 0 for all real numbers except λ. In this case, the scalar multiplication $\lambda\tilde{A}$ is defined by the well-known multiplication

$$\lambda\tilde{A} = \tilde{1}_{\{\lambda\}} \otimes \tilde{A},$$

where the membership function of $\lambda\tilde{A}$ is given by

$$\begin{aligned}\xi_{\lambda\tilde{A}}(z) = \xi_{\tilde{1}_{\{\lambda\}}\otimes\tilde{A}}(z) &= \sup_{\{(x,y):z=x*y\}} \min\left\{\xi_{\tilde{A}}(x), \tilde{1}_{\{\lambda\}}(y)\right\} \\ &= \sup_{\{x:\lambda x=z\}} \xi_{\tilde{A}}(x) = \begin{cases} \xi_{\tilde{A}}(z/\lambda) & \text{if } \lambda \neq 0 \\ \sup\limits_{x\in U} \xi_{\tilde{A}}(x) & \text{if } \lambda = 0 = z. \end{cases}\end{aligned}$$

For $\lambda = 0$ and $z \neq 0$, we define $\xi_{\lambda\tilde{A}}(z) = 0$. In other words, we have

$$\xi_{\lambda\tilde{A}}(z) = \begin{cases} \xi_{\tilde{A}}(z/\lambda) & \text{if } \lambda \neq 0 \\ \sup\limits_{x\in U} \xi_{\tilde{A}}(x) & \text{if } \lambda = 0 = z \\ 0 & \text{if } \lambda = 0 \text{ and } z \neq 0. \end{cases}$$

When $\tilde{A}$ is a normal fuzzy set, we have

$$\xi_{\lambda\tilde{A}}(z) = \begin{cases} \xi_{\tilde{A}}(z/\lambda) & \text{if } \lambda \neq 0 \\ 0 & \text{if } \lambda = 0 \text{ and } z \neq 0 \\ 1 & \text{if } \lambda = 0 = z = 0 \end{cases} = \begin{cases} \xi_{\tilde{A}}(z/\lambda) & \text{if } \lambda \neq 0 \\ \chi_{\{0\}}(z) & \text{if } \lambda = 0. \end{cases}$$

Under the addition and scalar multiplication described above, the space $\mathcal{F}(U)$ still cannot be a vector space. The topic of functional analysis is based on the concept of vector spaces. Since the space $\mathcal{F}(U)$ is not a vector space, we are not able to develop its properties in functional analysis. Also, the whole family cannot be embedded into a normed space. In order to remedy this inconvenience, we may try to embed some specific subfamilies of $\mathcal{F}(U)$ into a normed space or Banach space.

We denote by $\mathcal{P}(U)$ the family of all nonempty subsets of a normed space $(U, \|\cdot\|)$. Given any $A, B \in \mathcal{P}(U)$, the Hausdorff metric of A and B is defined by

$$d_H(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B} \| a-b \|, \sup_{b\in B}\inf_{a\in A} \| a-b \|\right\},$$

where $d_H(A,B)$ is an extended real number. By convention, we define

$$d_H(\emptyset,\emptyset) = 0 \text{ and } d_H(A,\emptyset) = \infty \text{ for } A \neq \emptyset.$$

We say that $\tilde{A}$ is a fuzzy interval in $\mathbb{R}$ when its α-level sets $\tilde{A}_\alpha$ are bounded closed intervals for all $\alpha \in [0,1]$, which is also denoted by

$$\tilde{A}_\alpha = \left[\tilde{A}_\alpha^L, \tilde{A}_\alpha^U\right] \text{ for all } \alpha \in [0,1]. \tag{2}$$

Given any two fuzzy intervals $\tilde{A}$ and $\tilde{B}$ in $\mathbb{R}$, it is clear that

$$d_H(\tilde{A}_\alpha, \tilde{B}_\alpha) = \max\left\{\left|\tilde{A}_\alpha^L - \tilde{B}_\alpha^L\right|, \left|\tilde{A}_\alpha^U - \tilde{B}_\alpha^U\right|\right\}$$

for all $\alpha \in [0,1]$.

Definition 1. *Let $\tilde{A}$ be a fuzzy set in a normed space $(U, \|\cdot\|)$. We consider the following function:*

$$\eta_{\tilde{A}} : [0,1] \to (\mathcal{P}(U), d_H) \text{ defined by } \alpha \mapsto \tilde{A}_\alpha,$$

where $\tilde{A}_\alpha$ denotes the α-level set of $\tilde{A}$ for $\alpha \in [0,1]$.

- *We say that $\eta_{\tilde{A}}$ is continuous with respect to the Hausdorff metric d_H when, given any $\epsilon > 0$, there exists $\delta > 0$ such that $|\alpha - \beta| < \delta$ implies $d_H(\tilde{A}_\alpha, \tilde{A}_\beta) < \epsilon$.*
- *We say that the function $\eta_{\tilde{A}}$ satisfies the uniform Lipschitz condition with respect to the Hausdorff metric d_H when there exists a constant $K > 0$ satisfying*

$$d_H(\tilde{A}_\alpha, \tilde{A}_\beta) \leq K|\alpha - \beta| \text{ for every } \alpha, \beta \in [0,1].$$

Different subfamilies of $\mathcal{F}(U)$, which depend on the α-level sets of fuzzy sets, will be considered in this paper. We adopt the following notations:

- We denote by $\mathcal{F}_l(U)$ the family of all fuzzy sets in U with nonempty closed α-level sets for $\alpha \in [0,1]$.
- We denote by $\mathcal{F}_k(U)$ the family of all fuzzy sets in U with nonempty compact α-level sets for $\alpha \in [0,1]$.
- We denote by $\mathcal{F}_{lc}(U)$ the family of all fuzzy sets in U with nonempty closed and convex α-level sets for $\alpha \in [0,1]$.
- We denote by $\mathcal{F}_{kc}(U)$ the family of all fuzzy sets in U with nonempty compact and convex α-level sets for $\alpha \in [0,1]$.
- We denote by $\mathcal{F}_k^C(U)$ the set of elements $\tilde{A}$ in $\mathcal{F}_k(U)$ such that the function $\eta_{\tilde{A}}$ is continuous.
- We denote by $\mathcal{F}_{kc}^C(U)$ the set of elements $\tilde{A}$ in $\mathcal{F}_{kc}(U)$ such that the function $\eta_{\tilde{A}}$ is continuous.
- We denote by $\mathcal{F}_{kc}^L(U)$ the set of elements $\tilde{A}$ in $\mathcal{F}_{kc}(U)$ such that the function $\eta_{\tilde{A}}$ satisfies the uniform Lipschitz condition.

Example 1. *Let $U = \mathbb{R}$. It is clear that each fuzzy interval $\tilde{A}$ is in $\mathcal{F}_{kc}(\mathbb{R})$. By referring to (2), we can define*

$$l(\alpha) = \tilde{A}_\alpha^L \text{ and } u(\alpha) = \tilde{A}_\alpha^U \text{ for } \alpha \in [0,1].$$

If the real-valued functions l and u are continuous on $[0,1]$, then $\tilde{A} \in \mathcal{F}_{kc}^C(\mathbb{R})$.

Since each compact set in a normed space is also a closed set, we have the following inclusions:

$$\mathcal{F}_k(U) \subseteq \mathcal{F}_l(U) \text{ and } \mathcal{F}_{kc}(U) \subseteq \mathcal{F}_{lc}(U).$$

When the function $\eta_{\tilde{A}}$ satisfies the uniform Lipschitz condition with respect to d_H, it is also continuous with respect to d_H. Therefore, we have the inclusion $\mathcal{F}_{kc}^L(U) \subseteq \mathcal{F}_{kc}^C(U)$.

In order to define the differentials of fuzzy functions, Puri and Ralescu [1] developed an embedding theorem to embed the space $\mathcal{F}_{kc}(U)$ into a normed space by following the approach of Rådström [2]. Also, Kaleva [3,4] and Román-Flores and Rojas-Medar [5] established different kinds of embedding theorems to study fuzzy differential equations. On the other hand, the embedding theorem can also be applied to study fuzzy random variables by referring to Puri and Ralescu [6,7] and Klement et al. [8].

Since the detailed structure of the normed space presented by Puri and Ralescu [1] is not clear, it cannot be efficiently used to study some interesting topics. By considering the support functions of the fuzzy set, Puri and Ralescu [6] presented another embedding theorem to embed the space $\mathcal{F}_{kc}^L(U)$ into a Banach space. In this case, the concrete Banach space can be realized through the support functions of fuzzy sets. Diamond and Kloeden [9,10], Ma [11], Wu and Ma [12–14] and Wu [15] also presented embedding theorems to embed the space $\mathcal{F}_{kc}^C(U)$ into different kinds of Banach spaces, which are all based on the support functions of fuzzy sets. Although these concrete Banach spaces can be realized in this approach via the role of support functions of fuzzy sets, many important properties of these kinds of Banach spaces are still not clear. Instead of using the support functions of fuzzy sets, this paper considers the concept of Cauchy sequences. In this case, a different Banach space can be established such that many important properties can be obtained.

Since the support functions of fuzzy sets are defined using the concept of the supremum, solving the supremum is equivalent to solving an optimization problem, which is not an easy task. Therefore, one of the advantages of this paper is its use of the concept of the Cauchy sequence, which is more useful than using the support functions of fuzzy sets.

This paper considers a general subfamily $\mathcal{F}$ of $\mathcal{F}(U)$ by imposing some conditions on $\mathcal{F}$ to establish embedding theorems. This means that any subfamilies like the ones described above can be embedded into suitable Banach spaces when the desired conditions are satisfied. A lot of important structural properties of this Banach space, like the denseness, completeness, separability and uniqueness in the sense of isometry, will be provided. In the literature, when the desired spaces are embedded into the corresponding Banach spaces, the issue of uniqueness for these Banach spaces is not addressed. This important uniqueness issue is studied in this paper.

In Section 2, we present many kinds of metric spaces of fuzzy sets, including the properties of completeness and separability. In Section 3, given a general subfamily $\mathcal{F}$ of $\mathcal{F}(U)$, an embedding theorem is established to embed the family $\mathcal{F}$ into a normed space in which a concrete normed space with a detailed structure can be realized. In Section 4, by introducing Cauchy sequences, the general subfamily $\mathcal{F}$ of $\mathcal{F}(U)$ can be embedded into a Banach space in which a concrete Banach space with a detailed structure can be realized. In particular, uniqueness in the sense of isometry will be presented. In Section 5, by using the completeness and separability obtained in Section 3, many important specific families of fuzzy sets can be embedded into a normed space or Banach space.

2. Metric Spaces of Fuzzy Sets

Let $(U, \|\cdot\|)$ be a normed space. Given any subsets A and B of U, we have

$$d_H(\lambda A, \lambda B) = |\lambda| \cdot d_H(A, B) \text{ for any } \lambda \in \mathbb{R} \tag{3}$$

and

$$d_H(A, B) = 0 \text{ if and only if } \mathrm{cl}(A) = \mathrm{cl}(B), \tag{4}$$

where $\mathrm{cl}(A)$ denotes the closure of A. Also, given any subsets A_1, A_2, B_1, B_2 of U, we have

$$d_H(A_1 + A_2, B_1 + B_2) \leq d_H(A_1, B_1) + d_H(A_2, B_2). \tag{5}$$

The above properties will be used to study the completeness and separability of different metric spaces of fuzzy sets.

Let $\mathcal{F}$ be a subfamily of $\mathcal{F}(U)$. In order to embed $\mathcal{F}$ into a normed space or Banach space, we need to guarantee that the space $\mathcal{F}$ is closed under addition and scalar multiplication. In other words, given any $\tilde{A}, \tilde{B} \in \mathcal{F}$ and any $\lambda \in \mathbb{R}$, we need to ensure $\tilde{A} \oplus \tilde{B} \in \mathcal{F}$ and $\lambda\tilde{A} \in \mathcal{F}$. The following proposition provides some of these subfamilies.

Proposition 1. *We have the following properties:*

(i) *Let U be a Hausdorff topological vector space. Then, the family $\mathcal{F}_{kc}(U)$ is closed under addition and scalar multiplication.*

(ii) *Let $(U, \|\cdot\|)$ be a normed space. Then, the families $\mathcal{F}_{kc}^{L}(U)$ and $\mathcal{F}_{kc}^{C}(U)$ are closed under addition and scalar multiplication.*

Moreover, we have

$$\left(\tilde{A} \oplus \tilde{B}\right)_\alpha = \tilde{A}_\alpha + \tilde{B}_\alpha \text{ and } \left(\lambda\tilde{A}\right)_\alpha = \lambda\tilde{A}_\alpha$$

for any $\lambda \in \mathbb{R}$ and for all $\alpha \in [0, 1]$.

Proof. To prove part (i), given any $\tilde{A}, \tilde{B} \in \mathcal{F}_{kc}(U)$, from Wu [16,17], we can obtain

$$\left(\tilde{A} \oplus \tilde{B}\right)_\alpha = \tilde{A}_\alpha + \tilde{B}_\alpha \text{ and } \left(\lambda\tilde{A}\right)_\alpha = \lambda\tilde{A}_\alpha$$

for any $\lambda \in \mathbb{R}$ and for all $\alpha \in [0,1]$. Since $\tilde{A}_\alpha + \tilde{B}_\alpha$ and $\lambda\tilde{A}_\alpha$ are compact and convex sets for all $\alpha \in [0,1]$, the family $\mathcal{F}_{kc}(U)$ is closed under addition and scalar multiplication.

To prove part (ii), given any $\tilde{A}, \tilde{B} \in \mathcal{F}_{kc}^{L}(U)$, it follows that the functions $\eta_{\tilde{A}}$ and $\eta_{\tilde{B}}$ are continuous. We want to claim that the functions $\eta_{\tilde{A}\oplus\tilde{B}}$ and $\eta_{\lambda\tilde{A}}$ are continuous for any $\lambda \in \mathbb{R}$. Now, we have

$$\begin{aligned} d_H\big((\tilde{A}\oplus\tilde{B})_\alpha, (\tilde{A}\oplus\tilde{B})_\beta\big) &= d_H\big(\tilde{A}_\alpha + \tilde{B}_\alpha, \tilde{A}_\beta + \tilde{B}_\beta\big) \\ &\le d_H\big(\tilde{A}_\alpha, \tilde{A}_\beta\big) + d_H\big(\tilde{B}_\alpha, \tilde{B}_\beta\big) \text{ (by (3))} \end{aligned}$$

and

$$\begin{aligned} d_H\big((\lambda\tilde{A})_\alpha, (\lambda\tilde{A})_\beta\big) &= d_H\big(\lambda\tilde{A}_\alpha, \lambda\tilde{A}_\beta\big) \\ &= |\lambda| d_H\big(\tilde{A}_\alpha, \tilde{A}_\beta\big) \text{ (by (5))}. \end{aligned}$$

The continuities of $\eta_{\tilde{A}}$ and $\eta_{\tilde{B}}$ mean that the functions $\eta_{\tilde{A}\oplus\tilde{B}}$ and $\eta_{\lambda\tilde{A}}$ are indeed continuous. This shows that the family $\mathcal{F}_{kc}^{C}(U)$ is closed under addition and scalar multiplication.

Given any $\tilde{A}, \tilde{B} \in \mathcal{F}_{kc}^{L}(U)$, it follows that the functions $\eta_{\tilde{A}}$ and $\eta_{\tilde{B}}$ satisfy the uniform Lipschitz condition. Therefore, there exist constants $M_1, M_2 > 0$ satisfying

$$d_H(\tilde{A}_\alpha, \tilde{A}_\beta) \le M_1 \cdot |\alpha - \beta| \text{ and } d_H(\tilde{B}_\alpha, \tilde{B}_\beta) \le M_2 \cdot |\alpha - \beta|.$$

Now, we have

$$d_H\big((\tilde{A}\oplus\tilde{B})_\alpha, (\tilde{A}\oplus\tilde{B})_\beta\big) \le d_H\big(\tilde{A}_\alpha, \tilde{A}_\beta\big) + d_H\big(\tilde{B}_\alpha, \tilde{B}_\beta\big) \le (M_1 + M_2) \cdot |\alpha - \beta|$$

and

$$d_H\big((\lambda\tilde{A})_\alpha, (\lambda\tilde{A})_\beta\big) = |\lambda| d_H\big(\tilde{A}_\alpha, \tilde{A}_\beta\big) \le |\lambda| \cdot M_1 \cdot |\alpha - \beta|.$$

which means that the functions $\eta_{\tilde{A}\oplus\tilde{B}}$ and $\eta_{\lambda\tilde{A}}$ also satisfy the uniform Lipschitz condition. This shows that the family $\mathcal{F}_{kc}^{L}(U)$ is closed under addition and scalar multiplication. □

Example 2. *Proposition 1 says that the family of all fuzzy intervals in $\mathbb{R}$ is closed under addition and scalar multiplication satisfying*

$$\big(\tilde{A}\oplus\tilde{B}\big)_\alpha = \tilde{A}_\alpha + \tilde{B}_\alpha = \Big[\tilde{A}_\alpha^L + \tilde{B}_\alpha^L, \tilde{A}_\alpha^U + \tilde{B}_\alpha^U\Big] \tag{6}$$

and

$$\big(\lambda\tilde{A}\big)_\alpha = \lambda\tilde{A}_\alpha = \lambda\Big[\tilde{A}_\alpha^L, \tilde{A}_\alpha^U\Big] = \begin{cases} \big[\lambda\tilde{A}_\alpha^L, \lambda\tilde{A}_\alpha^U\big] & \text{if } \lambda \ge 0 \\ \big[\lambda\tilde{A}_\alpha^U, \lambda\tilde{A}_\alpha^L\big] & \text{if } \lambda < 0 \end{cases}$$

for any $\lambda \in \mathbb{R}$ and for all $\alpha \in [0,1]$.

Let $(U, \|\cdot\|)$ be a normed space. Puri and Ralescu [1] considered the function $\hat{d}_{\mathcal{F}}$ defined on $\mathcal{F}(U) \times \mathcal{F}(U)$ by

$$\hat{d}_{\mathcal{F}}\big(\tilde{A}, \tilde{B}\big) = \sup_{\alpha\in(0,1]} d_H\big(\tilde{A}_\alpha, \tilde{B}_\alpha\big).$$

In this paper, we define

$$d_{\mathcal{F}}\big(\tilde{A}, \tilde{B}\big) = \sup_{\alpha\in[0,1]} d_H\big(\tilde{A}_\alpha, \tilde{B}_\alpha\big), \tag{7}$$

where $\alpha = 0$ is considered.

Example 3. *Given any two fuzzy intervals $\tilde{A}, \tilde{B} \in \mathcal{F}_{kc}(\mathbb{R})$, we have*

$$\hat{d}_{\mathcal{F}}\big(\tilde{A}, \tilde{B}\big) = \sup_{\alpha\in(0,1]} \max\Big\{\Big|\tilde{A}_\alpha^L - \tilde{B}_\alpha^L\Big|, \Big|\tilde{A}_\alpha^U - \tilde{B}_\alpha^U\Big|\Big\}$$

and

$$d_{\mathcal{F}}(\tilde{A},\tilde{B}) = \sup_{\alpha\in[0,1]} \max\left\{\left|\tilde{A}_\alpha^L - \tilde{B}_\alpha^L\right|, \left|\tilde{A}_\alpha^U - \tilde{B}_\alpha^U\right|\right\} \tag{8}$$

Moreover, given any two fuzzy intervals $\tilde{A},\tilde{B} \in \mathcal{F}_{kc}^C(\mathbb{R})$*, we can obtain*

$$d_{\mathcal{F}}(\tilde{A},\tilde{B}) = \hat{d}_{\mathcal{F}}(\tilde{A},\tilde{B}).$$

More precisely, we also have

$$d_{\mathcal{F}}(\tilde{A},\tilde{B}) = \max_{\alpha\in[0,1]} \max\left\{\left|\tilde{A}_\alpha^L - \tilde{B}_\alpha^L\right|, \left|\tilde{A}_\alpha^U - \tilde{B}_\alpha^U\right|\right\},$$

where the supremum in (8) is attained by considering the continuous functions $l(\alpha) = \tilde{A}_\alpha^L$ *and* $u(\alpha) = \tilde{A}_\alpha^L$ *on* $[0,1]$*.*

Remark 1. *With the consideration of* $d_{\mathcal{F}}$*, given any* $\tilde{A},\tilde{B},\tilde{C} \in \mathcal{F}_{kc}(U)$*, we can obtain*

$$d_{\mathcal{F}}(\lambda\tilde{A},\lambda\tilde{B}) = |\lambda| \cdot d_{\mathcal{F}}(\tilde{A},\tilde{B}) \text{ for } \lambda \in \mathbb{R}, \tag{9}$$

and

$$d_{\mathcal{F}}(\tilde{A}\oplus\tilde{C},\tilde{B}\oplus\tilde{C}) = d_{\mathcal{F}}(\tilde{A},\tilde{B}). \tag{10}$$

which also means that the metric $d_{\mathcal{F}}$ *is homogeneous and invariant under translation.*

Remark 2. *Let* $\{\tilde{A}^{(n)}\}_{n=1}^{\infty}$ *be a sequence in* $\mathcal{F}(U)$ *satisfying* $\tilde{A}^{(n)} \xrightarrow{d_{\mathcal{F}}} \tilde{A}$ *as* $n \to \infty$*. Given any* $\epsilon > 0$*, there exists an integer* n_0 *satisfying* $d_{\mathcal{F}}(\tilde{A}^{(n)},\tilde{A}) < \epsilon$ *for all* $n \geq n_0$*, i.e.,* $d_H(\tilde{A}_\alpha^{(n)},\tilde{A}_\alpha) < \epsilon$ *for all* $\alpha \in [0,1]$ *and* $n \geq n_0$*. In other words, we have* $\tilde{A}_\alpha^{(n)} \xrightarrow{d_H} \tilde{A}_\alpha$ *as* $n \to \infty$ *for each* $\alpha \in [0,1]$*, which is also a uniform convergence with respect to* α *on* $[0,1]$*.*

Proposition 2. *Let* $(U,\|\cdot\|)$ *be a normed space, and let* $\mathcal{F}$ *be a subset of* $\mathcal{F}_l(U)$*. Then,* $(\mathcal{F},d_{\mathcal{F}})$ *is a metric space.*

Proof. Given any $\tilde{A},\tilde{B} \in \mathcal{F}$, it is clear that

$$d_{\mathcal{F}}(\tilde{A},\tilde{B}) = 0 \text{ implies } d_H(\tilde{A}_\alpha,\tilde{B}_\alpha) = 0 \text{ for each } \alpha \in [0,1].$$

Therefore, we obtain $\mathrm{cl}(\tilde{A}_\alpha) = \mathrm{cl}(\tilde{B}_\alpha)$ for each $\alpha \in [0,1]$ by (4). Since $\tilde{A}_\alpha$ and $\tilde{B}_\alpha$ are closed for each $\alpha \in [0,1]$, it follows that $\tilde{A}_\alpha = \tilde{B}_\alpha$ for each $\alpha \in [0,1]$. Therefore, we obtain $\tilde{A} = \tilde{B}$. It is clear that $\tilde{A} = \tilde{B}$ implies $d_{\mathcal{F}}(\tilde{A},\tilde{B}) = 0$.

The symmetry $d_{\mathcal{F}}(\tilde{A},\tilde{B}) = d_{\mathcal{F}}(\tilde{B},\tilde{A})$ follows from the symmetry for d_H. To check the triangle inequality, we have

$$\begin{aligned} d_{\mathcal{F}}(\tilde{A},\tilde{B}) &= \sup_{\alpha\in[0,1]} d_H(\tilde{A}_\alpha,\tilde{B}_\alpha) \\ &\leq \sup_{\alpha\in[0,1]} \left[d_H(\tilde{A}_\alpha,\tilde{C}_\alpha) + d_H(\tilde{C}_\alpha,\tilde{B}_\alpha)\right] \text{ (using the triangle inequality of } d_H) \\ &\leq \sup_{\alpha\in[0,1]} d_H(\tilde{A}_\alpha,\tilde{C}_\alpha) + \sup_{\alpha\in[0,1]} d_H(\tilde{C}_\alpha,\tilde{B}_\alpha) \\ &= d_{\mathcal{F}}(\tilde{A},\tilde{C}) + d_{\mathcal{F}}(\tilde{C},\tilde{B}). \end{aligned}$$

This completes the proof. □

Remark 3. *Let* $(U,\|\cdot\|)$ *be a Banach space. Puri and Ralescu [1,7] have shown that the metric space* $(\mathcal{F}_l(U),\hat{d}_{\mathcal{F}})$ *is complete. We can similarly show that the metric space* $(\mathcal{F}_l(U),d_{\mathcal{F}})$ *is also complete. Since each closed subspace of a complete metric space is also a complete metric*

space according to Royden [18], it follows that each closed subspace of $(\mathcal{F}_l(U), d_{\mathcal{F}})$ is a complete metric space.

Remark 4. *Let $(U, \|\cdot\|)$ be a separable normed space. Román-Flores [19] has shown that the space $(\mathcal{F}_k^C(U), d_{\mathcal{F}})$ is separable. Since every subspace of a separable metric space is separable according to Royden [18], it follows that every subspace of $(\mathcal{F}_k^C(U), d_{\mathcal{F}})$ is separable.*

Proposition 3. *Let $(U, \|\cdot\|)$ be a Banach space. Then, we have the following properties:*

(i) *The spaces $(\mathcal{F}_k(U), d_{\mathcal{F}})$ and $(\mathcal{F}_{lc}(U), d_{\mathcal{F}})$ are closed subsets of $(\mathcal{F}_l(U), d_{\mathcal{F}})$, and the space $(\mathcal{F}_{kc}(U), d_{\mathcal{F}})$ is a closed subset of $(\mathcal{F}_{lc}(U), d_{\mathcal{F}})$.*

(ii) *The spaces $(\mathcal{F}_k(U), d_{\mathcal{F}})$, $(\mathcal{F}_{lc}(U), d_{\mathcal{F}})$ and $(\mathcal{F}_{kc}(U), d_{\mathcal{F}})$ are complete metric spaces.*

Proof. Given any $\tilde{A} \in \mathrm{cl}(\mathcal{F}_{lc}(U))$, there exists a sequence $\{\tilde{A}^{(n)}\}_{n=1}^{\infty}$ in $\mathcal{F}_{lc}(U)$ satisfying $\tilde{A}^{(n)} \xrightarrow{d_{\mathcal{F}}} \tilde{A}$ as $n \to \infty$. Using Remark 2, we have $\tilde{A}_\alpha^{(n)} \xrightarrow{d_H} \tilde{A}_\alpha$ as $n \to \infty$ for each $\alpha \in [0,1]$. Since each α-level set $\tilde{A}_\alpha^{(n)}$ is a closed and convex subset of U, the routine argument can show that $\tilde{A}_\alpha$ is also a closed and convex subset of U for each $\alpha \in [0,1]$, i.e., $\tilde{A} \in \mathcal{F}_{lc}(U)$. This shows that $\mathcal{F}_{lc}(U)$ is a closed subset of $\mathcal{F}_l(U)$. Since $(\mathcal{F}_l(U), d_{\mathcal{F}})$ is a complete metric space according to Remark 3, it follows that the closed subspace $(\mathcal{F}_{lc}(U), d_{\mathcal{F}})$ is also a complete metric space.

Since each compact set in the normed space U is a closed set, we have the inclusion $\mathcal{F}_{kc}(U) \subseteq \mathcal{F}_{lc}(U)$. Given any $\tilde{A} \in \mathrm{cl}(\mathcal{F}_{kc}(U))$, there exists a sequence $\{\tilde{A}^{(n)}\}_{n=1}^{\infty}$ in $\mathcal{F}_{kc}(U)$ satisfying $\tilde{A}^{(n)} \xrightarrow{d_{\mathcal{F}}} \tilde{A}$ as $n \to \infty$. Using Remark 2, we have $\tilde{A}_\alpha^{(n)} \xrightarrow{d_H} \tilde{A}_\alpha$ as $n \to \infty$ for each $\alpha \in [0,1]$. Since each $\tilde{A}_\alpha^{(n)}$ is a compact and convex subset of U, the routine argument can show that $\tilde{A}_\alpha$ is also a compact and convex subset of U for each $\alpha \in [0,1]$, i.e., $\tilde{A} \in \mathcal{F}_{kc}(U)$. This shows that $\mathcal{F}_{kc}(U)$ is a closed subset of $\mathcal{F}_{lc}(U)$. Since $(\mathcal{F}_{lc}(U), d_{\mathcal{F}})$ is a complete metric space, it follows that the closed subspace $(\mathcal{F}_{kc}(U), d_{\mathcal{F}})$ is also a complete metric space.

We can similarly show that the space $(\mathcal{F}_k(U), d_{\mathcal{F}})$ is a closed subset of $(\mathcal{F}_l(U), d_{\mathcal{F}})$ such that $(\mathcal{F}_k(U), d_{\mathcal{F}})$ is a complete metric space. This completes the proof. □

Proposition 4. *Let $(U, \|\cdot\|)$ be Banach space. We have the following properties:*

(i) *The space $\mathcal{F}_k^C(U)$ is a closed subset of $(\mathcal{F}_k(U), d_{\mathcal{F}})$, and the space $\mathcal{F}_{kc}^C(U)$ is a closed subset of $(\mathcal{F}_{kc}(U), d_{\mathcal{F}})$.*

(ii) *The spaces $(\mathcal{F}_k^C(U), d_{\mathcal{F}})$ and $(\mathcal{F}_{kc}^C(U), d_{\mathcal{F}})$ are complete metric spaces.*

Proof. Given any $\tilde{A} \in \mathrm{cl}(\mathcal{F}_k^C(U))$, there exists a sequence $\{\tilde{A}^{(n)}\}_{n=1}^{\infty}$ in $\mathcal{F}_k^C(U)$ satisfying $\tilde{A}^{(n)} \xrightarrow{d_{\mathcal{F}}} \tilde{A}$. Since $\tilde{A}^{(n)} \in \mathcal{F}_k(U)$ for all n, the routine argument can show $\tilde{A} \in \mathcal{F}_k(U)$. Now, given any $\epsilon > 0$, there exists an integer n_0 satisfying $d_{\mathcal{F}}(\tilde{A}^{(n)}, \tilde{A}) < \epsilon/3$ for all $n \geq n_0$. This also means $d_H(\tilde{A}_\alpha^{(n)}, \tilde{A}_\alpha) < \epsilon/3$ for all $n \geq n_0$ and all $\alpha \in [0,1]$. Since $\tilde{A}^{(n_0)} \in \mathcal{F}_k^C(U)$, there exists $\delta > 0$ such that

$$|\alpha - \beta| < \delta \text{ implies } d_H(\tilde{A}_\alpha^{(n_0)}, \tilde{A}_\beta^{(n_0)}) < \frac{\epsilon}{3}.$$

Therefore, we obtain

$$d_H(\tilde{A}_\alpha, \tilde{A}_\beta) \leq d_H\left(\tilde{A}_\alpha, \tilde{A}_\alpha^{(n_0)}\right) + d_H\left(\tilde{A}_\alpha^{(n_0)}, \tilde{A}_\beta^{(n_0)}\right) + d_H\left(\tilde{A}_\beta^{(n_0)}, \tilde{A}_\beta\right) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

which means that the mapping $\alpha \mapsto \tilde{A}_\alpha$ is continuous, i.e., $\tilde{A} \in \mathcal{F}_k^C(U)$. This shows that $\mathcal{F}_k^C(U)$ is a closed subset of $(\mathcal{F}_k(U), d_{\mathcal{F}})$. Since $(\mathcal{F}_k(U), d_{\mathcal{F}})$ is complete by Proposition 3, $(\mathcal{F}_k^C(U), d_{\mathcal{F}})$ is also complete.

Given any $\tilde{A} \in \mathrm{cl}(\mathcal{F}_{kc}^C(U))$, there exists a sequence $\{\tilde{A}^{(n)}\}_{n=1}^{\infty}$ in $\mathcal{F}_{kc}^C(U)$ satisfying $\tilde{A}^{(n)} \xrightarrow{d_{\mathcal{F}}} \tilde{A}$. The above proof has shown $\tilde{A} \in \mathcal{F}_k^C(U)$. From the proof of Proposition 3, we

can also realize $\tilde{A} \in \mathcal{F}_c(U)$. Therefore, we obtain $\tilde{A} \in \mathcal{F}_{kc}^C(U)$, which shows that $\mathcal{F}_{kc}^C(U)$ is a closed subset of $(\mathcal{F}_{kc}(U), d_{\mathcal{F}})$. The completeness of $(\mathcal{F}_{kc}(U), d_{\mathcal{F}})$ by Proposition 3 shows that the space $(\mathcal{F}_{kc}^C(U), d_{\mathcal{F}})$ is a complete metric space. This completes the proof. □

Example 4. *Since the space $\mathbb{R}$ is complete, it follows that the spaces $(\mathcal{F}_{kc}(\mathbb{R}), d_{\mathcal{F}})$ and $(\mathcal{F}_{kc}^C(\mathbb{R}), d_{\mathcal{F}})$ are complete. In other words, the space of all fuzzy intervals in $\mathbb{R}$ endowed with the metric $d_{\mathcal{F}}$ is complete. The space of all fuzzy intervals in $\mathbb{R}$ is endowed with the metric $d_{\mathcal{F}}$ such that, for each fuzzy interval $\tilde{A}$, the real-valued functions*

$$l(\alpha) = \tilde{A}_\alpha^L \text{ and } u(\alpha) = \tilde{A}_\alpha^U$$

are continuous on $[0,1]$ is also complete.

3. Embedding the Family of Fuzzy Sets into a Normed Space

Let $(U, \|\cdot\|)$ be a normed space, and let $\mathcal{F}$ be a subset of $\mathcal{F}(U)$. When $\mathcal{F}$ is not closed under addition and scalar multiplication, we cannot embed it into a normed space. Proposition 1 shows many types of families that are closed under addition and scalar multiplication. However, even though those families are closed under addition and scalar multiplication, they still cannot be vector spaces since the additive inverse element cannot be well defined. In other words, we cannot define some norms to make them normed spaces. However, they can be embedded into a normed space isometrically and isomorphically. Puri and Ralescu [1] have shown that the space $(\mathcal{F}_{kc}(U), \hat{d}_{\mathcal{F}})$ can be embedded into a normed space isomorphically and isometrically, where $(U, \|\cdot\|)$ is assumed to be a reflexive Banach space. In this paper, we are going to consider the embedding theorem of the space $(\mathcal{F}_{kc}(U), d_{\mathcal{F}})$. As a matter of fact, more general and interesting results will be given below.

Proposition 5 (Rådström [2])**.** *Let A, B, C be subsets of a vector space U. Then, we have the following properties:*

(i) $(A + B) + C = A + (B + C)$;
(ii) $A + B = B + A$;
(iii) $\lambda(A + B) = \lambda A + \lambda B$ *for* $\lambda \in \mathbb{R}$;
(iv) $\lambda_1(\lambda_2 A) = (\lambda_1\lambda_2)A$ *for* $\lambda_1, \lambda_2 \in \mathbb{R}$;
(v) *If A is a convex subset of X and $\lambda_1 \cdot \lambda_2 \geq 0$, then we have*

$$(\lambda_1 + \lambda_2)A = \lambda_1 A + \lambda_2 A.$$

Proposition 6. *Given any $\tilde{A}, \tilde{B} \in \mathcal{F}_{kc}(U)$, we have the following properties:*

(i) $\lambda(\tilde{A} \oplus \tilde{B}) = \lambda\tilde{A} \oplus \lambda\tilde{B}$ *for* $\lambda \geq 0$;
(ii) $(\lambda_1 + \lambda_2)\tilde{A} = \lambda_1\tilde{A} \oplus \lambda_2\tilde{A}$ *for* $\lambda_1, \lambda_2 \geq 0$;
(iii) $\lambda_1(\lambda_2\tilde{A}) = (\lambda_1\lambda_2)\tilde{A}$ *for* $\lambda_1, \lambda_2 \geq 0$.

Proof. Given any $\tilde{A}, \tilde{B} \in \mathcal{F}_{kc}(U)$ and $\lambda \geq 0$, Proposition 1 says that

$$\left(\tilde{A} \oplus \tilde{B}\right)_\alpha = \tilde{A}_\alpha + \tilde{B}_\alpha \text{ and } \left(\lambda\tilde{A}\right)_\alpha = \lambda\tilde{A}_\alpha$$

for all $\alpha \in [0,1]$. Using Proposition 5, for any $\alpha \in [0,1]$, we have

$$\begin{aligned}\left(\lambda(\tilde{A} \oplus \tilde{B})\right)_\alpha &= \lambda\left(\tilde{A} \oplus \tilde{B}\right)_\alpha = \lambda\left(\tilde{A}_\alpha + \tilde{B}_\alpha\right) = \lambda\tilde{A}_\alpha + \lambda\tilde{B}_\alpha \\ &= \left(\lambda\tilde{A}\right)_\alpha + \left(\lambda\tilde{B}\right)_\alpha = \left(\lambda\tilde{A} \oplus \lambda\tilde{B}\right)_\alpha,\end{aligned}$$

which shows part (i). Parts (ii) and (iii) can be similarly obtained using Proposition 5. This completes the proof. □

The spaces given in Proposition 1 are closed under addition and scalar multiplication. To establish embedding theorems, it suffices to consider nonnegative scalar multiplication. Let $\mathcal{F}$ be a subset of $\mathcal{F}(U)$ such that $\mathcal{F}$ is closed under addition and nonnegative scalar multiplication, which means $\lambda\tilde{A} \in \mathcal{F}$ for $\lambda \geq 0$. We are going to define the equivalence relation on the product set $\mathcal{F} \times \mathcal{F}$. Given two pairs $(\tilde{A}, \tilde{B})$ and $(\tilde{C}, \tilde{D})$ of elements of $\mathcal{F} \times \mathcal{F}$, the equivalence relation is defined by

$$(\tilde{A}, \tilde{B}) \sim (\tilde{C}, \tilde{D}) \text{ if and only if } \tilde{A} \oplus \tilde{D} = \tilde{B} \oplus \tilde{C}.$$

The equivalence class containing the pair $(\tilde{A}, \tilde{B})$ is denoted by $\langle \tilde{A}, \tilde{B} \rangle$. It is clear that

$$(\tilde{C}, \tilde{D}) \in \langle \tilde{A}, \tilde{B} \rangle \text{ implies } \langle \tilde{A}, \tilde{B} \rangle = \langle \tilde{C}, \tilde{D} \rangle.$$

Example 5. *Let $\tilde{A}, \tilde{B}, \tilde{C}$ and $\tilde{D}$ be fuzzy intervals in $\mathbb{R}$. According to (6), it is clear that $(\tilde{A}, \tilde{B}) \sim (\tilde{C}, \tilde{D})$ if and only if*

$$\left[\tilde{A}_\alpha^L + \tilde{D}_\alpha^L, \tilde{A}_\alpha^U + \tilde{D}_\alpha^U\right] = \left[\tilde{B}_\alpha^L + \tilde{C}_\alpha^L, \tilde{B}_\alpha^U + \tilde{C}_\alpha^U\right] \text{ for all } \alpha \in [0,1]. \tag{11}$$

The equivalence class $\langle \tilde{A}, \tilde{B} \rangle$ is the family of all pairs $(\tilde{C}, \tilde{D})$ such that equality (11) is satisfied for all $\alpha \in [0,1]$.

We denote by $\mathcal{N}$ the family of all those equivalence classes. The addition in $\mathcal{N}$ is defined by

$$\langle \tilde{A}, \tilde{B} \rangle + \langle \tilde{C}, \tilde{D} \rangle = \langle \tilde{A} \oplus \tilde{C}, \tilde{B} \oplus \tilde{D} \rangle. \tag{12}$$

We need to claim that this addition is well defined. Given any $(\tilde{A}', \tilde{B}') \in \langle \tilde{A}, \tilde{B} \rangle$ and $(\tilde{C}', \tilde{D}') \in \langle \tilde{C}, \tilde{D} \rangle$, we need to show

$$\langle \tilde{A}, \tilde{B} \rangle + \langle \tilde{C}, \tilde{D} \rangle = \langle \tilde{A} \oplus \tilde{C}, \tilde{B} \oplus \tilde{D} \rangle = \langle \tilde{A}' \oplus \tilde{C}', \tilde{B}' \oplus \tilde{D}' \rangle = \langle \tilde{A}', \tilde{B}' \rangle + \langle \tilde{C}', \tilde{D}' \rangle.$$

By definition, we have

$$\tilde{A}' \oplus \tilde{B} = \tilde{B}' \oplus \tilde{A} \text{ and } \tilde{C}' \oplus \tilde{D} = \tilde{D}' \oplus \tilde{C}.$$

Therefore, we obtain

$$\tilde{A}' \oplus \tilde{B} \oplus \tilde{C}' \oplus \tilde{D} = \tilde{B}' \oplus \tilde{A} \oplus \tilde{D}' \oplus \tilde{C},$$

which means

$$(\tilde{A}' \oplus \tilde{C}', \tilde{B}' \oplus \tilde{D}') \sim (\tilde{A} \oplus \tilde{C}, \tilde{B} \oplus \tilde{D}).$$

Therefore, we obtain

$$\langle \tilde{A}' \oplus \tilde{C}', \tilde{B}' \oplus \tilde{D}' \rangle = \langle \tilde{A} \oplus \tilde{C}, \tilde{B} \oplus \tilde{D} \rangle.$$

This means that the addition is indeed well defined.

Throughout this paper, we assume that $\mathcal{F}$ contains the crisp element $\tilde{1}_{\{\theta\}}$, where θ is the zero element of the normed space $(U, \|\cdot\|)$. We also write $\tilde{\theta} \equiv \tilde{1}_{\{\theta\}}$. Given any $\tilde{A} \in \mathcal{F}$, it is clear that

$$\tilde{A} = \tilde{A} \oplus \tilde{1}_{\{\theta\}} = \tilde{A} \oplus \tilde{\theta} = \tilde{\theta} \oplus \tilde{A},$$

which means that $\tilde{\theta}$ is the zero element of $\mathcal{F}$.

Proposition 7. *Let $(\mathcal{N}, +)$ be the set defined above. Then, we have the following properties:*

(i) *Given any $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{F}$, we have*

$$\langle \tilde{A}, \tilde{A} \rangle = \langle \tilde{B}, \tilde{B} \rangle \text{ and } \langle \tilde{A}, \tilde{B} \rangle = \langle \tilde{A} \oplus \tilde{C}, \tilde{B} \oplus \tilde{C} \rangle.$$

(ii) *We define*

$$\Theta = \langle \tilde{A}, \tilde{A} \rangle = \{(\tilde{A}, \tilde{A}) : \tilde{A} \in \mathcal{F}\}.$$

Then, Θ is the zero element of $\mathcal{N}$.

(iii) *The additive inverse of $\langle \tilde{A}, \tilde{B} \rangle$ is $\langle \tilde{B}, \tilde{A} \rangle$. In this case, we write*

$$\langle \tilde{A}, \tilde{B} \rangle - \langle \tilde{C}, \tilde{D} \rangle = \langle \tilde{A}, \tilde{B} \rangle + \langle \tilde{D}, \tilde{C} \rangle.$$

In particular, we have

$$\langle \tilde{A}, \tilde{\theta} \rangle - \langle \tilde{B}, \tilde{\theta} \rangle = \langle \tilde{A}, \tilde{\theta} \rangle + \langle \tilde{\theta}, \tilde{B} \rangle = \langle \tilde{A}, \tilde{B} \rangle.$$

(iv) *$(\mathcal{N}, +)$ forms an abelian group.*

(v) *Given any fixed $\tilde{A} \in \mathcal{F}$, we have*

$$\langle \tilde{A} \oplus \tilde{B}', \tilde{B}' \rangle = \langle \tilde{A} \oplus \tilde{B}'', \tilde{B}'' \rangle$$

for any $\tilde{B}', \tilde{B}'' \in \mathcal{F}$. Therefore, we can identify any element $\tilde{A} \in \mathcal{F}$ with the class $\langle \tilde{A} \oplus \tilde{B}, \tilde{B} \rangle$ for some $\tilde{B} \in \mathcal{F}$. In particular, any element $\tilde{A} \in \mathcal{F}$ can be identified with the class $\langle \tilde{A}, \tilde{\theta} \rangle$.

(vi) *We define the function*

$$\pi : (\mathcal{F}, \oplus) \to (\mathcal{N}, +) \text{ by } \pi(\tilde{A}) = \langle \tilde{A}, \tilde{\theta} \rangle.$$

Then, the following statements hold true:

(a) *We have*

$$\pi(\tilde{A} \oplus \tilde{B}) = \pi(\tilde{A}) + \pi(\tilde{B}).$$

(b) *The function π is one-to-one.*

(c) *We have*

$$\pi(\tilde{\theta}) = \Theta \text{ and } \langle \tilde{A}, \tilde{B} \rangle = \pi(\tilde{A}) - \pi(\tilde{B}).$$

The image $\pi(\mathcal{F})$ spans $\mathcal{N}$ in the sense of

$$\mathcal{N} = \pi(\mathcal{F}) - \pi(\mathcal{F}).$$

In this case, we say that the space $\mathcal{F}$ spans $\mathcal{N}$ via the embedding π.

(vii) *The space $(\mathcal{F}, \oplus)$ can be embedded into $(\mathcal{N}, +)$ isomorphically such that $\mathcal{F}$ spans $\mathcal{N}$ via the embedding.*

Proof. It is clear that

$$\tilde{A} \oplus (\tilde{B} \oplus \tilde{C}) = \tilde{B} \oplus (\tilde{A} \oplus \tilde{C}).$$

Therefore, we have

$$(\tilde{A}, \tilde{B}) \sim (\tilde{A} \oplus \tilde{C}, \tilde{B} \oplus \tilde{C}), \text{ i.e., } \langle \tilde{A}, \tilde{B} \rangle = \langle \tilde{A} \oplus \tilde{C}, \tilde{B} \oplus \tilde{C} \rangle.$$

Since $\tilde{A} \oplus \tilde{B} = \tilde{B} \oplus \tilde{A}$, we also have $(\tilde{A}, \tilde{A}) \sim (\tilde{B}, \tilde{B})$, i.e., $\langle \tilde{A}, \tilde{A} \rangle = \langle \tilde{B}, \tilde{B} \rangle$, which also means $(\tilde{B}, \tilde{B}) \in \langle \tilde{A}, \tilde{A} \rangle$. Let $\Theta = \langle \tilde{C}, \tilde{C} \rangle$. Given any $\langle \tilde{A}, \tilde{B} \rangle \in \mathcal{N}$, we have

$$\langle \tilde{A}, \tilde{B} \rangle + \Theta = \langle \tilde{A}, \tilde{B} \rangle + \langle \tilde{C}, \tilde{C} \rangle = \langle \tilde{A} \oplus \tilde{C}, \tilde{B} \oplus \tilde{C} \rangle = \langle \tilde{A}, \tilde{B} \rangle = \Theta + \langle \tilde{A}, \tilde{B} \rangle,$$

which shows that Θ is the zero element of $\mathcal{N}$. Since

$$\langle \tilde{A}, \tilde{B} \rangle + \langle \tilde{B}, \tilde{A} \rangle = \langle \tilde{B}, \tilde{A} \rangle + \langle \tilde{A}, \tilde{B} \rangle = \langle \tilde{A} \oplus \tilde{B}, \tilde{A} \oplus \tilde{B} \rangle = \langle \tilde{C}, \tilde{C} \rangle = \Theta,$$

the additive inverse of $\langle \tilde{A}, \tilde{B} \rangle$ is $\langle \tilde{B}, \tilde{A} \rangle$. Therefore, we conclude that $(\mathcal{N}, +)$ forms an abelian group.

Given any fixed $\tilde{A} \in \mathcal{F}$, since

$$(\tilde{A} \oplus \tilde{B}') \oplus \tilde{B}'' = \tilde{B}' \oplus (\tilde{A} \oplus \tilde{B}''),$$

we have

$$(\tilde{A} \oplus \tilde{B}', \tilde{B}') \sim (\tilde{A} \oplus \tilde{B}'', \tilde{B}''), \text{ i.e., } \langle \tilde{A} \oplus \tilde{B}', \tilde{B}' \rangle = \langle \tilde{A} \oplus \tilde{B}'', \tilde{B}'' \rangle$$

for any $\tilde{B}', \tilde{B}'' \in \mathcal{F}$. This means that $\tilde{A}$ can be identified with the class $\langle \tilde{A} \oplus \tilde{B}, \tilde{B} \rangle$ for some $\tilde{B} \in \mathcal{F}$. In particular, we can take $\tilde{B} = \tilde{\theta}$. This means that $\tilde{A}$ can be identified with the class $\langle \tilde{A} \oplus \tilde{\theta}, \tilde{\theta} \rangle = \langle \tilde{A}, \tilde{\theta} \rangle$.

We know that

$$\langle \tilde{A}', \tilde{\theta} \rangle = \langle \tilde{A}'', \tilde{\theta} \rangle \text{ implies } \tilde{A}' \oplus \tilde{\theta} = \tilde{A}'' \oplus \tilde{\theta}, \text{ i.e., } \tilde{A}' = \tilde{A}''$$

which shows that π is one-to-one. Given any $\tilde{A}, \tilde{B} \in \mathcal{F}$, we have

$$\pi(\tilde{A} \oplus \tilde{B}) = \langle \tilde{A} \oplus \tilde{B}, \tilde{\theta} \rangle = \langle \tilde{A}, \tilde{\theta} \rangle + \langle \tilde{B}, \tilde{\theta} \rangle = \pi(\tilde{A}) + \pi(\tilde{B}),$$

which shows that π is an isomorphism.

Now, we also have

$$\langle \tilde{A}, \tilde{B} \rangle = \langle \tilde{A}, \tilde{\theta} \rangle + \langle \tilde{\theta}, \tilde{B} \rangle = \langle \tilde{A}, \tilde{\theta} \rangle - \langle \tilde{B}, \tilde{\theta} \rangle = \pi(\tilde{A}) - \pi(\tilde{B}).$$

Therefore, we conclude that $\mathcal{F}$ can be embedded into $\mathcal{N}$ isomorphically such that $\mathcal{F}$ spans $\mathcal{N}$ via the embedding in the sense of

$$\mathcal{N} = \pi(\mathcal{F}) - \pi(\mathcal{F}).$$

Finally, we also have

$$\pi(\tilde{\theta}) = \langle \tilde{\theta}, \tilde{\theta} \rangle = \Theta.$$

This completes the proof. □

Next, we are going to consider scalar multiplication in $\mathcal{N}$. Now, given any $\tilde{A}, \tilde{B} \in \mathcal{F}$, we assume that the following conditions are satisfied:

- $\lambda(\tilde{A} \oplus \tilde{B}) = \lambda\tilde{A} \oplus \lambda\tilde{B}$ for $\lambda \geq 0$;
- $(\lambda_1 + \lambda_2)\tilde{A} = \lambda_1\tilde{A} \oplus \lambda_2\tilde{A}$ for $\lambda_1, \lambda_2 \geq 0$;
- $\lambda_1(\lambda_2\tilde{A}) = (\lambda_1\lambda_2)\tilde{A}$ for $\lambda_1, \lambda_2 \geq 0$.

We define the scalar multiplication in $\mathcal{N}$ by

$$\lambda\langle \tilde{A}, \tilde{B} \rangle = \begin{cases} \langle \lambda\tilde{A}, \lambda\tilde{B} \rangle, & \text{if } \lambda \geq 0 \\ \langle |\lambda|\tilde{B}, |\lambda|\tilde{A} \rangle = \langle -\lambda\tilde{B}, -\lambda\tilde{A} \rangle, & \text{if } \lambda < 0. \end{cases} \tag{13}$$

It is necessary to claim that this definition is well defined. Given any $(\tilde{A}', \tilde{B}') \in \langle \tilde{A}, \tilde{B} \rangle$, we need to show

$$\lambda\langle \tilde{A}, \tilde{B} \rangle = \begin{cases} \langle \lambda\tilde{A}', \lambda\tilde{B}' \rangle, & \text{if } \lambda \geq 0 \\ \langle |\lambda|\tilde{B}', |\lambda|\tilde{A}' \rangle, & \text{if } \lambda < 0. \end{cases}$$

By definition, we have $\tilde{A} \oplus \tilde{B}' = \tilde{B} \oplus \tilde{A}'$. Two cases are considered as follows.

- For $\lambda \geq 0$, we have

$$\lambda\tilde{A} \oplus \lambda\tilde{B}' = \lambda(\tilde{A} \oplus \tilde{B}') = \lambda(\tilde{B} \oplus \tilde{A}') = \lambda\tilde{B} \oplus \lambda\tilde{A}',$$

  which means

$$(\lambda\tilde{A}, \lambda\tilde{B}) \sim (\lambda\tilde{A}', \lambda\tilde{B}'), \text{ i.e., } \langle \lambda\tilde{A}, \lambda\tilde{B} \rangle = \langle \lambda\tilde{A}', \lambda\tilde{B}' \rangle.$$

- For $\lambda < 0$, we have

$$|\lambda|\tilde{A} \oplus |\lambda|\tilde{B}' = |\lambda|(\tilde{A} \oplus \tilde{B}') = |\lambda|(\tilde{B} \oplus \tilde{A}') = |\lambda|\tilde{B} \oplus |\lambda|\tilde{A}',$$

which means

$$(|\lambda|\tilde{B}, |\lambda|\tilde{A}) \sim (|\lambda|\tilde{B}', |\lambda|\tilde{A}'), \text{ i.e., } \langle |\lambda|\tilde{B}, |\lambda|\tilde{A}\rangle = \langle |\lambda|\tilde{B}', |\lambda|\tilde{A}'\rangle.$$

This shows that the scalar multiplication is indeed well defined. Using Proposition 7, the additive inverse of $\langle \tilde{A}, \tilde{B}\rangle$ is given by $\langle \tilde{B}, \tilde{A}\rangle$, i.e., $\langle \tilde{A}, \tilde{B}\rangle = -\langle \tilde{B}, \tilde{A}\rangle$. Also, the additive inverse of $\lambda\langle \tilde{A}, \tilde{B}\rangle$ is written as $-(\lambda\langle \tilde{A}, \tilde{B}\rangle)$. Then, we have some interesting results.

Proposition 8. *With the above settings, we have the following properties:*

(i) *We have* $-\langle \tilde{B}, \tilde{A}\rangle = -1\langle \tilde{B}, \tilde{A}\rangle$. *This means that the inverse element* $-\langle \tilde{B}, \tilde{A}\rangle$ *is equal to the element of* $\langle \tilde{B}, \tilde{A}\rangle$ *multiplied by* -1.

(ii) *We have* $-(\lambda\langle \tilde{A}, \tilde{B}\rangle) = (-\lambda)\langle \tilde{A}, \tilde{B}\rangle$. *This means that the inverse element* $-(\lambda\langle \tilde{A}, \tilde{B}\rangle)$ *is equal to the element of* $\langle \tilde{B}, \tilde{A}\rangle$ *multiplied by* $-\lambda$. *In this case, we also write* $-\lambda\langle \tilde{B}, \tilde{A}\rangle = (-\lambda)\langle \tilde{A}, \tilde{B}\rangle$.

(iii) *Given any* $\lambda_1, \lambda_2 \in \mathbb{R}$, *we have*

$$\lambda_1(\lambda_2\langle \tilde{A}, \tilde{B}\rangle) = (\lambda_1\lambda_2)\langle \tilde{A}, \tilde{B}\rangle.$$

Proof. We have

$$-1\langle \tilde{B}, \tilde{A}\rangle = \langle |-1|\tilde{A}, |-1|\tilde{B}\rangle = \langle \tilde{A}, \tilde{B}\rangle = -\langle \tilde{B}, \tilde{A}\rangle$$

and

$$\begin{aligned} -(\lambda\langle \tilde{A}, \tilde{B}\rangle) &= \begin{cases} -\langle \lambda\tilde{A}, \lambda\tilde{B}\rangle & \text{if } \lambda \geq 0 \\ -\langle |\lambda|\tilde{B}, |\lambda|\tilde{A}\rangle & \text{if } \lambda < 0 \end{cases} \\ &= \begin{cases} \langle \lambda\tilde{B}, \lambda\tilde{A}\rangle & \text{if } \lambda \geq 0 \\ \langle |\lambda|\tilde{A}, |\lambda|\tilde{B}\rangle & \text{if } \lambda < 0. \end{cases} = (-\lambda)\langle \tilde{A}, \tilde{B}\rangle. \end{aligned}$$

Now, given any $\lambda_1, \lambda_2 \in \mathbb{R}$, we have

$$\begin{aligned} \lambda_1(\lambda_2\langle \tilde{A}, \tilde{B}\rangle) &= \begin{cases} \lambda_1\langle \lambda_2\tilde{A}, \lambda_2 b\rangle & \text{if } \lambda_2 \geq 0 \\ \lambda_1\langle |\lambda_2|\tilde{B}, |\lambda_2|\tilde{A}\rangle & \text{if } \lambda_2 < 0 \end{cases} \\ &= \begin{cases} \langle (\lambda_1\lambda_2)\tilde{A}, (\lambda_1\lambda_2)\tilde{B}\rangle & \text{if } \lambda_1 \geq 0 \text{ and } \lambda_2 \geq 0 \\ \langle (|\lambda_1|\lambda_2)\tilde{A}, (|\lambda_1|\lambda_2)\tilde{B}\rangle & \text{if } \lambda_1 < 0 \text{ and } \lambda_2 \geq 0 \\ \langle (\lambda_1|\lambda_2|)\tilde{B}, (\lambda_1|\lambda_2|)\tilde{A}\rangle & \text{if } \lambda_1 \geq 0 \text{ and } \lambda_2 < 0 \\ \langle (|\lambda_1\lambda_2|)\tilde{B}, (|\lambda_1\lambda_2|)\tilde{A}\rangle & \text{if } \lambda_1 < 0 \text{ and } \lambda_2 < 0 \end{cases} \\ &= \begin{cases} \langle (\lambda_1\lambda_2)\tilde{A}, (\lambda_1\lambda_2)\tilde{B}\rangle & \text{if } \lambda_1\lambda_2 \geq 0 \\ \langle (|\lambda_1\lambda_2|)\tilde{A}, (|\lambda_1\lambda_2|)\tilde{B}\rangle & \text{if } \lambda_1\lambda_2 < 0 \end{cases} \\ &= (\lambda_1\lambda_2)\langle \tilde{A}, \tilde{B}\rangle. \end{aligned}$$

This completes the proof. □

Proposition 9. *Assume that addition and scalar multiplication in* $\mathcal{N}$ *are defined by* (12) *and* (13), *respectively. Then, we have the following properties:*

(i) $\mathcal{N}$ *is a vector space.*

(ii) *We define the function*

$$\pi : \mathcal{F} \to \mathcal{N} \text{ by } \pi(\tilde{A}) = \langle \tilde{A}, \tilde{\theta}\rangle.$$

Then, the image $\pi(\mathcal{F})$ *is a convex cone in the vector space* $\mathcal{N}$ *with vertex* Θ, *where* Θ *is a zero element of* $\mathcal{N}$.

(iii) *The space* $\mathcal{F}$ *can be embedded into* $\mathcal{N}$ *isomorphically such that* $\mathcal{F}$ *spans* $\mathcal{N}$ *via the embedding in the sense of*

$$\mathcal{N} = \pi(\mathcal{F}) - \pi(\mathcal{F}).$$

Proof. With the above settings, it is not difficult to show that $\mathcal{N}$ is a vector space. For example, we consider the special case $\lambda_1 > 0$, $\lambda_2 < 0$ and $\lambda_1 + \lambda_2 < 0$. Then, we have

$$\begin{aligned}(\lambda_1+\lambda_2)\langle \tilde{A}, \tilde{B}\rangle &= \langle (-\lambda_1-\lambda_2)\tilde{B}, (-\lambda_1-\lambda_2)\tilde{A}\rangle \\ &= \langle (-\lambda_1-\lambda_2)\tilde{B} \oplus \lambda_1\tilde{A} \oplus \lambda_1\tilde{B}, (-\lambda_1-\lambda_2)\tilde{A} \oplus \lambda_1\tilde{A} \oplus \lambda_1\tilde{B}\rangle \\ &= \langle [(-\lambda_1-\lambda_2)+\lambda_1]\tilde{B} \oplus \lambda_1\tilde{A}, [(-\lambda_1-\lambda_2)+\lambda_1]\tilde{A} \oplus \lambda_1\tilde{B}\rangle \\ &\quad \text{(since } \lambda_1 > 0 \text{ and } -\lambda_1-\lambda_2 > 0\text{)} \\ &= \langle -\lambda_2\tilde{B} \oplus \lambda_1\tilde{A}, -\lambda_2\tilde{A} \oplus \lambda_1\tilde{B}\rangle \\ &= \langle -\lambda_2\tilde{B}, -\lambda_2\tilde{A}\rangle + \langle \lambda_1\tilde{A}, \lambda_1\tilde{B}\rangle = \lambda_2\langle \tilde{A}, \tilde{B}\rangle + \lambda_1\langle \tilde{A}, \tilde{B}\rangle.\end{aligned}$$

Proposition 7 has shown the isomorphism

$$\pi(\tilde{A} \oplus \tilde{B}) = \pi(\tilde{A}) + \pi(\tilde{B}).$$

Since $\mathcal{F}$ is assumed to be closed under addition and nonnegative scalar multiplication,

$$\lambda\tilde{A} \oplus (1-\lambda)\tilde{B} \in \mathcal{F}$$

for any $\lambda \in (0,1)$ and $\tilde{A}, \tilde{B} \in \mathcal{F}$. Given any $\lambda \in (0,1)$ and $x, y \in \pi(\mathcal{F})$, we have $x = \pi(\tilde{A})$ and $y = \pi(\tilde{B})$ for some $\tilde{A}, \tilde{B} \in \mathcal{F}$. Therefore, we obtain

$$\lambda x + (1-\lambda)y = \lambda\pi(\tilde{A}) + (1-\lambda)\pi(\tilde{B}) = \pi(\lambda\tilde{A} \oplus (1-\lambda)\tilde{B}) \in \pi(\mathcal{F}),$$

which shows that $\pi(\mathcal{F})$ is a convex subset of $\mathcal{N}$. For $\lambda \geq 0$, we also have $\lambda\tilde{A} \in \mathcal{F}$ and

$$\lambda x = \lambda\pi(\tilde{A}) = \pi(\lambda\tilde{A}) \in \pi(\mathcal{F}).$$

Therefore, we conclude that $\pi(\mathcal{F})$ is indeed a convex cone in the vector space $\mathcal{N}$. This completes the proof. □

Theorem 1 (Embedding Theorem)**.** *Let $(U, \|\cdot\|)$ be a normed space, and let $\mathcal{F}$ be a subset of $\mathcal{F}(U)$ such that the following conditions are satisfied:*

- *The space $\mathcal{F}$ is closed under addition and nonnegative scalar multiplication.*
- *The space $\mathcal{F}$ contains the zero element $\tilde{\theta}$.*
- *Given any $\lambda \geq 0$ and $\tilde{A}, \tilde{B} \in \mathcal{F}$, we have $\lambda(\tilde{A} \oplus \tilde{B}) = \lambda\tilde{A} \oplus \lambda\tilde{B}$.*
- *Given any $\lambda_1, \lambda_2 \geq 0$, we have $(\lambda_1+\lambda_2)\tilde{A} = \lambda_1\tilde{A} \oplus \lambda_2\tilde{A}$.*
- *Given any $\lambda_1, \lambda_2 \geq 0$, we have $\lambda_1(\lambda_2\tilde{A}) = (\lambda_1\lambda_2)\tilde{A}$.*
- *Given any $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{F}$, we have*

$$d_{\mathcal{F}}(\tilde{A}, \tilde{B}) \leq d_{\mathcal{F}}(\tilde{A}, \tilde{C}) + d_{\mathcal{F}}(\tilde{C}, \tilde{B}),$$

  i.e., $d_{\mathcal{F}}$ satisfies the triangle inequality.
- *Given any $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{F}$, we have*

$$d_{\mathcal{F}}(\tilde{A} \oplus \tilde{C}, \tilde{B} \oplus \tilde{C}) = d_{\mathcal{F}}(\tilde{A}, \tilde{B}),$$

  i.e., $d_{\mathcal{F}}$ is invariant under translation.
- *Given any $\tilde{A}, \tilde{B} \in \mathcal{F}$ and $\lambda \geq 0$, we have*

$$d_{\mathcal{F}}(\lambda\tilde{A}, \lambda\tilde{B}) = \lambda d_{\mathcal{F}}(\tilde{A}, \tilde{B}),$$

  i.e., $d_{\mathcal{F}}$ is positively homogeneous.

We also assume that addition and scalar multiplication in $\mathcal{N}$ are defined by (12) and (13), respectively. Then, we have the following properties.

(i) *We define $d_{\mathcal{N}}$ on $\mathcal{N}$ by*

$$d_{\mathcal{N}}\big(\langle \tilde{A}, \tilde{B}\rangle, \langle \tilde{C}, \tilde{D}\rangle\big) = d_{\mathcal{F}}\big(\tilde{A} \oplus \tilde{D}, \tilde{B} \oplus \tilde{C}\big).$$

Then, $(\mathcal{N}, d_{\mathcal{N}})$ forms a metric space such that the metric $d_{\mathcal{N}}$ is homogeneous and invariant under translation; that is,

$$d_{\mathcal{N}}\big(\lambda\langle \tilde{A}, \tilde{B}\rangle, \lambda\langle \tilde{C}, \tilde{D}\rangle\big) = |\lambda| d_{\mathcal{N}}\big(\langle \tilde{A}, \tilde{B}\rangle, \langle \tilde{C}, \tilde{D}\rangle\big)$$

and

$$d_{\mathcal{N}}\big(\langle \tilde{A}, \tilde{B}\rangle + \langle \tilde{E}, \tilde{F}\rangle, \langle \tilde{C}, \tilde{D}\rangle + \langle \tilde{E}, \tilde{F}\rangle\big) = d_{\mathcal{N}}\big(\langle \tilde{A}, \tilde{B}\rangle, \langle \tilde{C}, \tilde{D}\rangle\big).$$

(ii) *Since $\mathcal{N}$ is a vector space with the zero element Θ, we define*

$$\| \langle \tilde{A}, \tilde{B}\rangle \|_{\mathcal{N}} = d_{\mathcal{N}}\big(\langle \tilde{A}, \tilde{B}\rangle, \Theta\big). \tag{14}$$

Then, we have

$$\| \langle \tilde{A}, \tilde{B}\rangle \|_{\mathcal{N}} = d_{\mathcal{F}}\big(\tilde{A}, \tilde{B}\big)$$

and $(\mathcal{N}, \|\cdot\|_{\mathcal{N}})$ forms a normed space. Moreover, we have

$$\| \langle \tilde{A}, \tilde{B}\rangle - \langle \tilde{C}, \tilde{D}\rangle \|_{\mathcal{N}} = \| \langle \tilde{A}, \tilde{B}\rangle + \langle \tilde{D}, \tilde{C}\rangle \|_{\mathcal{N}} = d_{\mathcal{N}}\big(\langle \tilde{A}, \tilde{B}\rangle, \langle \tilde{C}, \tilde{D}\rangle\big).$$

(iii) *We define the function*

$$\pi : (\mathcal{F}, d_{\mathcal{F}}) \to (\mathcal{N}, \|\cdot\|_{\mathcal{N}}) \text{ by } \pi(\tilde{A}) = \langle \tilde{A}, \tilde{\theta}\rangle.$$

Then, the following statements hold true:

(a) *The function π is one-to-one.*

(b) *Given any $\tilde{A}, \tilde{B} \in \mathcal{F}$ and $\lambda \geq 0$, we have*

$$\pi(\tilde{A} \oplus \tilde{B}) = \pi(\tilde{A}) + \pi(\tilde{B}) \text{ and } \pi(\lambda\tilde{A}) = \lambda\pi(\tilde{A}).$$

(c) *Given any $\tilde{A}, \tilde{B} \in \mathcal{F}$, we have*

$$d_{\mathcal{F}}\big(\tilde{A}, \tilde{B}\big) = d_{\mathcal{N}}\big(\pi(\tilde{A}), \pi(\tilde{B})\big) = \| \pi(\tilde{A}) - \pi(\tilde{B}) \|_{\mathcal{N}},$$

In other words, the image $\pi(\mathcal{F})$ and the space $\mathcal{F}$ are isometric. It also means that the function π and its inverse function $\pi^{-1} : \pi(\mathcal{F}) \to \mathcal{F}$ are continuous.

(d) *We have*

$$\langle \tilde{A}, \tilde{B}\rangle = \pi(\tilde{A}) - \pi(\tilde{B}).$$

In other words, the image $\pi(\mathcal{F})$ spans $\mathcal{N}$ in the sense of

$$\mathcal{N} = \pi(\mathcal{F}) - \pi(\mathcal{F}).$$

(e) *The image $\pi(\mathcal{F})$ is a convex cone in the vector space $\mathcal{N}$ with vertex Θ.*

(iv) *The space $(\mathcal{F}, d_{\mathcal{F}})$ can be embedded into the normed space $(\mathcal{N}, \|\cdot\|_{\mathcal{N}})$ isomorphically and isometrically such that $\mathcal{F}$ spans $\mathcal{N}$ via the embedding in the sense of*

$$\mathcal{N} = \pi(\mathcal{F}) - \pi(\mathcal{F}).$$

(v) *Suppose that the space $(\mathcal{F}, d_{\mathcal{F}})$ is complete. Then, the image $\pi(\mathcal{F})$ is complete; that is, every Cauchy sequence in $\pi(\mathcal{F})$ converges to an element in $\pi(\mathcal{F})$. Moreover, the image $\pi(\mathcal{F})$ is a closed convex cone in the vector space $\mathcal{N}$ with vertex Θ.*

(vi) *Suppose that the space $(\mathcal{F}, d_{\mathcal{F}})$ is separable. Then, the normed space $(\mathcal{N}, \|\cdot\|_{\mathcal{N}})$ is also separable.*

Proof. We want to claim that $d_{\mathcal{N}}$ is well defined. Given any

$$(\tilde{A}', \tilde{B}') \in \langle \tilde{A}, \tilde{B} \rangle \text{ and } (\tilde{C}', \tilde{D}') \in \langle \tilde{C}, \tilde{D} \rangle,$$

we need to show

$$d_{\mathcal{F}}(\tilde{A}' \oplus \tilde{D}', \tilde{B}' \oplus \tilde{C}') = d_{\mathcal{F}}(\tilde{A} \oplus \tilde{D}, \tilde{B} \oplus \tilde{C}).$$

By definition, we have

$$\tilde{A}' \oplus \tilde{B} = \tilde{B}' \oplus \tilde{A} \text{ and } \tilde{C}' \oplus \tilde{D} = \tilde{D}' \oplus \tilde{C}.$$

Since $d_{\mathcal{F}}$ is assumed to be invariant under translation, by adding $\tilde{A}' \oplus \tilde{B}' \oplus \tilde{C}' \oplus \tilde{D}'$, we obtain

$$\begin{aligned} d_{\mathcal{F}}(\tilde{A} \oplus \tilde{D}, \tilde{B} \oplus \tilde{C}) &= d_{\mathcal{F}}\big(\tilde{A} \oplus \tilde{D} \oplus \tilde{A}' \oplus \tilde{B}' \oplus \tilde{C}' \oplus \tilde{D}', \tilde{B} \oplus \tilde{C} \oplus \tilde{A}' \oplus \tilde{B}' \oplus \tilde{C}' \oplus \tilde{D}'\big) \\ &= d_{\mathcal{F}}\big((\tilde{A} \oplus \tilde{B}') \oplus (\tilde{C}' \oplus \tilde{D}) \oplus \tilde{A}' \oplus \tilde{D}', (\tilde{A}' \oplus \tilde{B}) \oplus (\tilde{C} \oplus \tilde{D}') \oplus \tilde{B}' \oplus \tilde{C}'\big) \\ &= d_{\mathcal{F}}\big((\tilde{A}' \oplus \tilde{B}) \oplus (\tilde{C} \oplus \tilde{D}') \oplus \tilde{A}' \oplus \tilde{D}', (\tilde{A}' \oplus \tilde{B}) \oplus (\tilde{C} \oplus \tilde{D}') \oplus \tilde{B}' \oplus \tilde{C}'\big) \\ &= d_{\mathcal{F}}\big(\tilde{A}' \oplus \tilde{D}', \tilde{B}' \oplus \tilde{C}'\big). \end{aligned}$$

This shows that $d_{\mathcal{N}}$ is well defined. Now, we want to show that $d_{\mathcal{N}}$ is a metric.

- Suppose that $d_{\mathcal{N}}(\langle \tilde{A}, \tilde{B} \rangle, \langle \tilde{C}, \tilde{D} \rangle) = 0$. This means $d_{\mathcal{F}}(\tilde{A} \oplus \tilde{D}, \tilde{B} \oplus \tilde{C}) = 0$, and thus,

  $$\tilde{A} \oplus \tilde{D} = \tilde{B} \oplus \tilde{C}, \text{ i.e., } (\tilde{A}, \tilde{B}) \sim (\tilde{C}, \tilde{D}).$$

  Therefore, we obtain $\langle \tilde{A}, \tilde{B} \rangle = \langle \tilde{C}, \tilde{D} \rangle$. On the other hand, given any $\langle \tilde{A}, \tilde{B} \rangle = \langle \tilde{C}, \tilde{D} \rangle$, i.e., $\tilde{A} \oplus \tilde{D} = \tilde{B} \oplus \tilde{C}$, we have

  $$d_{\mathcal{N}}\big(\langle \tilde{A}, \tilde{B} \rangle, \langle \tilde{C}, \tilde{D} \rangle\big) = d_{\mathcal{F}}\big(\tilde{A} \oplus \tilde{D}, \tilde{B} \oplus \tilde{C}\big) = 0.$$

- We have

  $$d_{\mathcal{N}}\big(\langle \tilde{A}, \tilde{B} \rangle, \langle \tilde{C}, \tilde{D} \rangle\big) = d_{\mathcal{F}}\big(\tilde{A} \oplus \tilde{D}, \tilde{B} \oplus \tilde{C}\big) = d_{\mathcal{F}}\big(\tilde{B} \oplus \tilde{C}, \tilde{A} \oplus \tilde{D}\big) = d_{\mathcal{N}}\big(\langle \tilde{C}, \tilde{D} \rangle, \langle \tilde{A}, \tilde{B} \rangle\big).$$

- We have

  $$\begin{aligned} d_{\mathcal{N}}\big(\langle \tilde{A}, \tilde{B} \rangle, \langle \tilde{E}, \tilde{F} \rangle\big) &= d_{\mathcal{F}}\big(\tilde{A} \oplus \tilde{F}, \tilde{B} \oplus \tilde{E}\big) \\ &= d_{\mathcal{F}}\big((\tilde{A} \oplus \tilde{D}) \oplus (\tilde{C} \oplus \tilde{F}), (\tilde{B} \oplus \tilde{C}) \oplus (\tilde{D} \oplus \tilde{E})\big) \\ &\quad \text{(using the invariance of translation)} \\ &\leq d_{\mathcal{F}}\big((\tilde{A} \oplus \tilde{D}) \oplus (\tilde{C} \oplus \tilde{F}), (\tilde{B} \oplus \tilde{C}) \oplus (\tilde{C} \oplus \tilde{F})\big) \\ &\quad + d_{\mathcal{F}}\big((\tilde{B} \oplus \tilde{C}) \oplus (\tilde{C} \oplus \tilde{F}), (\tilde{B} \oplus \tilde{C}) \oplus (\tilde{D} \oplus \tilde{E})\big) \\ &\quad \text{(using the triangle inequality)} \\ &= d_{\mathcal{F}}\big(\tilde{A} \oplus \tilde{D}, \tilde{B} \oplus \tilde{C}\big) + d_{\mathcal{F}}\big(\tilde{C} \oplus \tilde{F}, \tilde{D} \oplus \tilde{E}\big) \\ &\quad \text{(using the invariance of translation)} \\ &= d_{\mathcal{N}}\big(\langle \tilde{A}, \tilde{B} \rangle + \langle \tilde{C}, \tilde{D} \rangle\big) + d_{\mathcal{N}}\big(\langle \tilde{C}, \tilde{D} \rangle + \langle \tilde{E}, \tilde{F} \rangle\big). \end{aligned}$$

- Since $d_{\mathcal{F}}$ is positively homogeneous, we have

$$\begin{aligned} d_{\mathcal{N}}\big(\lambda\langle\tilde{A},\tilde{B}\rangle,\lambda\langle\tilde{C},\tilde{D}\rangle\big) &= \begin{cases} d_{\mathcal{N}}\big(\langle\lambda\tilde{A},\lambda\tilde{B}\rangle,\langle\lambda\tilde{C},\lambda\tilde{D}\rangle\big), & \text{if } \lambda\geq 0 \\ d_{\mathcal{N}}\big(\langle|\lambda|\tilde{B},|\lambda|\tilde{A}\rangle,\langle|\lambda|\tilde{D},|\lambda|\tilde{C}\rangle\big), & \text{if } \lambda<0 \end{cases} \\ &= \begin{cases} d_{\mathcal{F}}\big(\lambda\tilde{A}\oplus\lambda\tilde{D},\lambda\tilde{B}\oplus\lambda\tilde{C}\big), & \text{if } \lambda\geq 0 \\ d_{\mathcal{F}}\big(|\lambda|\tilde{B}\oplus|\lambda|\tilde{C},|\lambda|\tilde{A}\oplus|\lambda|\tilde{D}\big), & \text{if } \lambda<0 \end{cases} \\ &= \begin{cases} d_{\mathcal{F}}\big(\lambda(\tilde{A}\oplus\tilde{D}),\lambda(\tilde{B}\oplus\tilde{C})\big), & \text{if } \lambda\geq 0 \\ d_{\mathcal{F}}\big(|\lambda|(\tilde{B}\oplus\tilde{C}),|\lambda|(\tilde{A}\oplus\tilde{D})\big), & \text{if } \lambda<0 \end{cases} \\ &= \begin{cases} \lambda d_{\mathcal{F}}\big(\tilde{A}\oplus\tilde{D},\tilde{B}\oplus\tilde{C}\big), & \text{if } \lambda\geq 0 \\ |\lambda|d_{\mathcal{F}}\big(\tilde{B}\oplus\tilde{C},\tilde{A}\oplus\tilde{D}\big)=|\lambda|d_{\mathcal{F}}\big(\tilde{A}\oplus\tilde{D},\tilde{B}\oplus\tilde{C}\big), & \text{if } \lambda<0 \end{cases} \\ &= |\lambda|d_{\mathcal{N}}\big(\langle\tilde{A},\tilde{B}\rangle,\langle\tilde{C},\tilde{D}\rangle\big). \end{aligned}$$

- Since $d_{\mathcal{F}}$ is invariant under translation, we have

$$\begin{aligned} d_{\mathcal{N}}\big(\langle\tilde{A},\tilde{B}\rangle+\langle\tilde{E},\tilde{F}\rangle,\langle\tilde{C},\tilde{D}\rangle+\langle\tilde{E},\tilde{F}\rangle\big) &= d_{\mathcal{N}}\big(\langle\tilde{A}\oplus\tilde{E},\tilde{B}\oplus\tilde{F}\rangle,\langle\tilde{C}\oplus\tilde{E},\tilde{D}\oplus\tilde{F}\rangle\big) \\ &= d_{\mathcal{F}}\big(\tilde{A}\oplus\tilde{E}\oplus\tilde{D}\oplus\tilde{F},\tilde{B}\oplus\tilde{F}\oplus\tilde{C}\oplus\tilde{E}\big) \\ &= d_{\mathcal{F}}\big(\tilde{A}\oplus\tilde{D},\tilde{B}\oplus\tilde{C}\big) \\ &= d_{\mathcal{N}}\big(\langle\tilde{A},\tilde{B}\rangle,\langle\tilde{C},\tilde{D}\rangle\big). \end{aligned}$$

Therefore, we conclude that $(\mathcal{N},d_{\mathcal{N}})$ is a metric space such that $d_{\mathcal{N}}$ is homogeneous and invariant under translation. Moreover, we have

$$\begin{aligned} d_{\mathcal{N}}\big(\langle\tilde{A},\tilde{B}\rangle,\langle\tilde{C},\tilde{D}\rangle\big) &= d_{\mathcal{N}}\big(\langle\tilde{A},\tilde{B}\rangle+\langle\tilde{D},\tilde{C}\rangle,\langle\tilde{C},\tilde{D}\rangle+\langle\tilde{D},\tilde{C}\rangle\big) \\ &= d_{\mathcal{N}}\big(\langle\tilde{A},\tilde{B}\rangle+\langle\tilde{D},\tilde{C}\rangle,\Theta\big) \\ &= \|\ \langle\tilde{A},\tilde{B}\rangle+\langle\tilde{D},\tilde{C}\rangle\ \|_{\mathcal{N}}=\|\ \langle\tilde{A},\tilde{B}\rangle-\langle\tilde{C},\tilde{D}\rangle\ \|_{\mathcal{N}}\,. \end{aligned} \tag{15}$$

Considering the norm defined in (14), we want to show that the axioms of the norm are satisfied.

- We have

$$\|\ \langle\tilde{A},\tilde{B}\rangle\ \|_{\mathcal{N}}= d_{\mathcal{F}}\big(\langle\tilde{A},\tilde{B}\rangle,\Theta\big)\geq 0.$$

- Suppose that

$$\|\ \langle\tilde{A},\tilde{B}\rangle\ \|_{\mathcal{N}}= 0 = d_{\mathcal{N}}\big(\langle\tilde{A},\tilde{B}\rangle,\Theta\big).$$

  Since $d_{\mathcal{N}}$ is a metric, we have $\langle\tilde{A},\tilde{B}\rangle=\Theta$. On the other hand, for $\langle\tilde{A},\tilde{B}\rangle=\Theta$, we have

$$\|\ \langle\tilde{A},\tilde{B}\rangle\ \|_{\mathcal{N}}= d_{\mathcal{N}}\big(\langle\tilde{A},\tilde{B}\rangle,\Theta\big)=0.$$

- Since $d_{\mathcal{N}}$ is homogeneous, we have

$$\begin{aligned} \|\ \lambda\langle\tilde{A},\tilde{B}\rangle\ \|_{\mathcal{N}} &= d_{\mathcal{N}}\big(\lambda\langle\tilde{A},\tilde{B}\rangle,\Theta\big)=d_{\mathcal{N}}\big(\lambda\langle\tilde{A},\tilde{B}\rangle,\lambda\Theta\big) \\ &= |\lambda|d_{\mathcal{N}}\big(\langle\tilde{A},\tilde{B}\rangle,\Theta\big)=|\lambda|\ \|\ \langle\tilde{A},\tilde{B}\rangle\ \|_{\mathcal{N}}\,. \end{aligned}$$

- We have

$$\begin{aligned} \|\ \langle\tilde{A},\tilde{B}\rangle+\langle\tilde{C},\tilde{D}\rangle\ \|_{\mathcal{N}} &= d_{\mathcal{N}}\big(\langle\tilde{A},\tilde{B}\rangle+\langle\tilde{C},\tilde{D}\rangle,\Theta\big) \\ &\leq d_{\mathcal{N}}\big(\langle\tilde{A},\tilde{B}\rangle+\langle\tilde{C},\tilde{D}\rangle,\langle\tilde{C},\tilde{D}\rangle\big)+d_{\mathcal{N}}\big(\langle\tilde{C},\tilde{D}\rangle,\Theta\big) \\ &= d_{\mathcal{N}}\big(\langle\tilde{A},\tilde{B}\rangle,\Theta\big)+d_{\mathcal{N}}\big(\langle\tilde{C},\tilde{D}\rangle,\Theta\big) \\ &= \|\ \langle\tilde{A},\tilde{B}\rangle\ \|_{\mathcal{N}}+\|\ \langle\tilde{C},\tilde{D}\rangle\ \|_{\mathcal{N}}\,. \end{aligned}$$

Therefore, we conclude that $(\mathcal{N},\|\cdot\|_{\mathcal{N}})$ is a normed space. By referring to (15), we can similarly obtain

$$d_{\mathcal{F}}(\tilde{A},\tilde{B})=d_{\mathcal{N}}\big(\langle\tilde{A},\tilde{\theta}\rangle,\langle\tilde{B},\tilde{\theta}\rangle\big)=d_{\mathcal{N}}\big(\pi(\tilde{A}),\pi(\tilde{B})\big)=\|\ \pi(\tilde{A})-\pi(\tilde{B})\ \|_{\mathcal{N}}\,.$$

This shows that the space $\mathcal{F}$ can be embedded into the normed space $(\mathcal{N}, \|\cdot\|_{\mathcal{N}})$ isomorphically and isometrically.

Assume that the space $(\mathcal{F}, d_{\mathcal{F}})$ is complete. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\pi(\mathcal{F})$. Given any $\epsilon > 0$, there exists an integer n_0 satisfying $\| x_n - x_m \|_{\mathcal{N}} < \epsilon$ for $n, m > n_0$. We write $x_n = \pi(\tilde{A}^{(n)})$ for some $\tilde{A}^{(n)} \in \mathcal{F}$. Then, we have

$$d_{\mathcal{F}}\left(\tilde{A}^{(n)}, \tilde{A}^{(m)}\right) = \| \pi(\tilde{A}^{(n)}) - \pi(\tilde{A}^{(m)}) \|_{\mathcal{N}} = \| x_n - x_m \|_{\mathcal{N}} < \epsilon,$$

which means that $\{\tilde{A}^{(n)}\}_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathcal{F}, d_{\mathcal{F}})$. The completeness of $(\mathcal{F}, d_{\mathcal{F}})$ means that there exists $\tilde{A} \in \mathcal{F}$ satisfying $d_{\mathcal{F}}(\tilde{A}^{(n)}, \tilde{A}) \to 0$ as $n \to \infty$. Let $x = \pi(\tilde{A})$. Then, we obtain

$$\| x_n - x \|_{\mathcal{N}} = \| \pi(\tilde{A}^{(n)}) - \pi(\tilde{A}) \|_{\mathcal{N}} = d_{\mathcal{F}}\left(\tilde{A}^{(n)}, \tilde{A}\right) \to 0 \text{ as } n \to \infty.$$

This shows that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to $x = \pi(\tilde{A}) \in \pi(\mathcal{F})$.

Under the completeness of $(\mathcal{F}, d_{\mathcal{F}})$, we want to show that $\pi(\mathcal{F})$ is also a closed subset of $\mathcal{N}$. Given any $x \in \mathrm{cl}(\pi(\mathcal{F}))$, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $\pi(\mathcal{F})$ such that $\{x_n\}_{n=1}^{\infty}$ converges to x. This also means that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\pi(\mathcal{F})$. The completeness of $\pi(\mathcal{F})$ means that there exists $\hat{x} \in \pi(\mathcal{F})$ such that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to $\hat{x} \in \pi(\mathcal{F})$. The uniqueness of the limit means that $x = \hat{x} \in \pi(\mathcal{F})$. This shows that $\pi(\mathcal{F})$ is a closed subset of $\mathcal{N}$.

Assume that the space $(\mathcal{F}, d_{\mathcal{F}})$ is separable. Let $\{\tilde{A}^{(n)}\}_{n=1}^{\infty}$ be a countable dense subset of $\mathcal{F}$. We want to show that the countable set $\{x_{ij}\}_{i,j=1}^{\infty}$ of the form

$$x_{ij} = \pi(\tilde{A}^{(i)}) - \pi(\tilde{A}^{(j)})$$

is also dense in $\mathcal{N}$. Given any $x \in \mathcal{N}$ and $\epsilon > 0$, since $\pi(\mathcal{F})$ spans $\mathcal{N}$, the element x can be written as $x = \pi(\tilde{A}) - \pi(\tilde{B})$ for some $\tilde{A}, \tilde{B} \in \mathcal{F}$. Using the separability of $(\mathcal{F}, d_{\mathcal{F}})$ and the isometric embedding, there exist $\tilde{A}^{(i)}, \tilde{A}^{(j)} \in \{\tilde{A}^{(n)}\}_{n=1}^{\infty}$ satisfying

$$\| \pi(\tilde{A}) - \pi(\tilde{A}^{(i)}) \|_{\mathcal{N}} = d_{\mathcal{F}}\left(\tilde{A}, \tilde{A}^{(i)}\right) < \frac{\epsilon}{2} \text{ and } \| \pi(\tilde{B}) - \pi(\tilde{A}^{(j)}) \|_{\mathcal{N}} = d_{\mathcal{F}}\left(\tilde{B}, \tilde{A}^{(j)}\right) < \frac{\epsilon}{2}.$$

Therefore, we obtain

$$\begin{aligned} \| x - x_{ij} \|_{\mathcal{N}} &= \| [\pi(\tilde{A}) - \pi(\tilde{B})] - [\pi(\tilde{A}^{(i)}) - \pi(\tilde{A}^{(j)})] \|_{\mathcal{N}} \\ &= \| [\pi(\tilde{A}) - \pi(\tilde{A}^{(i)})] + [\pi(\tilde{A}^{(j)}) - \pi(\tilde{B})] \|_{\mathcal{N}} \\ &\leq \| \pi(\tilde{A}) - \pi(\tilde{A}^{(i)}) \|_{\mathcal{N}} + \| \pi(\tilde{B}) - \pi(\tilde{A}^{(j)}) \|_{\mathcal{N}} < \epsilon. \end{aligned}$$

This shows that the countable set $\{x_{ij}\}_{i,j=1}^{\infty}$ is dense in $\mathcal{N}$, and the proof is complete. □

4. Embedding the Family of Fuzzy Sets into a Banach Space

From Theorem 1, we have constructed the normed space $(\mathcal{N}, \|\cdot\|_{\mathcal{N}})$. Let

$$\{x_n\}_{n=1}^{\infty} = \left\{\langle \tilde{A}^{(n)}, \tilde{B}^{(n)} \rangle\right\}_{n=1}^{\infty} \text{ and } \{y_n\}_{n=1}^{\infty} = \left\{\langle \tilde{C}^{(n)}, \tilde{D}^{(n)} \rangle\right\}_{n=1}^{\infty}$$

be two Cauchy sequences in the normed space $(\mathcal{N}, \|\cdot\|_{\mathcal{N}})$, where

$$x_n \equiv \langle \tilde{A}^{(n)}, \tilde{B}^{(n)} \rangle \text{ and } y_n \equiv \langle \tilde{C}^{(n)}, \tilde{D}^{(n)} \rangle.$$

We say that the Cauchy sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are equivalent when

$$\lim_{n\to\infty} \| x_n - y_n \|_{\mathcal{N}} = \lim_{n\to\infty} \| \langle \tilde{A}^{(n)}, \tilde{B}^{(n)} \rangle - \langle \tilde{C}^{(n)}, \tilde{D}^{(n)} \rangle \|_{\mathcal{N}} = 0.$$

Let $\mathcal{B}$ be the family of all equivalence classes of Cauchy sequences obtained in this way. More precisely, given any $\hat{x} = \langle \{x_n\}_{n=1}^{\infty} \rangle \in \mathcal{B}$, it means that the Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ is a representative of the equivalence class $\hat{x} \in \mathcal{B}$, where $x_n \in \mathcal{N}$ for all n.

In order to make $\mathcal{B}$ a vector space, vector addition and scalar multiplication are defined coordinatewise. More precise descriptions are given below.

- Regarding vector addition, given any

$$\hat{x} = \langle \{x_n\}_{n=1}^{\infty} \rangle \in \mathcal{B} \text{ and } \hat{y} = \langle \{y_n\}_{n=1}^{\infty} \rangle \in \mathcal{B},$$

we set $z_n = x_n + y_n$, where $x_n, y_n \in \mathcal{N}$ for all n. Since

$$\| z_n - z_m \|_{\mathcal{N}} = \| x_n + y_n - (x_m + y_m) \|_{\mathcal{N}} \leq \| x_n - x_m \|_{\mathcal{N}} + \| y_n - y_m \|_{\mathcal{N}},$$

$\{z_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{N}$. Then, we define

$$\hat{x} + \hat{y} = \hat{z} = \langle \{z_n\}_{n=1}^{\infty} \rangle.$$

We still need to claim that this definition is independent of the particular choice of representatives. Given any $\{x_n'\}_{n=1}^{\infty} \in \hat{x}$ and $\{y_n'\}_{n=1}^{\infty} \in \hat{y}$, the equivalence between $\{x_n\}_{n=1}^{\infty}$ and $\{x_n'\}_{n=1}^{\infty}$ and the equivalence between $\{y_n\}_{n=1}^{\infty}$ and $\{y_n'\}_{n=1}^{\infty}$ mean

$$\lim_{n\to\infty} \| x_n - x_n' \|_{\mathcal{N}} = 0 = \lim_{n\to\infty} \| y_n - y_n' \|_{\mathcal{N}}.$$

The inequality

$$\| x_n + y_n - (x_n' + y_n') \|_{\mathcal{N}} \leq \| x_n - x_n' \|_{\mathcal{N}} + \| y_n - y_n' \|_{\mathcal{N}}$$

means that sequences $\{x_n + y_n\}_{n=1}^{\infty}$ and $\{x_n' + y_n'\}_{n=1}^{\infty}$ are equivalent, which shows that the addition $\hat{x} + \hat{y} = \hat{z}$ is well defined.

- Regarding the scalar multiplication $\lambda\hat{x}$ for $\lambda \in \mathbb{R}$ and $\hat{x} = \langle \{x_n\}_{n=1}^{\infty} \rangle \in \mathcal{B}$, we define

$$\lambda\hat{x} = \langle \{\lambda x_n\}_{n=1}^{\infty} \rangle.$$

We need to claim that this definition is independent of the particular choice of representatives. Given any $\{x_n'\}_{n=1}^{\infty} \in \hat{x}$, the equivalence between $\{x_n\}_{n=1}^{\infty}$ and $\{x_n'\}_{n=1}^{\infty}$ means

$$\lim_{n\to\infty} \| x_n - x_n' \|_{\mathcal{N}} = 0.$$

Then, we have

$$\lim_{n\to\infty} \| \lambda x_n - \lambda x_n' \|_{\mathcal{N}} = 0,$$

which means that the sequences $\{\lambda x_n\}_{n=1}^{\infty}$ and $\{\lambda x_n'\}_{n=1}^{\infty}$ are equivalent. This shows that the scalar multiplication is well defined.

Let $\Theta_{\mathcal{N}}$ be the zero element of $\mathcal{N}$. The element $\Theta_{\mathcal{B}} = \langle \{\theta_n\}_{n=1}^{\infty} \rangle \in \mathcal{B}$ is defined to be the equivalence class containing all Cauchy sequences in $\mathcal{N}$, which converges to $\Theta_{\mathcal{N}}$. This means that $\{\theta_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{N}$ satisfying $\theta_n \to \Theta_{\mathcal{N}}$ as $n \to \infty$. Then, we want to claim that $\Theta_{\mathcal{B}}$ is the zero element of $\mathcal{B}$. Given any $\hat{x} = \langle \{x_n\}_{n=1}^{\infty} \rangle \in \mathcal{B}$, the addition $\hat{x} + \Theta_{\mathcal{B}}$ is an equivalent class containing the sequence $\{x_n + \theta_n\}_{n=1}^{\infty}$. Now, we have

$$\lim_{n\to\infty} \| (x_n + \theta_n) - x_n \|_{\mathcal{N}} = \lim_{n\to\infty} \| \theta_n \|_{\mathcal{N}} = 0,$$

which means that the sequences $\{x_n + \theta_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ are equivalent, i.e., $\{x_n\}_{n=1}^{\infty} \in \hat{x} + \Theta_{\mathcal{B}}$. In other words, we obtain

$$\hat{x} + \Theta_{\mathcal{B}} = \langle \{x_n + \theta_n\}_{n=1}^{\infty} \rangle = \langle \{x_n\}_{n=1}^{\infty} \rangle = \hat{x},$$

which shows that $\Theta_{\mathcal{B}}$ is the zero element of $\mathcal{B}$. With the above settings, it is not difficult to show that $\mathcal{B}$ is a vector space under the above addition and scalar multiplication.

Proposition 10. *Given any* $\hat{x} = \langle \{x_n\}_{n=1}^{\infty} \rangle \in \mathcal{B}$, *where* $x_n \in \mathcal{N}$ *for all* n, *we define*

$$\| \hat{x} \|_{\mathcal{B}} = \lim_{n \to \infty} \| x_n \|_{\mathcal{N}} . \tag{16}$$

Then, $(\mathcal{B}, \| \cdot \|_{\mathcal{B}})$ *is a normed space.*

Proof. We first claim that the limit exists. Now, we have

$$\| x_n \|_{\mathcal{N}} \leq \| x_n - x_m \|_{\mathcal{N}} + \| x_m \|_{\mathcal{N}},$$

which implies

$$\| x_n \|_{\mathcal{N}} - \| x_m \|_{\mathcal{N}} \leq \| x_n - x_m \|_{\mathcal{N}} .$$

By interchanging the roles of n and m, we can similarly obtain

$$\| x_m \|_{\mathcal{N}} - \| x_n \|_{\mathcal{N}} \leq \| x_n - x_m \|_{\mathcal{N}} .$$

Therefore, we obtain

$$| \| x_n \|_{\mathcal{N}} - \| x_m \|_{\mathcal{N}} | \leq \| x_n - x_m \|_{\mathcal{N}} . \tag{17}$$

Since $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{N}$, given any $\epsilon > 0$, we have

$$| \| x_n \|_{\mathcal{N}} - \| x_m \|_{\mathcal{N}} | < \epsilon \text{ for sufficiently large } n \text{ and } m,$$

which means that $\{ \| x_n \|_{\mathcal{N}} \}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. The completeness of $\mathbb{R}$ means that the limit in (16) exists.

We need to claim that the limit in (16) is independent of the particular choice of representatives. Given any $\{x_n^{(1)}\}_{n=1}^{\infty}, \{x_n^{(2)}\}_{n=1}^{\infty} \in \hat{x}$, the definition says

$$\lim_{n \to \infty} \| x_n^{(1)} - x_n^{(2)} \|_{\mathcal{N}} = 0.$$

By referring to (17), we can similarly obtain

$$\left| \| x_n^{(1)} \|_{\mathcal{N}} - \| x_n^{(2)} \|_{\mathcal{N}} \right| \leq \| x_n^{(1)} - x_n^{(2)} \|_{\mathcal{N}} .$$

Therefore, we obtain

$$\lim_{n \to \infty} \| x_n^{(1)} \|_{\mathcal{N}} = \lim_{n \to \infty} \| x_n^{(2)} \|_{\mathcal{N}} = \| \hat{x} \|_{\mathcal{B}},$$

which means that the limit in (16) is well defined.

Now, we want to show that $(\mathcal{B}, \| \cdot \|_{\mathcal{B}})$ is a normed space.

- The zero element $\Theta_{\mathcal{B}} = \langle \{\theta_n\}_{n=1}^{\infty} \rangle$ of $\mathcal{B}$ satisfying $\| \theta_n \|_{\mathcal{N}} \to 0$ as $n \to \infty$. From (16), it follows that $\| \Theta_{\mathcal{B}} \|_{\mathcal{B}} = 0$. On the other hand, assume

  $$0 = \| \hat{x} \|_{\mathcal{B}} = \lim_{n \to \infty} \| x_n \|_{\mathcal{N}},$$

  which means that the Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{N}$ converges to the zero element $\Theta_{\mathcal{N}}$ of $\mathcal{N}$. Therefore, we obtain $\hat{x} = \Theta_{\mathcal{B}}$.
- Given any $\hat{x} = \langle \{x_n\}_{n=1}^{\infty} \rangle \in \mathcal{B}$, we have $\lambda \hat{x} = \langle \{\lambda x_n\}_{n=1}^{\infty} \rangle$ and

  $$\| \lambda \hat{x} \|_{\mathcal{B}} = \lim_{n \to \infty} \| \lambda x_n \|_{\mathcal{N}} = |\lambda| \lim_{n \to \infty} \| x_n \|_{\mathcal{N}} = |\lambda| \hat{x}.$$

- Given any

$$\hat{x} = \langle \{x_n\}_{n=1}^{\infty} \rangle \in \mathcal{B}, \quad \hat{y} = \langle \{y_n\}_{n=1}^{\infty} \rangle \in \mathcal{B} \text{ and } \hat{z} = \langle \{z_n\}_{n=1}^{\infty} \rangle \in \mathcal{B},$$

we have

$$\hat{x} - \hat{y} = \langle \{x_n - y_n\}_{n=1}^{\infty} \rangle, \quad \hat{x} - \hat{z} = \langle \{x_n - z_n\}_{n=1}^{\infty} \rangle \text{ and } \hat{z} - \hat{y} = \langle \{z_n - y_n\}_{n=1}^{\infty} \rangle.$$

From (16), it is clear that

$$\| \hat{x} - \hat{y} \|_{\mathcal{B}} = \lim_{n \to \infty} \| x_n - y_n \|_{\mathcal{N}}. \tag{18}$$

Therefore, we obtain

$$\begin{aligned} \| \hat{x} - \hat{y} \|_{\mathcal{B}} = \lim_{n \to \infty} \| x_n - y_n \|_{\mathcal{N}} &\leq \lim_{n \to \infty} [\| x_n - z_n \|_{\mathcal{N}} + \| z_n - y_n \|_{\mathcal{N}}] \\ &= \| \hat{x} - \hat{z} \|_{\mathcal{B}} + \| \hat{z} - \hat{y} \|_{\mathcal{B}}. \end{aligned}$$

This completes the proof. □

Given any $\tilde{A} \in \mathcal{F}$, we associate the equivalence class $\langle \tilde{A}, \tilde{\theta} \rangle \in \mathcal{N}$, which can further associate $\hat{x} \in \mathcal{B}$ such that the equivalent class $\hat{x}$ contains the constant Cauchy sequence $\{x_n = \langle \tilde{A}, \tilde{\theta} \rangle\}_{n=1}^{\infty}$ in $\mathcal{N}$. In this case, we define the function

$$\Gamma : (\mathcal{F}, d_{\mathcal{F}}) \to (\mathcal{B}, \| \cdot \|_{\mathcal{B}}) \text{ by } \tilde{A} \mapsto \hat{x} = \langle \{ \langle \tilde{A}, \tilde{\theta} \rangle \}_{n=1}^{\infty} \rangle.$$

In particular, the equivalence class $\langle \{x\}_{n=1}^{\infty} \rangle$ containing the constant sequence $\{x_n = x\}_{n=1}^{\infty}$ means that each sequence in the equivalence class $\langle \{x\}_{n=1}^{\infty} \rangle$ converges to x. Then, we have the following useful results.

Proposition 11. *With the above settings, we have*

$$\langle \{x + y\}_{n=1}^{\infty} \rangle = \langle \{x\}_{n=1}^{\infty} + \{y\}_{n=1}^{\infty} \rangle = \langle \{x\}_{n=1}^{\infty} \rangle + \langle \{y\}_{n=1}^{\infty} \rangle \tag{19}$$

and

$$\langle \lambda \{x\}_{n=1}^{\infty} \rangle = \lambda \langle \{x\}_{n=1}^{\infty} \rangle \text{ for } \lambda \in \mathbb{R}. \tag{20}$$

Proof. It is clear that

$$\{x + y\}_{n=1}^{\infty} = \{x\}_{n=1}^{\infty} + \{y\}_{n=1}^{\infty}.$$

Given any

$$\{z_n\}_{n=1}^{\infty} \in \langle \{x\}_{n=1}^{\infty} + \{y\}_{n=1}^{\infty} \rangle = \langle \{x + y\}_{n=1}^{\infty} \rangle,$$

it follows that $z_n \to x + y$ as $n \to \infty$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{N}$ satisfying $x_n \to x$ as $n \to \infty$. Then, we have $\{x_n\}_{n=1}^{\infty} \in \langle \{x\}_{n=1}^{\infty} \rangle$. In this case, we define $y_n = z_n - x_n$. Then, we have $y_n \to y$ as $n \to \infty$, i.e., $\{y_n\}_{n=1}^{\infty} \in \langle \{y\}_{n=1}^{\infty} \rangle$. Since $z_n = x_n + y_n$, we obtain the following inclusion:

$$\langle \{x\}_{n=1}^{\infty} + \{y\}_{n=1}^{\infty} \rangle \subseteq \langle \{x\}_{n=1}^{\infty} \rangle + \langle \{y\}_{n=1}^{\infty} \rangle.$$

On the other hand, given any $\{x_n\}_{n=1}^{\infty} \in \langle \{x\}_{n=1}^{\infty} \rangle$ and $\{y_n\}_{n=1}^{\infty} \in \langle \{y\}_{n=1}^{\infty} \rangle$, let $z_n = x_n + y_n$ for all n. Then, we have $\{z_n\}_{n=1}^{\infty} \in \langle \{x + y\}_{n=1}^{\infty} \rangle$, which shows the following inclusion:

$$\langle \{x\}_{n=1}^{\infty} \rangle + \langle \{y\}_{n=1}^{\infty} \rangle \subseteq \langle \{x\}_{n=1}^{\infty} + \{y\}_{n=1}^{\infty} \rangle.$$

Therefore, we obtain equality (19).

For $\lambda \neq 0$, given any

$$\{z_n\}_{n=1}^{\infty} \in \langle \lambda \{x\}_{n=1}^{\infty} \rangle = \langle \{\lambda x\}_{n=1}^{\infty} \rangle,$$

it follows that $z_n \to \lambda x$ as $n \to \infty$. Let $x_n = z_n/\lambda$ for all n. Then, we have $x_n \to x$ as $n \to \infty$, which shows $\{x_n\}_{n=1}^{\infty} \in \langle \{x\}_{n=1}^{\infty} \rangle$, i.e.,

$$\{z_n\}_{n=1}^{\infty} = \{\lambda x_n\}_{n=1}^{\infty} = \lambda \{x_n\}_{n=1}^{\infty} \in \lambda \langle \{x\}_{n=1}^{\infty} \rangle.$$

Therefore, we obtain $z_n \in \lambda \langle \{x\}_{n=1}^{\infty} \rangle$ and the following inclusion:

$$\langle \lambda \{x\}_{n=1}^{\infty} \rangle \subseteq \lambda \langle \{x\}_{n=1}^{\infty} \rangle.$$

On the other hand, given any $\{z_n\}_{n=1}^{\infty} \in \lambda \langle \{x\}_{n=1}^{\infty} \rangle$, it follows that $(1/\lambda)\{z_n\}_{n=1}^{\infty} \in \langle \{x\}_{n=1}^{\infty} \rangle$. Let $x_n = z_n/\lambda$ for all n. Then, we have $\{x_n\}_{n=1}^{\infty} \in \langle \{x\}_{n=1}^{\infty} \rangle$, i.e., $x_n \to x$ as $n \to \infty$. It follows that $z_n = \lambda x_n \to \lambda x$ as $n \to \infty$, which means $\{z_n\}_{n=1}^{\infty} \in \langle \{\lambda x\}_{n=1}^{\infty} \rangle$. This shows the following inclusion:

$$\lambda \langle \{x\}_{n=1}^{\infty} \rangle \subseteq \langle \lambda \{x\}_{n=1}^{\infty} \rangle.$$

Therefore, we obtain equality (20). For $\lambda = 0$, equality (20) is obvious. This completes the proof. □

Given any $x = \langle \tilde{A}, \tilde{B} \rangle \in \mathcal{N}$, we associate the equivalence class $\hat{x} \in \mathcal{B}$ such that the equivalent class $\hat{x}$ contains the constant Cauchy sequence $\{x = \langle \tilde{A}, \tilde{B} \rangle\}_{n=1}^{\infty}$ in $\mathcal{N}$. In this case, we define the function

$$\mu : (\mathcal{N}, \|\cdot\|_{\mathcal{N}}) \to (\mathcal{B}, \|\cdot\|_{\mathcal{B}}) \text{ by } x = \langle \tilde{A}, \tilde{B} \rangle \mapsto \hat{x} = \langle \{x = \langle \tilde{A}, \tilde{B} \rangle\}_{n=1}^{\infty} \rangle. \tag{21}$$

Recall the function

$$\pi : (\mathcal{F}, d_{\mathcal{F}}) \to (\mathcal{N}, \|\cdot\|_{\mathcal{N}}) \text{ defined by } \pi(\tilde{A}) = \langle \tilde{A}, \tilde{\theta} \rangle$$

in Theorem 1. It is clear that $\Gamma = \mu \circ \pi$.

Proposition 12. *The function μ is a continuous, one-to-one and linear isometry such that $\mu(\mathcal{N})$ and $\mathcal{N}$ are isometric. Moreover, we have*

$$\mu(\mathcal{N}) = \Gamma(\mathcal{F}) - \Gamma(\mathcal{F}).$$

Proof. Given any $x = \langle \tilde{A}, \tilde{B} \rangle \in \mathcal{N}$ and $y = \langle \tilde{C}, \tilde{D} \rangle \in \mathcal{N}$, using (19), we have

$$\mu(x+y) = \langle \{x+y\}_{n=1}^{\infty} \rangle = \langle \{x\}_{n=1}^{\infty} + \{y\}_{n=1}^{\infty} \rangle = \langle \{x\}_{n=1}^{\infty} \rangle + \langle \{y\}_{n=1}^{\infty} \rangle = \mu(x) + \mu(y).$$

Given any $\lambda \in \mathbb{R}$, using (20), we have

$$\mu(\lambda x) = \langle \{\lambda x\}_{n=1}^{\infty} \rangle = \langle \lambda \{x\}_{n=1}^{\infty} \rangle = \lambda \langle \{x\}_{n=1}^{\infty} \rangle = \lambda \mu(x)$$

Therefore, the function μ is linear. For

$$\mu(x) = \hat{x} = \langle \{x_n = x\}_{n=1}^{\infty} \rangle,$$

using (16), we have

$$\| \mu(x) \|_{\mathcal{B}} = \| \hat{x} \|_{\mathcal{B}} = \lim_{n \to \infty} \| x_n \|_{\mathcal{N}} = \| x \|_{\mathcal{N}},$$

which means that μ is an isometry. The isometry also means that μ is continuous. Since any isometry is one-to-one, it follows that $\mu(\mathcal{N})$ and $\mathcal{N}$ are isometric. Theorem 1 says that $\mathcal{N} = \pi(\mathcal{F}) - \pi(\mathcal{F})$. Since $\Gamma = \mu \circ \pi$, using the linearity of μ, we have

$$\mu(\mathcal{N}) = \mu(\pi(\mathcal{F}) - \pi(\mathcal{F})) = \mu(\pi(\mathcal{F})) - \mu(\pi(\mathcal{F})) = \Gamma(\mathcal{F}) - \Gamma(\mathcal{F}).$$

This completes the proof. □

Proposition 13. *With the above settings, we have the following properties:*

(i) *The function*

$$\Gamma : (\mathcal{F}, d_{\mathcal{F}}) \to (\mathcal{B}, \|\cdot\|_{\mathcal{B}}) \text{ defined by } \tilde{A} \mapsto \hat{x} = \langle \{\langle \tilde{A}, \tilde{\theta}\rangle\}_{n=1}^{\infty}\rangle$$

is one-to-one and satisfies $\Gamma(\tilde{\theta}) = \Theta_{\mathcal{B}}$ *and*

$$d_{\mathcal{F}}(\tilde{A}, \tilde{B}) = \| \hat{x} - \hat{y} \|_{\mathcal{B}} = \| \Gamma(\tilde{A}) - \Gamma(\tilde{B}) \|_{\mathcal{B}},$$

where $\hat{x} = \Gamma(\tilde{A})$ *and* $\hat{y} = \Gamma(\tilde{B})$. *In other words, the image* $\Gamma(\mathcal{F})$ *and the space* $\mathcal{F}$ *are isometric.*

(ii) *The function* Γ *satisfies*

$$\Gamma(\tilde{A} \oplus \tilde{B}) = \Gamma(\tilde{A}) + \Gamma(\tilde{B}) \text{ and } \Gamma(\lambda \tilde{A}) = \lambda \Gamma(\tilde{A})$$

for any $\tilde{A}, \tilde{B} \in \mathcal{F}$ *and* $\lambda \geq 0$.

(iii) *The set* $\Gamma(\mathcal{F}) - \Gamma(\mathcal{F})$ *is dense in* $\mathcal{B}$. *In other words, we have* $\mathrm{cl}(\Gamma(\mathcal{F}) - \Gamma(\mathcal{F})) = \mathcal{B}$.

(iv) $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ *is a Banach space.*

Proof. It is clear that $\Gamma(\tilde{\theta})$ is the zero element $\Theta_{\mathcal{B}}$ of $\mathcal{B}$. Given any $\tilde{A}, \tilde{B} \in \mathcal{F}$,

$$\Gamma(\tilde{A}) = \hat{x} = \langle \{x_n = \langle \tilde{A}, \tilde{\theta}\rangle\}_{n=1}^{\infty}\rangle \text{ and } \Gamma(\tilde{B}) = \hat{y} = \langle \{y_n = \langle \tilde{B}, \tilde{\theta}\rangle\}_{n=1}^{\infty}\rangle.$$

Using (18) and part (c) of part (iii) of Theorem 1, we have

$$\begin{aligned} d_{\mathcal{F}}(\tilde{A}, \tilde{B}) &= \| \langle \tilde{A}, \tilde{\theta}\rangle - \langle \tilde{B}, \tilde{\theta}\rangle \|_{\mathcal{N}} = \lim_{n\to\infty} \| \langle \tilde{A}, \tilde{\theta}\rangle - \langle \tilde{B}, \tilde{\theta}\rangle \|_{\mathcal{N}} \\ &= \lim_{n\to\infty} \| x_n - y_n \|_{\mathcal{N}} = \| \hat{x} - \hat{y} \|_{\mathcal{B}}, \end{aligned}$$

which means that Γ is an isometry. Since any isometry is one-to-one, it follows that $\Gamma(\mathcal{F})$ and $\mathcal{F}$ are isometric. Now, we have

$$\begin{aligned} \Gamma(\tilde{A} \oplus \tilde{B}) &= \langle \{z_n = \langle \tilde{A} \oplus \tilde{B}, \tilde{\theta}\rangle\}_{n=1}^{\infty}\rangle = \langle \{z_n = \langle \tilde{A}, \tilde{\theta}\rangle + \langle \tilde{B}, \tilde{\theta}\rangle\}_{n=1}^{\infty}\rangle \\ &= \langle \{x_n = \langle \tilde{A}, \tilde{\theta}\rangle\}_{n=1}^{\infty} + \{y_n = \langle \tilde{B}, \tilde{\theta}\rangle\}_{n=1}^{\infty}\rangle \text{ (where } z_n = x_n + y_n \text{ for all } n) \\ &= \langle \{x_n = \langle \tilde{A}, \tilde{\theta}\rangle\}_{n=1}^{\infty}\rangle + \langle \{y_n = \langle \tilde{B}, \tilde{\theta}\rangle\}_{n=1}^{\infty}\rangle \text{ (using Proposition 11)} \\ &= \Gamma(\tilde{A}) + \Gamma(\tilde{B}) \end{aligned}$$

and

$$\begin{aligned} \Gamma(\lambda \tilde{A}) &= \langle \{z_n = \langle \lambda \tilde{A}, \tilde{\theta}\rangle\}_{n=1}^{\infty}\rangle = \langle \{z_n = \lambda \langle \tilde{A}, \tilde{\theta}\rangle\}_{n=1}^{\infty}\rangle \text{ (since } \lambda \geq 0) \\ &= \langle \lambda \{x_n = \langle \tilde{A}, \tilde{\theta}\rangle\}_{n=1}^{\infty}\rangle \text{ (where } z_n = \lambda x_n \text{ for all } n) \\ &= \lambda \langle \{x_n = \langle \tilde{A}, \tilde{\theta}\rangle\}_{n=1}^{\infty}\rangle \text{ (using Proposition 11)} \\ &= \lambda \Gamma(\tilde{A}). \end{aligned}$$

We want to claim that the set $\Gamma(\mathcal{F}) - \Gamma(\mathcal{F})$ is dense in $\mathcal{B}$. Given any $\hat{x} = \langle \{x_n\}_{n=1}^{\infty}\rangle \in \mathcal{B}$, it follows that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{N}$. Given any $\epsilon > 0$, there exists an integer n^* satisfying $\| x_n - x_{n^*} \|_{\mathcal{N}} < \epsilon/2$ for $n \geq n^*$. We consider the equivalence class $\hat{x}_{n^*}$ containing the constant Cauchy sequence $\{x_{n^*}\}_{n=1}^{\infty}$, i.e., $\hat{x}_{n^*} = \langle \{x_{n^*}\}_{n=1}^{\infty}\rangle$. We write $x_{n^*} = \langle \tilde{A}^{(n^*)}, \tilde{B}^{(n^*)}\rangle$. Then, we have

$$x_{n^*} = \langle \tilde{A}^{(n^*)}, \tilde{\theta}\rangle - \langle \tilde{B}^{(n^*)}, \tilde{\theta}\rangle \equiv y_{n^*} - z_{n^*},$$

where

$$y_{n^*} = \langle \tilde{A}^{(n^*)}, \tilde{\theta} \rangle \text{ and } z_{n^*} = \langle \tilde{B}^{(n^*)}, \tilde{\theta} \rangle.$$

We also consider the equivalence classes $\hat{y}_{n^*}$ and $\hat{z}_{n^*}$ containing constant Cauchy sequences as follows:

$$\hat{y}_{n^*} = \langle \{ y_{n^*} = \langle \tilde{A}^{(n^*)}, \tilde{\theta} \rangle \}_{n=1}^{\infty} \rangle \text{ and } \hat{z}_{n^*} = \langle \{ z_{n^*} = \langle \tilde{B}^{(n^*)}, \tilde{\theta} \rangle \}_{n=1}^{\infty} \rangle.$$

Then, we obtain

$$\begin{aligned} \hat{x}_{n^*} &= \langle \{ y_{n^*} - z_{n^*} \}_{n=1}^{\infty} \rangle \\ &= \langle \{ y_{n^*} \}_{n=1}^{\infty} \rangle - \langle \{ z_{n^*} \}_{n=1}^{\infty} \rangle \text{ (using Proposition 11)} \\ &= \hat{y}_{n^*} - \hat{z}_{n^*} = \langle \{ \langle \tilde{A}^{(n^*)}, \tilde{\theta} \rangle \}_{n=1}^{\infty} \rangle - \langle \{ \langle \tilde{B}^{(n^*)}, \tilde{\theta} \rangle \}_{n=1}^{\infty} \rangle \\ &= \Gamma(\tilde{A}^{(n^*)}) - \Gamma(\tilde{B}^{(n^*)}), \end{aligned}$$

which means $\hat{x}_{n^*} \in \Gamma(\mathcal{F}) - \Gamma(\mathcal{F})$. From (18), we have

$$\| \hat{x} - \hat{x}_{n^*} \|_{\mathcal{B}} = \lim_{n \to \infty} \| x_n - x_{n^*} \|_N \leq \frac{\epsilon}{2} < \epsilon,$$

which shows that the set $\Gamma(\mathcal{F}) - \Gamma(\mathcal{F})$ is dense in $\mathcal{B}$.

Now, we want to show the completeness of $\mathcal{B}$. Since $\Gamma(\mathcal{F}) - \Gamma(\mathcal{F})$ is dense in $\mathcal{B}$, Proposition 12 says that $\mu(\mathcal{N})$ is also dense in $\mathcal{B}$, where the function μ is given in (21). Given any Cauchy sequence $\{\hat{x}_n\}_{n=1}^{\infty}$ in $\mathcal{B}$, for each $\hat{x}_n \in \mathcal{B}$, the denseness means that there exists $\hat{z}_n \in \mu(\mathcal{N})$ satisfying

$$\| \hat{x}_n - \hat{z}_n \|_{\mathcal{B}} < \frac{1}{n}. \tag{22}$$

Then, we have

$$\begin{aligned} \| \hat{z}_m - \hat{z}_n \|_{\mathcal{B}} &\leq \| \hat{z}_m - \hat{x}_m \|_{\mathcal{B}} + \| \hat{x}_m - \hat{x}_n \|_{\mathcal{B}} + \| \hat{x}_n - \hat{z}_n \|_{\mathcal{B}} \\ &< \frac{1}{m} + \| \hat{x}_m - \hat{x}_n \|_{\mathcal{B}} + \frac{1}{n}. \end{aligned}$$

Since $\{\hat{x}_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{B}$, it follows that $\| \hat{z}_m - \hat{z}_n \|_{\mathcal{B}} < \epsilon$ for sufficiently large m and n, which also means that $\{\hat{z}_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{B}$. Since μ is one-to-one by Proposition 12 and $\hat{z}_n \in \mu(\mathcal{N})$, we define $z_n = \mu^{-1}(\hat{z}_n)$ for all n. Since μ is an isometry by Proposition 12 and $\{\hat{z}_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{B}$, it follows that $\{z_n\}_{n=1}^{\infty}$ is also a Cauchy sequence in $\mathcal{N}$. Let $\hat{x} = \langle \{z_n\}_{n=1}^{\infty} \rangle \in \mathcal{B}$. We want to show that $\hat{x}$ is the limit of the Cauchy sequence $\{\hat{x}_n\}_{n=1}^{\infty}$. Given any fixed n^*, using (22), we have

$$\| \hat{x}_{n^*} - \hat{x} \|_{\mathcal{B}} \leq \| \hat{x}_{n^*} - \hat{z}_{n^*} \|_{\mathcal{B}} + \| \hat{z}_{n^*} - \hat{x} \|_{\mathcal{B}} < \frac{1}{n^*} + \| \hat{z}_{n^*} - \hat{x} \|_{\mathcal{B}}. \tag{23}$$

Since $\mu(z_{n^*}) = \hat{z}_{n^*}$, it follows that $\hat{z}_{n^*} = \langle \{z_{n^*}\}_{n=1}^{\infty} \rangle$ is an equivalence class containing a constant Cauchy sequence. Using (18), we have

$$\| \hat{z}_{n^*} - \hat{x} \|_{\mathcal{B}} = \lim_{n \to \infty} \| z_{n^*} - z_n \|_{\mathcal{N}}.$$

Using (23), we obtain

$$\| \hat{x}_{n^*} - \hat{x} \|_{\mathcal{B}} < \frac{1}{n^*} + \lim_{n \to \infty} \| z_{n^*} - z_n \|_{\mathcal{N}}.$$

Given any $\epsilon > 0$, since $\{z_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{N}$, it follows that $\| \hat{x}_{n^*} - \hat{x} \|_{\mathcal{B}} < \epsilon$ for sufficiently large n^*, which shows that $\hat{x}$ is the limit of the Cauchy sequence $\{\hat{x}_n\}_{n=1}^{\infty}$ in $\mathcal{B}$. This completes the proof. □

Theorem 2 (Embedding Theorem). *Let $(U, \|\cdot\|)$ be a normed space, and let $\mathcal{F}$ be a subset of $\mathcal{F}(U)$ such that the following conditions are satisfied:*

- *The space $\mathcal{F}$ is closed under addition and nonnegative scalar multiplication.*
- *The space $\mathcal{F}$ contains the zero element $\tilde{\theta}$.*
- *Given any $\lambda \geq 0$ and $\tilde{A}, \tilde{B} \in \mathcal{F}$, we have $\lambda(\tilde{A} \oplus \tilde{B}) = \lambda\tilde{A} \oplus \lambda\tilde{B}$.*
- *Given any $\lambda_1, \lambda_2 \geq 0$, we have $(\lambda_1 + \lambda_2)\tilde{A} = \lambda_1\tilde{A} \oplus \lambda_2\tilde{A}$.*
- *Given any $\lambda_1, \lambda_2 \geq 0$, we have $\lambda_1(\lambda_2\tilde{A}) = (\lambda_1\lambda_2)\tilde{A}$.*
- *Given any $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{F}$, we have*

$$d_{\mathcal{F}}(\tilde{A}, \tilde{B}) \leq d_{\mathcal{F}}(\tilde{A}, \tilde{C}) + d_{\mathcal{F}}(\tilde{C}, \tilde{B}),$$

  i.e., $d_{\mathcal{F}}$ satisfies the triangle inequality.
- *Given any $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{F}$, we have*

$$d_{\mathcal{F}}(\tilde{A} \oplus \tilde{C}, \tilde{B} \oplus \tilde{C}) = d_{\mathcal{F}}(\tilde{A}, \tilde{B}),$$

  i.e., $d_{\mathcal{F}}$ is invariant under translation.
- *Given any $\tilde{A}, \tilde{B} \in \mathcal{F}$ and $\lambda \geq 0$, we have*

$$d_{\mathcal{F}}(\lambda\tilde{A}, \lambda\tilde{B}) = \lambda d_{\mathcal{F}}(\tilde{A}, \tilde{B}),$$

  i.e., $d_{\mathcal{F}}$ is positively homogeneous.

With the above settings, given any $\hat{x} = \langle\{x_n\}_{n=1}^{\infty}\rangle \in \mathcal{B}$, where $x_n \in \mathcal{N}$ for all n, we define

$$\| \hat{x} \|_{\mathcal{B}} = \lim_{n\to\infty} \| x_n \|_{\mathcal{N}}.$$

Then, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a Banach space. We also define the function

$$\Gamma : (\mathcal{F}, d_{\mathcal{F}}) \to (\mathcal{B}, \|\cdot\|_{\mathcal{B}}) \text{ by } \tilde{A} \mapsto \hat{x} = \langle\{\langle\tilde{A}, \tilde{\theta}\rangle\}_{n=1}^{\infty}\rangle.$$

Then, we have the following properties:

(i) *The function Γ is one-to-one and satisfies $\Gamma(\tilde{\theta}) = \Theta_{\mathcal{B}}$ and*

$$\Gamma(\tilde{A} \oplus \tilde{B}) = \Gamma(\tilde{A}) + \Gamma(\tilde{B}) \text{ and } \Gamma(\lambda\tilde{A}) = \lambda\Gamma(\tilde{A}) \tag{24}$$

  for any $\tilde{A}, \tilde{B} \in \mathcal{F}$ and $\lambda \geq 0$.

(ii) *The set $\Gamma(\mathcal{F}) - \Gamma(\mathcal{F})$ is dense in $\mathcal{B}$. In other words, we have $\mathrm{cl}(\Gamma(\mathcal{F}) - \Gamma(\mathcal{F})) = \mathcal{B}$.*

(iii) *Given any $\tilde{A}, \tilde{B} \in \mathcal{F}$, we have*

$$d_{\mathcal{F}}(\tilde{A}, \tilde{B}) = \| \Gamma(\tilde{A}) - \Gamma(\tilde{B}) \|_{\mathcal{B}}. \tag{25}$$

  In other words, the image $\Gamma(\mathcal{F})$ and the space $\mathcal{F}$ are isometric. It also means that the function Γ and its inverse function $\Gamma^{-1} : \Gamma(\mathcal{F}) \to \mathcal{F}$ are continuous.

(iv) *The image $\Gamma(\mathcal{F})$ is a convex cone in the Banach space $\mathcal{B}$ with vertex $\Theta_{\mathcal{B}}$.*

(v) *The space $\mathcal{F}$ can be embedded into the Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ isomorphically and isometrically.*

(vi) *Suppose that the space $(\mathcal{F}, d_{\mathcal{F}})$ is complete. Then, the image $\Gamma(\mathcal{F})$ is complete; that is, every Cauchy sequence in $\Gamma(\mathcal{F})$ converges to an element in $\Gamma(\mathcal{F})$. Moreover, the image $\Gamma(\mathcal{F})$ is a closed convex cone in the Banach space $\mathcal{B}$ with vertex $\Theta_{\mathcal{B}}$.*

(vii) *Suppose that the space $(\mathcal{F}, d_{\mathcal{F}})$ is separable. Then, the Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is also separable.*

(viii) *The Banach space $\mathcal{B}$ is unique in the following sense. Let $\bar{\mathcal{B}}$ be another Banach space such that the embedding function $\bar{\Gamma} : \mathcal{F} \to \bar{\mathcal{B}}$ satisfies the following conditions.*

- *The set $\bar{\Gamma}(\mathcal{F}) - \bar{\Gamma}(\mathcal{F})$ is dense in $\bar{\mathcal{B}}$.*
- *The sets $\Gamma(\mathcal{F}) - \Gamma(\mathcal{F})$ and $\bar{\Gamma}(\mathcal{F}) - \bar{\Gamma}(\mathcal{F})$ are isometric (resp. linearly isometric).*

  Then, the Banach spaces $\mathcal{B}$ and $\bar{\mathcal{B}}$ are isometric (resp. linearly isometric).

Proof. Proposition 13 says that $(\mathcal{B}, \| \cdot \|_{\mathcal{B}})$ is a Banach space. Parts (i)–(iii) can also be realized from Proposition 13.

Since $\mathcal{F}$ is assumed to be closed under addition and nonnegative scalar multiplication, it follows that

$$\lambda \tilde{A} \oplus (1-\lambda)\tilde{B} \in \mathcal{F}$$

for any $\lambda \in (0,1)$ and $\tilde{A}, \tilde{B} \in \mathcal{F}$. Given any $\lambda \in (0,1)$ and $\hat{x}, \hat{y} \in \Gamma(\mathcal{F})$, there exists $\tilde{A}, \tilde{B} \in \mathcal{F}$ satisfying

$$\Gamma(\tilde{A}) = \hat{x} = \langle \{x_n = \langle \tilde{A}, \tilde{\theta} \rangle\}_{n=1}^{\infty} \rangle \text{ and } \Gamma(\tilde{B}) = \hat{y} = \langle \{y_n = \langle \tilde{B}, \tilde{\theta} \rangle\}_{n=1}^{\infty} \rangle.$$

Using (24), we obtain

$$\lambda \hat{x} + (1-\lambda)\hat{y} = \lambda \Gamma(\tilde{A}) + (1-\lambda)\Gamma(\tilde{A}) = \Gamma(\lambda \tilde{A} \oplus (1-\lambda)\tilde{B}) \in \Gamma(\mathcal{F}),$$

which shows that $\Gamma(\mathcal{F})$ is a convex subset of $\mathcal{B}$. For $\lambda \geq 0$, we also have $\lambda \tilde{A} \in \mathcal{F}$ and

$$\lambda \hat{x} = \lambda \Gamma(\tilde{A}) = \Gamma(\lambda \tilde{A}) \in \Gamma(\mathcal{F}).$$

Therefore, we conclude that $\Gamma(\mathcal{F})$ is indeed a convex cone in the Banach space $\mathcal{B}$.

Assume that the space $(\mathcal{F}, d_{\mathcal{F}})$ is complete. Let $\{\hat{x}_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\Gamma(\mathcal{F})$. Given any $\epsilon > 0$, there exists an integer n_0 satisfying $\| \hat{x}_n - \hat{x}_m \|_{\mathcal{B}} < \epsilon$ for $n, m > n_0$. We write $\hat{x}_n = \Gamma(\tilde{A}^{(n)})$ for some $\tilde{A}^{(n)} \in \mathcal{F}$. Using (25), we obtain

$$d_{\mathcal{F}}\left(\tilde{A}^{(n)}, \tilde{A}^{(m)}\right) = \| \Gamma(\tilde{A}^{(n)}) - \Gamma(\tilde{A}^{(m)}) \|_{\mathcal{B}} = \| \hat{x}_n - \hat{x}_m \|_{\mathcal{B}} < \epsilon,$$

which means that $\{\tilde{A}^{(n)}\}_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathcal{F}, d_{\mathcal{F}})$. The completeness of $(\mathcal{F}, d_{\mathcal{F}})$ means that there exists $\tilde{A} \in \mathcal{F}$ satisfying $d_{\mathcal{F}}(\tilde{A}^{(n)}, \tilde{A}) \to 0$ as $n \to \infty$. Let $\hat{x} = \Gamma(\tilde{A})$. Then, we obtain

$$\| \hat{x}_n - \hat{x} \|_{\mathcal{B}} = \| \Gamma(\tilde{A}^{(n)}) - \Gamma(\tilde{A}) \|_{\mathcal{B}} = d_{\mathcal{F}}\left(\tilde{A}^{(n)}, \tilde{A}\right) \to 0 \text{ as } n \to \infty.$$

This shows that the sequence $\{\hat{x}_n\}_{n=1}^{\infty}$ converges to $\hat{x} = \Gamma(\tilde{A}) \in \Gamma(\mathcal{F})$. Therefore, the image set $\Gamma(\mathcal{F})$ is complete.

Under the completeness of $(\mathcal{F}, d_{\mathcal{F}})$, we want to show that $\Gamma(\mathcal{F})$ is also a closed subset of $\mathcal{B}$. Given any $\hat{x} \in \text{cl}(\Gamma(\mathcal{F}))$, there exists a sequence $\{\hat{x}_n\}_{n=1}^{\infty}$ in $\Gamma(\mathcal{F})$ such that $\{\hat{x}_n\}_{n=1}^{\infty}$ converges to $\hat{x}$. This also means that $\{\hat{x}_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\Gamma(\mathcal{F})$. The completeness of $\Gamma(\mathcal{F})$ says that there exists $\hat{x}^{\circ} \in \Gamma(\mathcal{F})$ such that the sequence $\{\hat{x}_n\}_{n=1}^{\infty}$ converges to $\hat{x}^{\circ} \in \Gamma(\mathcal{F})$. The uniqueness of the limit says $\hat{x} = \hat{x}^{\circ} \in \Gamma(\mathcal{F})$. This shows that $\Gamma(\mathcal{F})$ is a closed subset of $\mathcal{N}$.

Assume that the space $(\mathcal{F}, d_{\mathcal{F}})$ is separable. Let $\{\tilde{A}^{(mn)}\}_{m,n=1}^{\infty}$ be a countable dense subset of $\mathcal{F}$. We want to show that the countable set $\{\hat{x}_{ijk}\}_{i,j,k=1}^{\infty}$ of the form

$$\hat{x}_{ijk} = \Gamma(\tilde{A}^{(ik)}) - \Gamma(\tilde{A}^{(jk)})$$

is also dense in $\mathcal{B}$. Given any $\hat{x} \in \mathcal{B}$ and $\epsilon > 0$, since $\text{cl}(\Gamma(\mathcal{F}) - \Gamma(\mathcal{F})) = \mathcal{B}$, there exists a sequence $\{\hat{x}_k\}_{k=1}^{\infty}$ in $\Gamma(\mathcal{F}) - \Gamma(\mathcal{F})$ satisfying $\| \hat{x} - \hat{x}_k \|_{\mathcal{B}} < \epsilon/3$ for sufficiently large k. Therefore, the element $\hat{x}_k$ can be written as $\hat{x}_k = \Gamma(\tilde{B}^{(k)}) - \Gamma(\tilde{C}^{(k)})$ for some $\tilde{B}^{(k)}, \tilde{C}^{(k)} \in \mathcal{F}$. Using (25) and the separability of $(\mathcal{F}, d_{\mathcal{F}})$, there exist $\tilde{A}^{(ik)}, \tilde{A}^{(jk)} \in \{\tilde{A}^{(mn)}\}_{m,n=1}^{\infty}$ satisfying

$$\| \Gamma(\tilde{B}^{(k)}) - \Gamma(\tilde{A}^{(ik)}) \|_{\mathcal{B}} = d_{\mathcal{F}}\left(\tilde{B}^{(k)}, \tilde{A}^{(ik)}\right) < \frac{\epsilon}{3}$$

and

$$\| \Gamma(\tilde{C}^{(k)}) - \Gamma(\tilde{A}^{(jk)}) \|_{\mathcal{B}} = d_{\mathcal{F}}\left(\tilde{C}^{(k)}, \tilde{A}^{(jk)}\right) < \frac{\epsilon}{3}.$$

Therefore, we obtain

$$\begin{aligned}\| x - \hat{x}_{ijk} \|_{\mathcal{B}} \leq & \| \hat{x} - \hat{x}_k \|_{\mathcal{B}} + \| \hat{x}_k - x_{ijk} \|_{\mathcal{B}} \\ = & \| \hat{x} - \hat{x}_k \|_{\mathcal{B}} + \| [\Gamma(\tilde{B}^{(k)}) - \Gamma(\tilde{A}^{(k)})] - [\Gamma(\tilde{A}^{(ik)}) - \Gamma(\tilde{A}^{(jk)})] \|_{\mathcal{B}} \\ = & \| \hat{x} - \hat{x}_k \|_{\mathcal{B}} + \| [\Gamma(\tilde{B}^{(k)}) - \Gamma(\tilde{A}^{(ik)})] + [\Gamma(\tilde{A}^{(jk)}) - \Gamma(\tilde{C}^{(k)})] \|_{\mathcal{B}} \\ \leq & \| \hat{x} - \hat{x}_k \|_{\mathcal{B}} + \| \Gamma(\tilde{B}^{(k)}) - \Gamma(\tilde{A}^{(ik)}) \|_{\mathcal{B}} + \| \Gamma(\tilde{C}^{(k)}) - \Gamma(\tilde{A}^{(jk)}) \|_{\mathcal{B}} \\ < & \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

This shows that the countable set $\{x_{ijk}\}_{i,j,k=1}^{\infty}$ is dense in $\mathcal{B}$. In other words, the Banach space $\mathcal{B}$ is separable.

Finally, we want to claim that $\mathcal{B}$ is unique except for isometries in the described sense. Since $\bar{\mathcal{W}} \equiv \bar{\Gamma}(\mathcal{F}) - \bar{\Gamma}(\mathcal{F})$ is dense in $\bar{\mathcal{B}}$, given any $\bar{x}, \bar{y} \in \bar{\mathcal{B}}$, there exist sequences $\{\bar{x}_n\}_{n=1}^{\infty}$ and $\{\bar{y}_n\}_{n=1}^{\infty}$ in $\bar{\mathcal{W}}$ satisfying

$$\lim_{n\to\infty} \| \bar{x}_n - \bar{x} \|_{\bar{\mathcal{B}}} = 0 = \lim_{n\to\infty} \| \bar{y}_n - \bar{y} \|_{\bar{\mathcal{B}}} .$$

By referring to (17), we can similarly obtain

$$\begin{aligned}|\| \bar{x} - \bar{y} \|_{\bar{\mathcal{B}}} - \| \bar{x}_n - \bar{y}_n \|_{\bar{\mathcal{B}}}| \leq & \| (\bar{x} - \bar{y}) - (\bar{x}_n - \bar{y}_n) \|_{\bar{\mathcal{B}}} \\ = & \| (\bar{x} - \bar{x}_n) + (\bar{y}_n - \bar{y}) \|_{\bar{\mathcal{B}}} \\ \leq & \| \bar{x} - \bar{x}_n \|_{\bar{\mathcal{B}}} + \| \bar{y} - \bar{y}_n \|_{\bar{\mathcal{B}}},\end{aligned}$$

which implies

$$\| \bar{x} - \bar{y} \|_{\bar{\mathcal{B}}} = \lim_{n\to\infty} \| \bar{x}_n - \bar{y}_n \|_{\bar{\mathcal{B}}} . \tag{26}$$

Since $\mathcal{W} \equiv \Gamma(\mathcal{F}) - \Gamma(\mathcal{F})$ is dense in $\mathcal{B}$, given any $\hat{x}, \hat{y} \in \mathcal{B}$, we can similarly obtain

$$\| \hat{x} - \hat{y} \|_{\mathcal{B}} = \lim_{n\to\infty} \| \hat{x}_n - \hat{y}_n \|_{\mathcal{B}}, \tag{27}$$

where $\{\hat{x}_n\}_{n=1}^{\infty}$ and $\{\hat{y}_n\}_{n=1}^{\infty}$ are sequences in $\mathcal{W}$. Since $\mathcal{W}$ and $\bar{\mathcal{W}}$ are assumed to be isometric, we consider the isometry $\tau : \mathcal{W} \to \bar{\mathcal{W}}$ and the function $\bar{\tau} : \mathcal{B} \to \bar{\mathcal{B}}$, where the function τ is one-to-one. Given any $\hat{x} \in \mathcal{B}$, we want to define $\bar{\tau}(\hat{x})$. The denseness of $\mathcal{W}$ in $\mathcal{B}$ means that there exists a sequence $\{\hat{x}_n\}_{n=1}^{\infty}$ in $\mathcal{W}$ satisfying $\hat{x}_n \to \hat{x}$ as $n \to \infty$. Let $\bar{x}_n = \tau(\hat{x}_n)$ for all n. The isometry τ indicates that the sequence $\{\bar{x}_n\}_{n=1}^{\infty}$ is convergent. Therefore, we have $\bar{x}_n \to \bar{x}$ as $n \to \infty$ for some $\bar{x} \in \bar{\mathcal{B}}$. In this case, we define $\bar{\tau}(\hat{x}) = \bar{x}$. Similarly, given $\hat{y} \in \mathcal{B}$, we can obtain the sequences $\{\hat{y}_n\}_{n=1}^{\infty}$ and $\{\bar{y}_n\}_{n=1}^{\infty}$ in $\mathcal{W}$ and $\bar{\mathcal{W}}$, respectively, satisfying $\bar{y}_n \to \bar{y}$ as $n \to \infty$, $\bar{\tau}(\hat{y}) = \bar{y}$ and $\bar{y}_n = \tau(\hat{y}_n)$ for all n. On the other hand, given any $\bar{x} \in \bar{\mathcal{B}}$, the denseness of $\bar{\mathcal{W}}$ in $\bar{\mathcal{B}}$ means that there exists a sequence $\{\bar{x}_n\}_{n=1}^{\infty}$ in $\bar{\mathcal{W}}$ satisfying $\bar{x}_n \to \bar{x}$ as $n \to \infty$. Since τ is one-to-one, it follows that we can define $\hat{x}_n = \tau^{-1}(\bar{x}_n)$ for all n. The isometry means that the sequence $\{\hat{x}_n\}_{n=1}^{\infty}$ is convergent. Therefore, we have $\hat{x}_n \to \hat{x}$ as $n \to \infty$ for some $\hat{x} \in \mathcal{B}$, i.e., $\bar{x} = \bar{\tau}(\hat{x})$. This shows that the function $\bar{\tau}$ is one-to-one. The isometry τ also indicates

$$\| \bar{x}_n - \bar{y}_n \|_{\mathcal{B}} = \| \tau(\hat{x}_n) - \tau(\hat{y}_n) \|_{\mathcal{B}} = \| \hat{x}_n - \hat{y}_n \|_{\mathcal{B}} .$$

From (26) and (27), it follows that

$$\| \bar{\tau}(\hat{x}) - \bar{\tau}(\hat{y}) \|_{\bar{\mathcal{B}}} = \| \bar{x} - \bar{y} \|_{\bar{\mathcal{B}}} = \| \hat{x} - \hat{y} \|_{\mathcal{B}},$$

which shows that $\bar{\tau}$ is an isometry. Since the isometry is also one-to-one, it follows that $\mathcal{B}$ and $\bar{\mathcal{B}}$ are isometric. Assume that τ is linear. We want to show that $\bar{\tau}$ is also linear.

- Now, we have

$$\hat{x}_n + \hat{y}_n \to \hat{x} + \hat{y} \text{ and } \bar{x}_n + \bar{y}_n \to \bar{x} + \bar{y} \text{ as } n \to \infty.$$

Using the linearity of τ, we have

$$\tau(\hat{x}_n + \hat{y}_n) = \tau(\hat{x}_n) + \tau(\hat{y}_n) = \bar{x}_n + \bar{y}_n \to \bar{x} + \bar{y} \text{ as } n \to \infty.$$

The definition of $\bar{\tau}$ says

$$\bar{\tau}(\hat{x} + \hat{y}) = \bar{x} + \bar{y} = \bar{\tau}(\hat{x}) + \bar{\tau}(\hat{y}).$$

- Given any $\lambda \in \mathbb{R}$, we have

$$\lambda\hat{x}_n \to \lambda\hat{x} \text{ and } \lambda\bar{x}_n \to \lambda\bar{x} \text{ as } n \to \infty.$$

Using the linearity of τ, we have

$$\tau(\lambda\hat{x}_n) = \lambda\tau(\hat{x}_n) = \lambda\bar{x}_n \to \lambda\bar{x} \text{ as } n \to \infty.$$

The definition of $\bar{\tau}$ says

$$\bar{\tau}(\lambda\hat{x}) = \lambda\bar{x} = \lambda\bar{\tau}(\hat{x}).$$

This completes the proof. □

5. Embedding Theorems for Special Kinds of Families of Fuzzy Sets

Many special kinds of families of fuzzy sets, i.e., $\mathcal{F}_l(U)$, $\mathcal{F}_k(U)$, $\mathcal{F}_{lc}(U)$, $\mathcal{F}_{kc}(U)$, $\mathcal{F}_k^C(U)$ $\mathcal{F}_{kc}^L(U)$ and $\mathcal{F}_{kc}^C(U)$, are introduced in the previous section. However, only the families $\mathcal{F}_{kc}(U)$, $\mathcal{F}_{kc}^L(U)$ and $\mathcal{F}_{kc}^C(U)$ can be embedded into a normed space $\mathcal{N}$ or a Banach space $\mathcal{B}$. More detailed structures are presented below.

Theorem 3. *We have the following properties for embedding into a normed space $\mathcal{N}$:*

(i) *Let $(U, \|\cdot\|)$ be a normed space. The space $(\mathcal{F}_{kc}(U), d_{\mathcal{F}})$ can be embedded into a normed space $\mathcal{N}$ isomorphically and isometrically via the embedding function*

$$\pi : \mathcal{F}_{kc}(U) \to \mathcal{N} \text{ defined by } \pi(\tilde{A}) = \langle \tilde{A}, \tilde{\theta} \rangle \tag{28}$$

such that the image $\pi(\mathcal{F}_{kc}(U))$ is a convex cone with vertex $\Theta_{\mathcal{N}}$ in $\mathcal{N}$ and that the space $\mathcal{F}_{kc}(U)$ spans $\mathcal{N}$ in the sense of

$$\mathcal{N} = \pi(\mathcal{F}_{kc}(U)) - \pi(\mathcal{F}_{kc}(U)).$$

The spaces $(\mathcal{F}_{kc}^L(U), d_{\mathcal{F}})$ and $(\mathcal{F}_{kc}^C(U), d_{\mathcal{F}})$ can also be embedded as convex cones into the corresponding normed spaces $\mathcal{N}^L$ and $\mathcal{N}^C$ isomorphically and isometrically such that the following properties are satisfied:

- *The space $\mathcal{F}_{kc}^L(U)$ spans $\mathcal{N}^L$ via the embedding function $\pi^L : \mathcal{F}_{kc}^L(U) \to \mathcal{N}^L$ defined in (28), satisfying*

$$\mathcal{N}^L = \pi^L(\mathcal{F}_{kc}^L(U)) - \pi^L(\mathcal{F}_{kc}^L(U)).$$

- *The space $\mathcal{F}_{kc}^C(U)$ spans $\mathcal{N}^C$ via the embedding function $\pi^C : \mathcal{F}_{kc}^C(U) \to \mathcal{N}^C$ defined in (28), satisfying*

$$\mathcal{N}^C = \pi^C(\mathcal{F}_{kc}^C(U)) - \pi^C(\mathcal{F}_{kc}^C(U)).$$

(ii) *Let $(U, \|\cdot\|)$ be a Banach space. The space $(\mathcal{F}_{kc}(U), d_{\mathcal{F}})$ can be embedded into the normed space $\mathcal{N}$ isomorphically and isometrically via the embedding function $\pi : \mathcal{F}_{kc}(U) \to \mathcal{N}$*

defined in (28) such that the image $\pi(\mathcal{F}_{kc}(U))$ is complete and is a closed convex cone with vertex $\Theta_{\mathcal{N}}$ in $\mathcal{N}$ and that $\mathcal{F}_{kc}(U)$ spans $\mathcal{N}$ in the sense of

$$\mathcal{N} = \pi(\mathcal{F}_{kc}(U)) - \pi(\mathcal{F}_{kc}(U)).$$

The space $(\mathcal{F}_{kc}^{C}(U), d_{\mathcal{F}})$ can also be embedded as a closed convex cone into the normed space $\mathcal{N}^{C}$ isomorphically and isometrically such that the image $\pi(\mathcal{F}_{kc}^{C}(U))$ is complete and that the space $\mathcal{F}_{kc}^{C}(U)$ spans $\mathcal{N}^{C}$ via the embedding function $\pi^{C} : \mathcal{F}_{kc}^{C}(U) \to \mathcal{N}^{C}$ defined in (28), satisfying

$$\mathcal{N}^{C} = \pi^{C}(\mathcal{F}_{kc}^{C}(U)) - \pi^{C}(\mathcal{F}_{kc}^{C}(U)).$$

Proof. To prove part (i), Proposition 1 says that these spaces are closed under addition and nonnegative scalar multiplication. It is clear that these spaces contain a zero element. From Remark 1, we see that the metric $d_{\mathcal{F}}$ is homogeneous and invariant under translation. From Proposition 6, we see that all conditions in Theorem 1 are satisfied. Therefore, these spaces can be embedded as a convex cone into the normed space $\mathcal{N}$ isomorphically and isometrically. For example, the space $\mathcal{F}_{kc}(U)$ spans $\mathcal{N}$ via the embedding function π, satisfying

$$\mathcal{N} = \pi(\mathcal{F}_{kc}(U)) - \pi(\mathcal{F}_{kc}(U)).$$

To prove part (ii), Propositions 3 and 4 say that the spaces $(\mathcal{F}_{kc}(U), d_{\mathcal{F}})$ and $(\mathcal{F}_{kc}^{C}(U), d_{\mathcal{F}})$ are complete, which shows the closeness of the convex cone and the completeness of the images $\pi(\mathcal{F}_{kc}(U))$ and $\pi(\mathcal{F}_{kc}^{C}(U))$ by Theorem 1. This completes the proof. □

Example 6. *Let $\mathcal{N}$ be the family of all equivalence classes $\langle \tilde{A}, \tilde{B} \rangle$ for $\tilde{A}, \tilde{B} \in \mathcal{F}_{kc}(\mathbb{R})$. Since the space $\mathbb{R}$ is complete, Theorem 3 says that the space $\mathcal{F}_{kc}(\mathbb{R})$ can be embedded into the normed space $\mathcal{N}$ isomorphically and isometrically via the embedding function*

$$\pi : \mathcal{F}_{kc}(\mathbb{R}) \to \mathcal{N} \text{ defined by } \pi(\tilde{A}) = \langle \tilde{A}, \tilde{\theta} \rangle$$

such that the image $\pi(\mathcal{F}_{kc}(\mathbb{R}))$ is complete and is a closed convex cone with vertex $\Theta_{\mathcal{N}}$ in $\mathcal{N}$ and that $\mathcal{F}_{kc}(\mathbb{R})$ spans $\mathcal{N}$ in the sense of

$$\mathcal{N} = \pi(\mathcal{F}_{kc}(\mathbb{R})) - \pi(\mathcal{F}_{kc}(\mathbb{R})).$$

More precisely, given any $\tilde{A}, \tilde{B} \in \mathcal{F}_{kc}(\mathbb{R})$ and $\lambda \geq 0$, we have

$$\pi(\tilde{A} \oplus \tilde{B}) = \pi(\tilde{A}) + \pi(\tilde{B}) = \langle \tilde{A}, \tilde{\theta} \rangle + \langle \tilde{B}, \tilde{\theta} \rangle = \langle \tilde{A} \oplus \tilde{B}, \tilde{\theta} \rangle$$

and

$$\pi(\lambda \tilde{A}) = \lambda \pi(\tilde{A}) = \lambda \langle \tilde{A}, \tilde{\theta} \rangle = \begin{cases} \langle \lambda \tilde{A}, \tilde{\theta} \rangle & \text{if } \lambda \geq 0 \\ \langle \tilde{\theta}, -\lambda \tilde{A} \rangle & \text{if } \lambda < 0. \end{cases}$$

The isometry means that

$$d_{\mathcal{F}}(\tilde{A}, \tilde{B}) = d_{\mathcal{N}}(\pi(\tilde{A}), \pi(\tilde{B})) = \| \pi(\tilde{A}) - \pi(\tilde{B}) \|_{\mathcal{N}} = \| \langle \tilde{A}, \tilde{B} \rangle \|_{\mathcal{N}}$$

for any $\tilde{A}, \tilde{B} \in \mathcal{F}_{kc}(\mathbb{R})$.

Theorem 4. *By considering separability, we have the following properties for embedding into a normed space $\mathcal{N}$:*

(i) *Let $(U, \| \cdot \|)$ be a separable normed space. The space $(\mathcal{F}_{kc}^{C}(U), d_{\mathcal{F}})$ can be embedded into a separable normed space $\mathcal{N}^{C}$ isomorphically and isometrically via the embedding function $\pi : \mathcal{F}_{kc}^{C}(U) \to \mathcal{N}^{C}$ such that the image $\pi(\mathcal{F}_{kc}^{C}(U))$ is a convex cone with vertex $\Theta_{\mathcal{N}}$ in $\mathcal{N}^{C}$ and that the space $\mathcal{F}_{kc}^{C}(U)$ spans $\mathcal{N}^{C}$ in the sense of*

$$\mathcal{N}^{C} = \pi^{C}(\mathcal{F}_{kc}^{C}(U)) - \pi^{C}(\mathcal{F}_{kc}^{C}(U)).$$

(ii) *Let $(U, \|\cdot\|)$ be a separable Banach space. The space $(\mathcal{F}_{kc}^{C}(U), d_{\mathcal{F}})$ can be embedded into a separable normed space $\mathcal{N}^{C}$ isomorphically and isometrically via the embedding function $\pi : \mathcal{F}_{kc}^{C}(U) \to \mathcal{N}^{C}$ such that the image $\pi(\mathcal{F}_{kc}^{C}(U))$ is complete and is a closed convex cone with vertex $\Theta_{\mathcal{N}}$ in $\mathcal{N}^{C}$ and that the space $\mathcal{F}_{kc}^{C}(U)$ spans $\mathcal{N}^{C}$ in the sense of*

$$\mathcal{N}^{C} = \pi^{C}(\mathcal{F}_{kc}^{C}(U)) - \pi^{C}(\mathcal{F}_{kc}^{C}(U)).$$

Proof. Since the normed space U is separable, Remark 4 says that the space $(\mathcal{F}_{kc}^{C}(U), d_{\mathcal{F}})$ is separable. Using Theorem 1, the normed space $\mathcal{N}$ is also separable. The desired results follow immediately from the same arguments of Theorem 3. This completes the proof. □

Example 7. *Let $\mathcal{N}^{C}$ be the family of all equivalence classes $\langle \tilde{A}, \tilde{B} \rangle$ for $\tilde{A}, \tilde{B} \in \mathcal{F}_{kc}^{C}(\mathbb{R})$. Since the space $\mathbb{R}$ is separable and complete, Theorem 4 says that the space $\mathcal{F}_{kc}^{C}(\mathbb{R})$ can be embedded into the separable normed space $\mathcal{N}^{C}$ isomorphically and isometrically via the embedding function*

$$\pi^{C} : \mathcal{F}_{kc}^{C}(\mathbb{R}) \to \mathcal{N}^{C} \text{ defined by } \pi^{C}(\tilde{A}) = \langle \tilde{A}, \tilde{\theta} \rangle$$

such that the image $\pi^{C}(\mathcal{F}_{kc}^{C}(\mathbb{R}))$ is complete and is a closed convex cone with vertex $\Theta_{\mathcal{N}}^{C}$ in $\mathcal{N}^{C}$ and that $\mathcal{F}_{kc}^{C}(\mathbb{R})$ spans $\mathcal{N}^{C}$ in the sense of

$$\mathcal{N}^{C} = \pi^{C}(\mathcal{F}_{kc}^{C}(\mathbb{R})) - \pi^{C}(\mathcal{F}_{kc}^{C}(\mathbb{R})).$$

Next, we are going to apply Theorem 2 to embed the desired spaces into the Banach spaces.

Theorem 5. *We have the following properties for embedding into a Banach space $\mathcal{B}$:*

(i) *Let $(U, \|\cdot\|)$ be a normed space. The space $(\mathcal{F}_{kc}(U), d_{\mathcal{F}})$ can be embedded into a Banach space $\mathcal{B}$ isomorphically and isometrically via the embedding function $\Gamma : \mathcal{F}_{kc}(U) \to \mathcal{B}$ such that the image $\Gamma(\mathcal{F}_{kc}(U))$ is a convex cone with vertex $\Theta_{\mathcal{B}}$ in $\mathcal{B}$ and that the space $\mathcal{F}_{kc}(U)$ spans $\mathcal{B}$ in the sense of*

$$\mathcal{B} = \mathrm{cl}(\Gamma(\mathcal{F}_{kc}(U)) - \Gamma(\mathcal{F}_{kc}(U))).$$

The Banach space $\mathcal{B}$ is unique in the following sense. Let $\bar{\mathcal{B}}$ be another Banach space such that the embedding function $\bar{\Gamma} : \mathcal{F}_{kc}(U) \to \bar{\mathcal{B}}$ satisfies the following conditions.

- *The set $\bar{\Gamma}(\mathcal{F}_{kc}(U)) - \bar{\Gamma}(\mathcal{F}_{kc}(U))$ is dense in $\bar{\mathcal{B}}$.*
- *The sets $\Gamma(\mathcal{F}_{kc}(U)) - \Gamma(\mathcal{F}_{kc}(U))$ and $\bar{\Gamma}(\mathcal{F}_{kc}(U)) - \bar{\Gamma}(\mathcal{F}_{kc}(U))$ are isometric (resp. linearly isometric).*

Then, the Banach spaces $\mathcal{B}$ and $\bar{\mathcal{B}}$ are isometric (resp. linearly isometric).

(ii) *Let $(U, \|\cdot\|)$ be a normed space. The spaces $(\mathcal{F}_{kc}^{L}(U), d_{\mathcal{F}})$ and $(\mathcal{F}_{kc}^{C}(U), d_{\mathcal{F}})$ can be embedded as convex cones into the corresponding Banach spaces $\mathcal{B}^{L}$ and $\mathcal{B}^{C}$ isomorphically and isometrically such that the following properties are satisfied:*

- *The space $\mathcal{F}_{kc}^{L}(U)$ spans $\mathcal{B}^{L}$ via the embedding function $\Gamma^{L} : \mathcal{F}_{kc}^{L}(U) \to \mathcal{B}^{L}$, satisfying*

$$\mathcal{B}^{L} = \mathrm{cl}\Big(\Gamma^{L}(\mathcal{F}_{kc}^{L}(U)) - \Gamma^{L}(\mathcal{F}_{kc}^{L}(U))\Big).$$

  Let $\bar{\mathcal{B}}^{L}$ be another Banach space such that the embedding function $\bar{\Gamma}^{L} : \mathcal{F}_{kc}^{L}(U) \to \bar{\mathcal{B}}^{L}$ satisfies conditions similar to those described in part (i). Then, the Banach spaces $\mathcal{B}^{L}$ and $\bar{\mathcal{B}}^{L}$ are isometric (resp. linearly isometric).
- *The space $\mathcal{F}_{kc}^{C}(U)$ spans $\mathcal{B}^{C}$ via the embedding function $\Gamma^{C} : \mathcal{F}_{kc}^{C}(U) \to \mathcal{B}^{C}$, satisfying*

$$\mathcal{B}^{C} = \mathrm{cl}\Big(\Gamma^{C}(\mathcal{F}_{kc}^{C}(U)) - \Gamma^{C}(\mathcal{F}_{kc}^{C}(U))\Big).$$

Let $\bar{\mathcal{B}}^C$ be another Banach space such that the embedding function $\bar{\Gamma}^C : \mathcal{F}_{kc}^C(U) \to \bar{\mathcal{B}}^C$ satisfies conditions similar to those described in part (i). *Then, the Banach spaces $\mathcal{B}^C$ and $\bar{\mathcal{B}}^C$ are isometric* (*resp. linearly isometric*).

(iii) *Let $(U, \|\cdot\|)$ be a Banach space. The space $(\mathcal{F}_{kc}(U), d_{\mathcal{F}})$ can be embedded into the Banach space $\mathcal{B}$ isomorphically and isometrically via the embedding function $\Gamma : \mathcal{F}_{kc}(U) \to \mathcal{B}$ such that the image $\Gamma(\mathcal{F}_{kc}(U))$ is complete and is a closed convex cone with vertex $\Theta_{\mathcal{B}}$ in $\mathcal{B}$ and that the space $\mathcal{F}_{kc}(U)$ spans $\mathcal{B}$ in the sense of*

$$\mathcal{B} = \mathrm{cl}(\Gamma(\mathcal{F}_{kc}(U)) - \Gamma(\mathcal{F}_{kc}(U))).$$

Let $\bar{\mathcal{B}}$ be another Banach space such that the embedding function $\bar{\Gamma} : \mathcal{F}_{kc}(U) \to \bar{\mathcal{B}}$ satisfies conditions similar to those described in part (i). *Then, the Banach spaces $\mathcal{B}$ and $\bar{\mathcal{B}}$ are isometric* (*resp. linearly isometric*).

(iv) *Let $(U, \|\cdot\|)$ be a Banach space. The space $(\mathcal{F}_{kc}^C(U), d_{\mathcal{F}})$ can be embedded as a closed convex cone into the Banach space $\mathcal{B}^C$ isomorphically and isometrically such that the image $\Gamma(\mathcal{F}_{kc}^C(U))$ is complete and that the space $\mathcal{F}_{kc}^C(U)$ spans $\mathcal{B}^C$ via the embedding function $\Gamma^C : \mathcal{F}_{kc}^C(U) \to \mathcal{B}^C$, satisfying*

$$\mathcal{B}^C = \mathrm{cl}\Big(\Gamma^C(\mathcal{F}_{kc}^C(U)) - \Gamma^C(\mathcal{F}_{kc}^C(U))\Big).$$

Let $\bar{\mathcal{B}}^C$ be another Banach space such that the embedding function $\bar{\Gamma}^C : \mathcal{F}_{kc}^C(U) \to \bar{\mathcal{B}}^C$ satisfies conditions similar to those described in part (i). *Then, the Banach spaces $\mathcal{B}^C$ and $\bar{\mathcal{B}}^C$ are isometric* (*resp. linearly isometric*).

Proof. The results follow immediately by applying Theorem 2 to the proof of Theorem 3. □

Example 8. *From Example 6, we have constructed the normed space $(\mathcal{N}, \|\cdot\|_{\mathcal{N}})$. Let*

$$\{x_n\}_{n=1}^{\infty} = \Big\{\langle \tilde{A}^{(n)}, \tilde{B}^{(n)}\rangle\Big\}_{n=1}^{\infty} \text{ and } \{y_n\}_{n=1}^{\infty} = \Big\{\langle \tilde{C}^{(n)}, \tilde{D}^{(n)}\rangle\Big\}_{n=1}^{\infty}$$

be two Cauchy sequences in the normed space $(\mathcal{N}, \|\cdot\|_{\mathcal{N}})$, where

$$x_n \equiv \langle \tilde{A}^{(n)}, \tilde{B}^{(n)}\rangle \text{ and } y_n \equiv \langle \tilde{C}^{(n)}, \tilde{D}^{(n)}\rangle.$$

We say that the Cauchy sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are equivalent when

$$\lim_{n\to\infty} \| x_n - y_n \|_{\mathcal{N}} = \lim_{n\to\infty} \| \langle \tilde{A}^{(n)}, \tilde{B}^{(n)}\rangle - \langle \tilde{C}^{(n)}, \tilde{D}^{(n)}\rangle \|_{\mathcal{N}} = 0.$$

Let $\mathcal{B}$ be the family of all equivalence classes of Cauchy sequences obtained in this way. More precisely, given any $\hat{x} = \langle \{x_n\}_{n=1}^{\infty}\rangle \in \mathcal{B}$, it follows that the Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ is a representative of the equivalence class $\hat{x} \in \mathcal{B}$, where $x_n \in \mathcal{N}$ for all n. Since the space $\mathbb{R}$ is complete, Theorem 5 says that the space $\mathcal{F}_{kc}(\mathbb{R})$ can be embedded into the Banach space $\mathcal{B}$ isomorphically and isometrically via the embedding function

$$\Gamma : \mathcal{F}_{kc}(\mathbb{R}) \to \mathcal{B} \text{ defined by } \Gamma(\tilde{A}) = \langle \tilde{A}, \tilde{\theta}\rangle$$

such that the image $\Gamma(\mathcal{F}_{kc}(\mathbb{R}))$ is complete and is a closed convex cone with vertex $\Theta_{\mathcal{B}}$ in $\mathcal{B}$ and that $\mathcal{F}_{kc}(\mathbb{R})$ spans $\mathcal{B}$ in the sense of

$$\mathcal{B} = cl(\Gamma(\mathcal{F}_{kc}(\mathbb{R})) - \Gamma(\mathcal{F}_{kc}(\mathbb{R}))).$$

The uniqueness of this Banach space $\mathcal{B}$ can also be realized from Theorem 5.

Theorem 6. *By considering separability, we have the following properties for embedding into a Banach space $\mathcal{B}$:*

(i) *Let $(U, \|\cdot\|)$ be a separable normed space. The space $(\mathcal{F}_{kc}^{C}(U), d_{\mathcal{F}})$ can be embedded into a separable Banach space $\mathcal{B}^{C}$ isomorphically and isometrically via the embedding function $\Gamma : \mathcal{F}_{kc}^{C}(U) \to \mathcal{B}^{C}$ such that the image $\Gamma(\mathcal{F}_{kc}^{C}(U))$ is a convex cone with vertex $\Theta_{\mathcal{B}}$ in $\mathcal{B}^{C}$ and that the space $\mathcal{F}_{kc}^{C}(U)$ spans $\mathcal{B}^{C}$ in the sense of*

$$\mathcal{B}^{C} = \mathrm{cl}\Big(\Gamma^{C}(\mathcal{F}_{kc}^{C}(U)) - \Gamma^{C}(\mathcal{F}_{kc}^{C}(U))\Big).$$

The Banach space $\mathcal{B}^{C}$ is unique in the following sense. Let $\bar{\mathcal{B}}^{C}$ be another Banach space such that the embedding function $\bar{\Gamma}^{C} : \mathcal{F}_{kc}^{C}(U) \to \bar{\mathcal{B}}^{C}$ satisfies the following conditions.

- *The set $\bar{\Gamma}^{C}(\mathcal{F}_{kc}^{C}(U)) - \bar{\Gamma}^{C}(\mathcal{F}_{kc}^{C}(U))$ is dense in $\bar{\mathcal{B}}^{C}$.*
- *The sets $\Gamma^{C}(\mathcal{F}_{kc}^{C}(U)) - \Gamma^{C}(\mathcal{F}_{kc}^{C}(U))$ and $\bar{\Gamma}^{C}(\mathcal{F}_{kc}^{C}(U)) - \bar{\Gamma}^{C}(\mathcal{F}_{kc}^{C}(U))$ are isometric (resp. linearly isometric).*

Then, the Banach spaces $\mathcal{B}^{C}$ and $\bar{\mathcal{B}}^{C}$ are isometric (resp. linearly isometric).

(ii) *Let $(U, \|\cdot\|)$ be a separable Banach space. The space $(\mathcal{F}_{kc}^{C}(U), d_{\mathcal{F}})$ can be embedded into a separable Banach space $\mathcal{B}^{C}$ isomorphically and isometrically via the embedding function $\Gamma : \mathcal{F}_{kc}^{C}(U) \to \mathcal{B}^{C}$ such that the image $\Gamma(\mathcal{F}_{kc}^{C}(U))$ is complete and is a closed convex cone with vertex $\Theta_{\mathcal{B}}$ in $\mathcal{B}^{C}$ and that the space $\mathcal{F}_{kc}^{C}(U)$ spans $\mathcal{B}^{C}$ in the sense of*

$$\mathcal{B}^{C} = \mathrm{cl}\Big(\Gamma^{C}(\mathcal{F}_{kc}^{C}(U)) - \Gamma^{C}(\mathcal{F}_{kc}^{C}(U))\Big).$$

Let $\bar{\mathcal{B}}^{C}$ be another Banach space such that the embedding function $\bar{\Gamma}^{C} : \mathcal{F}_{kc}^{C}(U) \to \bar{\mathcal{B}}^{C}$ satisfies conditions similar to those described in part (i)*. Then, the Banach spaces $\mathcal{B}^{C}$ and $\bar{\mathcal{B}}^{C}$ are isometric (resp. linearly isometric).*

Proof. The results follow immediately by applying Theorem 2 to the proof of Theorem 4. □

Example 9. *From Example 7, we have constructed the normed space $(\mathcal{N}^{C}, \|\cdot\|_{\mathcal{N}^{C}})$. According to Example 8, we can similarly construct the Banach space $\mathcal{B}^{C}$. Since the space $\mathbb{R}$ is separable and complete, Theorem 6 says that the space $\mathcal{F}_{kc}^{C}(\mathbb{R})$ can be embedded into the separable Banach space $\mathcal{B}^{C}$ isomorphically and isometrically via the embedding function*

$$\Gamma^{C} : \mathcal{F}_{kc}^{C}(\mathbb{R}) \to \mathcal{B}^{C} \text{ defined by } \Gamma^{C}(\tilde{A}) = \langle \tilde{A}, \tilde{\theta} \rangle$$

such that the image $\Gamma^{C}(\mathcal{F}_{kc}^{C}(\mathbb{R}))$ is complete and is a closed convex cone with vertex $\Theta_{\mathcal{B}}^{C}$ in $\mathcal{B}^{C}$ and that $\mathcal{F}_{kc}^{C}(\mathbb{R})$ spans $\mathcal{B}^{C}$ in the sense of

$$\mathcal{B}^{C} = \Gamma^{C}(\mathcal{F}_{kc}^{C}(\mathbb{R})) - \Gamma^{C}(\mathcal{F}_{kc}^{C}(\mathbb{R})).$$

The uniqueness of this Banach space $\mathcal{B}$ can also be realized from Theorem 6.

6. Conclusions

The main purpose of this paper is to embed the families of fuzzy sets into a normed space or Banach space. This paper shows that the families $\mathcal{F}_{kc}(U)$, $\mathcal{F}_{kc}^{L}(U)$ and $\mathcal{F}_{kc}^{C}(U)$ can be embedded into a normed space $\mathcal{N}$ or a Banach space $\mathcal{B}$. When the spaces are embedded into the corresponding Banach spaces $\mathcal{B}$, these Banach spaces can be guaranteed to be unique except for isometries in the sense described by Theorems 5 and 6. However, when the spaces are embedded into the corresponding normed spaces $\mathcal{N}$, the issue of uniqueness is unknown. Let $(U, \|\cdot\|)$ be a normed space. The embedding properties are summarized below.

Regarding the space $(\mathcal{F}_{kc}(U), d_{\mathcal{F}})$, given any $\tilde{A} \in \mathcal{F}_{kc}(U)$, the α-level sets $\tilde{A}_{\alpha}$ are compact and convex sets in U for all $\alpha \in [0, 1]$. The space $(\mathcal{F}_{kc}(U), d_{\mathcal{F}})$ can be embedded into a normed space $\mathcal{N}$ or Banach space $\mathcal{B}$ isomorphically and isometrically:

- Consider the embedding function $\pi : \mathcal{F}_{kc}(U) \to \mathcal{N}$. The image $\pi(\mathcal{F}_{kc}(U))$ is a convex cone with vertex $\Theta_{\mathcal{N}}$ in $\mathcal{N}$, and the space $\mathcal{F}_{kc}(U)$ spans $\mathcal{N}$ in the sense of

$$\mathcal{N} = \pi(\mathcal{F}_{kc}(U)) - \pi(\mathcal{F}_{kc}(U)).$$

  When $(U, \|\cdot\|)$ is a Banach space, the image $\pi(\mathcal{F}_{kc}(U))$ is complete and is a closed convex cone with vertex $\Theta_{\mathcal{N}}$ in $\mathcal{N}$.
- Consider the embedding function $\Gamma : \mathcal{F}_{kc}(U) \to \mathcal{B}$. The image $\Gamma(\mathcal{F}_{kc}(U))$ is a convex cone with vertex $\Theta_{\mathcal{B}}$ in $\mathcal{B}$, and the space $\mathcal{F}_{kc}(U)$ spans $\mathcal{B}$ in the sense of

$$\mathcal{B} = \mathrm{cl}(\Gamma(\mathcal{F}_{kc}(U)) - \Gamma(\mathcal{F}_{kc}(U))).$$

  When $(U, \|\cdot\|)$ is a Banach space, the image $\Gamma(\mathcal{F}_{kc}(U))$ is complete and is a closed convex cone with vertex $\Theta_{\mathcal{B}}$ in $\mathcal{B}$. The Banach space $\mathcal{B}$ is unique except for isometries in the sense described by Theorem 5.

Regarding the space $(\mathcal{F}^L_{kc}(U), d_{\mathcal{F}})$, given any $\tilde{A} \in \mathcal{F}^L_{kc}(U)$, the α-level sets $\tilde{A}_\alpha$ are compact and convex sets in U for all $\alpha \in [0,1]$, and function $\eta_{\tilde{A}}$ satisfies the uniform Lipschitz condition with respect to the Hausdorff metric d_H. The space $(\mathcal{F}^L_{kc}(U), d_{\mathcal{F}})$ can be embedded into a normed space $\mathcal{N}^L$ or Banach space $\mathcal{B}^L$ isomorphically and isometrically:

- Consider the embedding function $\pi : \mathcal{F}^L_{kc}(U) \to \mathcal{N}^L$. The image $\pi(\mathcal{F}^L_{kc}(U))$ is a convex cone with vertex $\Theta_{\mathcal{N}^L}$ in $\mathcal{N}^L$, and the space $\mathcal{F}^L_{kc}(U)$ spans $\mathcal{N}^L$ in the sense of

$$\mathcal{N}^L = \pi(\mathcal{F}^L_{kc}(U)) - \pi(\mathcal{F}^L_{kc}(U)).$$

- Consider the embedding function $\Gamma : \mathcal{F}^L_{kc}(U) \to \mathcal{B}^L$. The image $\Gamma(\mathcal{F}^L_{kc}(U))$ is a convex cone with vertex $\Theta_{\mathcal{B}^L}$ in $\mathcal{B}^L$, and the space $\mathcal{F}^L_{kc}(U)$ spans $\mathcal{B}^L$ in the sense of

$$\mathcal{B}^L = \mathrm{cl}\Big(\Gamma(\mathcal{F}^L_{kc}(U)) - \Gamma(\mathcal{F}^L_{kc}(U))\Big).$$

  The Banach space $\mathcal{B}^L$ is unique except for isometries in the sense described by Theorem 5.

Regarding the space $(\mathcal{F}^C_{kc}(U), d_{\mathcal{F}})$, given any $\tilde{A} \in \mathcal{F}^C_{kc}(U)$, the α-level sets $\tilde{A}_\alpha$ are compact and convex sets in U for all $\alpha \in [0,1]$, and the function $\eta_{\tilde{A}}$ is continuous with respect to the Hausdorff metric d_H. The space $(\mathcal{F}^C_{kc}(U), d_{\mathcal{F}})$ can be embedded into a normed space $\mathcal{N}^C$ or Banach space $\mathcal{B}^C$ isomorphically and isometrically:

- Consider the embedding function $\pi : \mathcal{F}^C_{kc}(U) \to \mathcal{N}^C$. The image $\pi(\mathcal{F}^C_{kc}(U))$ is a convex cone with vertex $\Theta_{\mathcal{N}^C}$ in $\mathcal{N}^C$, and the space $\mathcal{F}^C_{kc}(U)$ spans $\mathcal{N}^C$ in the sense of

$$\mathcal{N}^C = \pi(\mathcal{F}^C_{kc}(U)) - \pi(\mathcal{F}^C_{kc}(U)).$$

  When $(U, \|\cdot\|)$ is a Banach space, the image $\pi(\mathcal{F}^C_{kc}(U))$ is complete and is a closed convex cone with vertex $\Theta_{\mathcal{N}^C}$ in $\mathcal{N}^C$. We further assume that the space U is separable. Then, the normed space $\mathcal{N}^C$ is also separable.
- Consider the embedding function $\Gamma : \mathcal{F}^C_{kc}(U) \to \mathcal{B}^C$. The image $\Gamma(\mathcal{F}^C_{kc}(U))$ is a convex cone with vertex $\Theta_{\mathcal{B}^C}$ in $\mathcal{B}^C$, and the space $\mathcal{F}^C_{kc}(U)$ spans $\mathcal{B}^C$ in the sense of

$$\mathcal{B}^C = \mathrm{cl}\Big(\Gamma(\mathcal{F}^C_{kc}(U)) - \Gamma(\mathcal{F}^C_{kc}(U))\Big).$$

  When $(U, \|\cdot\|)$ is a Banach space, the image $\Gamma(\mathcal{F}^C_{kc}(U))$ is complete and is a closed convex cone with vertex $\Theta_{\mathcal{B}^C}$ in $\mathcal{B}^C$. The Banach space $\mathcal{B}^C$ is unique except for isometries in the sense described by Theorem 5. We further assume that the space U is separable. Then, the Banach space $\mathcal{B}^C$ is also separable.

In future research, we may try to look for some other families of fuzzy sets that can still be embedded into some corresponding Banach spaces such that the useful properties

presented in the topic of functional analysis can be borrowed to study practical applications using fuzzy sets.

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