

Article

Distribution Concepts Under Duplicated Structure

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Abstract: In the context of standard states, contributors are typically either fully engaged or not engaged at all under interactions with other contributors. However, in real-world situations, contributors may exhibit varying degrees of involvement and operational capacities. Additionally, related influence may differ across various situations. Therefore, this paper introduces distribution concepts utilizing duplicated structures and weights to address these complexities. To examine the mathematical precision and practical applicability of these distribution concepts, we introduce a specific reduction and its analogues, which provide axiomatic characterizations and dynamic processes.

Keywords: distribution concept; duplicated structure; weight; axiomatic characterization; dynamic process

MSC: 91A12; 91A40; 91B06; 91B16

1. Introduction

In real-world situations, most interaction models consist of numerous participating elements whose utilities are interdependent and often mutually influential. In this study, these participating elements are referred to as *contributors*. Under standard game-theoretical states, a contributor either fully participates or does not participate in the overall interaction model. However, in many practical interaction models, it is more logical and realistic for contributors to engage at varying levels based on differing conditions. Therefore, within game-theoretical *multi-choice states*, contributors can operate at different levels of participation, termed *manipulating capacities* in this study. Consider a situation where multiple firms collaborate on a complex software development project with strict deadlines and penalties for delays. The project's completion depends on each firm's contribution level: the greater the combined efforts, the sooner the project is completed. Here, each firm operates with a distinct level of resource allocation, reflecting a multi-choice state. The coalition's value is defined as the negative of the delay penalty, proportional to each firm's investment toward timely completion.

Calvo and Santos [1] proposed that each manipulating capacity of a contributor within a multi-choice state be viewed as a replicated or *duplicate contributor*. Consequently, interactions within multi-choice states can be understood as interactions among these duplicate contributors, with the resulting benefits reflecting those generated when each contributor participates at a specific manipulating capacity. In essence, this approach establishes a *duplicated structure* relative to multi-choice states. After contributors interact towards certain objectives, the ultimate goal is to assess the value or allocate the resulting benefits fairly among them. In this study, the methods for such evaluations or allocations are termed *distribution concepts*. Based on the notion of duplicated structures under multi-choice states, Calvo and Santos [1] characterized the extended Shapley value [2] developed by van den Nouweland et al. [3]. Inspired by results due to Brink and Lann [4], Driessen and Funakp [5], Moulin [6], Ju and Wettstein [7] and Ransmeier [8], Hwang and Liao [9] and Liao [10] further applied duplicated structures to introduce an extended core and an extended equal allocation of non-separable costs (EANSC) with similar results.



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In game-theoretical analysis, evaluating a distribution concept often involves verifying its unique conformity to a set of logically essential and irreducible axioms, known as *axiomatic processes*. *Consistency*, first explored by Harsanyi [11] through the concept of bilateral equilibrium, is a critical feature in axiomatic processes for distribution concepts. This concept suggests that contributors in a given state form expectations about the state, and use these to guide the calculation of payments. A distribution concept is deemed consistent if it provides identical payments in both the original state and its corresponding reduction, thereby indicating the model's internal "robustness". Consistency has been extensively studied across various fields using reduction analysis, including bankruptcy, taxation, negotiation, fair allocation of indivisible goods, and resource allocation. Different definitions of reduction have emerged depending on how contributors outside the sub-coalition are treated.

The influence contributors exert may vary based on objective and subjective factors. For example, factors like the electoral district weight of a congress member, the dedication of a company employee, or the negotiation skills of a business team can all differ significantly. Additionally, asymmetry can arise when contributors have varying negotiating abilities. Given these considerations, one might expect resources to be allocated proportionally according to the weights assigned to each contributor and their respective levels of effort. *Weights* play a crucial role in resource allocating situations; for instance, in investment strategy allocation, weights might correspond to the profitability of different options. Similarly, a company's year-end bonus distribution could assign weights to departments based on their relative contributions.

Dynamic results from Stearns [12] established a foundational framework in game-theoretical analysis, using convergent transfer schemes to achieve iterative equilibrium through incremental resource adjustments. His approach, which stabilizes interactions without computing entire solution sets, has informed dynamic game theory and influenced fields like economic modeling and evolutionary theory, where sequential adjustments and iterative transfers promote stable outcomes. Building on this, Hwang and Liao [13] explored consistency and the dynamic approach of indexes, highlighting the iterative recalibration needed for balanced allocation in multi-choice settings. Maschler and Owen [14] further extended dynamic principles with the consonant Shapley value for hyperplane games, proposing a consensus-based solution that aligns contributions with underlying game structures. These advancements collectively emphasize the versatility of dynamic approaches in areas such as strategic planning and biology, where iterative processes reflect evolving interactions among contributors. Relevant studies, including those by Benthem et al. [15], Friesz and Mookherjee [16], Ranjbar-Sahraei et al. [17], and Shakiba and Autio [18], further illustrate how dynamic methods capture the adaptability essential to cooperative and competitive contexts alike.

The concept of an *individual index* is a commonly used assessment tool in allocation processes, typically used to evaluate the impact of an individual, factor, or variable's specific change on the overall environment. Hsieh and Liao [19] were the first to adopt the individual index to propose the *pseudo-equal allocation of non-separable costs* (PEANSC), later applying a specific reduction and related consistency to show that PEANSC is a stable distribution concept. Further, Hsieh and Liao [19] developed the weighted distribution concept, known as the *pseudo-weighted allocation of non-separable costs* (PWANSC).

Based on the preceding discussion, the following motivations emerge:

- To explore whether the concept of duplicated structures can extend the distribution concepts of PEANSC and PWANSC to multi-choice state frameworks.
- To first employ axiomatic processes to validate the mathematical accuracy and practical applicability of these two extended distribution concepts. Furthermore, to apply dynamic processes to describe how any efficient outcome vector might progressively approach these extended distribution concepts through specific procedures.

This paper aims to address these motivations, presenting several main results:

1. In Section 2, this study proposes that the *duplicated PEANSC* of a multi-choice state corresponds to the PEANSC of the corresponding duplicated structures. Additionally, we introduce a novel generalization of the PWANSC, termed the *duplicated PWANSC*, which incorporates weights and duplicated structures under multi-choice states.
2. In order to present the mathematical validity and practical applicability, this study defines a specific reduction to characterize these extended distribution concepts in Section 3.
3. Within the framework of multi-choice states, this study defines two correction functions in Section 4. Through related dynamic processes derived from these correction functions, any efficient outcome vector can gradually converge toward these two distribution concepts.

2. Duplicated Distribution Concepts

2.1. Preliminaries

Let $\overline{UC}$ be the universe of all potential contributors (for instance, the set formed by people in the world). Any $p \in \overline{UC}$ is a contributor (for instance, a person in the world). For $p \in \overline{UC}$, $\overline{M}_p = \{0, 1, \dots, \zeta_p\}$ can be treated as the set of all manipulating capacities of contributor p , where 0 means not acting. Suppose that $\overline{C} \subseteq \overline{UC}$, $\overline{C} \neq \emptyset$ is the entire grand collection of contributors to a specific interactive surrounding (for example, all members in an institution). For $\overline{C} \subseteq \overline{UC}$, $\overline{C} \neq \emptyset$, let $\overline{M}^{\overline{C}} = \prod_{p \in \overline{C}} \overline{M}_p$ be the product set of the capacity collections for contributors $\overline{C}$. Denote the zero vector under $\mathbb{R}^{\overline{C}}$ to be $0_{\overline{C}}$.

A **multi-choice state** is denoted by $(\overline{C}, \zeta, v)$, where $\overline{C} \neq \emptyset$ is finite collection of contributors, $\zeta = (\zeta_p)_{p \in \overline{C}}$ is the vector that shows the amount of manipulating capacities for each contributor, and $v : \overline{M}^{\overline{C}} \rightarrow \mathbb{R}$ is a function which assigns to each participating vector $\rho = (\rho_p)_{p \in \overline{C}} \in \overline{M}^{\overline{C}}$ the worth that the contributors can gain if each contributor p operates at manipulating capacity $\rho_p \in \overline{M}_p$ with $v(0_{\overline{C}}) = 0$. In multi-choice states, contributors can adopt varying manipulating capacities, and the potential value a coalition can achieve depends on the manipulating capacity chosen by each contributor. We denote the family of all multi-choice states as $[\mathbf{MCS}]$.

Let $(\overline{C}, \zeta, v) \in [\mathbf{MCS}]$. A **unit capacity outcome vector** of $(\overline{C}, \zeta, v)$ is a vector $x = (x_p)_{p \in \overline{C}}$ where the x_p means the **unit outcome** that contributor p obtains for each $p \in \overline{C}$, hence $\zeta_p \cdot x_p$ is the **accumulation outcome** that contributor p obtains at $(\overline{C}, \zeta, v)$. For simpleness, one could write “outcome vector” instead of “unit capacity outcome vector”. A **distribution concept** on $[\mathbf{MCS}]$ is a mapping γ appointing to every $(\overline{C}, \zeta, v) \in [\mathbf{MCS}]$ an outcome

$$\gamma(\overline{C}, \zeta, v) = (\gamma_p(\overline{C}, \zeta, v))_{p \in \overline{C}} \in \mathbb{R}^{\overline{C}}.$$

For convenience, one could define that $\gamma_p(\overline{C}, \zeta, v) = 0$ for each $p \in \overline{C}$ with $\zeta_p = 0$.

Given that $(\overline{C}, \zeta, v) \in [\mathbf{MCS}]$, $\rho \in \overline{M}^{\overline{C}}$ and $H \subseteq \overline{C}$, $|H|$ is the amount of the elements in H , $\rho_H \in \mathbb{R}^H$ is the restriction of ρ to H , $\rho(H) = \sum_{p \in H} \rho_p$ and $Z(\rho) = \{k \in \overline{C} \mid \rho_k \neq 0\}$. Let $\rho, \iota \in \mathbb{R}^{\overline{C}}$; we define $\iota \leq \rho$ if $\iota_p \leq \rho_p$ for all $p \in \overline{C}$.

Consider $(\overline{C}, \zeta, v) \in [\mathbf{MCS}]$. One defines a function $w : \overline{C} \rightarrow \mathbb{R}^+$ as a **weight function** if w is a non-negative mapping. Additionally, one defines $|H|_w = \sum_{p \in H} w(p)$ and $\rho(w) = \sum_{p \in \overline{C}} w(p) \rho_p$ for all weight functions w , for all $\rho \in \overline{M}^{\overline{C}}$, and for all $H \subseteq \overline{C}$. Depending on the circumstances, contributors in $\overline{C}$ can be assigned different weights through weight functions. These weights can be viewed as *a priori measures of importance*, representing factors not captured by the characteristic mapping. For example, when allocating travel expenses among various sites, weights could correspond to the number of days spent at each location (cf. Shapley [2]).

A **standard state** is a pair $(\overline{C}, V)$ where $\overline{C} \neq \emptyset$ is a finite coalition of contributor and V is a function $V : 2^{\overline{C}} \rightarrow \mathbb{R}$ with $V(\emptyset) = 0$. Without loss of generality, one could suppose

that $Z(\zeta) = \bar{\mathbf{C}}$. Denote the class of all standard states to be $[\mathbf{SS}]$. A **distribution concept** on $[\mathbf{SS}]$ is a map Γ which appoints an element $\Gamma(\bar{\mathbf{C}}, V)$ of $\mathbb{R}^{\bar{\mathbf{C}}}$ to every $(\bar{\mathbf{C}}, V) \in [\mathbf{SS}]$.

Given $(\bar{\mathbf{C}}, \zeta, v) \in [\mathbf{MCS}]$, let $\bar{\mathbf{C}}^\zeta$ be a collection of **duplicated contributors** as follows:

$$\bar{\mathbf{C}}^\zeta = \bigcup_{p \in \bar{\mathbf{C}}} \bar{\mathbf{C}}_p^\zeta,$$

where, for all $p \in Z(\zeta)$, $\bar{\mathbf{C}}_p^\zeta = \{p_1, \dots, p_{\zeta_p}\}$, and for all $p \notin Z(\zeta)$, $\bar{\mathbf{C}}_p^\zeta = \emptyset$. Now, for any $H \subseteq \bar{\mathbf{C}}^\zeta$, we define the action vector $\Omega^H \in \bar{\mathbf{M}}^{\bar{\mathbf{C}}}$ by for all $p \in \bar{\mathbf{C}}$,

$$\Omega_p^H = |H \cap \bar{\mathbf{C}}_p^\zeta|.$$

Then, we define the **duplicated state** $(\bar{\mathbf{C}}^\zeta, V^d)$ for all $H \subseteq \bar{\mathbf{C}}^\zeta$,

$$V^d(H) = v(\Omega^H).$$

Definition 1.

- The **pseudo-equal allocation of non-separable costs (PEANSC, Hsieh and Liao [19])** Ψ is defined by

$$\Psi_p(\bar{\mathbf{C}}, V) = V(\{p\}) + \frac{1}{|\bar{\mathbf{C}}|} \cdot \left[V(\bar{\mathbf{C}}) - \sum_{q \in \bar{\mathbf{C}}} V(\{q\}) \right] \quad (1)$$

for each $(\bar{\mathbf{C}}, V) \in [\mathbf{SS}]$ and for each $p \in \bar{\mathbf{C}}$. Based on the distributing notion of Ψ , contributors firstly partake its marginal utility, and then distribute uniformly the rest of utility. Further, the **duplicated PEANSC** ψ is defined by

$$\psi_p(\bar{\mathbf{C}}, \zeta, v) = \Psi_q(\bar{\mathbf{C}}^\zeta, V^d) = \frac{1}{\zeta_p} \cdot \sum_{t \in \bar{\mathbf{C}}_p^\zeta} \Psi_t(\bar{\mathbf{C}}^\zeta, V^d) \quad (2)$$

for each $(\bar{\mathbf{C}}, \zeta, v) \in [\mathbf{MCS}]$, for each weight function w , for each $p \in \bar{\mathbf{C}}$ and for each $q \in \bar{\mathbf{C}}_p^\zeta$, where $(\bar{\mathbf{C}}^\zeta, V^d)$ is the corresponding duplicated state. Equations (1) and (2) assert that the duplicated PEANSC of a multi-choice state is the PEANSC of a corresponding duplicated state.

- The **pseudo-weighted allocation of non-separable costs (PWANSC, Hsieh and Liao [19])** Ψ^w is defined by

$$\Psi_p^w(\bar{\mathbf{C}}, V) = V(\{p\}) + \frac{w(p)}{|\bar{\mathbf{C}}|_w} \cdot \left[V(\bar{\mathbf{C}}) - \sum_{q \in \bar{\mathbf{C}}} V(\{q\}) \right] \quad (3)$$

for each $(\bar{\mathbf{C}}, V) \in [\mathbf{SS}]$, for each weight function w , and for each $p \in \bar{\mathbf{C}}$. Based on the allocating notion of Ψ^w , contributors firstly partake its marginal contribution, and then distribute proportionally the rest of utility by weights. Further, the **duplicated PWANSC** ψ^w is defined by

$$\psi_p^w(\bar{\mathbf{C}}, \zeta, v) = \Psi_q^w(\bar{\mathbf{C}}^\zeta, V^d) = \frac{1}{\zeta_p} \cdot \sum_{t \in \bar{\mathbf{C}}_p^\zeta} \Psi_t^w(\bar{\mathbf{C}}^\zeta, V^d) \quad (4)$$

for each $(\bar{\mathbf{C}}, \zeta, v) \in [\mathbf{MCS}]$, for each weight function w , for each $p \in \bar{\mathbf{C}}$ and for each $q \in \bar{\mathbf{C}}_p^\zeta$, where $(\bar{\mathbf{C}}^\zeta, V^d)$ is the corresponding duplicated state. Equations (3) and (4) assert that the duplicated PWANSC of a multi-choice state is the PWANSC of a corresponding duplicated state.

2.2. Practical Examples

In modern organizations, contributors often operate with varying levels of responsibility and influence, depending on the specific context of their roles. This example will demonstrate how the multi-choice allocation framework proposed in this study can be applied to a real-world scenario, such as project management within a utility company.

- **Utility company project management:** Consider a utility company managing a large-scale infrastructure project that requires input from different departments such as engineering, logistics, and regulatory compliance. Let us define the set of contributors as $\bar{C}$, representing these key departments. The function v acts as a utility assessment function, which measures the overall value generated by the combined efforts of the contributors when operating with their designated capacities, represented by the vector $\rho = (\rho_p)_{p \in \bar{C}}$. Here, ρ_p indicates the capacity of each department p .
- **Multi-choice states and utility allocation:** In practice, each department's contribution is not necessarily equal; thus, assigning appropriate weights to reflect the importance of each department's role is crucial. For example, engineering may play a more pivotal role in certain stages of the project, whereas regulatory compliance may become more significant during the final approval phase.

Under this allocation framework, the utility function v assesses the combined utility based on the operational capacities of each department. This function considers interactions between departments that may lead to synergistic benefits or, conversely, friction and inefficiencies. The multi-choice allocation model accounts for these dynamics by evaluating the contributions of each department under various operational situations.

- **Numerical example of allocation:** To illustrate, assume the project involves two primary contributors: the Engineering department (i) and the Regulatory department (j). Let the capacity vector be $\zeta = (2, 1)$, representing the levels of operational capacity for these departments. The utility assessments are as follows:
 - $v(2, 1) = 9$ (full operational capacity);
 - $v(1, 0) = -3$ (engineering alone at reduced capacity);
 - $v(1, 1) = 12$ (both departments at moderate capacity);
 - $v(0, 1) = 3$ (regulatory alone at capacity);
 - $v(2, 0) = -1$ (engineering at full capacity but regulatory inactive);
 - $v(0, 0) = 0$ (no contribution).

Breaking down the contributors:

- $\bar{C}_i^\zeta = i_1, i_2$ (Engineering contributors);
- $\bar{C}_j^\zeta = j_1$ (Regulatory contributor);
- $\bar{C}^\zeta = i_1, i_2, j_1$ (total contributor set).

Utility under different sub-coalitions:

- $V^d(i_1, i_2, j_1) = v(2, 1)$;
- $V^d(i_1, j_1) = V^d(i_2, j_1) = v(1, 1)$;
- $V^d(i_1, i_2) = v(2, 0)$;
- $V^d(i_1) = V^d(i_2) = v(1, 0)$;
- $V^d(j_1) = v(0, 1)$;
- $V^d(\emptyset) = v(0, 0)$.

Assume weight functions where $w(i_1) = w(i_2) = 1$ and $w(j_1) = 8$. According to the proposed allocation framework, the contribution of each department can be assessed as follows:

- $\psi_i(\bar{C}, \zeta, v) = \Psi_{i_1}(\bar{C}^\zeta, V^d) = \Psi_{i_2}(\bar{C}^\zeta, V^d) = 1$;
- $\psi_j(\bar{C}, \zeta, v) = \Psi_{j_1}(\bar{C}^\zeta, V^d) = 7$.

Weighted contributions:

- $\psi_i^w(\bar{C}, \zeta, v) = \Psi_{i_1}^w(\bar{C}^\zeta, V^d) = \Psi_{i_2}^w(\bar{C}^\zeta, V^d) = -1.8$;

$$- \quad \psi_j^w(\bar{\mathbf{C}}, \zeta, v) = \Psi_{j_1}^w(\bar{\mathbf{C}}^\zeta, V^d) = 12.6.$$

- Conclusion: this practical example demonstrates how the allocation framework can be applied to real-world project management, showing the significance of considering multiple contributors, their varying capacities, and weighted influences to achieve an optimal utility assessment.

3. Axiomatic Characterization

Based on a reduced state, several axiomatic results of the duplicated PEANSC and the duplicated PWANSC will be provided in this section.

The following reduction is rooted in the concept of “re-allocating upon disagreement”. In a multi-choice TU state (K, ζ_K, v_K^γ) , if a group or individual K within $\bar{\mathbf{C}}$ disputes the distribution concept γ , a re-allocating is initiated. Should none of the dissenting parties wish to operate in this re-allocating, the resulting utility is zero. In situations where multiple parties dissent but only one seeks re-allocating, the process is simplified to avoid unnecessary complexity, allotting that party the utility it would receive by operating independently. When multiple dissenting parties operate in the re-allocating, all agreeing parties fully engage in the process. Upon completion, the agreeing parties receive its share as per the original outcomes assigned by γ and subsequently exit, leaving the remaining utility to be distributed among the dissenting parties according to the outcome under re-allocating.

- Given a distribution concept γ , $(\bar{\mathbf{C}}, \zeta, v) \in [\mathbf{MCS}]$ and $K \subseteq \bar{\mathbf{C}}$, the **multi-choice revised-reduced state** (K, ζ_K, v_K^γ) with respect to γ and K is defined by for all $\rho \in \bar{\mathbf{M}}^K$,

$$v_K^\gamma(\rho) = \begin{cases} 0 & , \rho = 0_K, \\ v(\rho) & , |K| \geq 2 \text{ and } \rho = (t, 0_{K \setminus \{i\}}) \text{ for some } t \in \bar{\mathbf{M}}_i^+, \\ v(\rho, \zeta_{\bar{\mathbf{C}} \setminus K}) - \sum_{p \in \bar{\mathbf{C}} \setminus K} \zeta_p \cdot \gamma_p(\bar{\mathbf{C}}, \zeta, v) & , \text{ otherwise.} \end{cases}$$

- The requirement of *multi-choice revised-consistency* can be understood in the following way. For any coalition of contributors within a state, a “reduced state” can be defined by considering the amounts left after the other contributors have received its outcomes determined by the distribution concept γ . The distribution concept γ is deemed *multi-choice revised-consistent* if, when applied to every reduced state, it always produces outcomes that coincide with those in the initial state. Formally, a distribution concept γ satisfies **multi-choice revised-consistency (MCRCON)** if $\gamma_p(K, \zeta_K, v_K^\gamma) = \gamma_p(\bar{\mathbf{C}}, \zeta, v)$ for each $(\bar{\mathbf{C}}, \zeta, v) \in [\mathbf{MCS}]$, for each $K \subseteq \bar{\mathbf{C}}$ and for each $p \in K$.

Remark 1.

1. Hsieh and Liao [19] defined a specific reduction on $[\mathbf{SS}]$ as follows. Given distribution concept Γ , $(\bar{\mathbf{C}}, V) \in [\mathbf{SS}]$ and $K \subseteq \bar{\mathbf{C}}$, the **revised reduced state** $(K, (V)_K^\Gamma)$ is defined for each $H \subseteq K$ by

$$(V)_K^\Gamma(H) = \begin{cases} 0 & , H = \emptyset, \\ V(H) & , |K| \geq 2 \text{ and } H = \{i\} \text{ for some } i \in H, \\ V(H \cup (\bar{\mathbf{C}} \setminus K)) - \sum_{t \in \bar{\mathbf{C}} \setminus K} \Gamma_t(\bar{\mathbf{C}}, V) & , \text{ otherwise.} \end{cases}$$

Hsieh and Liao [19] examined the revised reduction $(K, (V)_K^\Gamma)$, where each coalition within the subgroup K can distribute outcomes to its contributors only if these outcomes are revised consistent with the initial outcomes given to the contributors in $\bar{\mathbf{C}} \setminus K$. Absolutely, the multi-choice revised-reduction defined in this paper is a multi-choice generalization of the revised reduction.

2. Γ satisfies **revised consistency (RECON)** if $\Gamma_p(\bar{\mathbf{C}}, V) = \Gamma_p(K, (V)_K^\Gamma)$ for each $(\bar{\mathbf{C}}, V) \in [\mathbf{SS}]$, for each $K \subseteq \bar{\mathbf{C}}$ and for each $p \in K$. Hsieh and Liao [19] characterized that the PEANSC and the PWANSC by means of RECON.

Let γ be a distribution concept on $[\mathbf{MCS}]$ and Γ be a distribution concept on $[\mathbf{SS}]$. Suppose that, for all $(\bar{\mathbf{C}}, \zeta, v) \in [\mathbf{MCS}]$, and the corresponding duplicated states $(\bar{\mathbf{C}}^\zeta, V^d)$, for all $p \in \bar{\mathbf{C}}$, and for all $q \in \bar{\mathbf{C}}_p^\zeta$,

$$\gamma_p(\bar{\mathbf{C}}, \zeta, v) = \Gamma_q(\bar{\mathbf{C}}^\zeta, V^d) = \frac{1}{\zeta_p} \cdot \sum_{t \in \bar{\mathbf{C}}_p^\zeta} \Gamma_t(\bar{\mathbf{C}}^\zeta, V^d). \quad (5)$$

In fact, “the corresponding duplicated state of the revised reduction of a multi-choice state” coincides with “the revised reduction of the corresponding duplicated state of a multi-choice state”, i.e., the order among “revised reduction” and “duplication” does not matter.

Lemma 1. Let $(\bar{\mathbf{C}}, \zeta, v) \in [\mathbf{MCS}]$ and $(\bar{\mathbf{C}}^\zeta, V^d)$ be the corresponding duplicated state. Let γ be a distribution concept on $[\mathbf{MCS}]$ and Γ be a distribution concept on $[\mathbf{SS}]$. Suppose that γ and Γ satisfy Equation (5). Let $K \subseteq \bar{\mathbf{C}}$ and $\bigcup_{p \in K} \bar{\mathbf{C}}_p^\zeta = K^\zeta = K^{\zeta_K}$. Then,

$$(K^{\zeta_K}, (v_K^\gamma)_K^d) = (K^\zeta, (V^d)_{K^\zeta}^\Gamma), \quad (6)$$

where $(K^{\zeta_K}, (v_K^\gamma)_K^d)$ is the corresponding duplicated state of (K, ζ_K, v_K^γ) . Furthermore, for all $p \in Z(\zeta_K)$, and for all $p_q \in \bar{\mathbf{C}}_p^\zeta$,

$$\gamma_p(K, \zeta_K, v_K^\gamma) = \Gamma_{p_q}(K^\zeta, (V^d)_{K^\zeta}^\Gamma) = \frac{1}{\zeta_p} \cdot \sum_{p_k \in \bar{\mathbf{C}}_p^\zeta} \Gamma_{p_k}(K^\zeta, (V^d)_{K^\zeta}^\Gamma). \quad (7)$$

Proof. To verify Equation (6), let $K \subseteq \bar{\mathbf{C}}$ and $\bigcup_{p \in K} \bar{\mathbf{C}}_p^\zeta = K^\zeta = K^{\zeta_K}$. Let $H \subseteq K^\zeta$. Two situations can be distinguished:

- **Case 1:** if $H = K^\zeta$, then

$$\begin{aligned} (v_K^\gamma)_K^d(K^\zeta) &= v_K^\gamma(\Omega^{K^\zeta}) \\ &= v_K^\gamma(\zeta_K) \\ &= v(\zeta) - \sum_{p \in \bar{\mathbf{C}} \setminus K} \zeta_p \cdot \gamma_p(\bar{\mathbf{C}}, \zeta, v) \\ &= V^d(\bar{\mathbf{C}}^\zeta) - \sum_{p \in \bar{\mathbf{C}} \setminus K} \zeta_p \cdot \left[\frac{1}{\zeta_p} \sum_{p_k \in \bar{\mathbf{C}}_p^\zeta} \Gamma_{p_k}(\bar{\mathbf{C}}^\zeta, V^d) \right] \\ &= V^d(\bar{\mathbf{C}}^\zeta) - \sum_{t \in \bar{\mathbf{C}}^\zeta \setminus K^\zeta} \Gamma_t(\bar{\mathbf{C}}^\zeta, V^d) \\ &= (V^d)_{K^\zeta}^\Gamma(K^\zeta). \end{aligned}$$

- **Case 2:** if $H \subseteq K^\zeta, H \neq K^\zeta$, then

$$\begin{aligned} (v_K^\gamma)_K^d(H) &= v_K^\gamma(\Omega^H) \\ &= v(\Omega^H, \zeta_{\bar{\mathbf{C}} \setminus K}) - \sum_{p \in \bar{\mathbf{C}} \setminus K} \zeta_p \cdot \gamma_p(\bar{\mathbf{C}}, \zeta, v) \\ &= V^d(H \cup (\bar{\mathbf{C}}^\zeta \setminus K^\zeta)) - \sum_{p \in \bar{\mathbf{C}} \setminus K} \zeta_p \cdot \left[\frac{1}{\zeta_p} \sum_{p_k \in \bar{\mathbf{C}}_p^\zeta} \Gamma_{p_k}(\bar{\mathbf{C}}^\zeta, V^d) \right] \\ &= V^d(H \cup (\bar{\mathbf{C}}^\zeta \setminus K^\zeta)) - \sum_{t \in \bar{\mathbf{C}}^\zeta \setminus K^\zeta} \Gamma_t(\bar{\mathbf{C}}^\zeta, V^d) \\ &= (V^d)_{K^\zeta}^\Gamma(H). \end{aligned}$$

Finally, Equation (7) follows via Equations (5) and (6). $\square$

Lemma 2. *The duplicated PEANSC ψ and the duplicated PWANSC ψ^w satisfy MCRCON.*

Proof. This lemma immediately follows via Definition 1 and Lemma 1. $\square$

To characterize the duplicated PEANSC and the duplicated PWANSC, one would make use of some more properties. Let γ be a distribution concept on $[\mathbf{MCS}]$. γ satisfies **MCS-efficiency (MCSEFF)** if, for each $(\bar{C}, \zeta, v) \in [\mathbf{MCS}]$, $\sum_{p \in \bar{C}} \zeta_p \cdot \gamma_p(\bar{C}, \zeta, v) = v(\zeta)$. γ satisfies **weak MCS-efficiency (WMCSFF)** if, for each $(\bar{C}, \zeta, v) \in [\mathbf{MCS}]$ with $|\bar{C}| = 1$, γ satisfies MCSFF under $(\bar{C}, \zeta, v)$. γ satisfies **MCS-standard for two-person states (MCSTS)** if, for each $(\bar{C}, \zeta, v) \in [\mathbf{MCS}]$ with $|\bar{C}| = 2$, $\gamma_p(\bar{C}, \zeta, v) = \psi_p(\bar{C}, \zeta, v)$. γ satisfies **weighted MCS-standard for two-person states (WMCSTS)** if, for each $(\bar{C}, \zeta, v) \in [\mathbf{MCS}]$ with $|\bar{C}| = 2$, and for each weight function w , $\gamma_p(\bar{C}, \zeta, v) = \psi_p^w(\bar{C}, \zeta, v)$.

MCSFF asserts that all contributors should fully allocate the entire utility. WMCSFF states that single-person states should be resolved efficiently. MCSTS and WMCSTS are analogues of the two-person standardness notion developed by Hart and Mas-Colell [20]. MCSTS asserts that in two-person states, all contributors allocate the entire utility through the distribution concept ψ . Similarly, WMCSTS asserts that in two-person states, all contributors allocate the entire utility through the weighted distribution concept ψ^w .

Remark 2. *By Definition 1, the duplicated PEANSC (PWANSC) satisfies MCSFF and MCSTS (WMCSTS) absolutely. It is also trivial to originate that a distribution concept γ satisfies WMCSFF if γ satisfies MCSTS (WMCSTS) and CON. Indeed, $\zeta_p \cdot \gamma(\{p\}, \zeta_p, v) = v(\zeta_p) = \zeta_p \cdot \psi(\{p\}, \zeta_p, v) (= \psi^w(\{p\}, \zeta_p, v))$ for each $(\{p\}, \zeta_p, v) \in [\mathbf{MCS}]$. Related proofs are coincident with Hart and Mas-Colell [20].*

Inspired by Hart and Mas-Colell [20], the duplicated PEANSC and the duplicated PWANSC would be axiomatized by means of consistency and relative two-person standardness.

Lemma 3. *On $[\mathbf{MCS}]$, a distribution concept γ satisfies MCSFF if it satisfies WMCSFF and MCRCON.*

Proof. Let $(\bar{C}, \zeta, v) \in [\mathbf{MCS}]$, and γ be a distribution concept on $[\mathbf{MCS}]$ satisfying WMCSFF and MCRCON. It is trivial if $|\bar{C}| = 1$ via WMCSFF. Assume that $|\bar{C}| \geq 2$. Let $q \in \bar{C}$, consider the reduction $(\{q\}, \zeta_q, v_{\{q\}}^\gamma)$ of $(\bar{C}, \zeta, v)$. By definition of $v_{\{q\}}^\gamma$,

$$v_{\{q\}}^\gamma(\zeta_q) = v(\zeta) - \sum_{p \in \bar{C} \setminus \{q\}} \zeta_p \cdot \gamma_p(\bar{C}, \zeta, v).$$

Since γ satisfies MCRCON,

$$\gamma_q(\bar{C}, \zeta, v) = \gamma_q(\{q\}, \zeta_q, v_{\{q\}}^\gamma).$$

By WMCSFF,

$$\gamma_q(\bar{C}, \zeta, v) = v_{\{q\}}^\gamma(\zeta_q).$$

Thence, $\sum_{p \in \bar{C}} \zeta_p \cdot \gamma_p(\bar{C}, \zeta, v) = v(\zeta)$, i.e., γ satisfies MCSFF. $\square$

Theorem 1.

1. On $[\mathbf{MCS}]$, a distribution concept γ satisfies MCSTS and MCRCON if and only if $\gamma = \Psi$.
2. On $[\mathbf{MCS}]$, a distribution concept γ satisfies WMCSTS and MCRCON if and only if $\gamma = \Psi^w$.

Proof. Clearly, Ψ and Ψ^w satisfy MCSTS and WMCSTS, respectively. By Lemma 2, Ψ and Ψ^w satisfy MCRCON simultaneously.

To present the uniqueness of result 1, suppose γ satisfies MCSTS and MCRCON. Let $(\bar{\mathbf{C}}, \zeta, v) \in [\mathbf{MCS}]$. If $|\bar{\mathbf{C}}| \leq 2$, then by Remark 2 and MCSTS of γ , $\gamma(\bar{\mathbf{C}}, \zeta, v) = \Psi(\bar{\mathbf{C}}, \zeta, v)$. The condition $|\bar{\mathbf{C}}| > 2$: Let $p \in \bar{\mathbf{C}}$ and $K = \{p, k\}$ for some $k \in \bar{\mathbf{C}} \setminus \{p\}$. Then,

$$\begin{aligned}
 & \gamma_p(\bar{\mathbf{C}}, \zeta, v) - \psi_p(\bar{\mathbf{C}}, \zeta, v) \\
 &= \gamma_p(K, \zeta_K, v_K^\gamma) - \psi_p(K, \zeta_K, v_K^\psi) \quad (\text{by MCRCON of } \gamma) \\
 &= \psi_p(K, \zeta_K, v_K^\gamma) - \psi_p(K, \zeta_K, v_K^\psi) \quad (\text{by MCSTS of } \gamma) \\
 &= \frac{1}{\zeta_p} \sum_{p_q \in \bar{\mathbf{C}}_p^\zeta} \Psi_{p_q}(K^\zeta, (V^d)_{K^\zeta}^\Gamma) - \frac{1}{\zeta_p} \sum_{p_q \in \bar{\mathbf{C}}_p^\zeta} \Psi_{p_q}(K^\zeta, (V^d)_{K^\zeta}^\Psi) \quad (\text{by Lemma 1}) \\
 &= \frac{1}{\zeta_p} \sum_{p_q \in \bar{\mathbf{C}}_p^\zeta} \left[(V^d)_{K^\zeta}^\Gamma(\{p_q\}) + \frac{1}{|K^\zeta|} \cdot [(V^d)_{K^\zeta}^\Gamma(K^\zeta) - \sum_{t \in K^\zeta} (V^d)_{K^\zeta}^\Gamma(\{t\})] \right] \\
 &\quad - \frac{1}{\zeta_p} \sum_{p_q \in \bar{\mathbf{C}}_p^\zeta} \left[(V^d)_{K^\zeta}^\Psi(\{p_q\}) + \frac{1}{|K^\zeta|} \cdot [(V^d)_{K^\zeta}^\Psi(K^\zeta) - \sum_{t \in K^\zeta} (V^d)_{K^\zeta}^\Psi(\{t\})] \right] \\
 &= \frac{1}{\zeta_p} \sum_{p_q \in \bar{\mathbf{C}}_p^\zeta} \frac{1}{|K^\zeta|} \left[(V^d)_{K^\zeta}^\Gamma(K^\zeta) - (V^d)_{K^\zeta}^\Psi(K^\zeta) \right] \quad (\text{by definitions of } (V^d)_{K^\zeta}^\Gamma \text{ and } (V^d)_{K^\zeta}^\Psi) \\
 &= \frac{|\bar{\mathbf{C}}_p^\zeta|}{\zeta_p \cdot |K^\zeta|} \cdot [(V^d)_{K^\zeta}^\Gamma(K^\zeta) - (V^d)_{K^\zeta}^\Psi(K^\zeta)].
 \end{aligned}$$

Similarly,

$$\gamma_k(\bar{\mathbf{C}}, \zeta, v) - \psi_k(\bar{\mathbf{C}}, \zeta, v) = \frac{|\bar{\mathbf{C}}_k^\zeta|}{\zeta_k \cdot |K^\zeta|} \cdot [(V^d)_{K^\zeta}^\Gamma(K^\zeta) - (V^d)_{K^\zeta}^\Psi(K^\zeta)].$$

Clearly, γ and Ψ satisfy MCSFF via Remark 2 and Lemma 3. By MCSFF of γ and ψ ,

$$\begin{aligned}
 [\gamma_p(\bar{\mathbf{C}}, \zeta, v) - \psi_p(\bar{\mathbf{C}}, \zeta, v)] \cdot \sum_{k \in \bar{\mathbf{C}}} \frac{\zeta_k}{|\bar{\mathbf{C}}_k^\zeta|} &= \frac{\zeta_p}{|\bar{\mathbf{C}}_p^\zeta|} \sum_{k \in \bar{\mathbf{C}}} [\gamma_k(\bar{\mathbf{C}}, \zeta, v) - \psi_k(\bar{\mathbf{C}}, \zeta, v)] \\
 &= \frac{\zeta_p}{|\bar{\mathbf{C}}_p^\zeta|} [v(\zeta) - v(\zeta)] \\
 &= 0.
 \end{aligned}$$

That is, $\gamma_p(\bar{\mathbf{C}}, \zeta, v) = \psi_p(\bar{\mathbf{C}}, \zeta, v)$ for all $p \in \bar{\mathbf{C}}$. The proof of the uniqueness of result 2 is similar, and is omitted. $\square$

4. Dynamic Approach

Under the framework of standard states, Maschler and Owen [14] introduced a specific reduced state to present dynamic processes that lead the contributors to the Shapley value [2] as follows.

- Let $(\bar{\mathbf{C}}, V) \in [\mathbf{SS}]$. An **outcome vector** of $(\bar{\mathbf{C}}, V)$ is a vector $(X_p)_{p \in \bar{\mathbf{C}}} \in \mathbb{R}^{\bar{\mathbf{C}}}$ where X_p denotes the outcome to the contributor p for all $p \in \bar{\mathbf{C}}$. An outcome vector $X \in \mathbb{R}^{\bar{\mathbf{C}}}$ is **efficient** under $(\bar{\mathbf{C}}, V)$ if $\sum_{p \in \bar{\mathbf{C}}} X_p = V(\bar{\mathbf{C}})$. Define that $E(\bar{\mathbf{C}}, V) = \{X \in \mathbb{R}^{\bar{\mathbf{C}}} | X \text{ is efficient under } (\bar{\mathbf{C}}, V)\}$. Let $(\bar{\mathbf{C}}, V) \in [\mathbf{SS}]$, $K \subseteq \bar{\mathbf{C}}$, $X \in E(\bar{\mathbf{C}}, V)$ and Γ be a distribution concept. The (X, Γ) -**dependent standard reduced state** $(K, (V)_K^X)$ for each $H \subseteq K$ is defined by

$$(V)_K^X(H) = \begin{cases} V(\bar{\mathbf{C}}) - \sum_{p \in \bar{\mathbf{C}} \setminus K} X_p & , \text{ if } H = K, \\ (V)_K^\Gamma(H) & , \text{ otherwise.} \end{cases}$$

The (X, Γ) -dependent standard reduction is defined based on the following concept. Suppose there exists a coalition K of contributors who are dissatisfied with an efficient outcome vector X ; In this case, a re-assessing process is initiated. If all dissatisfied contributors would like to operate in the re-assessing, then all contributors who are content

with X also join in the re-assessing. Once the process is completed, contributors who were initially satisfied with X retain their original outcome from X and exit the process. If not all dissatisfied contributors opt to operate, however, a revised reduction is selected based on a reasonable distribution concept Γ to guide the re-assessing.

Next, the (X, Γ) -dependent standard reduced state would be extended to multi-choice states as follows.

- Let $(\bar{C}, \zeta, v) \in [\mathbf{MCS}]$. A vector $x \in \mathbb{R}^{\bar{C}}$ is **efficient** under $(\bar{C}, \zeta, v)$ if $\sum_{p \in \bar{C}} \zeta_p x_p = v(\zeta)$. Define $E(\bar{C}, \zeta, v) = \{x \in \mathbb{R}^{\bar{C}} | x \text{ is efficient under } (\bar{C}, \zeta, v)\}$. Let $(\bar{C}, \zeta, v) \in [\mathbf{MCS}]$, $K \subseteq \bar{C}$, $x \in E(\bar{C}, \zeta, v)$ and γ be a distribution concept. The (x, γ) -**dependent reduced state** $(K, \zeta_K, v_K^{x, \gamma})$ is defined by for each $\rho \in \bar{M}^S$,

$$v_K^{x, \gamma}(\rho) = \begin{cases} v(\zeta) - \sum_{p \in \bar{C} \setminus K} \zeta_p \cdot x_p & , \text{ if } \rho = \zeta_K, \\ v_K^\psi(\rho) & , \text{ otherwise.} \end{cases}$$

Lemma 4. Let $(\bar{C}, \zeta, v) \in [\mathbf{MCS}]$, $x \in E(\bar{C}, \zeta, v)$ and $(\bar{C}^\zeta, V^d)$ be the corresponding duplicated state. And let $X \in E(\bar{C}^\zeta, V^d)$ with $X_{p_q} = x_p$ for all $p \in \bar{C}$ and for all $p_q \in \bar{C}_p^\zeta$. Let $K \subseteq \bar{C}$ and $\bigcup_{p \in K} \bar{C}_p^\zeta = K^\zeta = K^{\zeta_K}$. Then,

$$(K^{\zeta_K}, (v_K^{x, \gamma})_{SS}^d) = (K^\zeta, (V^d)_{K^\zeta}^{X, \Gamma}),$$

where $(K^{\zeta_K}, (v_K^{x, \gamma})_{SS}^d)$ is the corresponding duplicated state of $(K, \zeta_K, v_K^{x, \gamma})$. That is, $(K^\zeta, (V^d)_{K^\zeta}^{X, \Gamma})$ be the corresponding duplicated state of $(K, \zeta_K, v_K^{x, \gamma})$.

Proof. The proof is coincident with Lemma 1, and is omitted. $\square$

Similarly to Maschler and Owen [14], Hwang and Liao [13] presented a dynamic process leading to the EANSC. First, we recall the results of Hwang and Liao [13].

Given $(\bar{C}, V) \in [\mathbf{SS}]$ with $|\bar{C}| \geq 3$ and $X \in E(\bar{C}, V)$. Define the SS-correction function $F : E(\bar{C}, V) \rightarrow E(\bar{C}, V)$ and the WSS-correction function $F^w : E(\bar{C}, V) \rightarrow E(\bar{C}, V)$ for each $p \in \bar{C}$ by

$$\begin{cases} F_p(X) = X_p + \lambda \cdot \sum_{q \in \bar{C} \setminus \{p\}} (\Psi_p(\{p, q\}, (V)_{\{p, q\}}^{X, \Psi}) - X_p), \\ F_p^w(X) = X_p + \lambda \cdot \sum_{q \in \bar{C} \setminus \{p\}} (\Psi_p^w(\{p, q\}, (V)_{\{p, q\}}^{X, \Psi^w}) - X_p). \end{cases} \quad (8)$$

The SS-correction (WSS-correction) functions described above are adjusted based on the following process. Each contributor individually engages in a re-assessing process, seeking cooperation from all others using the (X, Ψ) -dependent ((X, Ψ^w) -dependent) standard reduction and applying distribution concept Ψ (Ψ^w). After this re-assessing, any differences between the newly obtained outcome and the original outcome X are identified and adjusted through λ , ultimately refining the initial outcome provided by X .

Consider the dynamic sequences $\{X^n\}_{n=1}^\infty$ and $\{X^{wn}\}_{n=1}^\infty$ for each $n \in \mathbb{N}$ by

$$\begin{cases} X^0 = X, X^1 = F(X^0), X^2 = F(X^1), \dots, X^n = F(X^{n-1}), \\ X^{w0} = X, X^{w1} = F^w(X^{w0}), X^{w2} = F^w(X^{w1}), \dots, X^{wn} = F^w(X^{w(n-1)}). \end{cases}$$

Theorem 2 (Hwang and Liao [13]). 1. If $0 < \lambda < \frac{2}{|\bar{C}|}$, then $\{X^n\}_{n=1}^\infty$ converges to $\Psi(\bar{C}, V)$ for each $X \in E(\bar{C}, V)$.

2. If $0 < \lambda < \frac{2}{|\bar{C}|^w}$, then $\{X^{wn}\}_{n=1}^\infty$ converges to $\Psi^w(\bar{C}, V)$ for each $X \in E(\bar{C}, V)$.

By Lemma 1, it is known that the duplicated PEANSC ψ (the duplicated PWANSC ψ^w) of a multi-choice state is the PEANSC Ψ (the PWANSC Ψ^w) of a corresponding duplicated

state. In order to provide relative dynamic result for ψ (ψ^w), it is reasonable that the SS-correction (the WSS-correction) function could be extended to the multi-choice states as follows. Let $(\bar{\mathbf{C}}, \zeta, v) \in [\mathbf{MCS}]$ with $|\bar{\mathbf{C}}| \geq 3$ and $x \in E(\bar{\mathbf{C}}, \zeta, v)$. One define the MCS-correction function $f : E(\bar{\mathbf{C}}, \zeta, v) \rightarrow E(\bar{\mathbf{C}}, \zeta, v)$ and the WMCS-correction function $f^w : E(\bar{\mathbf{C}}, \zeta, v) \rightarrow E(\bar{\mathbf{C}}, \zeta, v)$ for each $p \in \bar{\mathbf{C}}$ by

$$\begin{cases} f_p(x) = x_p + \lambda \cdot \sum_{q \in \bar{\mathbf{C}} \setminus \{p\}} \frac{\zeta_p + \zeta_q}{2} (\psi_p(\{p, q\}, \zeta_{\{p, q\}}, v_{\{p, q\}}^{x, \psi}) - x_p) \\ f_p^w(x) = x_p + \lambda \cdot \sum_{q \in \bar{\mathbf{C}} \setminus \{p\}} \frac{\zeta_p + \zeta_q}{2} (\psi_p^w(\{p, q\}, \zeta_{\{p, q\}}, v_{\{p, q\}}^{x, \psi^w}) - x_p). \end{cases} \quad (9)$$

Based on definition of (x, ψ) -dependent reduced state, (x, ψ^w) -dependent reduced state and MCSFF of ψ and ψ^w ,

$$\begin{cases} \zeta_p \psi_p(\{p, q\}, \zeta_{\{p, q\}}, v_{\{p, q\}}^{x, \psi}) + \zeta_q \psi_q(\{p, q\}, \zeta_{\{p, q\}}, v_{\{p, q\}}^{x, \psi}) = \zeta_p x_p + \zeta_q x_q, \\ \zeta_p \psi_p^w(\{p, q\}, \zeta_{\{p, q\}}, v_{\{p, q\}}^{x, \psi^w}) + \zeta_q \psi_q^w(\{p, q\}, \zeta_{\{p, q\}}, v_{\{p, q\}}^{x, \psi^w}) = \zeta_p x_p + \zeta_q x_q, \end{cases} \quad (10)$$

and it is easy to verify that $f(x) \in E(\bar{\mathbf{C}}, \zeta, v)$ and $f^w(x) \in E(\bar{\mathbf{C}}, \zeta, v)$ if x is MCSFF.

Consider the dynamic sequences $\{x^n\}_{n=1}^\infty$ and $\{x^{wn}\}_{n=1}^\infty$ for each $n \in \mathbb{N}$ by

$$\begin{cases} x^0 = x, x^1 = f(x^0), x^2 = f(x^1), \dots, x^n = f(x^{n-1}), \\ x^{w0} = x, x^{w1} = f^w(x^{w0}), x^{w2} = f^w(x^{w1}), \dots, x^{wn} = f^w(x^{w(n-1)}). \end{cases}$$

Based on the notion of the duplication and Lemma 1, it is clear that the number of manipulating capacities for each contributor in a multi-choice state could be treated as the number of duplicated contributors in a corresponding duplicated state, and the duplicated PEANSC (PWANSC) of a multi-choice state is the PEANSC (PWANSC) of a corresponding duplicated state. Thus, it is reasonable that Theorem 2 could be extended to the dynamic result for the duplicated PEANSC (PWANSC) as follows.

Theorem 3. Given $(\bar{\mathbf{C}}, \zeta, v) \in [\mathbf{MCS}]$ with $|\bar{\mathbf{C}}| \geq 3$.

1. If $0 < \lambda < \frac{2}{\zeta(\bar{\mathbf{C}})}$, then $\{x^n\}_{n=1}^\infty$ converges to $\psi(\bar{\mathbf{C}}, \zeta, v)$ for each $x \in E(\bar{\mathbf{C}}, \zeta, v)$.
2. If $0 < \lambda < \frac{2}{\zeta(w)}$, then $\{x^{wn}\}_{n=1}^\infty$ converges to $\psi^w(\bar{\mathbf{C}}, \zeta, v)$ for each $x \in E(\bar{\mathbf{C}}, \zeta, v)$.

Proof. Let $(\bar{\mathbf{C}}, \zeta, v) \in [\mathbf{MCS}]$, $x \in E(\bar{\mathbf{C}}, \zeta, v)$ and $(\bar{\mathbf{C}}^\zeta, V^d)$ be the corresponding duplicated state. And let $X \in E(\bar{\mathbf{C}}^\zeta, V^d)$ with $X_{p_k} = x_p$ for all $p \in \bar{\mathbf{C}}$ and for all $p_k \in \bar{\mathbf{C}}^\zeta$. Let $p, q \in \bar{\mathbf{C}}$ and $\bar{\mathbf{C}}_p^\zeta \cup \bar{\mathbf{C}}_q^\zeta = \{p, q\}^\zeta = \{p, q\}^{\zeta_{\{p, q\}}}$. Let $p \in \bar{\mathbf{C}}_p^\zeta$ and $q \in \bar{\mathbf{C}}_q^\zeta$. We will show that

$$\begin{aligned} \Psi_p(\{p, q\}, \zeta_{\{p, q\}}, v_{\{p, q\}}^{x, \psi}) - x_p &= \Psi_p(\{p, q\}^\zeta, (v_{\{p, q\}}^{x, \psi})_{SS}^d) - x_p \\ &= \Psi_p(\{p, q\}^\zeta, (V^d)_{\{p, q\}^\zeta}^X) - x_p \\ &= \frac{2\zeta_q}{\zeta_p + \zeta_q} \{\Psi_p(\{p, q\}, ((V^d)_{\{p, q\}^\zeta}^X)_{\{p, q\}}^X) - x_p\} \\ &= \frac{2\zeta_q}{\zeta_p + \zeta_q} \{\Psi_p(\{p, q\}, (V^d)_{\{p, q\}}^{X, \Psi}) - x_p\}. \end{aligned} \quad (11)$$

If Equation (11) holds, then Equation (9) becomes

$$\begin{aligned} f_p(x) &= x_p + \lambda \cdot \sum_{q \in \bar{\mathbf{C}} \setminus \{p\}} \frac{\zeta_p + \zeta_q}{2} (\Psi_p(\{p, q\}, \zeta_{\{p, q\}}, v_{\{p, q\}}^{x, \psi}) - x_p) \\ &= x_p + \lambda \cdot \sum_{q \in \bar{\mathbf{C}} \setminus \{p\}} \zeta_q \{\Psi_p(\{p, q\}, (V^d)_{\{p, q\}}^{X, \Psi}) - x_p\}. \end{aligned} \quad (12)$$

Consider the corresponding duplicated state $(\overline{\mathbf{C}}^\zeta, V^d)$ and the SS-correction function $F : E(\overline{\mathbf{C}}^\zeta, V^d) \rightarrow E(\overline{\mathbf{C}}^\zeta, V^d)$ by

$$F_p(X) = X_p + \lambda \cdot \sum_{q \in \overline{\mathbf{C}}^\zeta \setminus \{p\}} (\Psi_p(\{p, q\}, (V)_{\{p, q\}}^X) - X_p). \quad (13)$$

It is easy to see that $\Psi_p(\{p, q\}, (V^d)_{\{p, q\}}^{X, \Psi}) = x_p$ if $p, q \in \overline{\mathbf{C}}^\zeta$. So, Equation (12) coincides with Equation (13). Therefore, Theorem 3 immediately follows via Theorem 2. It remains to prove Equation (11). The proof of Equation (11) could be generated from the following three steps. **Step 1:** By Lemma 4,

$$(\{p, q\}^\zeta, (v_{\{p, q\}}^{x, \psi})_{SS}^d) = (\{p, q\}^\zeta, (V^d)_{\{p, q\}^\zeta}^X). \quad (14)$$

Based on Definition 1,

$$\psi_p(\{p, q\}, \zeta_{\{p, q\}}, v_{\{p, q\}}^{x, \psi}) = \Psi_p(\{p, q\}^\zeta, (v_{\{p, q\}}^{x, \psi})_{SS}^d). \quad (15)$$

Combining Equation (14) with Equation (15),

$$\psi_p(\{p, q\}, \zeta_{\{p, q\}}, v_{\{p, q\}}^{x, \psi}) = \Psi_p(\{p, q\}^\zeta, (V^d)_{\{p, q\}^\zeta}^X). \quad (16)$$

Step 2: By MCRCON of Ψ and the equal treatment property due to Moulin [21],

$$\Psi_p(\{p, q\}^\zeta, (V^d)_{\{p, q\}^\zeta}^X) = \Psi_p(\{p, q\}, ((V^d)_{\{p, q\}^\zeta}^X)_{\{p, q\}}^\Psi). \quad (17)$$

$$\Psi_q(\{p, q\}^\zeta, (V^d)_{\{p, q\}^\zeta}^X) = \Psi_q(\{p, q\}, ((V^d)_{\{p, q\}^\zeta}^X)_{\{p, q\}}^\Psi). \quad (18)$$

It is also straightforward to verify that

$$(\{p, q\}, ((V^d)_{\{p, q\}^\zeta}^X)_{\{p, q\}}^\Psi) = (\{p, q\}, (V^d)_{\{p, q\}}^{X, \Psi}), \quad (19)$$

and

$$((V^d)_{\{p, q\}^\zeta}^X)_{\{p, q\}}^\Psi(\{k\}) = ((V^d)_{\{p, q\}^\zeta}^X)_{\{p, q\}}^X(\{k\}) \text{ for } k = p, q. \quad (20)$$

Based on Equations (19) and (20),

$$\begin{aligned} & \Psi_p(\{p, q\}, ((V^d)_{\{p, q\}^\zeta}^X)_{\{p, q\}}^\Psi) - \Psi_q(\{p, q\}, ((V^d)_{\{p, q\}^\zeta}^X)_{\{p, q\}}^\Psi) \\ &= \Psi_p(\{p, q\}, ((V^d)_{\{p, q\}^\zeta}^X)_{\{p, q\}}^X) - \Psi_q(\{p, q\}, ((V^d)_{\{p, q\}^\zeta}^X)_{\{p, q\}}^X) \\ &= \Psi_p(\{p, q\}, (V^d)_{\{p, q\}}^{X, \Psi}) - \Psi_q(\{p, q\}, (V^d)_{\{p, q\}}^{X, \Psi}). \end{aligned} \quad (21)$$

Based on Equations (16)–(18) and (21),

$$\begin{aligned} & \psi_p(\{p, q\}, \zeta_{\{p, q\}}, v_{\{p, q\}}^{x, \psi}) - \psi_q(\{p, q\}, \zeta_{\{p, q\}}, v_{\{p, q\}}^{x, \psi}) \\ &= \Psi_p(\{p, q\}, (V^d)_{\{p, q\}}^{X, \Psi}) - \Psi_q(\{p, q\}, (V^d)_{\{p, q\}}^{X, \Psi}). \end{aligned} \quad (22)$$

Based on Equations (10) and (22), it is easy to see that

$$\begin{aligned} & \psi_p(\{p, q\}, \zeta_{\{p, q\}}, v_{\{p, q\}}^{x, \psi}) \\ &= \frac{1}{\zeta_p + \zeta_q} [\zeta_p x_p + \zeta_q x_q + \zeta_q \Psi_p(\{p, q\}, (V^d)_{\{p, q\}}^{X, \Psi}) - \zeta_q \Psi_q(\{p, q\}, (V^d)_{\{p, q\}}^{X, \Psi})]. \end{aligned} \quad (23)$$

Step 3:

$$\begin{aligned}
& \psi_p(\{p, q\}, \zeta_{\{p, q\}}, v_{\{p, q\}}^{x, \psi}) - x_p \\
= & \frac{1}{\zeta_p + \zeta_q} [\zeta_p x_p + \zeta_q x_q + \zeta_q \Psi_p(\{p, q\}, (V^d)_{\{p, q\}}^{X, \Psi}) - \zeta_q \Psi_q(\{p, q\}, (V^d)_{\{p, q\}}^{X, \Psi})] - x_p \\
& \quad \text{(by Equation (23))} \\
= & \frac{1}{\zeta_p + \zeta_q} [\zeta_q x_q + \zeta_q \Psi_p(\{p, q\}, (V^d)_{\{p, q\}}^{X, \Psi}) - \zeta_q \Psi_q(\{p, q\}, (V^d)_{\{p, q\}}^{X, \Psi}) - \zeta_q x_p] \\
= & \frac{\zeta_q}{\zeta_p + \zeta_q} [x_q + \Psi_p(\{p, q\}, (V^d)_{\{p, q\}}^{X, \Psi}) - \Psi_q(\{p, q\}, (V^d)_{\{p, q\}}^{X, \Psi}) - x_p] \\
= & \frac{\zeta_q}{\zeta_p + \zeta_q} [2(\Psi_p(\{p, q\}, (V^d)_{\{p, q\}}^{X, \Psi}) - X_p)] \\
& \quad \text{(by } x_p = X_p, x_q = X_q, \Psi_p(\{p, q\}, (V^d)_{\{p, q\}}^{X, \Psi}) + \Psi_q(\{p, q\}, (V^d)_{\{p, q\}}^{X, \Psi}) = X_p + X_q) \\
= & \frac{2\zeta_q}{\zeta_p + \zeta_q} [\Psi_p(\{p, q\}, (V^d)_{\{p, q\}}^{X, \Psi}) - X_p].
\end{aligned}$$

The proof of result 2 is similar, and is omitted. $\square$

5. Comparisons and Conclusions

5.1. Comparisons

Inspired by the existing results on distribution concepts in both standard and multi-choice states, this paper proposes a novel distribution concept and related outcomes. A comparison with existing results highlights several key comparisons:

1. Standard states have focused solely on the binary participation status of contributors. However, it is reasonable for each contributor to employ different manipulating capacities, and contributors may have varying influences related to different situations. Thus, unlike existing distribution concepts, the duplicated PEANSC and the duplicated PWANSC are introduced to extend the PEANSC and the PWEANSC by simultaneously incorporating duplicating structure and weights under multi-choice states.
2. Based on duplicated structures under multi-choice states, Calvo and Santos [1], Hwang and Liao [9], and Liao [10] each proposed distinct outcomes. Below is a comparison of this document with the three prior studies:
 - Calvo and Santos [1] introduced multi-choice states by treating each participation level as a separate entity, creating a duplicated structure. This document builds on their model by incorporating weights, enabling differentiated utility distribution based on contributors' importance.
 - Hwang and Liao [9] focused on fair core allocations within multi-choice states, using duplicated structures. The present study advances this by adding weighted contributions and axiomatic characterizations, making the model applicable in practical, weighted multi-choice situations.
 - Liao [10] proposed the duplicate extension for equal allocation in non-separable costs, aiming for fair distribution across manipulation capacities. This paper generalizes Liao's framework by incorporating weights into duplicated structures, enhancing dynamic consistency and robustness across diverse multi-choice states.

In sum, this document extends prior models by integrating weights into duplicated structures, broadening both theoretical depth and practical applicability in cooperative multi-choice game settings.

3. Hart and Mas-Colell's [20] defined a specific consistency and two-person standardness to characterize the Shapley value [2]. Hsieh and Liao [19] extended axiomatic techniques of Hart and Mas-Colell's [20] work to characterize the PEANSC. This study extends Hsieh and Liao's [19] approach by incorporating duplicated structures and weights, thus generalizing the pseudo-EANSC model to handle a wider range of multi-choice situations.

While both studies apply analogues of related axiomatic properties due to Hart and Mas-Colell, this study advances these ideas by integrating dynamic processes with weights, allowing for more complex allocating situations. This distinction enables this

study to address a broader range of applications where contributors' influence varies systematically, reflecting practical enhancements to related game-theoretical model introduced by Hsieh and Liao [19].

4. The following compares several studies on dynamic processes mentioned in this paper:
 - Hwang and Liao [13] applied Maschler and Owen's [14] dynamic principles from the consonant Shapley value [2] to stabilize cost allocations as contributors' roles evolve, using dynamic adjustments for equilibrium in multi-choice states. Similarly, Hsieh and Liao [19] built on these dynamic results through an iterative approach to the pseudo-EANSC, adjusting allocations based on changes in contributors' influence.
 - This study advances these frameworks by combining Hwang and Liao's [13] iterative consistency with the axiomatic basis of the pseudo-EANSC, while introducing duplicated structures and weighted contributions for situations with varied contributor influence. Unlike Hsieh and Liao's [19] model, it employs weights, allowing precise allocation adjustments.
 - This model aligns with other dynamic applications: van Benthem et al. [15] uses logic-based adjustments similar to the attached paper's iterative approach; Friesz and Mookherjee [16] focus on day-to-day equilibrium in traffic networks, mirroring the stability goals of the pseudo-EANSC; and Ranjbar-Sahraei et al. [17] along with Shakiba and Autio [18] examine role adaptations in real-time systems, paralleling the weighted dynamics in this paper's model.

In summary, this study differentiates itself by adapting foundational dynamic notions with practical weights, broadening applicability across various situations.

5.2. Conclusions

This study develops and extends distribution concepts by embedding weights and duplicated structures into multi-choice states, leading to comprehensive and adaptable solutions for resource allocation among contributors with varying levels of influence. The main contributions of this research are as follows.

1. **Introduction of the duplicated distribution concepts:** This work introduces the *duplicated PEANSC* and the *duplicated PWANSC*, which generalize the foundational concepts of the PEANSC and the PWANSC to accommodate multi-choice frameworks. These extensions enable a more precise assessment of each contributor's value by aggregating its marginal impacts across all possible manipulating capacities. Through this approach, contributors are valued not only by their presence, but also by its varying degrees of engagement, providing a robust model for diverse real-world scenarios.
2. **Axiomatic foundations and validation:** To ensure that these distribution concepts are mathematically sound and practically applicable, this study establishes an axiomatic basis for the duplicated PEANSC and the duplicated PWANSC under multi-choice settings. Key axioms such as consistency, efficiency, and two-person standardness underpin these models, ensuring that the distributing aligns with contributors' relative individual contributions. This axiomatic validation is crucial for applying the duplicated structures across different fields, confirming that these models are both theoretically rigorous and capable of supporting reasonable allocating under complex situations.
3. **Dynamic processes for outcome convergence:** Recognizing the need for dynamic adaptability under multi-choice states, this study defines two correction functions and its corresponding dynamic processes to guide efficient outcome convergence. These processes allow contributors to achieve the duplicated PEANSC and the duplicated PWANSC from any initial efficient outcome vector. By iteratively adjusting allocations based on each contributor's manipulating capacities, these dynamic processes facilitate a stepwise approach toward a stable and fair distribution, thus expanding the practical utility of the model for evolving cooperative settings.

4. **Future research directions and motivations:** This study opens pathways for exploring further distribution concepts and refining axiomatic frameworks within multi-choice states. The findings highlight the potential to develop alternative models and dynamic processes that cater to increasingly complex interaction situations, particularly in applications involving continuous adaptation and varying influence levels among contributors. These areas remain relatively unexplored, presenting opportunities for extending dynamic game-theoretical models and integrating them into broader fields such as economics, network theory, and ecosystem management.

In summary, the proposed duplicated PEANSC and duplicated PWANSC, grounded in strong axiomatic and dynamic frameworks, provide versatile and reliable distribution concepts for multi-choice states. This study advances the field by offering adaptable tools for equitable resource allocation, laying a foundation for further advancements in game theory.

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