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On the Evolution Operators of a Class of Linear Time-Delay Systems

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Abstract: This paper studies the properties of the evolution operators of a class of time-delay systems with linear delayed dynamics. The considered delayed dynamics may, in general, be time-varying and associated with a finite set of finite constant point delays. Three evolution operators are defined and characterized. The basic evolution operator is the so-called point delay operator, which generates the solution trajectory under point initial conditions at $t_0 = 0$. Furthermore, this paper also considers the whole evolution operator and the delay strip evolution operator, which define the solution trajectory, respectively, at any time instant and along a strip of time whose size is that of the maximum delay. These operators are defined for any given bounded piecewise continuous function of initial conditions on an initialization time interval of measure being identical to the maximum delay. It is seen that the semigroup property of the time-invariant undelayed dynamics, which is generated by a C_0-semigroup, becomes lost by the above evolution operators in the presence of the delayed dynamics. This fact means that the point evolution operator is not a strongly and uniformly continuous one-parameter semigroup, even if its undelayed part has a time-invariant associated dynamics. The boundedness and the stability properties of the time-delay system, as well as the strong and uniform continuity properties of the evolution operators, are also discussed.

Keywords: delay differential systems; point delays; evolution operator; C_0-semigroup; strongly continuous operator; uniformly continuous operator

MSC: 34C40; 34D20; 34K12; 37L05

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1. Introduction

Time-delay differential systems are very common in real life, for instance, they are present in war/peace models; biological problems like, for instance, the sun flower equation, logistic equations, epidemic models, etc.; electrical transmission lines, economic models, tele-operation processes; Volterra-type integro-differential equations; etc. The reason for this is that such delayed dynamics often influences the solution trajectory at each current time instant. In some control theory problems, delays can be introduced by the controller even if the controlled system is delay-free (for instance, refs. [1–10], and some of the references therein).

Delays can be of different types, which sometimes operate in a combined manner on delayed dynamic systems. In particular, delays can be internal, that is, in the state, or external, that is, in the forcing inputs or controls or in the outputs, that is, measurable combinations of the state components. They can also be constant or time-varying and point delays or distributed delays. These last ones are typical in certain deterministic or stochastic classes of Volterra-type integro-differential equations [11–14]. Also, point delays can be commensurate, with all of them being integer multiples of a base delay or incommensurate [1–4]. On the other hand, time-delay systems are also sometimes subject to impulsive effects [12,13]. A major technical problem of time-invariant time-delay systems

is that they are infinite-dimensional because of time-interval-dependent memory. As a result, they have infinitely many characteristic zeros, contrary to the case of undelayed time-invariant systems. Fortunately, infinitely many of the characteristic zeros lie in the stable complex half-plane, with real parts tending towards minus infinity; the number of characteristic zeros within any finite band of the complex plane is finite, and also, the larger unstable characteristic zero, if present, has a finite stability abscissa [1,4,11].

This paper is devoted to the explicit definitions and the study of the main properties of three "ad hoc" evolution operators for the solutions of a class of time-varying time-delay differential systems of n-th order, that is, those being equivalent to a functional differential equation of order n. The evolution operators are calculated based on the fundamental matrix of the differential system. They allow for the global generation of the solution in an explicit form for any bounded piecewise continuous function of initial conditions on an initialization interval of measure equal to that of the size of the maximum delay. Both the undelayed and the delayed dynamics of the differential system associated with the delays are allowed to be time-varying in the most general case. Such a class of time-delay systems has a finite set of constant point delays with associated respective matrices of dynamics, which can have, in general, bounded piecewise continuous entries. The delay-free part can also be time-varying while defining a bounded piecewise continuous matrix function of order n. Such undelayed dynamics is split into two parts, one of them being time-invariant and the other one being eventually time-varying, in general. The delays are assumed, in general, to be incommensurate in the sense that they are not necessarily integer multiples of a base minimum delay and constant. The case of commensurate delays might be dealt with directly as an obvious particular case so that it is not explicitly considered. The initial conditions of the differential time-delay systems are defined by any given bounded absolutely continuous real vector function defined on an initialization interval of measure to be identical to that of the maximum delay h. The trajectory solutions of the considered time-delay systems are generated through "ad hoc" obtained and analyzed evolution operators, which are defined explicitly in an analytic way. The characterized evolution operators are of three types, namely, the so-called point evolution operator, the whole evolution operator, and the delay strip evolution operator. The point evolution operator generates the solution trajectory from point initial conditions in the case when the vector function of initial conditions is zeroed on its definition domain, except at the zero initial time instant. The whole evolution operator considers any admissible, i.e., bounded piecewise continuous, real vector function defined on an initialization interval of measure equal to that of the maximum delay. On the other hand, the so-called delay strip evolution operator generates the solution on any time interval of measure equal to the maximum delay size.

It might be noticed that, if the initial conditions are nonzero only at the initial time instant $t = 0$ but identically zero on the time interval $[-h, 0)$, where h is the maximum delay, then both operators give the same solution through time for any later time instant. Also, the point evolution operator only gives a partial solution trajectory for initializations at any other initialization closed real-time interval whose right extreme is equal or larger than the size of the minimum delay due to the memory action of delays.

The paper gives relevant importance to the point evolution operator since, in some practical problems, the initialization is performed at the zero time instant with zeroed initial conditions in the interval $[-h, 0)$. This initialization does not imply an essential loss in generality since the delayed dynamics operate to generate the solution through the operator itself, which contains the information of all the delayed dynamics associated with the whole set of delays. On the other hand, it is seen that, if there is no time-varying dynamics in the delay-free part and there is no undelayed dynamics, then the resulting particular differential system has an evolution operator, which is a C_0-semigroup, of time-invariant infinitesimal generator [15–17] (see also, refs. [18–20]), which is then known to be a one-parameter semigroup, and strongly continuous and uniformly continuous as a result. It is seen that the above three operators generate the solution trajectory of the

time-delay differential system. However, it is also proved that they lose the one-parameter semigroup property of the linear time-invariant undelayed case, although the evolution operators are still strongly and uniformly continuous under some more conditions on the time-varying and on the delayed dynamics since they are closed and bounded. We can now recall that the one-parameter semigroup property of a C_0-semigroup allows the generation of the solution trajectory from any time instant to another one only depending on the interval length between both of them. This fact allows for constructing the whole solution trajectory by successive composition of the evolution operator evaluated between adjacent time intervals (from the semigroup associative property) and independently of the order of the above composition (from the commutative property from a commutative semigroup). Those properties are lost, in general, due to the presence of delayed dynamics and to the fact of the time-varying characteristics of the dynamics.

This paper is organized as follows. Section 2 defines the linear differential system of n coupled first-order differential functional (because of the delays' presence) equations to be studied and the solution trajectory for any given initialization function of bounded piecewise continuous vector functions on $[-h\ ,\ 0]$. In general, it is assumed a finite number of constant point delays whose associated dynamics are, in general, time-varying defined by matrix functions of bounded piecewise continuous entries. The delay-free part can also be assumed to be time-varying under similar hypotheses. Section 3 characterizes explicitly the trajectory solution of the differential via the above-mentioned evolution operators. The main results and some worked illustrative examples are given in this section. Explicit sufficiency-type conditions are given for global stability and for global exponential stability of the dynamic system. Those conditions rely on either the asymptotic stability of exponential type and/or on the critical stability of the time-invariant delay-free part of the dynamics and the sufficiently smallness of the time-varying delay-free and delayed dynamics. The main obtained results are mainly focused on the properties of the point evolution operator that is also represented, by convenience, as an exponential matrix function, which does not have an associated infinitesimal generator and is not already a (one-parameter) C_0-semigroup either, as it would be the case in the absence of time-varying and delayed dynamics, due to the presence of time-varying delay-free and/or the presence of delayed dynamics. It is also proved that under some weak conditions, the point evolution operator is strongly and uniformly continuous. The above loss of the semigroup property of the point evolution operator also translates into a loss of the property by the other two operators. Finally, some conclusions end this paper.

Basic Nomenclature to Be Used in This Manuscript

The subsequent notation will be used throughout this manuscript.

A set of consecutive p positive integer numbers starting with unity is denoted as $\overline{p} = \{1, 2, \cdots,\ p\}$.

The so-called p commensurate point delays h_i, ordered according to their increasing sizes as $h_m = h_1 < h_2 = 2h_m < \cdots < h_p = ph_m = h$, are those which are integer multiples of a minimum common multiple, which is the minimum, or base, minimum delay $h_m = h_1$, namely, $h_i = ih_m$ for $i \in \overline{p}$ so that the biggest delay is $h = h_p = ph_m$.

The so-called incommensurate p point delays, ordered according to their increasing sizes as $h_m = h_1 < h_2 < \cdots < h_p = h$, are those that are not necessarily integer multiples of a minimum common multiple.

$\overline{f}_{ta}$ denotes the truncation on the strip $[t-a\ ,\ t]$ of the function $f : \boldsymbol{R} \to \boldsymbol{R}^n$, which is defined as a function from $\boldsymbol{R}$ to $\boldsymbol{R}^n$ by $\overline{f}_{ta}(\tau) = f(\tau);\ \tau \in [t-a\ ,\ t]$, $\overline{f}_{ta}(\tau) = 0;\ \tau \in \boldsymbol{R} \backslash [t-a,\ t]$ for all $t \in \boldsymbol{R}$ in such a way that the support of $\overline{f}_{ta}$ is either a proper or improper subset of $[t-a\ ,\ t]$. In this way, $\overline{x}_{th}$ is the strip of the solution $x(t)$ on $[t-h\ ,\ t]$ of a n-th order time-delay system on a maximum delay period of size h. To abbreviate the notation, $\overline{x}_{th}$ is simply referred to as $\overline{x}_t$ when no confusion is expected.

I_n is the n-th identity matrix.

$$\boldsymbol{R}_{0+} = \boldsymbol{R}_{+} \cup \{0\} = \{r \in \boldsymbol{R} : r \geq 0\};\ \boldsymbol{R}_{+} = \{r \in \boldsymbol{R} : r > 0\},$$

$$\boldsymbol{R}_{-0} = \boldsymbol{R}_{-} \cup \{0\} = \{r \in \boldsymbol{R} : r \leq 0\};\ \boldsymbol{R}_{-} = \{r \in \boldsymbol{R} : r < 0\},$$

where $\boldsymbol{R}$ is the set of real numbers.

In the same way, we can define "mutatis mutandis" as the respective subsets $\boldsymbol{Z}_{0+}$, $\boldsymbol{Z}_{+}$, $\boldsymbol{Z}_{-0}$ and $\boldsymbol{Z}_{-}$ of the set $\boldsymbol{Z}$ of integer numbers.

The closures of the above sets are $cl\ \boldsymbol{R} = \boldsymbol{R} \cup \{\pm\infty\}$, and

$$cl\ \boldsymbol{R}_{0+} = \boldsymbol{R}_{0+} \cup \{+\infty\};\ cl\ \boldsymbol{R}_{+} = \boldsymbol{R}_{+} \cup \{0, +\infty\}$$

$$cl\ \boldsymbol{R}_{-0} = \boldsymbol{R}_{-0} \cup \{-\infty\};\ cl\ \boldsymbol{R}_{-} = \boldsymbol{R}_{-} \cup \{0, -\infty\}$$

$BPC([-h, 0]\ ; \boldsymbol{R}^n)$ denotes the set of bounded piecewise continuous functions from $[-h\ ,\ 0]$ to $\boldsymbol{R}^n$.

$PC^1(\boldsymbol{R}_{+}\ ;\ \boldsymbol{R}^n)$ is the continuous differentiable class of n-real vector functions from $\boldsymbol{R}_{+}$ to $\boldsymbol{R}^n$ whose first- derivative is piecewise continuous but non-necessarily continuous. Note that this set contains that of continuously differentiable vector functions from $\boldsymbol{R}_{+}$ to $\boldsymbol{R}^n$, that is, $C^1(\boldsymbol{R}_{+}\ ;\ \boldsymbol{R}^n) \subset PC^1(\boldsymbol{R}_{+}\ ;\ \boldsymbol{R}^n)$.

$\boldsymbol{R}_{0+h}(t) = \{[t-h\ ,\ t] : t \in \boldsymbol{R}_{0+}\} \subset \boldsymbol{R}$; $\boldsymbol{R}_{+h}(t) = \{[t-h\ ,\ t] : t \in \boldsymbol{R}_{+}\} \subset \boldsymbol{R}$ are respective non-negative and positive strips of real intervals of Lebesgue measure h up to the time instant "t".

2. Time-Delay Systems and the Evolution Operators

This section presents the linear time-varying differential system involved both undelayed and delayed dynamics to be analyzed through this manuscript. All the dynamics are allowed to be time-varying, in the most general case. The set of finite point delays can be incommensurate in the general case, that is, they do not necessarily follow rules of integer proportionality with respect to the minimum, or base, delay. Also, the point, whole and delay strip evolution operators are defined explicitly to generate the whole solution. The point evolution operator generates the solution at any time instant $t > 0$ for arbitrary finite point initial conditions at the zero initial time. In this case, the piecewise bounded function of initial conditions is assumed to be null on the initialization interval $[-h\ , 0)$, that is, in the whole initialization interval $[-h\ ,\ 0]$, except at $t = 0$, where h is the largest delay. The whole evolution operator generates the solution at any time instant $t > 0$ for any admissible bounded piecewise continuous function of initial conditions. The contribution of the function of initial conditions on $[-h\ , 0)$ is incorporated with an associated forcing term in this operator compared to the definition of the point evolution operator. The delay strip evolution operator is defined based on the whole evolution operator to generate a whole strip of solution on an interval whose measure is that of the maximum delay. Also, some particular cases are discussed via explicit remarks concerning the time-invariant case and when delays are zero.

Consider the following class of linear n-th order time-delay systems with eventual incommensurate delays:

$$\dot{x}(t) = Ax(t) + \sum_{i=0}^{p} A_{di}(t)x(t-h_i);\ t \in \boldsymbol{R}_{0+} \tag{1}$$

where

(a) $x(t) = \varphi(t)$ for $t \in [-h_p\ ,\ 0]$ where $\varphi : [-h_p,\ 0] \to \boldsymbol{R}^n$ is a given bounded piecewise continuous function of initial conditions (that is, $\varphi \in BPC([-h, 0]\ ; \boldsymbol{R}^n)$) of the state-trajectory solution. $x : \boldsymbol{R}_{0+} \to \boldsymbol{R}^n$ ($x(t) \in \boldsymbol{R}^n$ is said to be the state vector at the time instant t) with $x(0) = \varphi(0) = x_0$; $A \in \boldsymbol{R}^{n \times n}$ and $A_{di} : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n \times n}$; $i \in \overline{p} \cup \{0\}$ have bounded piecewise continuous entries.

(b) $\{h_i : i \in \overline{p} \cup \{0\}\}$ is an ordered finite set of finite point delays with $h_0 = 0$, $h_i < h_j$ if $i(< j), j \in \overline{p}$ and $h_p = h$ being the minimum delay and the maximum delay, respectively. Note that h_0 is not properly a delay as such so that, in fact, the matrix of delay-free dynamic is time-varying and defined by $(A + A_{d0}(t))$. In this way, it is considered that the system can also have a time-varying undelayed dynamics whose time-varying part is defined by $A_{d0}(t)$. The following is assumed with no loss in generality:

(a) If $A_{d0} \equiv 0$ so that the matrix of delay-free dynamics is constant, given by $A \in \boldsymbol{R}^{n\times n}$, then (1) becomes simplified as follows:

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{p} A_{di}(t)x(t-h_i);\ t \in \boldsymbol{R}_{0+} \tag{2}$$

(b) If A_{d0} is not identically zero for all time, then the matrix of delay-free dynamics is time-varying and defined by $(A + A_{d0}(t))$. Note that the amount $h_0 = 0$ is not in fact a real delay, but it is introduced to incorporate all the varying dynamics, including a time-varying part associated with the delay-free dynamics contribution, in a unique summation term in (1) when the delay-free dynamics is time-varying. In this context, description (1) also includes (2) as a particular case.

(c) The differential system (1) is said to be free of delayed dynamics if $A_{di}(t) \equiv 0$ for $i \in \overline{p}$.

One determines, by direct inspection of (1), that its unique solution trajectory for any given bounded piecewise continuous function of initial conditions $\varphi : [-h,\ 0] \to \boldsymbol{R}^n$ is:

$$\begin{array}{ll} x(t) & = e^{At}x_0 + \sum_{i=0}^{p} \int_0^t e^{A(t-\tau)} A_{di}(\tau)x(\tau - h_i)d\tau \\ & = e^{At}x_0 + \sum_{i=0}^{p} \int_{-h_i}^{0} e^{A(t-\tau-h_i)} A_{di}(\tau + h_i)\varphi(\tau)d\tau + \sum_{i=0}^{p} \int_0^{t-h_i} e^{A(t-\tau-h_i)} A_{di}(\tau + h_i)x(\tau)d\tau;\ t \in \boldsymbol{R}_{0+}. \end{array} \tag{3}$$

The matrix function $\Psi : \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}^{n\times n}$ satisfies the same differential system as the time-delay system, that is:

$$\dot{\Psi}(t,0) = A\Psi(t,0) + \sum_{i=0}^{p} A_{di}(t)\Psi(t-h_i,\ 0);\ t \in \boldsymbol{R}_{0+} \tag{4}$$

$$\dot{\Psi}(t,\tau) = A\Psi(t,\tau) + \sum_{i=0}^{p} A_{di}(t)\Psi(t-h_i,\ \tau);\ \tau, t(\geq \tau) \in \boldsymbol{R}_{0+} \tag{5}$$

subject to $\Psi(t\ ,\ t) = I_n$ for $t \in \boldsymbol{R}_{0+}$; and $\Psi(t\ ,\tau) = 0$ for $\tau, t(< \tau) \in \boldsymbol{R}$; and for $t \in \boldsymbol{R}_-$ and $\tau \in \boldsymbol{R}$, so that one obtains by direct integration of (4) $\sigma\ ,\ \ t \in \boldsymbol{R}_{0+}$:

$$\Psi(t\ ,\ \sigma) = \begin{cases} 0 \quad for \quad t < \sigma \\ e^{A(t-\sigma)}\left(I_n + \sum_{i=0}^{p} \int_\sigma^t e^{-A\tau} A_{di}(\tau)\ \Psi(\tau - h_i\ ,\ \sigma)d\tau\right)\ \ for\ \ t \geq\ \sigma \end{cases} \tag{6}$$

Note that the above expression satisfies the claimed initial conditions $\Psi(0\ ,\ 0) = I_n$; $\Psi(t\ ,0) = 0$ for $t < 0$. Note that Equation (3) can be expressed equivalently by using (6) as follows:

$$x(t) = \Psi(t,0)x_0 + \sum_{i=0}^{p} \int_{-h_i}^{0} \Psi(t-\tau,0)\varphi(\tau)d\tau;\ t \in \boldsymbol{R}_{0+}. \tag{7}$$

To verify this property, take time-derivatives in (7) and then replace again Equation (7) in the obtained result, which leads directly to the differential system (2) since:

$$\begin{array}{l} \dot{x}(t) = \dot{\Psi}(t,0)x_0 + \sum_{i=0}^{p} \int_{-h_i}^{0} \dot{\Psi}(t-\tau,0)\varphi(\tau)d\tau \\ = \left[A\Psi(t,0) + \sum_{i=0}^{p} A_{di}(t)\Psi(t-h_i\ ,\ 0)\right]x_0 + \sum_{i=0}^{p} \int_{-h_i}^{0} \left[A\Psi(t-\tau,0) + \sum_{i=1}^{p} A_{di}(t)\Psi(t-\tau-h_i,\ 0)\right]\varphi(\tau)d\tau \\ = A\left[\Psi(t,0)x_0 + \sum_{i=0}^{p} \int_{-h_i}^{0} \Psi(t-\tau,0)\varphi(\tau)d\tau\right] + \sum_{i=0}^{p} A_{di}(t)\left[\Psi(t-h_i\ ,\ 0_i)x_0 + \int_{-h_i}^{0} \Psi(t-\tau-h_i,\ 0)\varphi(\tau)\ d\tau\right] \\ = Ax(t) + \sum_{i=0}^{p} A_{di}(t)x(t-h_i);\ t \in \boldsymbol{R}_{0+}. \end{array} \tag{8}$$

Definition 1. *The point evolution operator* $\boldsymbol{\Psi} : \boldsymbol{R}^n \to PC^1(\boldsymbol{R}\,;\,\boldsymbol{R}^n)$ *defines the solution trajectory as* $x(t) = (\boldsymbol{\Psi} x_0)(t) = \begin{cases} \Psi(t,0)x_0;\ t \in \boldsymbol{R}_{0+} \\ 0\,;\ t \in \boldsymbol{R}_- \end{cases}$ *under point initial conditions at* $t = 0$*, namely, if the function of initial conditions* $\varphi \in BPC([-h, 0]\,;\boldsymbol{R}^n)$ *is defined by* $\varphi(t) = 0$ *for* $t \in [-h\,,\,0)$ *and* $\varphi(0) = x(0) = x_0 \in \boldsymbol{R}^n$.

Note that the interest in our current context is the image restriction of the point evolution operator $\boldsymbol{\Psi}$ to non-negative time instants, that is, when it is restricted from $\boldsymbol{R}^n$ to $PC^1(\boldsymbol{R}_{0+}\,;\,\boldsymbol{R}^n)$, which maps each given point initial condition x_0 to the solution value $x(t)$ at any time $t \in \boldsymbol{R}_{0+}$ given by the matrix function $\Psi(t,0)$ so that $x(t) = (\boldsymbol{\Psi} x_0)(t) = \Psi(t,0)x_0$; $t \in \boldsymbol{R}_{0+}$. It would be seen that, if virtual point initial conditions are picked up at some $t_0 > t_1 > h_m$, while the real ones are also point ones at the zero time instant, then the point initial operator does not generate the whole trajectory solution as $(\boldsymbol{\Psi} x(t_0))(t) = \Psi(t,0)x(t_0)$ for $t > t_0$.

Definition 2. *The total evolution operator, simply referred to in the sequel as the evolution operator,* $\hat{\boldsymbol{\Psi}} : \big(BPC([-h\,,0]\,;\boldsymbol{R}^n) \cup PC^1(\boldsymbol{R}_{+h}(t)\,;\,\boldsymbol{R}^n)\,\big) \to PC^1(\boldsymbol{R}\,;\,\boldsymbol{R}^n)$ *generates the solution trajectory as follows:*

$$x(t) = (\hat{\boldsymbol{\Psi}}\overline{x}_\sigma)(t) = \begin{cases} (\boldsymbol{\Psi}\, x(\sigma))(t) + \sum_{i=0}^{p} \int_{t-\sigma-h_i}^{t} \Psi(t,\tau)x(\tau)d\tau = \Psi(t,\sigma)x(\sigma) + \sum_{i=0}^{p} \int_{-h_i}^{0} \Psi(t-\tau,\sigma)x(\tau+\sigma)d\tau;\ \sigma, t(\geq \sigma) \in \boldsymbol{R}_{0+} \\ 0\,;\,\sigma\,, t(<\sigma) \in \boldsymbol{R}_{0+} \text{ or if } t \in \boldsymbol{R}_- \end{cases}$$

for any given function of initial conditions under point initial conditions at $t = 0$*, namely, if the function of initial conditions* $\varphi \in BPC([-h, 0]\,;\boldsymbol{R}^n)$ *with* $\varphi(0) = x(0) = x_0 \in \boldsymbol{R}^n$.

In particular if $\sigma = 0$, the whole evolution operator generates the solution trajectory from admissible initial conditions on $[-h\,,\,0]$ as follows:

$$x(t) = (\hat{\boldsymbol{\Psi}}\overline{\varphi})(t) = \begin{cases} (\boldsymbol{\Psi}\, x_0)(t) + \sum_{i=0}^{p} \int_{t-h_i}^{t} \Psi(t,\tau)x(\tau)d\tau = \Psi(t,0)x_0 + \sum_{i=0}^{p} \int_{-h_i}^{0} \Psi(t-\tau,0)\varphi(\tau)d\tau;\ t \in \boldsymbol{R}_{0+} \\ 0\,;t \in \boldsymbol{R}_- \end{cases}$$

The interest of this operator is relevant when the image is restricted to $PC^1(\boldsymbol{R}_{0+}\,;\,\boldsymbol{R}^n)$ since it is identically zero for negative time instants.

The delay strip evolution operator generates a solution strip of size that of the maximum delay from an initialization interval of measure that of the maximum delay. It is defined as follows:

Definition 3. *The delay strip evolution operator of symbol* $\bar{\hat{\boldsymbol{\Psi}}}_t$ *from* $BPC([-h\,,0]\,;\boldsymbol{R}^n) \cup PC^1(\boldsymbol{R}_{0+h}(t)\,;\,\boldsymbol{R}^n)$ *to* $PC^1(\boldsymbol{R}\,;\,\boldsymbol{R}^n)$ *generates any strip of solution of size* h *from the evolution operator as follows:*

$$\overline{x}_t = \left(\bar{\hat{\boldsymbol{\Psi}}}_t \overline{x}_\sigma\right)(t) = \big(\ \big(\hat{\boldsymbol{\Psi}}\overline{x}_\sigma\big)(\tau) : \tau \in [t-h\,,\,t]\big);\ \sigma,\ t \in \boldsymbol{R}_{0+}.$$

The interest of the above three evolution operators is their image restriction to piecewise continuous vector functions for non-negative time instants, which are the generated solution trajectories of the differential system (1).

Remark 1. *Note that the evolution operator defines the solution trajectory for any admissible function of initial conditions as a superposition of two terms, one of them being the part of the solution associated with the point initial condition* $x(0) = \varphi(0) = x_0$ *at* $t = 0$*, given by the point evolution operator, and the second one being generated by the function of initial conditions on* $[-h\,,\,0)$.

Note that $\big(\hat{\boldsymbol{\Psi}}\ \varphi\big)(t) = (\boldsymbol{\Psi}\ x_0)(t)$ *if* $\varphi(\tau) = 0$ *for* $\tau \in [t-h\,,\,0)$.

It turns out that, if $t < h$, *then* $\overline{x}_t(\tau) = \begin{cases} x(\tau);\ \tau \in [0\ ,\ t] \\ x(\tau) \equiv \varphi(\tau)\ ;\ \tau \in [-h\ ,\ 0] \\ 0\ ;\ \tau \in (t, +\infty)\ \cup\ (-\infty\ ,\ -h) \end{cases}$.

Note that the point evolution operator generates the solution trajectory for point initial conditions $\varphi(0) = x(0) = x_0$ *at* $t_0 = 0$ *but only for such initial conditions* $t_0 = 0$ *since* $\varphi(t) = 0$; $t \in [t - h\ ,\ 0]$. *For any initial time instant* $t_0 > h_m = h_1$, *the virtual point initial condition, defined as* $x(t) = (\mathbf{\Psi} x(t_0))(t)$ *for any* $t_0,\ t(> t_0) \in \mathbf{R}_+$, *does not generate the whole solution trajectory for* $t > t_0$ *in the case when the point initialization of the system is* $x(0) = x_0$ *at the zero time instant. In that case, a forcing additive term must be added to construct the whole solution due to the delayed dynamics since the function of initial conditions is* $x : [t_0 - h_m\ ,\ t_0] \to \mathbf{R}^n$ *in this case. See, Equation (3) for a related illustration.*

Note also that the domains of the evolution operator and of the delay strip evolution operator include continuous time-differentiable functions from $\mathbf{R}_{0+}$ *to* $\mathbf{R}^n$ *since they generate the continuous time-differentiable solution trajectory of (1).*

A major relevance is given in this paper to the properties of the point evolution operator since it generates the whole solution trajectory for point initial conditions $x(0) = x_0$ at $t_0 = 0$ (although not for $t > t_0 = h_m$). This initialization is one of the common ones in time-delay systems.

Remark 2. *If* $\varphi(\tau) = 0$ *for* $\tau \in [-h\ ,\ 0)$ *and* $\varphi(0) = x(0) = x_0$, *with* $h = h_p$, *that is, if the initial condition is punctual, it follows from (7) and (3) that*

$$\begin{aligned} x(t) &= \Psi(t,\ 0)x_0 = e^{At}x_0 + \sum_{i=0}^{p} \int_{h_i}^{t} e^{A(t-\tau)} A_{di}(\tau) x(\tau - h_i) d\tau \\ &= e^{At}\left(I_n + \sum_{i=0}^{p} \int_{h_i}^{t} e^{-A\tau} A_{di}(\tau) \Psi(\tau - h_i\ , 0) d\tau\right) x_0;\ t \in \mathbf{R}_{0+} \end{aligned} \tag{9}$$

for arbitrary point initial condition x_0 *at* $t = 0$ *so that*

$$\Psi(t,\ 0) = e^{At}\left(I_n + \sum_{i=0}^{p} \int_{h_i}^{t} e^{-A\tau} A_{di}(\tau) \Psi(\tau - h_i\ , 0) d\tau\right);\ t \in \mathbf{R}_{0+} \tag{10}$$

as it is also concluded from (6) for $t \geq \sigma = 0$.

Remark 3. *If the differential system (1) is free of delayed dynamics but time-varying, that is, if* $A_{di}(t) \equiv 0$ *for* $i \in \overline{p}$ *, while at least one of the entries of* $A_{d0}(t)$ *has a compact support in some subset of* $\mathbf{R}_{0+}$ *of nonzero Lebesgue measure, then*

$$x(t) = \Psi(t,\ 0)x_0 = e^{At}x_0 + \int_0^t e^{A(t-\tau)} A_{d0}(\tau) x(\tau) d\tau;\ t \in \mathbf{R}_{0+} \tag{11}$$

for arbitrary x_0 *so that*

$$\Psi(t,\ 0) = e^{At}\left(I_n + \int_0^t e^{-A\tau} A_{d0}(\tau) \Psi(\tau,\ 0) d\tau\right);\ t \in \mathbf{R}_{0+} \tag{12}$$

Remark 4. *Assume that it is admitted that some delays can be eventually identical, that is,* $h_m = h_1 \leq h_2 \leq \cdots \leq h_p = h$. *In this case, the differential system (1) is said to be delay-free if* $h_i = 0$ *for* $i \in \overline{p} \cup \{0\}$, *resulting in:*

$$\dot{x}(t) = \left(A + \sum_{i=0}^{p} A_{di}(t)\right) x(t);\ t \geq 0 \tag{13}$$

and the point evolution operator satisfies the same differential system, that is,

$$\dot{\Psi}(t\ , 0) = \left(A + \sum_{i=0}^{p} A_{di}(t)\right) \Psi(t, 0),\ \Psi(0,\ 0) = I_n;\ t \in \mathbf{R}_{0+} \tag{14}$$

The solution-trajectory of (13) is then given by

$$\begin{array}{ll} x(t) & = \Psi(t,\,0)x_0 = e^{At}x_0 + \sum_{i=0}^{p}\int_0^t e^{A(t-\tau)}A_{di}(\tau)x(\tau)d\tau \\ & = e^{At}\Big(I_n + \sum_{i=0}^{p}\int_0^t e^{-A\tau}A_{di}(\tau)\Psi(\tau\,,0)d\tau\Big)x_0;\ t \in \boldsymbol{R}_{0+} \end{array} \tag{15}$$

for arbitrary finite x_0 so that

$$\Psi(t,\,0) = e^{At}\left(I_n + \sum_{i=0}^{p}\int_0^t e^{-A\tau}A_{di}(\tau)\Psi(\tau\,,0)d\tau\right);\ t \in \boldsymbol{R}_{0+}. \tag{16}$$

Remark 5. *If the differential system (1) is free of delayed dynamics and time-invariant, then $\dot{x}(t) = Ax(t)$; $t \geq 0$ with $x(0) = x_0 = \varphi(0)$, and then the solution is $x(t) = e^{At}x_0$; $t \in \boldsymbol{R}_{0+}$.*

3. The Loss of the Semigroup Property and Boundedness and Stability Properties

It can be pointed out that the solution of the particular linear time-invariant system without delays has the one-parameter semigroup property since the point evolution operator in this case is the fundamental matrix e^{At} of the resulting differential system for any $t \in \boldsymbol{R}$, which is also a C_0-semigroup of infinitesimal generator A. This section discusses the loss of the semigroup property due to the presence of the delays and the time-varying nature of the dynamics. Stability properties are also provided and demonstrated as well as some illustrative examples.

If the differential system (1) is free of delayed dynamics and time-invariant, then the point evolution operator e^{At} for any $t \in \boldsymbol{R}$ is bounded and $\left\{e^{At}\right\}_{t\in\boldsymbol{R}}$ is a family of bounded linear maps from $\boldsymbol{R}^n$ to itself, which satisfies $e^{A0} = \lim\limits_{t\to 0} e^{At} = I_n$; $e^{A(t+s)} = e^{At}e^{As} = e^{As}e^{As}$; and $\left(e^{At}\right)^{-1} = e^{-At}$ for any $t, s \in \boldsymbol{R}$ so that such a family is a multiplicative group. If the set is restricted for $t \in \boldsymbol{R}_{0+}$, then it is a multiplicative semigroup $\left\{e^{At}\right\}_{t\in\boldsymbol{R}_{0+}}$ of bounded linear operators. Also, the infinitesimal generator A of the differential system $\dot{x}(t) = Ax(t)$ with arbitrary initial conditions $x(0) = x_0 = \varphi(0)$; $t \in \boldsymbol{R}_{0+}$ satisfies $A = \lim\limits_{t\to 0^+}\left[t^{-1}\left(e^{At} - I_n\right)\right]$.

On the other hand, note also that e^{At} is a fundamental matrix of the differential system $\dot{x}(t) = Ax(t)$ (so that its columns are linearly independent functions which are particular solutions of it); hence, e^{At} is non-singular of inverse e^{-At} for all $t \in \boldsymbol{R}$. Also, e^{At} is normally referred to in control theory as the state transition matrix of the differential system since the solution trajectory satisfies from the C_0-semigroup structure the associative and commutative properties, in particular:

$$x(t+s) = e^{At}x(s) = e^{As}x(t) = e^{A(t+s)}x_0 = e^{A(s+t)}x_0;\ t, s \in \boldsymbol{R}_{0+}. \tag{17}$$

By noting that A is a bounded linear operator on X, the Banach space of the solutions of the differential system (1) for the set arbitrary finite initial conditions, of domain $D(A) \subset X$, one has that $\left\{e^{At}\right\}_{t\in\boldsymbol{R}_{0+}}$ is a family of bounded linear operators on X, which is a C_0-semigroup on X of infinitesimal generator A. For any given $v \in X$, one has that $Av = \lim\limits_{t\to 0^+}\left[t^{-1}\left(e^{At}v - I_n\right)\right]$ and $D(A) = \left\{v \in X : \lim\limits_{t\to 0^+}\left[t^{-1}\left(e^{At}v - I_n\right)\ exists\right]\right\}$. Note that $D(A) = X = R^n$. However, it could be possible that an unbounded operator A could generate a bounded family $\left\{e^{At}\right\}_{t\in\boldsymbol{R}_{0+}}$ of bounded linear operators in X, which has a semigroup structure.

The incorporation of delayed dynamics and time-varying terms in (1) related to the time-invariant one leads, in general, to the loss of the semigroup structure as it is discussed in the sequel.

First, note that there exist non-unique real constants $K \geq 1$ and non-unique real constants $K_1 \in (0\,,\,K)$, which are matrix norm-dependent, and ρ and ρ_1 such that

$K_1 e^{\rho_1 t} \leq \left\| e^{At} \right\| \leq K e^{\rho t}$ for all $t \in \boldsymbol{R}_{0+}$, where $\rho \geq \rho_0 = \underset{i \in sp(A)}{maxRe\lambda_i}$ and $0 \leq \rho_1 \leq \rho_0 = \underset{i \in sp(A)}{maxRe\lambda_i}$ such that equality holds for the above inequalities if the maximum eigenvalue of A is simple. Also note the following:

(1) All the entries of e^{At} are functions of exponential order. If the stability abscissa ρ_0 of A fulfills $\rho_0 > 0$, then A is unstable, if $\rho_0 = 0$, then A is critically stable, and if $\rho_0 < 0$, then A is stable, or also said to be a stability matrix;
(2) If $\rho_0 > 0$ (respectively, if $\rho_0 < 0$), then the maximum admissible value of ρ_1 and the minimum admissible value of ρ, being compatible with the constraints $K_1 e^{\rho_1 t} \leq \left\| e^{At} \right\| \leq K e^{\rho t}$ for all $t \in \boldsymbol{R}_{0+}$, are positive (respectively, negative);
(3) If $\rho_0 = 0$, then $\rho_1 = 0$ and $\rho \geq 0$.

To simplify the notation and exposition, in the sequel, we refer to the real constant ρ as the stability abscissa of the matrix A, while having in mind, as discussed above, that in some cases, its minimum admissible value is larger than (but arbitrarily close) to the real feasible stability abscissa ρ_0.

The first subsequent result gives sufficiency-type conditions for the non-singularity of the point evolution operator.

Theorem 1. *Assume that* $A_{di} : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n \times n}$; $i \in \overline{p} \cup \{0\}$ *are bounded with* $a_{di} = \underset{t \geq 0}{sup} \left\| A_{di}(t) \right\|$; $i \in \overline{p} \cup \{0\}$ *and* $a_d = \underset{i \in \overline{p} \cup \{0\}}{max} a_{di}$. *Then, the following properties hold:*

(i) *If, for given finite delays,* a_d *is small enough and* $\rho \leq 0$, *then* $\underset{t \in \boldsymbol{R}_{0+}}{sup} \left\| \Psi(t\,, 0) \right\| < +\infty$ *so that the differential system (1) is globally stable for any admissible function of the initial condition. If, in addition,* $\rho < 0$, *then the point evolution operator norm converges exponentially to zero with time so that the differential system (1) is globally exponentially stable for any given admissible function of the initial condition.*

(ii) *If* $\rho > 0$, *then the point evolution operator norm is unbounded at an exponential rate for* $p \geq 2$ *and also except for some particular cases with* $p = 0, 1$ *discussed in the subsequent proof.*

(iii) *The entries of the point evolution operator* $\Psi(t\,, 0)$ *are of exponential order with t and there exist non-unique real constants* $M > 0$ *(being norm-dependent) and* ω *such that*

$$\left\| \Psi(t\,, 0) \right\| \leq M e^{\omega t}; t \in \boldsymbol{R}_{0+}. \tag{18}$$

If a_d *is small enough and* $\rho \leq 0$, *then the minimum constant* ω *valid in the right-hand side of (18) is non-negative, and if, furthermore,* $\rho < 0$, *then the minimum* ω *valid in the right-hand side of (18) is negative.*

(iv) *If* $\left(I_n + \sum_{i=0}^{p} \int_{h_i}^{t} e^{-A\tau} A_{di}(\tau) \Psi(\tau - h_i\,, 0) d\tau \right)$ *is non-singular for all* $t \in \boldsymbol{R}_{0+}$, *then* $\left\| e^{-At} \Psi(t, 0) - I_n \right\|$ *is bounded and* $\Psi(t, 0)$ *is non-singular for all* $t \in \boldsymbol{R}_{0+}$.

If, for $i \in \overline{p} \cup \{0\}$ *and all* $t \in \boldsymbol{R}_{0+}$, $A_{di}(t) = F_{di}(t) \Psi^{-1}(t - h_i\,, 0)$ *with the matrix function* $F_{di}(t)$ *defined from* $\boldsymbol{R}_{0+}$ *to* $\boldsymbol{R}^{n \times n}$, *is of exponential order according to* $\left\| F_{di}(t) \right\| \leq K_{fdi} e^{\rho_{dfi} t}$ *with positive real constants* K_{dfi} *and* ρ_{dfi} *and* $\underset{0 \leq i \leq p}{max} \rho_{dfi} < \rho$, *then*

$$\left\| e^{-At} \Psi(t, 0) - I_n \right\| \leq K \sum_{i=0}^{p} K_{fdi} \frac{1 - e^{(\rho_{fdi} - \rho) h_i}}{\rho - \rho_{dfi}} \leq C < +\infty; t \in \boldsymbol{R}_{0+}. \tag{19}$$

Proof. Taking matrix norms in (10) yields, provided that $\|\Psi(t,\ 0)\| \leq Me^{\omega t}$; $t \in \boldsymbol{R}_{0+}$ and $\rho \neq 0$,

$$\begin{aligned} \|\Psi(t,\ 0)\| &\leq Ke^{\rho t} + K\sum_{i=0}^{p}\left(a_{di}\int_{h_i}^{t} e^{\rho(t-\tau)}\|\Psi(\tau-h_i\ ,0)\|d\tau\right) \\ &\leq Ke^{\rho t}\left(1+\sup_{0\leq\tau\leq t}\|\Psi(\tau-h\ ,0)\|\left(\sum_{i=0}^{p} a_{di}\frac{e^{-\rho h_i}-1}{|\rho|}\right)\right) \\ &\leq K\,e^{\rho t}\left(1+\sup_{0\leq\tau\leq t}\|\Psi(\tau-h\ ,0)\|\left(\sum_{i=0}^{p} a_{di}\frac{e^{-\rho h_i}-1}{|\rho|}\right)\right);\ t\in\boldsymbol{R}_{0+}. \end{aligned} \tag{20}$$

Since $\|\Psi(t,0)\| = 0$ for $t < 0$, the above inequality implies that

$$\sup_{0\leq\tau\leq t}\|\Psi(\tau\ ,0)\| \leq K\left(\sup_{0\leq\tau\ \leq t} e^{\rho\tau}\right)\left(1+\sup_{0\leq\tau\leq t}\|\Psi(\tau\ ,0)\|\left(\sum_{i=0}^{p} a_{di}\frac{e^{-\rho h_i}-1}{|\rho|}\right)\right);\ t\in\boldsymbol{R}_{0+}. \tag{21}$$

Consider three cases related to the stability abscissa of A, namely,

Case a: If $\rho = 0$, that is, if A is critically stable, and the parametric constraint $K\left(\sum_{i=0}^{p} a_{di}\right) < 1$ holds, then

$$\sup_{0\leq\tau\leq t}\|\Psi(\tau\ ,0)\| \leq \frac{K}{1-K\left(\sum_{i=0}^{p} a_{di}\right)} < +\infty;\ t\in\boldsymbol{R}_{0+}. \tag{22}$$

If the above parametric constraint is changed to the stronger one $K(p+1)a_d < 1$, then

$$\sup_{0\leq\tau\leq t}\|\Psi(\tau\ ,0)\| \leq \frac{K}{1-K(p+1)a_d} < +\infty;\ t\in\boldsymbol{R}_{0+}. \tag{23}$$

Case b: If $\rho < 0$, that is, if A is stable, and $K\left(\sum_{i=0}^{p} a_{di}\dfrac{\left|e^{|\rho|h_i}-1\right|}{|\rho|}\right) < 1$, then

$$\sup_{0\leq\tau\leq t}\|\Psi(\tau\ ,0)\| \leq \frac{K|\rho|}{|\rho|-K\left(\sum_{i=0}^{p} a_{di}\left(e^{|\rho|h_i}-1\right)\right)} \leq \overline{\Psi} < +\infty;\ t\in\boldsymbol{R}_{0+} \tag{24}$$

If the above parametric constraint is changed to the stronger one $K(p+1)\left|e^{|\rho|h}-1\right|a_d|\rho|^{-1} < 1$, then

$$\sup_{0\leq\tau\leq t}\|\Psi(\tau\ ,0)\| \leq \frac{K|\rho|}{|\rho|-K(p+1)\left|e^{|\rho|h}-1\right|a_d} \leq \overline{\Psi}_1 < +\infty;\ t\in\boldsymbol{R}_{0+}. \tag{25}$$

Also, replacing the upper-bound $\overline{\Psi}$ in (24) into the right-hand side of (20) leads to:

$$\|\Psi(t\ ,0)\| \leq K\,e^{\rho t}\left(1+\left(\sum_{i=0}^{p} a_{di}\frac{e^{-\rho h_i}-1}{|\rho|}\right)\overline{\Psi}\right);\ t\in\boldsymbol{R}_{0+} \tag{26}$$

so that $\lim_{t\to\infty}\|\Psi(t\ ,0)\| = 0$ and the point evolution operator norm decreases exponentially with time. A similar conclusion follows from the replacement of (24) into (20). Thus, Case a and Case b prove Property (i).

Case c: If $\rho > 0$, that is, if A is unstable, then there is some $t_0 \in \boldsymbol{R}_{0+}$ such that $K\left(\sum_{i=0}^{p} a_{di}\frac{\left|1-e^{\rho(t-h_i)}\right|}{\rho}\right) \geq 1$ for $t \geq t_0$. Note that, together with (20), one also obtains, by reversing such an inequality to look for a lower-bound, that:

$$\|\Psi(t\,,0)\| \geq aK_1\, e^{\rho_1 t}\left(\inf_{h<\tau\leq t}\|\Psi(\tau-h\,,0)\|\right);\ t(\geq h) \in \boldsymbol{R}_{0+} \tag{27}$$

where

$$a = \rho_1^{-1}\left|\min_{i\in\overline{p}\cup\{0\}}\inf_{t\in\boldsymbol{R}_{0+}}\left(\|A_{di}(t)\|\left|1-e^{\rho_1(t-h_i)}\right|\right) - \left(1+\sum_{j(\neq i)=0}^{p}\sup_{t\in\boldsymbol{R}_{0+}}\left(\left\|A_{dj}(t)\right\|\left|1-e^{\rho_1(t-h_j)}\right|\right)\right)\right| \tag{28}$$

which implies that $\lim_{t\to+\infty}\|\Psi(t\,,0)\| = +\infty$ if $a > 0$, which always holds for the following:

(a) The infimum of $A_{d0}(t)$ is not identically zero on $cl\boldsymbol{R}_{0+}$, $p = 1$ and $\min_{i\in\{0\,,1\}}\inf_{t\in\boldsymbol{R}_{0+}}$ $\left(\|A_{di}(t)\|\left|1-e^{\rho_1(t-h_1)}\right|\right) \neq 1$;
(b) $p \geq 2$;
(c) $p = 0$ and the infimum of $A_{d0}(t)$ is nonzero on $cl\,\boldsymbol{R}_{0+}$ and $\inf_{t\in\boldsymbol{R}_{0+}}\left(\|A_{d0}(t)\|\left|1-e^{\rho_1 t}\right|\right) \neq 1$.
In particular, if $A_{d0}(t)$ is zero on $cl\,\boldsymbol{R}_{0+}$, then $a = 0$, $\Psi(t,0)$ is exponentially unbounded given by $\Psi(t,0) = e^{At}$.

Furthermore, in the above cases, the above lower-bound (27), subject to (28), of the norm of the point evolution operator grows exponentially with time. Then, the differential system (1) is unstable with an exponential rate.

Property (ii) has been proved. Property (iii) is a direct consequence of Property (i) and Property (ii).

Property (iv) is proved as follows. Note from (10) that if $\left(I_n + \sum_{i=0}^{p}\int_{h_i}^{t} e^{-A\tau}A_{di}(\tau)\Psi(\tau-h_i\,,0)d\tau\right)$ is non-singular for all $t \in \boldsymbol{R}_{0+}$, then $\Psi(t,0)$ is non-singular since e^{At} is non-singular for all $t \in \boldsymbol{R}_{0+}$ and $\left\|e^{-At}\Psi(t,0) - I_n\right\| \leq C < +\infty$ for all $t \in \boldsymbol{R}_{0+}$. Now, if for $i \in \overline{p}\cup\{0\}$ and all $t \in \boldsymbol{R}_{0+}$, $A_{di}(t) = F_{di}(t)\Psi^{-1}(t-h_i\,,0)$ and, furthermore, the matrix function $F_{di}(t)$ defined from $\boldsymbol{R}_{0+}$ to $\boldsymbol{R}^{n\times n}$ is of exponential order according to $\|F_{di}(t)\| \leq K_{fdi}e^{\rho_{dfi}t}$, then the second part of the property is immediate since

$$\begin{aligned}\left\|e^{-At}\Psi(t,0) - I_n\right\| &= \left\|\sum_{i=0}^{p}\int_{h_i}^{t} e^{-A\tau}F_{di}(\tau)d\tau\right\| \leq K\left|\sum_{i=0}^{p}\int_{h_i}^{t} e^{(\rho_{fdi}-\rho)\tau}K_{fdi}d\tau\right| \\ &\leq K\sum_{i=0}^{p}K_{fdi}\frac{e^{(\rho_{fdi}-\rho)t} - e^{(\rho_{fdi}-\rho)h_i}}{\rho-\rho_{dfi}} \leq K\sum_{i=0}^{p}K_{fdi}\frac{1-e^{(\rho_{fdi}-\rho)h_i}}{\rho-\rho_{dfi}};\ t \in \boldsymbol{R}_{0+}.\end{aligned} \tag{29}$$

□

An example is now discussed, which visualizes that if a particular case of the differential system (1) is exponentially unstable in the absence of delayed dynamics, it is not necessarily concluded that it is also exponentially unstable in the presence of delayed dynamics since the exponential instability holds under some additional conditions. See Cases a-c in the proof of Property (ii) of Theorem 1 for illustration.

Example 1. *Consider the differential equation with a delay h:*

$$\dot{x}(t) = ax(t) + a_d(t)x(t-h) \tag{30}$$

with $a > 0$ punctual initial conditions $\varphi(0) = x(0) = x_0$. The solution is:

$$\begin{aligned} x(t) &= \Psi(t,0)x_0 = e^{at}\Big(x_0 + \int_0^t e^{-a\tau}a_d(\tau)x(\tau-h)d\tau\Big) \\ &= e^{at}\Big(1 + \int_0^t e^{-a\tau}a_d(\tau)\Psi(\tau-h\ ,\ 0)d\tau\Big)x_0,\ t \in \boldsymbol{R}_{0+}. \end{aligned} \tag{31}$$

The point evolution operator $\Psi(t,0)$ satisfies (30) and it is given by:

$$\begin{aligned} \Psi(t,\ 0) &= e^{at}\Big(1 + \int_0^t e^{-a\tau}a_d(\tau)\Psi(\tau-h\ ,\ 0)d\tau\Big) \\ &= e^{at}\Big(1 + \int_h^t e^{-a\tau}a_d(\tau)\Psi(\tau-h\ ,\ 0)d\tau\Big);\ \Psi(t,t) = 1,\ \Psi(t,\ \tau) = 0 \text{for } t(\geq h), \tau(> t) \in \boldsymbol{R}_{0+}. \end{aligned} \tag{32}$$

We now claim that $a_d(t) = k_d(t)e^{-\sigma t}$ for some real constant σ may lead to $\Psi(t,0) \leq k_\Psi e^{-\rho t}$ for $t \geq h$ and some real constants k_Ψ and $\rho \geq 0$. Assume that $k_d(t) = k_{dn}$ for $t \in [nh\ ,(n+1)h)$ so that it is piecewise continuous. Replacing those conditions in the right-hand side of (32) yields:

$$\begin{aligned} &\Psi((n+1)h,0) = e^{a(n+1)h} + k_d k_\Psi \int_h^{(n+1)h} e^{a(t-\tau)}e^{-\sigma\tau}e^{-\rho(\tau-h)}d\tau \\ &= e^{a(n+1)h} + \sum_{j=1}^{n-1} k_{dj}k_\Psi \int_{jh}^{(j+1)h} e^{a((n+1)h-\tau)}e^{-\sigma\tau}e^{-\rho(\tau-h)}d\tau + k_\Psi k_{dn}\int_{nh}^{(n+1)h} e^{a((n+1)h-\tau)}e^{-\sigma\tau}e^{-\rho(\tau-h)}d\tau \\ &= e^{ah}\Psi(nh,0) + k_\Psi k_{dn}e^{(a(n+1)+\rho)h}\int_{nh}^{(n+1)h} e^{-(a+\sigma+\rho)\tau}d\tau \\ &= e^{ah}\Psi(nh,0) + k_\Psi k_{dn}e^{(a(n+1)+\rho)h}\frac{e^{-(a+\sigma+\rho)nh} - e^{-(a+\sigma+\rho)(n+1)h}}{a+\sigma+\rho} \\ &= e^{ah}\Psi(nh,0) + k_\Psi k_{dn}e^{\rho h}\frac{e^{-(\sigma+\rho)nh}e^{ah} - e^{-(\sigma+\rho)(n+1)h}}{a+\sigma+\rho} \\ &= k_\Psi e^{-\rho(n+1)h}; n \in \boldsymbol{Z}_{0+} \end{aligned} \tag{33}$$

provided that

$$\begin{aligned} k_{dn} &= \Big(e^{-\rho(n+1)h} - k_\Psi^{-1}e^{ah}\Psi(nh,0)\Big)e^{-\rho h}\Big(\tfrac{a+\sigma+\rho}{e^{-(\sigma+\rho)nh}e^{ah} - e^{-(\sigma+\rho)(n+1)h}}\Big) \\ &= \Big(e^{-\rho(n+1)h} - k_\Psi^{-1}e^{ah}k_\Psi e^{-\rho nh}\Big)e^{-\rho h}\Big(\tfrac{a+\sigma+\rho}{e^{-(\sigma+\rho)nh}e^{ah} - e^{-(\sigma+\rho)(n+1)h}}\Big) \\ &= \Big(e^{-\rho h} - e^{ah}\Big)e^{-\rho nh}e^{-\rho h}\Big(\tfrac{a+\sigma+\rho}{e^{-(\sigma+\rho)nh}e^{ah} - e^{-(\sigma+\rho)(n+1)h}}\Big) \\ &= \Big(e^{-\rho h} - e^{ah}\Big)e^{-\rho(n+1)h}\Big(\tfrac{a+\sigma+\rho}{e^{-(\sigma+\rho)nh}e^{ah} - e^{-(\sigma+\rho)(n+1)h}}\Big) \\ &= \Big(e^{-\rho h} - e^{ah}\Big)e^{-\rho(n+1)h}e^{(\sigma+\rho)nh}\Big(\tfrac{a+\sigma+\rho}{e^{ah} - e^{-(\sigma+\rho)h}}\Big) \\ &= \Big(e^{-\rho h} - e^{ah}\Big)e^{-\rho h}e^{\sigma nh}\Big(\tfrac{a+\sigma+\rho}{e^{ah} - e^{-(\sigma+\rho)h}}\Big); n \in \boldsymbol{Z}_{0+}. \end{aligned} \tag{34}$$

Then, it has been proved by complete induction that $\Psi(nh,0) \leq k_\Psi e^{-\rho nh}$ for any $n \in \boldsymbol{Z}_+$ implies that $\Big\{\Psi(nh,0) - k_\Psi e^{-\rho nh}\Big\}_{n=0}^{\infty} \subseteq \boldsymbol{R}_{-0}$ if (34) holds.

It also follows, if (34) holds, that:

Case (a): the sequence $\{k_{dn}\}_{n=0}^{\infty}$ is negative and unbounded if $\sigma > 0$, while it is bounded if $\sigma \leq 0$ so that $a_d(t)$ is also bounded for all time according to $a_d(t) = k_{dn}e^{-|\sigma|nh} \leq k_{dn}$ for any $t \in [nh\ ,\ (n+1)h)$. If $\sigma < 0$, then $\{k_{dn}\}_{n=0}^{\infty} \to 0$ and $a_d(t) \to 0$ as $t \to \infty$.

Case (b): if $\rho \geq 0$, then $\Psi(nh,0) \leq k_\Psi$ for $n \in \boldsymbol{Z}_{0+}$ and, furthermore, if $\rho > 0$, then $\Psi(nh,0) \leq k_\Psi e^{-\rho nh}$ converges asymptotically to zero at an exponential rate as $n \to \infty$. The above properties hold even if the C_0-semigroup $\{e^{At}\}_{t\in\boldsymbol{R}_{0+}}$, which defines the solution of the delay-free system, is unbounded, that is, $\rho > 0$.

Case (c): if $\rho > 0$ and $\sigma \leq 0$, then $\Psi(t,0) \leq k_\Psi e^{-\rho t}$ for $t \in \boldsymbol{R}_{0+}$, and it converges exponentially to zero since $\Psi(nh,0)\Big\{\leq k_\Psi e^{-\rho nh}\Big\} \to 0$ as $n \to \infty$ from Case b, which leads $t_n(\tau) = nh + \tau$ for any $n \in \boldsymbol{Z}_{0+}$ and for any $\tau \in [0\ ,h)$ to the following expression:

$$\begin{aligned}
x(t_n(\tau)) &= e^{a\tau}\big(x(nh) + \textstyle\int_0^\tau e^{-a\tau\prime} a_d(nh+\tau\prime)x[(n-1)h+\tau\prime]d\tau\prime\big)\\
&= e^{a\tau}\big(x(nh) + x[(n-1)h]\; a_d(nh)\big(\textstyle\int_0^\tau e^{-a\tau\prime}\Psi[(n-1)h+\tau\prime,(n-1)h]d\tau\prime\big)\big)\\
&= e^{a\tau}\big(\Psi(nh\,,0) + \Psi((n-1h\,,0)\;)a_d(nh)\big(\textstyle\int_0^\tau e^{-a\tau\prime}\Psi[(n-1)h+\tau\prime,(n-1)h]d\tau\prime\big)\big)x_0\\
&\le k_\Psi e^{a\tau}\Big(e^{-\rho nh} + e^{-\rho(n-1)h}a_d(nh)\big(\textstyle\int_0^\tau e^{-a\tau\prime}\Psi[(n-1)h+\tau\prime,(n-1)h]d\tau\prime\big)\Big)x_0\\
&\le k_\Psi e^{a\tau}e^{-\rho(n-1)h}\Big(e^{-\rho h} + a_d(nh)\big(\textstyle\int_0^\tau e^{-a\tau\prime}\Psi[(n-1)h+\tau\prime,(n-1)h]d\tau\prime\big)\Big)x_0\\
&= k_\Psi e^{a\tau}e^{-\rho nh}\Big(1 + a_d(nh)e^{\rho h}\big(\textstyle\int_0^\tau e^{-a\tau\prime}\Psi[(n-1)h+\tau\prime,(n-1)h]d\tau\prime\big)\Big)x_0\\
&\le \Big[k_\Psi e^{(\rho+a)\tau}\Big(1 + a_d(nh)e^{\rho h}\big(\textstyle\int_0^\tau e^{-a\tau\prime}\Psi[(n-1)h+\tau\prime,(n-1)h]d\tau\prime\big)\Big)\Big]\; e^{-\rho(nh+\tau)}x_0\\
&= \bar{k}_\Psi\; e^{-\rho t_n(\tau)}x_0;\; n\in \mathbf{Z}_{0+}, \tau\in[0\,,h),
\end{aligned} \tag{35}$$

and then $\Psi(t_n(\tau)\;,\,0) \le \bar{k}_\Psi e^{-\rho t_n(\tau)}$; $n \in \mathbf{Z}_{0+}$, $\tau \in [0\,,h)$, *where*

$$\begin{aligned}
\bar{k}_\Psi &= k_\Psi e^{(\rho+a)h}\Big(1 + e^{\rho h}\sup_{n\in\mathbf{Z}_{0+}}(a_d(nh))\sup_{\tau\in[0\,,h]}\big(\textstyle\int_0^\tau e^{-a\tau\prime}\Psi[(n-1)h+\tau\prime,(n-1)h]d\tau\prime\big)\Big)\\
&= k_\Psi e^{(\rho+a)h}\Big(1 + a^{-1}e^{\rho h}\big(1-e^{-ah}\big)\sup_{n\in\mathbf{Z}_{0+}}(a_d(nh))\Big)\\
&= k_\Psi e^{(\rho+a)h}\Big(1 + a^{-1}e^{\rho h}\big(1-e^{-ah}\big)\sup_{n\in\mathbf{Z}_{0+}}k_{dn}\Big) < +\infty
\end{aligned} \tag{36}$$

since $a_d(nh) \le k_{dn} \le \sup_{n\in\mathbf{Z}_{0+}} k_{dn} < +\infty$; $n \in \mathbf{Z}_{0+}$ *from Case a since* $\sigma \le 0$. *Note that Equation (34) is also valid for* $\tau = 0$, *that is, for* $t_n(0) = nh$, *and note also that* $\bar{k}_\Psi > k_\Psi$. *Then,* $\Psi(t,0) \le \bar{k}_\Psi e^{-\rho t}$, *subject to (34) and (36), with* $\rho \ge 0$ *and* $\sigma \le 0$.

Based on Theorem 1, the subsequent result establishes some sufficiency-type conditions, which guarantee that the point evolution operator $\boldsymbol{\Psi}$ from $\boldsymbol{R}^n$ to $\boldsymbol{R}^n$ can also be represented as an exponential matrix function of order n from $\boldsymbol{R}_{0+}$ to $\boldsymbol{R}^{n\times n}$ if the matrix A is stable or critically stable.

Theorem 2. *The following properties hold:*

(i) *Assume that if A is critically stable, then either the constraint (22) holds or the stronger one (23) holds, and, if A is stable, then either the constraint (24) holds or the stronger one (25) holds. Then, the point evolution operator (10) of the differential system (1) is non-singular for all non-negative time if the stability abscissa of A is non-positive (that is, A is either critically stable or a stability matrix) and it can also be expressed as an exponential matrix function of the form* $\Psi(t,\,0) = e^{(A+\overline{A}(t))\;t}$; $t \in \boldsymbol{R}_{0+}$, *where* $\overline{A}(t)$ *is unique, continuous and time-differentiable in* $\boldsymbol{R}_{0+}$ *and defined by*

$$\overline{A}(t) = \int_{0^+}^t \left(\tau^{-1}\left[e^{-(A+\overline{A}(\tau))\tau}\left(\sum_{i=0}^p A_{di}(\tau)\; e^{(A+\overline{A}(\tau-h_i))\tau}\right) - \overline{A}(\tau)e^{-\overline{A}(\tau)\tau}\right]\right)d\tau;\; t\in\boldsymbol{R}_{0+} \tag{37}$$

$$= \int_{0^+}^t \left(\tau^{-1}\left[\Psi^{-1}(\tau\,,0)\left(\sum_{i=0}^p A_{di}(\tau)\Psi(\tau-h_i,0)\;\right) - \overline{A}(\tau)e^{-\overline{A}(\tau)\tau}\right]\right)d\tau;\; t\in\boldsymbol{R}_{0+}. \tag{38}$$

If the given conditions on stability or critical stability of A are removed, then the point evolution operator whose associated matrix function $\Psi(t,\,0) = e^{(A+\overline{A}(t))\;t}$ *is non-singular for any* $t \in [0,\,T)$ *and some* $T \in \boldsymbol{R}_+$.

(ii) *Define the truncated matrix function* $\overline{A} : [0\;,\,T] \to \boldsymbol{R}^{n\times n}$ *for some given* $0 < T \le +\infty$ *so that* $\overline{\boldsymbol{A}_{\boldsymbol{T}}} : \boldsymbol{R}^n \to \boldsymbol{R}^n$ *is an associated linear operator defined by* $\left(\overline{\boldsymbol{A}_{\boldsymbol{T}}}z\right)(t) = \overline{z}(t)$ *for any* $z \in D\left(\overline{\boldsymbol{A}_{\boldsymbol{T}}}\right) = \boldsymbol{R}^n$ *and* $t \in [0\;,\,T]$. *If* $\overline{\boldsymbol{A}_{\boldsymbol{T}}}$ *is closed and bounded, equivalently continuous since it*

is linear, then the point evolution operator (10) restricted to the interval $[0\ ,\ T]$ *is strongly continuous and also uniformly continuous in* $[0\ ,\ T]$ *with respect to the operator norm.*

(iii) *Under the conditions of Property (i), the point evolution operator (10) is strongly continuous and uniformly continuous in cl* $\boldsymbol{R}_{0+}$ *and*

$$A = \lim_{t\to 0^+}\left(t^{-1}(\Psi(t,\ 0) - I_n)\right) = \lim_{t\to 0^+}\left(t^{-1}\left(e^{At} - I_n\right)\right);\ \overline{A}(0^+) = 0$$

$$\overline{A}\left(t_0^+\right) = t_0^{-1}(\Psi(t_0\ ,\ 0) - I_n) - \sum_{j=0}^{\infty}\frac{\left(A + \overline{A}(t_0)\right)^{j+2}t_0^{j+1}}{(j+2)!} - \lim_{t\to 0^+}\left(t^{-1}\left(e^{At} - I_n\right)\right) \text{for } t_0 \in \boldsymbol{R}_{+}.$$

Proof. Note that if $\rho = 0$ and (22) or (23) holds, and if $\rho < 0$ and if (24) or (25) holds, then the matrix $\left(I_n + \sum_{i=0}^{p}\int_{h_i}^{t} e^{-A\tau}A_{di}(\tau)\Psi(\tau - h_i\ ,0)d\tau\right)$ is non-singular for all $t \in \boldsymbol{R}_{0+}$ and, therefore, from (10), $\Psi(t,0)$ is also non-singular, since e^{At} is non-singular, and bounded for all $t \in \boldsymbol{R}_{0+}$.

It is claimed that $\Psi(t,\ 0)$ is of the form $\Psi(t,\ 0) = e^{(A+\overline{A}(t))\ t}$ for $t \in \boldsymbol{R}_{0+}$ for some existing matrix function $\overline{A} : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n\times n}$ of continuous and differentiable entries on $\boldsymbol{R}_{+}$ so that

$$\Psi(t,0) = e^{At}\left(I_n + \sum_{i=0}^{p}\int_{h_i}^{t} e^{-A\tau}A_{di}(\tau)\Psi(\tau - h_i\ ,0)d\tau\right) = e^{(A+\overline{A}(t))\ t};\ t \in \boldsymbol{R}_{0+} \tag{39}$$

where

$$e^{\overline{A}(t)t} = I_n + \sum_{i=0}^{p}\int_{h_i}^{t} e^{-A\tau}A_{di}(\tau)\Psi(\tau - h_i\ ,0)d\tau;\ t \in \boldsymbol{R}_{0+}. \tag{40}$$

It is seen directly that (18) holds for $t = 0$ with $\overline{A}(0) = 0$. Since (39), subject to (40), is an identity for all $t \geq 0$, then the derivatives with respect to time of both right-hand-side terms have to be also identical on $\boldsymbol{R}_{0+}$. Then, one obtains the subsequent identity:

$$\frac{d}{dt}\left(e^{(A+\overline{A}(t))t}\right) = \left(t\dot{\overline{A}}\ (t)\ + \overline{A}(t)\right)e^{\overline{A}(t)t} = e^{-At}\left(\sum_{i=0}^{p}A_{di}(t)\Psi(t - h_i\ ,0)\right);\ t \in \boldsymbol{R}_{0+}. \tag{41}$$

Also, $e^{\overline{A}(t)t}$ is non-singular for all $t \in \boldsymbol{R}_{0+}$ under the given assumptions on the matrix A for zero and for negative stability abscissa of A and sufficiently smallness of the undelayed time-varying and delayed dynamics in terms of smallness of norms, which imply the boundedness of the matrix function $\overline{A}(t)$ for all $t \in \boldsymbol{R}_{0+}$. Then, $\Psi(t,\ 0) = e^{(A+\overline{A}(t))\ t}$ holds for all $t \in \boldsymbol{R}_{0+}$ if and only if

$$\dot{\overline{A}}\ (t)\ = t^{-1}e^{-\overline{A}(t)t}\left[e^{-At}\left(\sum_{i=0}^{p}A_{di}(t)\ \Psi(t - h_i\ ,0)\right) - \overline{A}(t)\right];\ t \in \boldsymbol{R}_{0+} \tag{42}$$

so that

$$\overline{A}(t) = \overline{A}\left(0^+\right) + \int_{0^+}^{t}\left(\tau^{-1}e^{-\overline{A}(\tau)\tau}\left[e^{-A\tau}\left(\sum_{i=0}^{p}A_{di}(\tau)\ \Psi(\tau - h_i\ ,0)\right) - \overline{A}(\tau)\right]\right)d\tau;\ t \in \boldsymbol{R}_{0+}, \tag{43}$$

which is identical to (35) and (36) and is unique for all $t \in \boldsymbol{R}_{0+}$. The first part of Property (i) has been proved for the case when A is either stable or critically stable. In any case, irrespective of the eigenvalues of A, there always exists some $T \in \boldsymbol{R}_{+}$ such that $e^{\overline{A}(t)t}$ is non-singular for $t \in [0\ ,T]$ from (40), the continuity of $\overline{A} : [0,T) \to \boldsymbol{R}^{n\times n}$ and the fact that $e^{\overline{A}(0).0} = I_n$ is non-singular. Property (i) has been fully proved.

To prove Property (ii), note that from Property (i), $\Psi(t,\ 0) = e^{(A+\overline{A}(t))\ t}$ holds for all $t \in [0\ ,T]$ and some (finite or infinite) $T \in \boldsymbol{R}_{+}$. It turns out that there always exists (at least) a finite $T \in \boldsymbol{R}_{0+}$ such that $\overline{A}_T{:}\boldsymbol{R}^n \to \boldsymbol{R}^n$ is bounded and then continuous in $[0\ ,\ T]$ since it is linear. Then, the point evolution operator $\boldsymbol{\Psi} : \boldsymbol{R}^n \to PC^1(\ [0,\ T]\ ;\ \boldsymbol{R}^n)$, whose associated matrix function is $\Psi(t,\ 0) = e^{(A+\overline{A}(t))\ t}; t \in [0\ ,\ T]$, is strongly continuous, since it is closed,

and uniformly continuous in $[0\ ,\ T]$, since it is closed and bounded, with respect to the operator norm. Thus, for any $z \in \boldsymbol{R}^n$ and any $t_0 \in [0\ ,\ T]$, one has that:

$$\lim_{t \to t_0^+} \|\Psi(t,0)z - \Psi(t_0,\ 0)z\| = \left\|e^{(A+\overline{A}(t_0^+))}z - e^{(A+\overline{A}(t_0))}z\right\| = \left\|e^{(A+\overline{A}(t_0))}z - e^{(A+\overline{A}(t_0))}z\right\| = 0$$

$$\lim_{t \to t_0^+} \|\Psi(t,0) - \Psi(t_0,\ 0)\| = \left\|e^{(A+\overline{A}(t_0^+))} - e^{(A+\overline{A}(t_0))}\right\| = \left\|e^{(A+\overline{A}(t_0))} - e^{(A+\overline{A}(t_0))}\right\| = 0$$

so that $\boldsymbol{\Psi} : \boldsymbol{R}^n \to PC^1(\ [0,\ T]\ ;\ \boldsymbol{R}^n)$ is strongly continuous, and it is also uniformly continuous. Property (ii) has been proved.

The first part of Property (iii) follows directly from Property (i) since Property (ii) holds for $0 \leq T \leq \infty$, that is, on $cl\ \boldsymbol{R}_{0+}$. Now, note that, for $t \in \boldsymbol{R}_+$:

$$\begin{aligned} t^{-1}\left(e^{At+\overline{A}(t)t} - I_n\right) &= t^{-1}\left(\sum_{j=0}^{\infty} \frac{\left(A+\overline{A}(t)\right)^j t^j}{j!} - I_n\right) \\ &= \sum_{j=1}^{\infty} \frac{\left(A+\overline{A}(t)\right)^j t^{j-1}}{j!} = A + \overline{A}(t) + \sum_{j=2}^{\infty} \frac{\left(A+\overline{A}(t)\right)^j t^{j-1}}{j!}, \end{aligned} \tag{44}$$

equivalently,

$$e^{(A+\overline{A}(t))t} - I_n = t\left(A+\overline{A}(t)\right) + \sum_{j=2}^{\infty} \frac{\left(A+\overline{A}(t)\right)^j t^j}{j!} \tag{45}$$

so that for $t_0 > 0$, one has

$$\begin{aligned} A + \overline{A}(t_0^+) &= \lim_{t \to t_0}\left(t^{-1}\left(e^{(A+\overline{A}(t))t} - I_n\right) - \sum_{j=2}^{\infty} \frac{\left(A+\overline{A}(t)\right)^j t^{j-1}}{j!}\right) \\ &= \lim_{t \to t_0^+}\left(t^{-1}(\Psi(t\ ,\ 0) - I_n) - \sum_{j=0}^{\infty} \frac{\left(A+\overline{A}(t)\right)^{j+2} t^{j+1}}{(j+2)!}\right) \\ &= t_0^{-1}(\Psi(t_0\ ,\ 0) - I_n) - \sum_{j=0}^{\infty} \frac{\left(A+\overline{A}(t_0)\right)^{j+2} t_0^{j+1}}{(j+2)!} \end{aligned} \tag{46}$$

and for $t_0 = 0$,

$$A + \overline{A}(0^+) = A = \lim_{t \to 0^+}\left(t^{-1}\left(e^{(A+\overline{A}(t))t} - I_n\right)\right) = \lim_{t \to 0^+}\left(t^{-1}(\Psi(t\ ,\ 0) - I_n)\right) = \lim_{t \to 0^+}\left(t^{-1}\left(e^{At} - I_n\right)\right). \tag{47}$$

The second part of Property (iii) follows from the above equations. □

Remark 6. *Concerning the uniform continuity of the point evolution operator [see Theorem 2 (iii)], note that $\overline{A}(t)$ is not guaranteed to be bounded for all $t \in cl\ \boldsymbol{R}_{0+}$ even if A is critically stable or stable, as illustrated in the subsequent example.*

Example 2. *Assume that $p = 1$, $A_{d0}(t) \equiv 0$, and that A and $A_d(t)$ are diagonal such that $A_{ij} = a$ and $(A_{d1})_{ii}(t) = a_d(t) < 0$ for all $t \in \boldsymbol{R}_{0+}$ are both diagonal with identical diagonal entries. It turns out that the point evolution operator (10) is diagonal with positive diagonal entries and that there exists a finite time instant T such that $e^{\overline{A}(T)T} = 0$ and $e^{\overline{A}(t)t}$ is non-singular for $t \in [0,\ T)$. Then, $\overline{A}(T)$ is diagonal with infinity diagonal entries, and this happens irrespective of A being stable or critically stable and $\overline{A}(t)$ and $\left(A+\overline{A}(t)\right)$ are both bounded for all $t \in [0,\ T)$. This results in $\overline{\boldsymbol{A}_{\boldsymbol{T}}}$ being unbounded in $[0,\ T]$ since $\overline{A}(T)$ has infinity diagonal entries. Therefore, $\Psi(t,\ 0) = e^{(A+\overline{A}(t))\ t}; t \in [0,\ T)$ is strongly and uniformly continuous in any closed subset $[0\ ,\ E]$ of $[0,\ T)$, since $\overline{\boldsymbol{A}_{\boldsymbol{E}}}$ is bounded, but not in $[0,\ T]$ nor in $cl\ \boldsymbol{R}_{0+}$ since $\overline{A}_{cl\ \boldsymbol{R}_{0+}}$ is unbounded since the operator $\overline{\boldsymbol{A}_{\boldsymbol{T}}}$ on $\boldsymbol{R}^n$ is unbounded.*

Example 3. *Assume that* $p = 1$, $A_{d0}(t) \equiv 0$, $A_{d1}(t)$ *is diagonal for all time, and that* A *is diagonal and either critically stable or stable and that the point evolution operator is bounded (for instance, under the conditions of Theorem 1 (i)) and non-singular for all* $t \in \boldsymbol{R}_{0+}$, *which holds if* $A_{d1}(t)$ *is such that all the eigenvalues of the matrix function of the right-hand side of (40) are in* $(0\,, 1)$ *for all time. Then,* $\overline{A}(t)$ *is bounded for all* $t \in cl\ \boldsymbol{R}_{0+}$ *and* $\Psi(t,\ 0) = e^{(A+\overline{A}(t))\ t}$ *is strongly continuous and uniformly continuous in cl* $\boldsymbol{R}_{0+}$.

Theorem 1(i) may be reformulated for any arbitrary stable or critically stable matrix $M \in \boldsymbol{R}^{n\times n}$ taking the role of the matrix A as follows. Note that for any $M \in \boldsymbol{R}^{n\times n}$, Equation (1) may be rewritten as follows:

$$\dot{x}(t) = Mx(t) + (A - M + A_{d0}(t))x(t) + \sum_{i=1}^{p} A_{di}(t)x(t - h_i);\ t \in \boldsymbol{R}_{0+} \tag{48}$$

so that its unique solution (3), equivalently (7), for any admissible initial condition function $\varphi\ : [-h\ ,\ 0] \to \boldsymbol{R}^n$ can be rewritten equivalently as follows:

$$x(t) = e^{Mt}\left(x_0 + \int_0^t e^{-M\tau}\left((A - M + A_{d0}(\tau))x(\tau) + \textstyle\sum_{i=1}^{p} A_{di}(\tau)x(\tau - h_i)\right)d\tau\right)$$

and the point evolution operator (6) is also given equivalently by:

$$\Psi(t,\ 0) = e^{Mt}\left(I_n + \int_0^t e^{-M\tau}\left((A - M + A_{d0}(\tau))\Psi(\tau,0) + \textstyle\sum_{i=1}^{p} A_{di}(\tau)\Psi(\tau - h_i\ ,0)\right)d\tau\right);\ t \in \boldsymbol{R}_{0+} \tag{49}$$

with $\Psi(t,0) = 0$ for $t \in \boldsymbol{R}_{-}$. Redefine in the proof of Theorem 1 a_d as $a_d = \max\limits_{i\in\overline{p}\ \cup\{0\}} a_{di}$ according to:

$$a_{di} = \sup_{t\geq 0} \|A_{di}(t)\|; i \in \overline{p},\ a_{d0} = \sup_{t\geq 0} \|A - M + A_{d0}(t)\| \tag{50}$$

and let real numbers $\rho_M > \rho_{0M}$ and ρ_{0M} be the stability abscissa of M and let $K_M \geq 1$ be a real constant such that $\left\|e^{Mt}\right\| \leq K_M e^{\rho_M t};\ t \in \boldsymbol{R}_{0+}$. Note that if $\rho_M = 0$, that is if M is critically stable, and the parametric constraint $K_M\left(\sum_{i=0}^{p} a_{di}\right) < 1$, guaranteed by the stronger one $K_M(p+1)a_d < 1$, holds under the redefinitions (50), and the replacements $\rho \to \rho_M$, $K \to K_M$, then (22) holds, or respectively, (23) holds. Again with the redefinitions (44) and the replacements $\rho \to \rho_M$, $K \to K_M$, if $\rho_M < 0$, that is if M is stable, and $K_M\left(\sum_{i=0}^{p} a_{di}\dfrac{\left|e^{|\rho_M|h_i} - 1\right|}{|\rho_M|}\right) < 1$, then (24) holds. Also, (25) holds under the stronger condition $K_M(p+1)\left|e^{|\rho_M|h} - 1\right|a_d|\rho_N|^{-1} < 1$. The above considerations are direct counterparts of Cases a and b discussed in the proof of Theorem 1 (i). As a result, Theorem 1(i) can be generalized as follows by the replacement of $A \to M$ with M (then, under the replacements $\rho \to \rho_M$, $K \to K_M$) being either critically stable or stable:

Corollary 1. *Assume that* $A_{di} : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n\times n}; i \in \overline{p} \cup \{0\}$ *are bounded with the redefined* a_{d0} *as* $a_{d0} = \sup\limits_{t\geq 0} \|A - M + A_{d0}(t)\|$, $a_{di} = \sup\limits_{t\geq 0} \|A_{di}(t)\|; i \in \overline{p}$ *and* $a_d = \max\limits_{i\in\overline{p}\ \cup\{0\}} a_{di}$. *Then, if, for given finite delays,* a_d *is small enough and* $\rho_M \leq 0$, *then* $\sup\limits_{t\in\boldsymbol{R}_{0+}} \|\Psi(t\ ,0)\| < +\infty$ *so that the differential system (1) is stable. If, in addition,* $\rho_M < 0$, *then, furthermore, the point evolution operator norm converges exponentially to zero with time so that the differential system (1) is exponentially stable.*

Note that Theorem 1(i) can be viewed as a particular case of Corollary 1 when $A = M$ is critically stable or stable.

Remark 7. *It can be pointed out that Corollary 1 cannot be used as an alternative to Theorem 1(i) if A is unstable since a critically stable or stable matrix M does not exist fulfilling the necessary conditions on the parameters for its application. This concern is illustrated in the subsequent simple example.*

Example 4. *Consider the scalar case of order* $n = 1$*, which has no delay and no time-varying undelayed dynamics that* $p = 0$ *and* $a_{d0}(.) \equiv 0$*. Assume that* $A = a > 0$ *(instability assumption of the constant part of the undelayed dynamics). Then, in order to apply Corollary 1 for some* $M = m \leq 0$*, the necessary condition* $\|A - M\| = a + |m| < |m|$ *has to be fulfilled, which implies that* $a < 0$*, which contradicts the instability hypothesis* $a > 0$*. Thus, there is no real* $m < 0$ *valid to apply Corollary 1.*

A particular case of interest relies on the case when the delay-free system (13), that is, the particular case of (1) with all the delays being null, is time-invariant and either stable or critically stable. In this case, $A_{d0}(t) = 0$, $A_{di}(t) = A_{di}$ for $i \in \overline{p}$; $t \in \boldsymbol{R}_{0+}$, $\left(A + \sum_{i=0}^{p} A_{di}\right)$ is constant and critically stable or stable.

Applying Corollary 1 to the time-delay system (1) with $M - A = \sum_{i=1}^{p} A_{di}$ in order to give critical stability/stability conditions for the time-delay system provided that the delay-free particular system has the counterpart property. Thus, $a_{di} = \|A_{di}\|$; $i \in \overline{p}$, and

$$a_{d0} = \|A - M\| \leq \overline{a}_{d0} = \sum_{i=1}^{p} \|A_{di}\| = \sum_{i=1}^{p} a_{di} \tag{51}$$

$$\overline{a}_{di} = 2a_{di};\ i \in \overline{p},\ a_d = \max_{i \in \overline{p} \cup \{0\}} \overline{a}_{di} \tag{52}$$

Then, one obtains the subsequent particular result of Corollary 1:

Corollary 2. *Assume that* $A_{d0}(t) \equiv 0$, $A_{di}(t) = A_{di}$ *for* $i \in \overline{p}$; $t \in \boldsymbol{R}_{0+}$ *and that* $\left(A + \sum_{i=0}^{p} A_{di}\right)$ *is constant and either critically stable or stable. Assume the replacements* $a_{d0} \to \overline{a}_{d0} = \sum_{i=1}^{p} a_{di}$, $a_{di} \to \overline{a}_{di} = 2a_{di}$; $i \in \overline{p}$ *and* $a_d \to \overline{a}_d = \max_{i \in \overline{p} \cup \{0\}} \overline{a}_{di}$ *in Corollary 1. Then, if for given finite delays,* $\overline{a}_d$ *is small enough and* $\rho_M \leq 0$*, then* $\sup_{t \in \boldsymbol{R}_{0+}} \|\Psi(t, 0)\| < +\infty$ *so that the resulting differential system (1) is stable. If, in addition,* $\rho_M < 0$*, then, furthermore, the point evolution operator norm converges exponentially to zero with time so that the differential system (1) is exponentially stable.*

In particular, the said "small enough" conditions of Corollary 2 are directly addressed from the replacements of the statement Corollary 2, together with the replacements $\rho \to \rho_M$, $K \to K_M$, in the conditions (21)–(24).

Remark 8. *Note that the loss in the presence of delays of the semigroup property by the point evolution operator directly implies the associated loss of such a semigroup property if the evolution operator and the delay strip evolution operator. This property is simple to elucidate since for point initial conditions at the interval* $[h, 0]$*, that is, for* $\varphi(t) = 0$ *for* $t \in [-h, 0)$; $\varphi(0) = x(0) = x_0$*, the unique solution trajectory defined by the (whole) evolution operator is identical to that defined by the point evolution operator.*

It can be seen that Theorem 2(iii) can be extended to more general situations than the invoked given constraints in Theorem 2(i). For instance, $\overline{A}(t)$ is bounded in $cl\ \boldsymbol{R}_{0+}$, provided that $A_{di} : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n \times n}$; $i \in \overline{p} \cup \{0\}$ are bounded, since from (10) and (18), one has that

$$\begin{aligned} \left\| \sum_{i=0}^{p} \int_{h_i}^{t} e^{-A\tau} A_{di}(\tau) \Psi(\tau - h_i, 0) d\tau \right\| &= \left\| e^{\overline{A}(t)t} - I_n \right\| \\ &= \left\| \Psi(t,0) e^{-At} \right\| \leq \| \Psi(t,0) \| \left\| e^{-At} \right\| \leq MKe^{(\omega - \rho)t};\ t \in \boldsymbol{R}_{0+}. \end{aligned} \tag{53}$$

Then, $\left\|e^{\overline{A}(t)t}\right\| \leq 1 + MKe^{(\omega-\rho)t} \leq \hat{K}e^{\hat{\omega}t}$ for some finite positive real constants $\hat{K}$ and $\hat{\omega}$. Thus, the matrix function $\overline{A} : cl\, \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n\times n}$ is bounded. Thus, the following result extends Theorem 2(iii).

Corollary 3. *Assume that* $A_{di} : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n\times n}$; $i \in \overline{p} \cup \{0\}$ *are bounded. Then, the matrix function* $A + \overline{A} : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n\times n}$ *is bounded and the operator* $A + \boldsymbol{A} : cl\, \boldsymbol{R}_{0+} \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ *is closed and bounded since* $\boldsymbol{A} : cl\, \boldsymbol{R}_{0+} \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ *is closed and bounded.*

Remark 9. *Concerning the used hypothesis in several results that the matrix A of the time-invariant dynamics is a stability matrix, note that this concern is easily achievable through the incorporation of an additional linear state control term. Assume that a control term* $Bu(t)$ *is added to the right-hand side of (1), where the matrix* $B \in \boldsymbol{R}^{n\times m}$ *is the control matrix, where m is the dimension of the piecewise continuous control vector* $u : \boldsymbol{R}_{0+} \to \boldsymbol{R}^m$ *with* $1 \leq m \leq n$. *If the pair* $(A\,,\, B)$ *is controllable, that is, if rank* $\left[B\,, AB, \cdots,\, A^{n-1}B\right] = n$, *then there exists a linear control law* $u = K_c x(t)$ *with a controller gain* $K_c \in \boldsymbol{R}^{m\times n}$ *such that the closed-loop linear time-invariant matrix* $A_{c\updownarrow} = A + BK_c \in \boldsymbol{R}^{n\times n}$, *obtained after the feedback injection, is a stability matrix whose stability abscissa* $\rho_{c\updownarrow} < 0$ *may be chosen arbitrarily by the appropriate choice of such a controller gain* K_c *[21]. Note that all the former analyses in this article can be now re-formulated with the replacement* $A \to A_{c\updownarrow}$ *and real constants* $\rho \to \rho_{c\updownarrow}$ *and* $K(\geq 1) \to K_{c\updownarrow}(\geq 1)$ *such that the closed-loop point evolution operator* $\Psi_{cl}(t, 0)$ *is recalculated with the replacement* $e^{At} \to e^{A_{cl}t}$ *and* $\left\|e^{A_{cl}t}\right\| \leq K_{cl}e^{\rho_{cl}t}$ *for all* $t \in \boldsymbol{R}_{0+}$.

4. Conclusions

This paper has studied a functional linear time-varying time-delay differential system of n-th order under an initialization real vector function, which is piecewise continuous and bounded and defined on an interval whose measure is that of the maximum delay. The delays are constant point incommensurate delays, and their total number is finite while their respective associated dynamics are, in general, time-varying and parameterized by time-varying piecewise continuous real vector functions. The delay-free part can also be, in general, time-varying under similar hypotheses as the ones invoked above for the delayed dynamics. Later on, the unique solution trajectory for each given set of bounded piecewise continuous real vector functions is obtained by "ad hoc" defined evolution operators whose analytical closed form is given and whose evolution properties are characterized. Those evolution operators are the so-called point evolution operator, the total evolution operator, and the delay strip evolution operator.

The point evolution operator generates the whole state-trajectory solution at any time instant from any bounded point initial conditions at the zero time instant and, partially, for any potential later initialization interval. The whole evolution operator generates the whole solution at any time instant for any admissible initialization at any time interval of measure being that of the maximum delay size. Finally, the delay strip evolution operator generates the whole solution on any period of time of duration being that of the maximum delay from any admissible initialization.

It is proved that the point evolution operator is strongly continuous and uniformly continuous but it loses the C_0-semigroup property, which corresponds to a potential nominal linear differential system, and which is time-invariant and free of delayed dynamics.

In future subsequent research, it is foreseen to study the evolution equations of certain nonlinear problems of interest, eventually subject to the presence of internal delays, as well as some control-related issues under "ad hoc" controllability assumptions [22,23].

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References

1. Niculescu, S.I. *Delay Effects on Stability. A Robust Control Approach*; Springer: Berlin/Heidelberg, Germany, 2001.
2. De la Sen, M. Robust adaptive control of linear time-delay systems with point time-varying delays via multiestimation. *Appl. Math. Model.* **2009**, *33*, 959–977. [CrossRef]
3. De la Sen, M. About robust stability of Caputo linear fractional dynamic systems with time delays through fixed point theory. *Fixed Point Theory Appl.* **2011**, *2011*, 867932. [CrossRef]
4. Delasen, M. On some structures of stabilizing control laws for linear and time-invariant systems with bounded point delays and unmeasurable states. *Int. J. Control* **1994**, *59*, 5298–5541.
5. Zhang, X.M.; Han, Q.L.; Seuret, A.; Gouaisbaut, F.; He, Y. Overview of recent advances in stability of linear systems with time-varying delays. *IET Control Theory Appl.* **2019**, *13*, 1–16. [CrossRef]
6. Shen, J.; Lam, J. On ℓ_∞ and L_∞ gains for positive systems with bounded time-varying delays. *Int. J. Syst. Sci.* **2015**, *46*, 1953–1960. [CrossRef]
7. Guajardo-Benavides, E.J.; Arteaga, M.A. On the finite time force estimation for bilateral teleoperation of robot manipulators with time-varying delays. *Control Eng. Pract.* **2023**, *138*, 105622. [CrossRef]
8. Feng, Z.; Zhang, X.; Lam, J.; Fan, C. Estimation of reachable set for switched singular systems with time-varying delay and state jump. *Appl. Math. Comput.* **2023**, *456*, 128132. [CrossRef]
9. Qian, M.; Sun, C.; Wang, C.; Yan, X.; Bo, C. Control of time-delay interconnected nonlinear systems under input saturations: A broad learning system-based approach. *Int. J. Syst. Sci.* **2023**, *55*, 1037–1055. [CrossRef]
10. Zhang, C.-K.; He, Y.; Jiang, L.; Wang, Q.-G.; Wu, M. Stability analysis of discrete-time neural networks with time-varying delay via an extended reciprocally convex matrix inequality. *IEEE Trans. Cybern.* **2017**, *17*, 3040–3049. [CrossRef] [PubMed]
11. Burton, T.A. *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*; Dover Publications Inc.: Mineola, NY, USA, 1985.
12. De la Sen, M. A study of the stability of integro-differential Volterra-type systems of equations with impulsive effects and point delay dynamics. *Mathematics* **2024**, *12*, 960. [CrossRef]
13. Yenicerioglu, A.F. Stability of linear impulsive neutral delay differential equations with constant coefficients. *J. Math. Anal. Appl.* **2019**, *479*, 2196–2213. [CrossRef]
14. Jin, S. Existence and uniqueness of solutions for the stochastic Volterra-Levin equation with variable delays. *Open Math.* **2022**, *20*, 769–782. [CrossRef]
15. Pani, A.K. Semigroup Theory and Evolution Equations. Available online: https://www.researchgate.net/profile/Amiya-Pani/publication/303569377_Semigroup_Theory_and_Evolution_Equations/links/5748fd8208ae2e0dd30169c3/Semigroup-Theory-and-Evolution-Equations.pdf (accessed on 1 September 2024).
16. Beffa, F. *Weakly Nonlinear Systems with Applications in Communication Systems*; Springer: Cham, Switzerland, 2024.
17. Engel, K.G.; Nagel, R.; Brendle, S. *One-Parameter Semigroups for Linear Evolution Equations*; Springer: Berlin/Heidelberg, Germany, 2013.
18. Missier, S.P.; Anbarasi Rodrigo, P. Strongly α^*-continuous functions in topological spaces. *IOSR J. Math.* **2014**, *10*, 55–60. [CrossRef]
19. Jankovic, D.S.; Konstadilaki-Savvopoulou Thessaloniki, C. On α-continuous functions. *Math. Bohem.* **1992**, *117*, 259–270. [CrossRef]
20. Chang, S.S.; Cho, Y.J.; Zhou, H. Demi-closed principle and weak convergence problems for asymptotically nonexpansive mappings. *J. Korean Math. Soc.* **2001**, *38*, 1245–1260.
21. Kailath, T. *Linear Systems*; Prentice-Hall Ltd.: Englewood Cliffs, NJ, USA, 1980.
22. Kumar, A.; Pani, A.K.; Joshi, M.C. Approximate controllability of liner parabolic equation with memory. *Comput. Math. Appl.* **2022**, *128*, 320–330. [CrossRef]
23. Kumar, A.; Joshi, M.C.; Pani, A.K. On approximation theorems for controllability of nonlinear parabolic problems. *IMA J. Math. Control Inf.* **2007**, *24*, 115–136. [CrossRef]

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