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The Effect of Network Delay and Contagion on Mobile Banking Users: A Dynamical Analysis

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Abstract: This paper analyses how network contagion affects the acceptance or rejection of mobile banking, emphasizing the risk of contagion that hinders the growth of users among bank clients. As a research method, a mathematical model based on the Susceptible, Infected, and Recovered framework is used within a nonlinear dynamic system with a distributed time delay and the optimal control strategy. The approach aims to find the behaviour dynamics of mobile banking users, non-users, and undecided ones. Our model adds value to the existing literature by employing a qualitative dynamic analysis, as opposed to other studies that study contagion empirically. The steady states and their conditions for stability are found, and the critical value for time delay causing oscillatory is determined. The optimal control strategies are identified to enlarge the number of mobile banking users and decrease the number of non-users. Numerical simulations support the theoretical findings. The research has practical implications given that the model predicts mobile banking user behaviour in assisting policy decision-making and addresses the possible policy measures that aim to increase the number of mobile banking users. The conclusions suggest that financial inclusion can contribute to the expansion of mobile banking users.

Keywords: dynamic system; mathematical modelling; mobile banking; simulation; control strategy; contagion

MSC: 34D20; 34H05; 91-10

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1. Introduction

Financial transactions and payments have undergone tremendous changes over the years, consistent with digitalisation, internet banking, mobile banking, etc. Under such circumstances, banks and their clients were bound to follow suit, constantly adapting to the new requirements.

The aim of the paper is to highlight the behavioural changes regarding the use of mobile banking because of contagion in the frame of a nonlinear dynamical system with distributed time delay. In addition, the optimal control strategy can be identified to decrease the number of non-mobile banking users.

The main motivation of this paper is to highlight the significance of financial inclusion in the growth of mobile banking and to provide policy guidance for increasing the use of these tools. Additionally, the paper emphasizes the risk of contagion among bank clients, which hinders the expansion of mobile banking.

Hence, the purpose of the paper is to find whether network contagion exists among bank clients when they adopt or reject mobile banking.

Considering the larger spectrum of possible approaches, the authors of this paper fill a gap in the literature by opting for a mathematical model with a time delay. The employed model grants sufficient abstraction and generalization for various contexts, followed by numerical simulations that substantiate the theoretical findings.

Our research builds on the existing literature by highlighting the importance of behavioural patterns in mobile banking. We contribute to this field by using qualitative dynamic analysis, which distinguishes our study from other studies that primarily rely on empirical methods to investigate contagion. Additionally, we have outlined several policy measures aimed at increasing the number of mobile banking users. The current research addresses the issue of risk contagion in the mobile banking market using a mathematical model rooted in the Susceptible, Infected, and Recovered (SIR) model. This approach is distinct from most research models, which usually rely on empirical methods employing quantitative tools or questionnaires tailored to specific economic contexts. The reason for using this approach is that most scientific concepts in various research fields, including economics and finance, can be defined in terms of dynamical systems. The research method may appear abstract, but it offers a more comprehensive understanding of behavioural contagion in mobile banking. This qualitative-type analysis is suitable for depicting long-term behavioural trends under the impact of state intervention that supports financial inclusion by using either direct, administrative measures or indirect levers.

The paper proceeds as follows: The literature review is presented in Section 2; Section 3 refers to the Problem Discussion; Section 4 describes the Methodology. The optimal control strategy is provided in Section 5. Section 6 substantiates the theoretical findings using numerical simulations, followed by the Discussion in Section 7. The main results, limitations, and future directions are summarized in Section 8.

2. Literature Review

The literature has long been preoccupied with the topic of mobile banking and its economic and social effects. Our paper supports the idea that intervention policy measures can enhance the acceptance rate of mobile banking. However, we suggest that the intervention should be made through indirect channels, such as nudging, rather than enforced regulations.

Financial inclusion seems to play a supporting role in mobile banking by increasing the cohort of users. When people and organizations have access to cost-effective financial services and products that responsibly and sustainably meet their needs, they are said to be financially included. Therefore, a particular strand of literature refers to this nexus. In Table 1, the main works from the literature are selected and presented regarding financial inclusion and mobile banking.

Table 1. Major topics regarding financial inclusion and mobile banking.

Major Topic	Sub-Topic and Research Insights
Financial crisis	• Hannig and Jansen [1] show that the 2009 financial crisis, despite producing devastating economic effects mainly on low-income economies, witnessed a surge in the use of mobile phones. This trend offers multiple benefits by expanding the possibility to access financial services and alleviate poverty. • Rewilak [2] considers the role of the financial sector in reducing poverty in the case of developing countries. The author questions whether only adopting new technologies and digital currency is enough to reduce poverty, concluding that bank branches are still important to ensure financial flows and reduce personal economic hardships.

Table 1. *Cont.*

Major Topic	Sub-Topic and Research Insights
Technology and mobile banking design	• The influence of technology has, arguably, extensively contributed to the dissemination of information among bank clients. (Yumei et al. [3] and Ballestra et al. [4]). • It must be added, though, that such technology is either contributing to extending the cohort of mobile banking users or persuading others to renounce these services, leaving the system altogether. Steering bank clients towards using electronic payments, including mobile banking, through nudges may be more successful than introducing enforced measures. • Nevertheless, in doing so, authors (Ayodgan and Hove [5]) identify that the consumers' payment behaviour relating to socio-demographic characteristics is important in using cash or electronic payments. Pursuing the same behavioural path, Wijland et al. [6] show that a mobile banking app design that relies on an intuitive interface alongside nudges is very important, mainly for the younger generations.
Regulations of mobile banking and state intervention	• Regarding the literature that studies the regulations underlying mobile banking operations. Gutierrez and Sandeep [7] are concerned with the possible regulatory framework that would make mobile banking more attractive, considering the determinants underlying its acceptance. • Other authors argue that the main driver of economic growth is technology (Tridico [8]); hence, state intervention to support mobile banking is necessary. Considering this perspective, the question of whether regulations and stimuli can complement each other to achieve the same ends can be raised. • The question is relevant since behavioural sciences are used on a rather large scale, as a missing link, to explain the impact of accessible financial services on reducing poverty (Datta and Desai [9]).
Financial inclusion	• Nandhi [10] argues that mobile banking has positively impacted poor regions, increased savings, and enhanced financial inclusion, but efforts are needed for individuals to overcome their reluctance to use it. Financial inclusion is the ease of access to, and the availability of, basic financial services to all members of the population (Yoshino and Morgan [11] and Peterson [12]). The role of branchless banks was studied by Birochi and Pozzebon [13] as a prerequisite to the broader topic of financial inclusion. • Asongu and Odhiambo [14], by analyzing a limited set of data, demonstrated that mobile banking reduces the asymmetric information between clients and banks and increases the efficiency of information sharing. • Demirguc-Kunt et al. [15], starting from the relevant empirical analysis, review the possibilities of enhancing financial inclusion, concluding that technology has the potential to strengthen economic growth and the financial inclusion nexus. • According to Ernst and Young [16], 200 mil+ micro and small businesses worldwide are financially excluded, not being eligible to participate in the financial flow system for various reasons, including a lack of financial education, no credit history, expensive financial products, no identification, etc. Most of the unbanked agents are in emerging economies in Asia, Africa, and Latin America, where the intense preoccupation of academics, policymakers, and practitioners in these countries is on finding viable solutions for enlarging financial inclusion. • According to Yoshino and Morgan [11], financial inclusion has been successfully promoted in most economies through a variety of strategies, such as supporting institutions focused on financial inclusion, promoting financial innovation and technologies (primarily by subsidizing them), and facilitating savings and lending by offering a wide range of securities that are easily accessible. • Pesqué-Cela et al. [17] suggest that composite measures of financial inclusion should be checked using confirmatory factor analysis (CFA) to overcome the shortcomings of inadequately selected indicators. • Peterson [12] argues that financial inclusion should be considered a public good (i.e., non-rival and non-exclusive). Similarly, Mariscal and Rojas-Lozano [18] also support state intervention as a catalyst for financial inclusion by providing incentives for key stakeholders, including banks, n-tech companies, and mobile operators, to expand mobile banking users. • Morshadul et al. [19] have researched the contribution of technology to financial inclusion in China, concluding that remote, rural regions are much better interconnected with formal financial institutions that facilitate savings and lending. Yang et al. argue that digitalisation can be a strategic choice to improve productivity (Yang et al. [20]). Nevertheless, the conclusions of Liao et al. [21] can be extended to mobile banking so that greenwashing does not occur.

Table 1. *Cont.*

Major Topic	Sub-Topic and Research Insights
Financial inclusion	• Chen and Divanbeigi [22] conducted a comprehensive study and concluded that individuals in countries with high-quality regulatory frameworks tend to prefer addressing financial intermediaries. Al-Okaily et al. [23] seek to explore the elements that impact the rate of diffusion for digital financial services. • Mansoor and Amjad [24], in an extensive literature review on financial inclusion, argue that there is a need for cross-country and regional studies to better determine the state of the matter. They also show that most researchers rely on surveys that include only a small number of indicators to be referred to. Indeed, the financial aspects are often considered by the literature (Fu et al. [25]), stressing the reasons why mobile apps are often rejected. • The Global Findex Database released by the World Bank [26] employs empirical methods to demonstrate the correlation between mobile banking and financial inclusion.

Source: Authors' contributions based on the systematic literature review.

In Table 2, the main literature regarding network theory, the SIR model, bifurcation analysis, and distributed delays are selected and presented.

Table 2. Major topic of network theory, the SIR model, bifurcation analysis, and distributed delays.

Major Topic	Sub-Topic and Research Insights
Network analysis and financial risk contagion	• Network analysis has long been employed in economic analyses (Callon [27]). Network economies are studied by the authors Cossin and Schellhorn [28]. • Franklin and Carletti [29] refer to financial innovation and its consequences on financial diversification and welfare. • According to Haldane and May [30] and Saeed and Xu [31], systemic risk, in which contagion plays a critical role, contributes to triggering a chain reaction of financial crises.
SIR model	• The SIR model is part of a group of behavioural models that are commonly used in the mathematical modelling of infectious diseases. These models simplify the complexities of population dynamics by categorizing the population into distinct groups, usually labelled as S, I, and R. These labels represent different stages of disease progression: S for Susceptible, I for Infectious, and R for Recovered. • In this framework, each group represents a specific cohort of individuals based on their disease status. Susceptible individuals have not contracted the disease but are at risk of infection. They can transition to the infectious compartment upon contact with an infectious person. Infectious individuals are those who currently have the disease and can transmit it to others. Finally, recovered individuals are those who have either successfully recovered from the illness, typically gaining immunity, or have died because of the disease. • The transitions between these groups are governed by specific mathematical rules that reflect the dynamics of disease transmission. The models are often formulated using ordinary differential equations which provide a structured and deterministic view of how the disease spreads over time. In this deterministic framework, the progress of the illness is predictable based on the initial conditions. • To better capture the inherent randomness and variability present in real-world situations, these models can be adapted to a stochastic (random) framework. This adaptation allows for a more nuanced understanding of disease dynamics, as it considers random events and fluctuations in population behaviour. However, such models tend to be more complex in terms of analysis. • Behavioural models are invaluable tools in epidemiology, enabling the analysis of disease progression over time. They allow researchers to estimate key metrics such as the total number of infections, recoveries, and mortality rates. Additionally, these models assist in evaluating various epidemiological parameters, which can inform public health interventions and policy decisions. • The SIR model is applicable in various areas other than epidemiology, i.e., behavioural sciences, computer sciences and cybersecurity, education, psychology, economics and business, etc. • Considering the bank clients to be interconnected, the technological or societal contagion can be analyzed using the SIR model extensively used in the literature (Fanelli and Maddalena [32], Kyrychko and Blyuss [33], and Zhang et al. [34]). • Roukny et al. [35] show that the interdependencies within financial networks, including the interbank market, can be seen as transmission channels of contagion.

Table 2. *Cont.*

Major Topic	Sub-Topic and Research Insights
Bifurcation analysis and distributed delays	• Bifurcation analysis provides a framework for understanding qualitative changes in system behaviour due to parameter variations. It provides critical thresholds where stable behaviours may become unstable or periodic. Bifurcation analysis seeks to identify these critical points, providing insight into the system's long-term behaviour and sensitivity to parameter variations. This analysis is important for fields requiring reliable predictions of stability and transitions, such as neuroscience, ecology, engineering, and economics (Kuznetsov [36]). • Regarding the Hopf bifurcation, a stable steady state loses stability and a periodic orbit (oscillatory behaviour) occurs, often leading to oscillations. In systems with time delays, bifurcation analysis becomes more complex as delays can change the stability of solutions in non-trivial ways. The time delays often induce Hopf bifurcations, where the delay parameter controls the onset of oscillatory solutions (Hale and Verduyn Lunel [37]). • The distributed delays, which illustrate the impact of past events on the current state, were first introduced in mathematical biology and later applied to mathematical economics (Cushing [38], Invernizzi and Medio [39], Guerrini, Krawiec, and Szydłowski [40]).

Source: Authors' contribution based on the systematic literature review.

Table 2 shows that the literature addresses various mobile banking issues, primarily based on empirical evidence or focusing on specific problems faced by different countries.

Comparatively, our paper fills a gap in the existing literature as follows:

- It introduces a different research method based on a mathematical model with time delay.
- In our approach, we choose to illustrate the lag that separates the moment of contact between mobile banking users and non-users and the moment of contagion by considering the time delay.
- Our paper employs a mathematical model and simulation because they provide a more abstract analysis of behaviours backing contagion and, at the same time, grant a more forward-looking approach, allowing decision makers (either financial institutions or governments) to build reliable strategies in this field.
- It captures the behavioural stance of bank clients related to mobile banking usage.
- Our approach is more comprehensive because it employs a qualitative dynamic analysis, as opposed to other studies that approach contagion empirically, and demonstrates the contagion among mobile bank users.

Therefore, our method allows for generalization and applicability under any circumstances and context.

3. Problem Discussion

During the last few years, most countries have taken action to promote inclusive finance. This endeavour has been backed by the extended use of mobile technology that has enabled individuals, even in the remotest and poorest parts of the world, to use a bank account, transfer money, and save.

In this respect, among the most accepted measures are mobile communication and e-payments, non-invasive identification methods, promoting transparency and thus enabling banks to lower costs based on accurate client information, using behavioural finance principles and mechanisms to steer towards mobile payments, and extending financial services digitalisation.

According to the Global Findex Database released by the World Bank [26], 52% of adults currently engage in monetary transfers, the most dynamic market being in China, where 57% of account owners use the internet for various transactions, either financial or purchases; however, a staggering group of 200 million Chinese in rural areas do not have access to financial services.

Worldwide, decision makers have taken steps to ensure that women have the same access to financial services as men, overcoming traditional misconceptions of gender inequality, although a 7% gap is maintained. Disparities between rich and poor households remain a matter of concern, the former having much more information and access to financial services. Likewise, those living in urban areas are in a better position than those living in rural areas. On the same page, education remains one of the most studied determinants of financial inclusion. In low-income countries, most adults have received only primary education, which explains the difficulty of understanding and accessing financial services. Mobile phones have become increasingly popular for making payments, with 51% of adults in high-income countries compared to 19% of adults in developing economies (World Bank [26]).

A well-known fact is that not all mobile phone owners use their devices for financial transactions. While these devices are designed to be user-friendly, some individuals may lack the ability to operate them. Additionally, some people may not have access to the internet, network infrastructure, or electricity, preventing them from using digital payments. Furthermore, some merchants may not accept digital payments, distrusting their reliability. Those not using digital payments often cite concerns about fraudulent activity, lack of cybersecurity, poor quality bank services, distrust, lack of consumer protection, fear of bank failures, and other related issues.

The chart in Figure 1 compares selected countries, grouped according to income, mobile banking, and financial inclusion. Mobile bank users are classified based on employment status, education level (primary or secondary), gender, wealth status, and location (urban or rural). Across all these countries, education seems to be a crucial factor in financial inclusion. The data reveal that people who have completed at least secondary education are more likely to adopt mobile banking than those who have completed only primary education. The richer segment of society is more financially included, while the rural population appears largely excluded. Comparing the considered groups of countries, it is noticeable that the high-income countries show the highest degree of financial inclusion, with insignificant disparities among individuals.

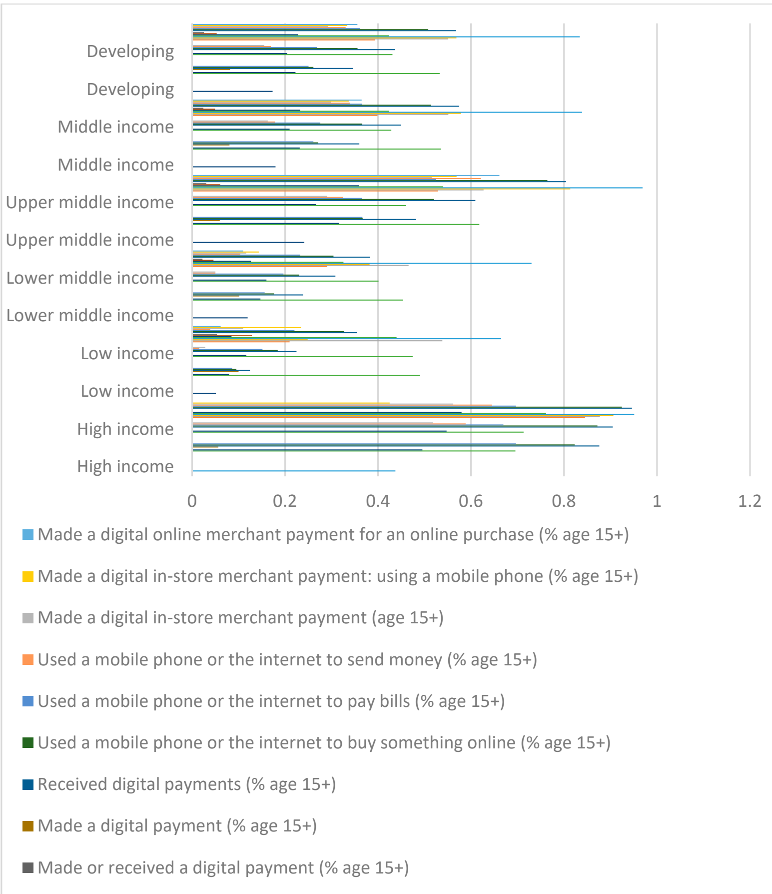

Figure 1. This is a figure. Schemes follow the same formatting. Source: Global Findex Database 2021.

As a preliminary conclusion, we can confidently say that in countries with significant incomes and a dynamic labour market, financial inclusion—particularly in mobile banking—is higher when several factors are present. These include strong and reliable banking systems, effective financial education, efficient investments in digital infrastructure and internet access, political stability, transparent and trustworthy public policies, social equity, and low levels of fraud and failures. Hence, when one or more of these determinants deteriorate, financial inclusion worsens.

4. Methodology

To study the dynamics of the phenomenon under investigation, we aim to analyze the stability of the system (1) consisting of three differential equations with distributed time delay. The methodology integrates a qualitative analysis with computational tools, offering a better understanding and managing the stability of systems with time delays. It employs a combination of bifurcation analysis and numerical simulations. The steady states are determined, and their stability is assessed through linear stability analysis, involving the computation of the eigenvalues of the Jacobian matrix associated with the system. Special attention is given to the conditions under which the steady states transition from stability to instability, often involving bifurcation analysis to identify critical points where qualitative changes in the system occur. Numerical simulations are important in validating the analytical results and providing a more comprehensive understanding of the system's behaviour over time. The control variable incorporation further contributes to exploring strategies for system control.

To build the mathematical model employed to illustrate mobile contagion, we use the following variables: M is the number of mobile banking users; N is the number of mobile banking non-users; and U stands for the undecided mobile phone owners (with mobile banking apps installed) who are reluctant to use it.

We consider that the growth rate of mobile bank users is a and b is the rejection rate of mobile banking usage following the interaction between mobile bank users and non-users. Also, we introduce k as the carrying capacity (the maximum population size that the bank can support). The inhibition effect, denoted by α, refers to the change in the behaviour of mobile banking users caused by an increase in their numbers or due to the presence of a large number of mobile banking non-users, commonly known as the crowding effect (Xu and Ma [41]). The nonlinear incidence rate is described by the function $f:[0,\infty)\to[0,\infty),\ f(x)=\frac{x}{1+\alpha x}$ with $\alpha>0$, which allows the inclusion of the behavioural changes in mobile banking users. It is a monotone increasing function, bounded by $\frac{1}{\alpha}$, and thus prevents unbounded contact rates (Fanelli and Maddalena [32]). The dynamics of mobile banking users are given by

$$\dot{M}(t)=aM(t)\left(1-\frac{M(t)}{k}\right)-bM(t)f\left(\int_{-\infty}^{t}N(t-s)h(s)ds\right),$$

where the function $h:[0,\infty)\to[0,\infty)$, called delay kernel, is a piecewise continuous function, bounded with $\int_0^\infty h(t)dt=1$.

The state may intervene to boost mobile banking by using either direct (administrative-type regulations) and/or indirect ones (mainly to support financial inclusion as a prerequisite to increase the number of mobile banking users) that may act as levers or nudges. Let c be the state intervention which allows the transition from non-users to undecided. The rate of bank clients that leave the system is denoted by d. The dynamics for the number of mobile banking non-users is described by

$$\dot{N}(t)=bM(t)f\left(\int_{-\infty}^{t}N(t-s)h(s)ds\right)-(c+d)N(t)$$

and the rate of change for the number of the undecided mobile phone owners is

$$\dot{U}(t) = cN(t) - dU(t).$$

Therefore, the mathematical model which illustrates the mobile contagion is

$$\begin{aligned} &\dot{M}(t) = aM(t)\left(1 - \tfrac{M(t)}{k}\right) - bM(t)f\left(\int_{-\infty}^{t} N(t-s)h(s)ds\right) \\ &\dot{N}(t) = bM(t)f\left(\int_{-\infty}^{t} N(t-s)h(s)ds\right) - (c+d)N(t) \\ &\dot{U}(t) = cN(t) - dU(t) \end{aligned} \tag{1}$$

The average time delay is given by

$$\tau = \int_0^\infty th(t)dt < \infty$$

and represents the average time lag in which mobile banking users become non-users because of the contagion.

In what follows, we are interested to study the dynamics of (1). First of all, we find the steady states of the system (1) as the solutions of the system as follows:

$$\begin{aligned} &aM\left(1 - \tfrac{M}{k}\right) - bM\tfrac{N}{1+\alpha N} = 0 \\ &bM\tfrac{N}{1+\alpha N} - (c+d)N = 0 \\ &cN - dU = 0. \end{aligned}$$

Three steady states can be obtained: the trivial equilibrium $E_0 = (0,0,0)$ is unstable; the mobile banking free $E_1 = (k,0,0)$, *i.e., all* means that bank clients own and use mobile banking; and E_2 is the endemic equilibrium.

To find the positive steady state $E_2 = (M_*, N_*, U_*)$, the threshold value $R_0 = \frac{bk}{c+d}$ (Zhang et al. [34]), which determines whether the risk contagion occurs, is used. If the condition

$$\mathrm{c + d < bk}, (R_0 > 1) \tag{2}$$

holds, there exists E_2, as given by

$$M_* = \frac{k(1+\alpha N_*)}{R_0},\ U_* = \frac{cN_*}{d},$$

where N_* is the positive solution of the following equation:

$$a\alpha^2 N^2 + (2a\alpha + R_0(b - a\alpha))N - a(R_0 - 1) = 0$$

given by

$$N_* = \frac{R_0(a\alpha - b) - 2a\alpha + \sqrt{\Delta}}{2a\alpha^2}$$

with $\Delta = (R_0(b - a\alpha) + 2a\alpha)^2 - 4a^2\alpha^2(R_0 - 1)$.

If the parameter $R_0 < 1$, then the risk of contagion among mobile bank users and non-users is overcome and the cohort of users increases. On the contrary, if $R_0 > 1$, the risk contagion occurs and the number of users of mobile banking does not improve.

To study the local stability of any steady state $E = \left(x_1^*,\ x_2^*,\ x_3^*\right)$, the roots of the following equation must be analyzed:

$$(\lambda - a_{33})[(\lambda - a_{11})(\lambda - a_{22}) + b_{12}H(\lambda)(\lambda - a_{11} - a_{22})] = 0 \tag{3}$$

here $H(\lambda) = \int_0^\infty e^{-\lambda t} h(t) dt$ and

$$a_{11} = a - \frac{2a}{k} x_1^* - b \frac{x_2^*}{1+\alpha x_2^*},\ b_{12} = b \frac{x_1^*}{\left(1+\alpha x_2^*\right)^2},\ a_{21} = b \frac{x_2^*}{1+\alpha x_2^*},\ a_{22} = -(c+d),$$
$$b_{22} = -b_{12},\ \ a_{32} = c,\ a_{33} = -d$$

If there is no delay ($H(\lambda) = 1$), $E_1 = (k, 0, 0)$ is locally asymptotically stable. For any delay, E_1 is locally asymptotically stable if $c + d > bk$ ($R_0 < 1$).

If there is no delay, ($H(\lambda) = 1$), $E_2 = (M_*, N_*, U_*)$ is locally asymptotically stable if $R_0 > 1$. For any delay, E_2 is locally asymptotically stable if

$$R_0 > \left(\frac{k}{M_*}\right)^2. \tag{4}$$

A necessary condition for the occurrence of Hopf bifurcation in a neighbourhood of the endemic equilibrium E_2 is

$$R_0 \leq \left(\frac{k}{M_*}\right)^2. \tag{5}$$

If we consider the discrete time delay τ and the inequalities in (5) hold, and

$$(M_*)^2 < (k - 2M_*)^2 (1 + \alpha N_*)^2 \tag{6}$$

then E_2 is locally asymptotically stable if and only if $\tau \in [0, \tau_0)$, where

$$\tau_0 = \frac{1}{\omega_0} \arccos\left[\frac{\omega_0^2\left(c + d - a + a\frac{M_*}{k}\right) + a^2\left(\frac{(c+d)M_*}{k}\right)\left(1 + \frac{2M_*}{k}\right)}{(c+d)^2(1+\alpha N_*)^2\left[\left(a - \frac{2aM_*}{k}\right)^2 + \omega_0^2\right]}\right] \tag{7}$$

and ω_0 is the positive solution of

$$\omega_0^4 + \omega_0^2\left(a_{22}^2 + a_{11}^2 - b_{12}^2\right) + a_{11}^2 a_{22}^2 - b_{12}^2 (a_{11} + a_{21})^2 = 0.$$

When the time delay crosses the critical value (7), a Hopf bifurcation occurs at E_2.

5. Optimal Control Strategy

In what follows, we aim to find the necessary and sufficient conditions of the optimal control strategy. We introduce the control variable $u(t)$, a continuous function defined as the control strategy, as effective tools targeting the undecided mobile users at time t. We consider that at time t, only $\epsilon u(t) U(t)$ become non-users due to the limitation of the control strategy with $\epsilon \in [0, 1]$, and $(1-\epsilon) u(t) U(t)$ become mobile banking users at time t.

The admissible control set is given by

$$\mho = \left\{u(t)\ is\ measurable,\ 0 \leq u(t) \leq \Delta,\ t \in \left[0, t_f\right]\right\},$$

with t_f and Δ being two given constants.

The optimal control problem minimizes the functional

$$J(u) = \int_0^{t_f} \left[U(t) + \frac{\delta}{2} u(t)^2\right] dt, \tag{8}$$

subject to

$$\begin{array}{l} \dot{M}(t) = aM(t)\left(1 - \frac{M(t)}{k}\right) - bM(t)f(N(t-\tau)) + (1-\epsilon)u(t)U(t) \\ \dot{N}(t) = bM(t)f(N(t-\tau)) - (c+d)N(t) + \epsilon u(t)U(t) \\ \dot{U}(t) = cN(t) - dU(t) - u(t)U(t) \end{array} \quad (9)$$

where τ is the time delay, $M(\sigma) > 0$, $N(\sigma) > 0$, $U(\sigma) > 0$, with $\sigma \in [-\tau, 0]$, δ is a positive constant to keep a balance in the size of $u(t)$, and $u(t)^2$ represents the impact of the control (Chen et al. [34]).

The Hamilton function is

$$H(M,N,U,u,\lambda_1,\ \lambda_2,\lambda_3,t) = U(t) + \frac{\delta}{2}u(t)^2 + \lambda_1(t)\frac{dM(t)}{dt} + \lambda_2(t)\frac{dN(t)}{dt} + \lambda_3(t)\frac{dU(t)}{dt} \quad (10)$$

where $\lambda_1,\ \lambda_2, \lambda_3$ are the adjoint functions.

Following the findings from (Chen et al. [42]), we deduce that there is an optimal control denoted by $u^*(t)$ such that $J(u^*(t)) = minJ(u(t))$, constrained to (9).

With the optimal control $u^*(t)$ and the corresponding variables $M^*(t)$, $N^*(t)$, and $U^*(t)$, the adjoint variables $\lambda_1(t),\ \lambda_2(t), \lambda_3(t)$ verify the following system:

$$\begin{array}{l} \dot{\lambda_1}(t) = -\frac{\partial H}{\partial M}(t) - \chi_{[0,t_f-\tau]}(t)\frac{\partial H}{\partial M_\tau}(t+\tau), \\ \dot{\lambda_2}(t) = -\frac{\partial H}{\partial N}(t) - \chi_{[0,t_f-\tau]}(t)\frac{\partial H}{\partial N_\tau}(t+\tau), \\ \dot{\lambda_3}(t) = -\frac{\partial H}{\partial U}(t) - \chi_{[0,t_f-\tau]}(t)\frac{\partial H}{\partial U_\tau}(t+\tau), \end{array} \quad (11)$$

with the transversality conditions

$$\lambda_i\left(t_f\right) = 0, i = 1,2,3$$

and

$$\chi_{[0,t_f-\tau]}(t) = \begin{cases} 1, \ t \in \left[0, t_f - \tau\right] \\ 0, \ otherwise \end{cases}$$

The system (11) is equivalent to

$$\begin{array}{c} \dot{\lambda_1}(t) = -1 + \left(-a + \frac{2a}{k}M^* + \frac{bU^*}{1+\alpha U^*}\right)\lambda_1(t) - \frac{bU^*}{1+\alpha U^*}\lambda_2(t), \\ \dot{\lambda_2}(t) = -(c+d)\lambda_2(t) + \lambda_3(t) - \lambda_1(t+\tau)\frac{bM^*}{(1+U^*)^2}\chi_{[0,t_f-\tau]}(t) + \\ \lambda_2(t+\tau)\frac{bM^*}{(1+U^*)^2}\chi_{[0,t_f-\tau]}(t), \\ \dot{\lambda_3}(t) = -1 + d\lambda_3(t) \end{array} \quad (12)$$

Using Theorem 2.1 from Chen et al. [42], the optimal control is given by

$$u^*(t) = min\left\{max\left(0, \frac{U^*(t)}{\delta}((1-\epsilon)\lambda_1(t) - \epsilon\lambda_2(t) + \lambda_3(t))\right), \Delta\right\}.$$

6. Numerical Simulation

In this section, we show the influence of the time delay and control on the dynamics of system (1) by visualizing the trajectories using the Mathematica 13. 1 and Matlab R2023b software. This approach supports the theoretical findings.

Drawing on Zhang et al. [34], a scenario is used to better capture the components of M, N, and U, in which the following parameters of the model are considered: the growth rate of mobile banking users is 10% (a = 0.1) and the rate of rejection of mobile banking usage following the interaction between mobile banking users and non-users is 1% (b = 0.01). We suppose that as a consequence of the financial inclusive policy intervention, 10% of the non-users of mobile banking join the undecided group (c = 0.1). The rate of bank clients that leave the undecided group is 20% (d = 0.2), and the maximum population size that the bank can support is 100% (k = 100). The positive parameter is $\alpha = 0.008$. We summarize the values of the parameters in Table 3.

Table 3. The values of parameters for system (1).

Symbol	Parameter Meaning	Value	Source
a	the growth rate of mobile bank users	0.1	Zhang et al. [34]
b	the rate of rejection of mobile banking usage following the interaction between mobile bank users and non-users	0.01	Zhang et al. [34]
c	the state intervention	0.1	Zhang et al. [34]
d	the rate of bank clients that leave the system	0.2	Zhang et al. [34]
k	the carrying capacity (the maximum population size that the bank can support)	100	Zhang et al. [34]
α	The inhibition effect of mobile banking users	0.008	Xu and Ma [41]

The condition (2) for the existence of the endemic equilibrium is satisfied and is given by

$$E_2 = (M_* = 31.73,\ N_* = 7.22,\ U_* = 14.44).$$

The value of the threshold is $R_0 = 3.33 > 1$. If there is no time delay, (Figure 2) E_2 is locally asymptotically stable. Taking the initial conditions near E_2, as time increases, the trajectories of the system move towards the steady state.

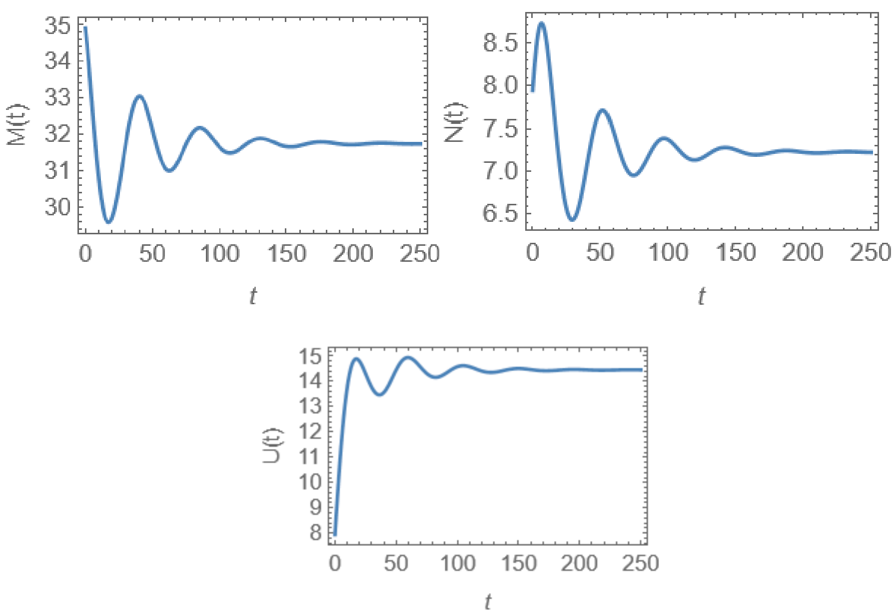

Figure 2. The orbits (t, M(t)), (t, N(t)), (t, U(t)) when there is no time delay; E_2 is locally asymptotically stable.

If there are small perturbations, either endogenous or exogenous, and no time delay exists, it can be noticed that, after some time, the system converges to a steady state (Figure 3).

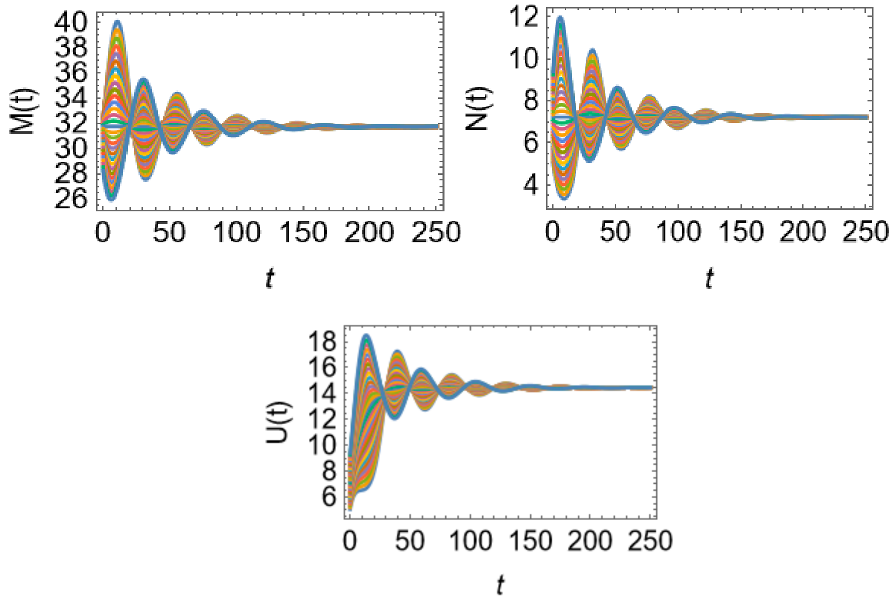

Figure 3. The orbits (t, M(t)), (t, N(t)), (t, U(t)) when there is no time delay and different initial conditions; E_2 is locally asymptotically stable.

If there is a discrete time delay, conditions (5), (6), and (7) are satisfied and a Hopf bifurcation occurs when the bifurcation parameter τ takes the critical value $\tau_0 = 28.94$. Figure 4 shows the system's stability changes and a periodic solution occurs.

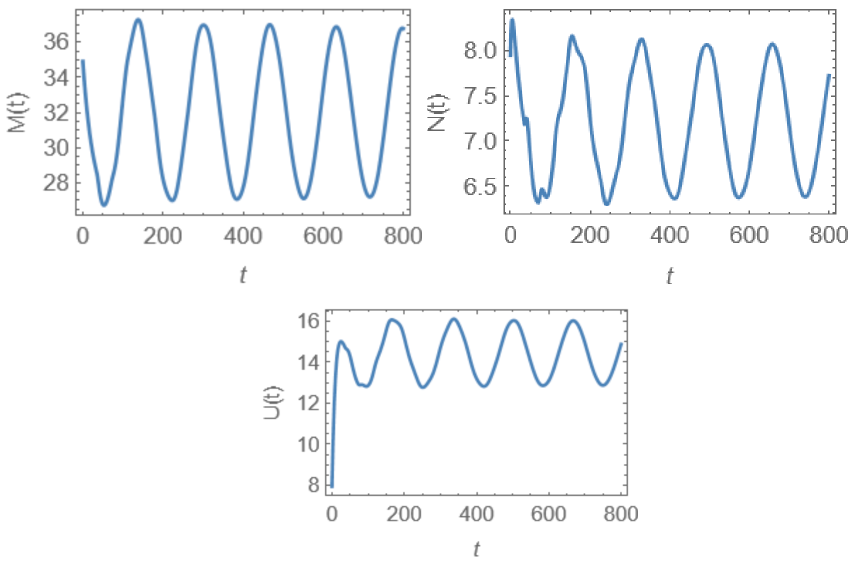

Figure 4. The oscillatory orbits (t, M(t)), (t, N(t)), (t, U(t)) when the critical time delay is $\tau_0 = 28.94$.

In the context of nonlinear systems with time delays, we consider the robustness of the findings with respect to the small changes in initial conditions. Based on the above numerical simulations (see Figures 3 and 5), the dynamics of system (1) (convergence to a limit cycle or steady state) are not sensitive to the exact starting point. It means that the system converges to the same behaviour even with small deviations in the initial conditions, making predictions more reliable.

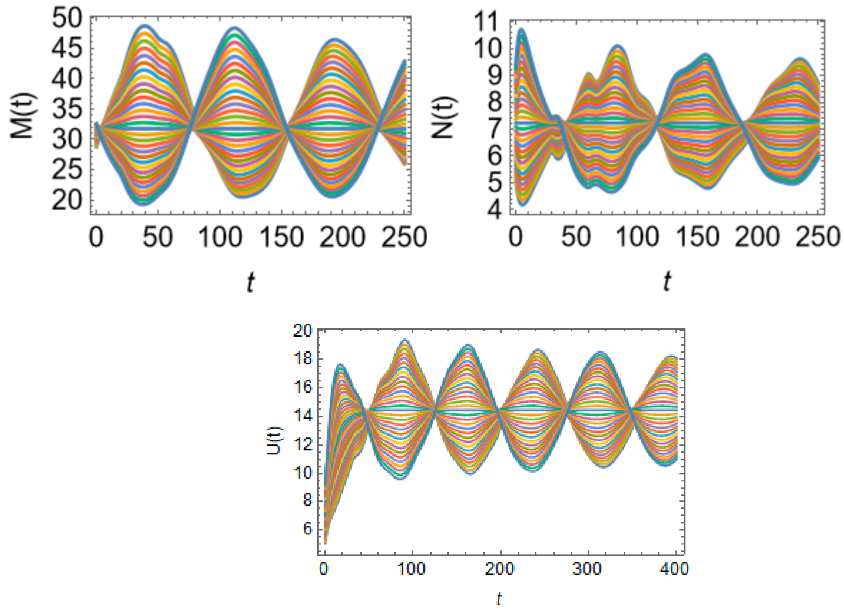

Figure 5. The oscillatory orbits (t, M(t)), (t, N(t)), (t, U(t)) with different initial conditions and the critical time delay is $\tau_0 = 28.94$.

If various initial conditions are considered, (Figure 5), the periodic solutions of the system can be observed:

Moreover, when the bifurcation parameter is less than the critical value, then E_2 is locally asymptotically stable for $\tau \in [0, \tau_0)$. In Figure 6, it can be observed that for different values of the time delay that are less than the critical value, the system converges to a steady state after a while.

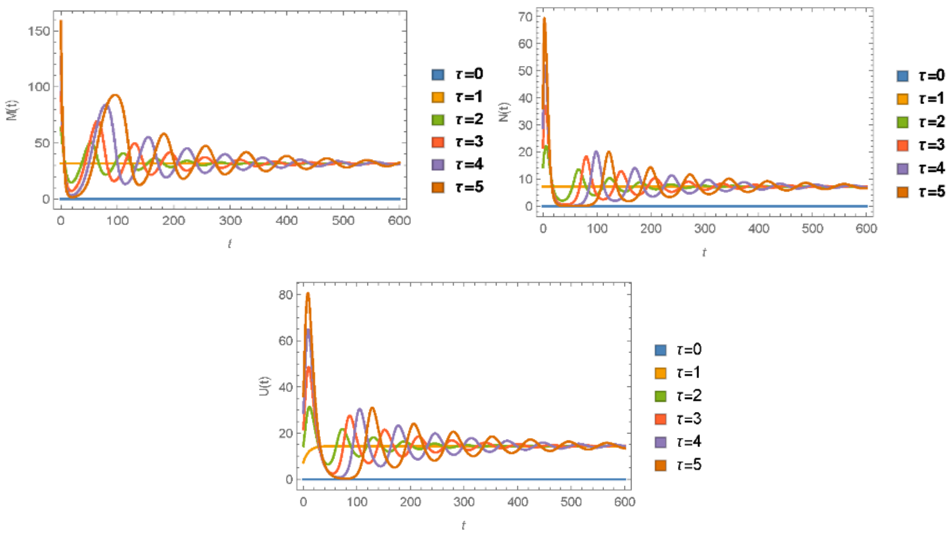

Figure 6. The orbits (t, M(t)), (t, N(t)), (t, U(t)) when the time delay is less than the critical value; E_2 is locally asymptotically stable.

To find the optimal strategy solution of (9) that minimizes the functional (8), we choose the following values of the parameters: $t_f = 500$, $\Delta = 0.8$, $\varepsilon = 0.8$, $\delta = 20$, $\tau = \text{mh}$, with the step size h = 0.01 and different values for m: 50, 200, 400.

Taking the time delay $\tau = 5$, in Figures 7 and 8, we represent the control variable *u(t)* and the evolution of the mobile banking users, the mobile banking non-users, and the undecided mobile phone owners (with mobile banking apps installed) who are reluctant to use it with constant control, without control, and with optimal control, respectively.

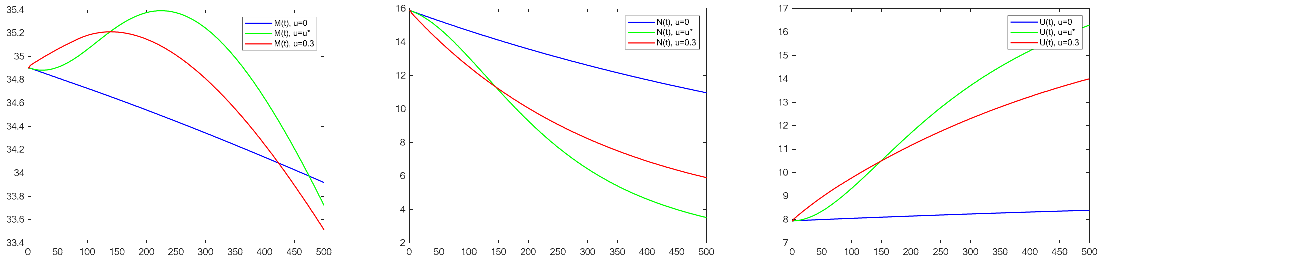

Figure 7. The orbits (t, M(t)), (t, N(t)), (t, U(t)) for the following control strategies: u = 0 (blue), u = u* (green), and u = 0.3 (red), and where the discrete time delay τ = 5.

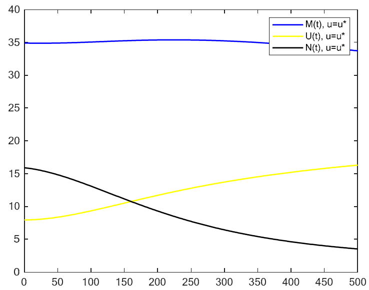

Figure 8. The orbits (t, M(t))-blue, (t, N(t))-black, (t, U(t))-yellow) with the optimal control and the discrete time delay τ = 5.

As Figures 7 and 8 show, for τ = 5, the number of mobile banking users (M) remains in the same range up to a certain moment in time, after which the control becomes ineffective requiring a reconsideration of the strategy. Figure 9 shows the control variable for τ = 5. The number of non-users (*N)* and the number of undecided (U) manifest opposite behaviours as the cohort of mobile banking non-users increases and the undecided mobile phone owners (with mobile banking apps installed) decrease. The control strategy is effective. For the time delay τ = 20, Figures 10 and 11 display the control variable *u(t)* and the evolution of the mobile banking users, the mobile banking non-users, and the undecided mobile phone owners (with mobile banking apps installed) who are reluctant to use it with constant control, with optimal control and without control, respectively.

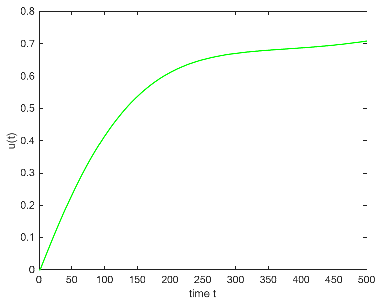

Figure 9. The control variable *u(t)* for the discrete time delay τ = 5.

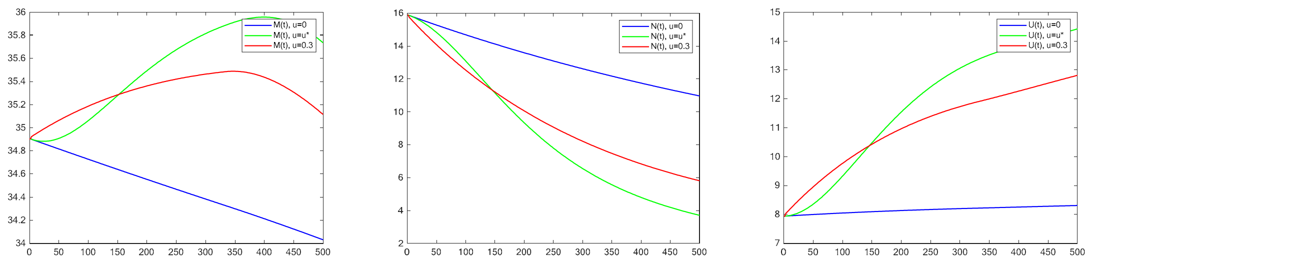

Figure 10. The orbits (t, M(t)), (t, N(t)), (t, U(t)) for the following control strategies: u = 0 (blue), u = u* (green), and u = 0.3 (red), and where the time delay τ = 20.

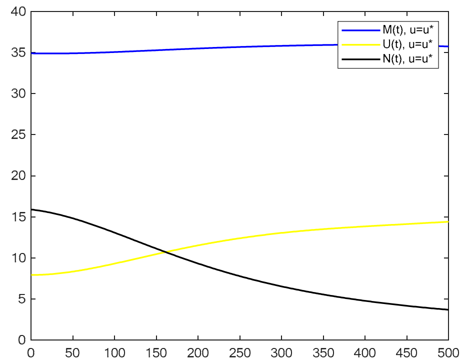

Figure 11. The evolutions of M(t)-blue, N(t)- yellow, and U(t)-black for the optimal control and the τ = 20 discrete time delay.

Figures 11 and 12 show that for τ = 20, the *M* number of mobile banking users slowly increases during a timeframe followed by a decreasing phase. The number of mobile banking non-users decreased, and the number of undecided mobile phone owners (with mobile banking apps installed) increased. It can be observed that the control is effective.

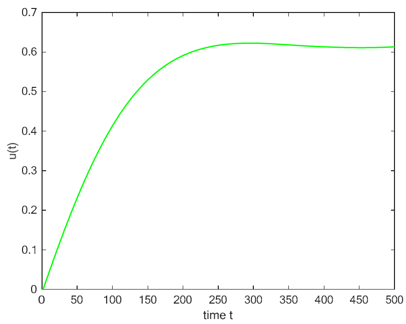

Figure 12. The control variable u(t) for the discrete time delay τ = 20.

For the time delay τ = 40, in Figure 13, we can visualize the evolution of mobile banking users, mobile banking non-users, and undecided mobile phone owners (with mobile banking apps installed) who are reluctant to use it with constant control, optimal control, and without control, respectively.

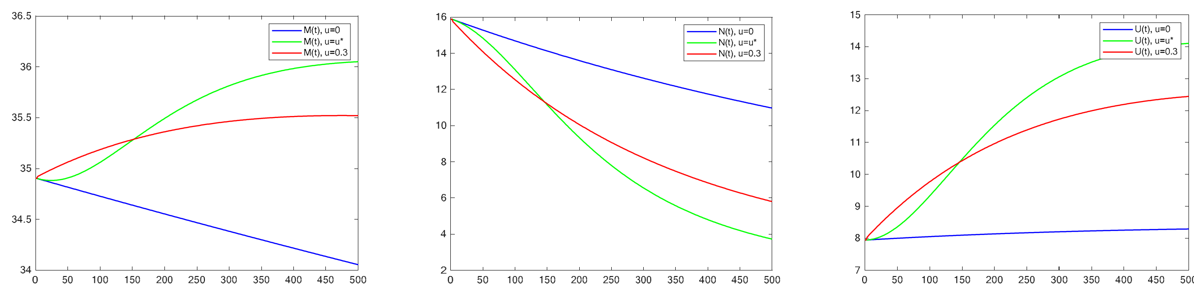

Figure 13. The orbits (t, M(t)), (t, N(t)), (t, U(t)) for the following control strategies: u = 0 (blue), u = u* (green), and u = 0.3 (red), and where the time delay τ = 40.

When τ = 40, Figure 13 shows that the number of mobile banking users slowly increases during a timeframe, followed by a decreasing phase. The number of mobile banking non-users decreases, and the number of undecided mobile phone owners (with mobile banking apps installed) sharply increases. The control strategy is effective.

7. Discussion

In this study, we employ a mathematical model, described by a system of differential equations with time delay, to show the contagion between mobile banking users. Our approach is a qualitative analysis, and it is different from the literature regarding the topic that generally investigates mobile contagion using empirical methods and/or is contextualized to specific countries.

In the present paper, the mathematical model and the numerical simulations consider, starting from reality, that there are three groups of bank customers related to mobile banking: 1. those who fully embrace and actively use mobile banking (M), 2. those who reject it (N)—despite downloading the apps on their mobile phones, and 3. the undecided ones (U) who ponder whether to use mobile banking or not. All these groups of agents interact and are subject to information sharing, influencing each other's behaviour and decisions in this matter. Through contagion, non-users (N) may influence mobile banking users (M) to renounce this banking service since they are most vulnerable to external influences. In addition, a percentage of the undecided group may opt to join the users or the non-users group. However, bank clients are often subjected to financial exclusion (because of their economic, financial, age, gender, etc., stances) or encounter cultural obstacles. Either voluntarily or involuntarily excluded, the mobile bank non-users and the undecided ones may reject the service and often choose to leave the system altogether (Figure 14).

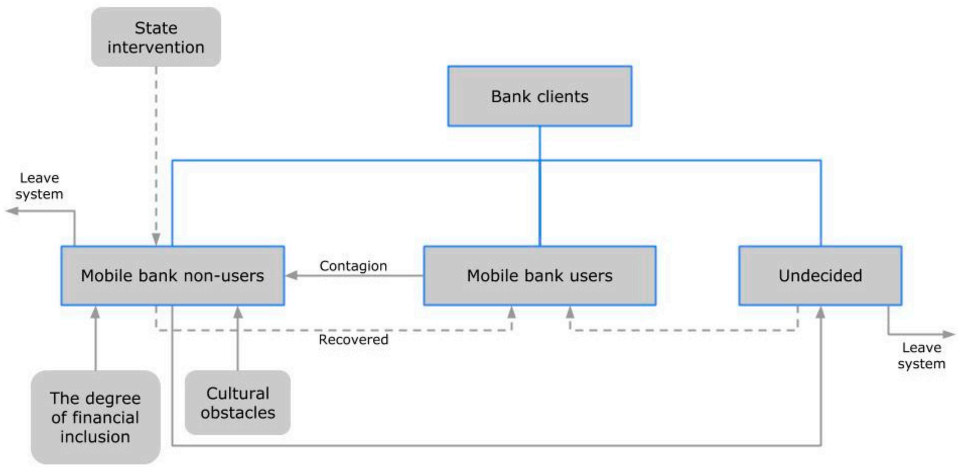

Figure 14. Mobile banking contagion and state intervention.

If mobile banking is regarded as a quasi-public good that induces positive externalities, then the state may intervene using specific levers to help non-users overcome the obstacles that prevent them from joining the mobile banking users' group and support the users' decision to remain in the system.

Therefore, the question is raised regarding the most appropriate public measures that might alleviate the contagion of mobile banking non-users among users and undecided ones. These measures embed stabilizers that either prevent the contagion of mobile banking non-users among users or can trigger the willingness of bank clients to shift towards mobile banking.

Following the above demonstration and numerical simulation, it can be argued that policy interventions described by the parameter c, from our mathematical model, could contribute to obtaining only one steady state E_1 or reduce the default level E_2. To overcome contagion ($R < 1$)—see Equation (2)—parameter c should grow at a higher rate than d while decreasing parameter b (including the carrying capacity k).

From the perspective of the present paper, financial inclusion can be targeted in line with Chen and Divanbeigi [22] and Mariscal and Rojas-Lozano [18], concentrating on the crowding effect as a prerequisite to embracing mobile banking.

It has already been proven that financial exclusion is extended to hundreds of millions of individuals who are unbanked or lack financial literacy. From an economic and social standpoint, low-income categories, retirees, the unemployed, those struck by poverty, rural and remote geographical area inhabitants, ethnic minorities, single parents, those who lack personal identity cards, and those who do not meet the requirements to open a bank account or find the cost of financial services too high contribute to financial exclusion.

For the considered mathematical model, we have determined the stable states and adequate conditions for their stability concerning the time delay. Moreover, changes in the equilibria and qualitative behaviour of solutions may result from political actions, as Fanelli and Maddalena concluded [32].

The model has practical implications, as it examines the potential contagion effect among mobile banking users, non-users, and those who are undecided. Based on this analysis, banks and policymakers can explore how to encourage undecided users to join the group of established users.

Since mobile banking is subject to market mechanisms and private financial institutions supply it, the legitimate question is whether the government can positively determine bank clients to actively use this service. Such political actions may be either direct (administrative regulations) or indirect. They are not deliberately possible on a private market, with the tools at hand being legislative regulations, promoting consumers' protection and cybersecurity, investments to extend secure grids and infrastructure, promoting and enhancing financial inclusion, nudging, etc. However, to be able to do so, mobile banking should be included in the quasi-public goods category, in line with Peterson [12].

Some of the possible policies are listed below.

a. *Legislative regulations*

Indeed, it seems that the legislative and regulatory framework enables bank customers to use mobile banking more intensely because it often supports digitalisation, equitable competition among providers, and alleviates asymmetric information risks. Therefore, communication and banking regulators, as well as competition authorities, can influence the industry. In this respect, regulations concerning electronic signatures become essential in a market where agents (providers or consumers) intervene remotely. In addition, personal identity cards are essential when opening an account, and, therefore, the legislation should support simplifying procedures to obtain such documents. Cybersecurity measures are also included in this set of measures, granting trustworthiness to use mobile banking securely.

b. *Consumer protection regulations*

Most mobile banking users have little understanding of the technical process that underlies mobile banking, and therefore, governments must step in by issuing consumer protection norms and providing users with the confidence that they are shielded against fraud and funds mismanagement.

Though the use of mobile banking has become increasingly extended, bank clients still lack trust. Their deepest concern is mainly related to the reliability of platforms that host these services. To enhance trust, public decision makers are responsible for issuing regulations and guidelines that grant sufficient transparency and security to support bank customers in their financial decisions.

c. *Infrastructure investments*

As rural and remote areas are most often affected by the lack of proper infrastructure for high-speed internet connections, the state authorities may support the extension of grids by being technically involved in these investments or granting subsidies to specialized companies.

d. *Economic and financial measures*

One of the obstacles often raised is the relatively high cost of bank services and the lack of funds that exclude large categories of low-income or economically vulnerable groups from using mobile banking. It is in the power of public authorities to implement measures to prevent unemployment, mainly in disadvantaged areas, promote minimum wages and productivity enhancement, improve self-employment, and regulate the labour market. When market failure occurs, state intervention can balance demand and supply until the equilibrium is restored and market mechanisms can effectively channel labour and financial resources.

Vulnerable groups (e.g., age, gender, low income/poverty) are most often excluded from the financial system, either because they are not eligible to open an account or because they encounter cultural/educational barriers. In this respect, financial literacy and general education may be drivers in adopting mobile banking.

Other obstacles relate to the following behaviours: fear of bank procedures, privacy issues, technical inability, fear of novelty, high costs of transactions, not understanding the terms and conditions of contracts, etc. To encourage reluctant individuals to use mobile banking, the state can design friendly, easy-to-understand payment platforms (either for payments made as governmental aids, subsidies, and welfare benefits or to facilitate tax payments and any funding initiated by the government) that can increase the sentiment of credibility and security.

Promoting financial inclusion through nudging, as psychological stimuli to extend the use of mobile banking, requires a good understanding of the economic and social context and the level of financial inclusion.

To introduce nudging-type measures, the authorities should understand each group's constraints that hinder them from embracing mobile banking and, thus, directly address the specific needs. Here, an interesting principle to stimulate mobile banking is applying the design of choice architecture (Thaler [43]) which helps beneficiaries to opt for the best possible alternative to mobile banking.

Hence, cooperation between government, banks, non-governmental organizations, and businesses is essential to target and solve the most distressing issues.

8. Conclusions, Limitations and Further Research

The rise in digital networks and internet connectivity has greatly facilitated bank payments, even in the remote rural areas of the world. Modern communication enhances social connections and drives economic growth, with the flow of information playing a vital role in this process. As a result, local collaboration enhances growth by boosting both individual and collective capabilities. Nevertheless, contagion via technology, information sharing and networks among mobile banking non-users, active users, and undecided people may prevent the extension of digital payments when the users or those undecided are negatively influenced by those who, for various reasons (either economic or cultural), do not find mobile banking appealing. Indeed, such contagion corroborated that the financial inclusion rate influences the overall behaviour and incentives towards mobile bank usage.

While there is a noticeable overall increase in the number of mobile banking users, behaviours among different segments of bank clients may vary for several reasons. These can include concerns about fraudulent transactions, inadequate cybersecurity, negative rumours, poor bank services, technology-related insecurities, mistrust, and lingering negative experiences. As a result, client behaviours can become more unpredictable. The existence of contagion may justify that approximately half of the high-income countries' populations, and even less of the developing economies, use mobile banking apps.

To achieve the paper's objective, we have opted for a qualitative analysis utilizing a mathematical model within the framework of nonlinear differential equations with distributed time delay, which demonstrates extensive applicability. Thus, bank managers can adopt a behavioural approach to implementing mobile banking based on inclusiveness. It may also contribute to profit increases following the sustainability requirements. The focus was on a dynamic analysis of the interactions among mobile banking users, non-users, and undecided ones. We have proceeded with numerical simulations to demonstrate the existence of behavioural changes in mobile banking usage due to contagion.

The mathematical model on which the study relies shows that the contagion risk may determine mobile banking users to become non-users. The undecided and non-users might either be recovered (by joining the users) or leave the system. Some policies can be identified to slow down these trends, either by inducing trustworthiness or supporting financial inclusion. Nevertheless, the process is a continuous one. The numerical simulation does not consider specific units of time and suggests that, in the long run, behavioural trends stabilize under the conditions of the parameters following our analysis. We have found two steady states: the first with all bank clients owning and using mobile banking and the second including all parties (M, N, U). Their stability has been discussed. The average distributed time delay was chosen as the bifurcation parameter, and the existence of the oscillatory solutions was investigated.

Moreover, we have identified the optimal control strategy that diminishes non-users of mobile banking and increases the undecided group at the first stage. Later, some of this group might shift toward mobile banking users.

The numerical simulations follow the theoretical guidelines. The robustness of the results exists in the context of the nonlinear dynamical system with time delay, which illustrates the mobile banking contagion. It shows that the system converges to the same behaviour even with small deviations in the initial conditions.

Since human behaviour is sensitive to information-driven contagion and rather unforeseeable under external influences, the oscillatory orbits resulting from the simulation predict its volatility. The paper does not consider an individual country or region but rather looks at mobile banking contagion, per se, more broadly.

The conclusions are consistent with adopting new technologies (including mobile banking) as a long-term process. Moreover, given that a percentage of the undecided are persuaded to leave the group, joining the users or the non-users group supports the oscillatory behaviours on the market, considering the time delay. The degree of persuasion depends very much on the personality and trustworthiness of the information messenger.

The question is whether banks alone can foster mobile banking or, in certain instances, the decision makers can intervene with concrete measures without disturbing the market mechanisms. It becomes clear that cooperation between public, financial, and academic institutions is essential for such an approach to be successful. On the one hand, financial institutions should carefully look at the economic stance of their clients, thus providing specific products to disadvantaged or low-educated groups. In contrast, academic institutions provide tailored financial courses to alleviate illiteracy in this field. On the other hand, the government should enforce the necessary laws and regulations and invest by extending the internet infrastructure and/or subsidizing connectivity costs for vulnerable groups. Promoting transparent, accessible, simple, and friendly platforms is essential in nudging people towards Internet banking. Undeniably, the state can play an important role in controlling the asymmetries between agents and channelling efforts towards financial inclusion. In this respect, regulations, investments, and financial education are most essential.

To summarize, the main findings of the study are the following:

1. The steady states and the conditions for their local asymptotic stability are found.
2. The critical value for the time delay is determined, which leads to the oscillatory behaviour.
3. The optimal control strategy is identified; it focuses on the undecided group of mobile users so that the non-user group diminishes in the first stage and, later, the group of mobile banking users might increase.

The main takeaways from the present research are

1. The adoption of mobile banking is highly subjected to contagion induced, mainly, by the non-users on the undecided ones. The latter segment is the most vulnerable to external influences when deciding whether to join the cohort of users or nonusers of mobile banking
2. By extending mobile banking, financial inclusivity can be improved.

3. A set of direct and indirect policy measures can be implemented to enlarge the cohort of mobile bank users

The present research covers several gaps in in the literature by

- introducing a different research method based on a mathematical model with time delay;
- highlighting the behavioural stance of bank clients related to mobile banking usage;
- being more comprehensive because it employs a qualitative dynamic analysis, as opposed to other studies that approach contagion empirically;
- demonstrating that contagion among mobile bank users exists;
- putting forward several policy measures to increase financial inclusion and foster mobile banking usage.

The caveats and limitations of this study have two directions as follows:

- The lack of recent and complete data regarding financial inclusion and mobile banking adoption;
- Different types of kernels can be employed to develop the present study;
- More scenarios with different parameters could further support our results.

Nevertheless, the mathematical model grants sufficient abstraction and generalization for various contexts. The effect of memory can be considered by fractional derivatives.

As more information and data become available, future research will concentrate on empirical analysis to depict the best practises in various countries that foster mobile banking.

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