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Solving Viscous Burgers' Equation: Hybrid Approach Combining Boundary Layer Theory and Physics-Informed Neural Networks

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Abstract: In this paper, we develop a hybrid approach to solve the viscous Burgers' equation by combining classical boundary layer theory with modern Physics-Informed Neural Networks (PINNs). The boundary layer theory provides an approximate analytical solution to the equation, particularly in regimes where viscosity dominates. PINNs, on the other hand, offer a data-driven framework that can address complex boundary and initial conditions more flexibly. We demonstrate that PINNs capture the key dynamics of the Burgers' equation, such as shock wave formation and the smoothing effects of viscosity, and show how the combination of these methods provides a powerful tool for solving nonlinear partial differential equations.

Keywords: boundary layer theory; Physics-Informed Neural Networks (PINNs); nonlinear partial differential equations; Burgers' equation; shock waves; traveling waves

MSC: 76D10; 35K55; 35Q35; 76L05; 35C07

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Citation: Ortiz Ortiz, R.D.; Martínez Núñez, O.; Marín Ramírez, A.M. Solving Viscous Burgers' Equation: Hybrid Approach Combining Boundary Layer Theory and Physics-Informed Neural Networks. *Mathematics* **2024**, *12*, 3430. https://doi.org/10.3390/math12213430

Academic Editors: Zhibin Du and Jonathan Blackledge

Received: 16 September 2024
Revised: 24 October 2024
Accepted: 30 October 2024
Published: 1 November 2024

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1. Introduction

Nonlinear partial differential equations (PDEs) play a fundamental role in modeling complex phenomena across various scientific disciplines, including fluid dynamics, turbulence, traffic flow, and environmental sciences [1,2]. Among these equations, the Burgers' equation is particularly significant as it encapsulates essential features of nonlinearity and diffusion, serving as a simplified model for the Navier–Stokes equations [3–8]. In its viscous form, the Burgers' equation captures the interplay between nonlinear convection and viscous diffusion, making it a valuable tool for studying shock waves, turbulence, and boundary layer behavior [9].

Traditional analytical methods, such as the Boundary Layer Theory (BLT) introduced by Prandtl in 1904 [10], have been instrumental in providing approximate solutions to the viscous Burgers' equation, especially in regions dominated by viscosity near boundaries where sharp gradients occur [11,12]. BLT offers a simplified yet powerful framework for analyzing viscous fluid flows by focusing on the behavior of thin layers adjacent to solid boundaries. However, while BLT is effective in handling viscous effects near boundaries, it faces limitations when addressing the complex nonlinear dynamics in the interior of the domain, particularly under complex boundary conditions or in high-dimensional settings.

Advancements in computational methods and machine learning have opened new avenues for solving nonlinear PDEs. High-accuracy numerical schemes, such as those used in Direct Numerical Simulation (DNS) and acoustics, have been developed to simulate complex PDEs with greater precision [13]. Additionally, methods based on vorticity and incompressible flow have provided significant insights into fluid dynamics, which are detailed by Majda and Bertozzi [14]. More recently, data-driven techniques have been

employed to develop discretizations for PDEs, with machine learning being applied to improve the accuracy of numerical solutions [15].

In particular, *Physics-Informed Neural Networks* (PINNs), introduced by Raissi et al. in 2019 [16], have emerged as a powerful tool for solving both ordinary and partial differential equations by incorporating the governing physical laws directly into the neural network's loss function. PINNs ensure that the learned solutions respect the underlying physics, allowing for accurate solutions to complex nonlinear PDEs [17]. The effectiveness of PINNs has been demonstrated in various studies, including applications to the viscous Burgers' equation [18,19], stochastic PDEs [20], and fractional equations [21,22].

Despite their success, PINNs face challenges related to convergence efficiency and the handling of complex initial and boundary conditions. Innovative approaches like the Deep Ritz method, introduced by E and Yu [23], extend the capabilities of neural networks in solving variational problems and represent a natural progression of the PINN framework. The potential of deep learning in solving PDEs is further highlighted in the work of Bishop and Bishop [24], which bridges traditional numerical analysis and modern machine learning techniques.

In recent years, neural networks have proven to be effective tools for solving ordinary and partial differential equations [25]. This approach has been significantly enhanced by deep learning techniques, such as the Deep Galerkin Method (DGM) proposed by Sirignano and Spiliopoulos [26]. Furthermore, Physics-Informed Neural Networks (PINNs) have been successfully applied to inverse problems, including those in nano-optics and metamaterials [27].

Another significant development is found in the study of stochastic differential equations in infinite dimensions, developed by Da Prato and Zabczyk [28], providing a solid theoretical framework for stochastic problems.

In this paper, we propose a hybrid method that combines the analytical strength of Boundary Layer Theory with the flexibility of Physics-Informed Neural Networks to solve the viscous Burgers' equation more effectively. By using BLT to provide an initial approximation of the solution in regions dominated by viscosity and employing PINNs to refine the solution in areas where nonlinearity and complex boundary conditions dominate, we leverage the strengths of both methods. This approach enhances the accuracy and robustness of the solutions.

The remainder of the paper is organized as follows. In Section 2, we provide a physical analysis of the viscous Burgers' equation. Section 3 discusses the analytical solution using Boundary Layer Theory. In Section 4, we describe the hybrid approach combining BLT and PINNs. Section 5 presents numerical results. Finally, we conclude with a discussion of the implications of our findings and suggestions for future research.

2. Target Audience

The target readers of this work are diverse and include the following:

- Researchers in Applied Mathematics: Especially those focused on numerical methods for solving partial differential equations (PDEs), like the Burgers' equation. This audience will be interested in their comparison with traditional methods such as Finite Difference Methods (FDM) and modern techniques like Physics-Informed Neural Networks (PINNs) and Boundary Layer Theory (BLT).
- Computational Scientists and Engineers: Professionals who work with simulations in fluid dynamics, heat transfer, or similar physical phenomena that are often modeled using PDEs will benefit from the insights provided in this work, particularly the application of Burgers' equation to model these processes.
- Graduate Students in Mathematics, Physics, or Engineering: Students who are learning about numerical methods, PDEs, or computational modeling may find this work helpful for understanding the strengths and limitations of different methods such as FDM and PINNs.

- Machine Learning Practitioners in Physics: With the growing use of machine learning in solving physical problems, researchers from the machine learning community may also find this work valuable, particularly in how Physics-Informed Neural Networks (PINNs) are applied to classical physical problems like Burgers' equation.

This work aims to bridge the gap between traditional numerical methods and modern machine learning approaches, making it relevant to a wide interdisciplinary audience.

3. Physical Analysis of Burgers' Equation with Viscosity

The viscous Burgers' equation models the competition between nonlinear convection and diffusion. It takes the following form:

$$u_t(x,t) + u(x,t)u_x(x,t) = \epsilon u_{xx}(x,t), \quad -1 \le x \le 1,\ \epsilon > 0,\ t > 0, \tag{1}$$

where $u(x,t)$ represents the velocity field, and ϵ is the viscosity. The nonlinear term $u(x,t)u_x(x,t)$ represents the convective transport, while the term $\epsilon u_{xx}(x,t)$ introduces diffusive effects.

In systems like fluid dynamics and traffic flow, the balance between these two terms determines the formation of shock waves or smooth transitions. When $\epsilon = 0$, the system behaves as inviscid, leading to the formation of sharp shocks. When $\epsilon > 0$, viscosity smooths the discontinuities, leading to smoother solutions.

4. Burgers' Equation with Viscosity: Analytical Solution

The stationary version of Burgers' equation is expressed as follows:

$$\epsilon u_{xx}(x) - u(x)u_x(x) = 0. \tag{2}$$

We now seek a solution $U(x,\epsilon)$ that represents the stationary velocity field under the influence of both convection and diffusion. The solution $U(x,\epsilon)$ provides a steady-state approximation that depends only on the spatial variable x and the viscosity ϵ.

Theorem 1. *The solution of* (2)*, which is subject to the boundary conditions*

$$u(-1) = \alpha, \quad u(1) = \beta,$$

is given by the following:

$$U(x,\epsilon) = -\alpha \tanh\Big[\frac{\alpha}{2}\Big(\frac{x}{\epsilon} + k\Big)\Big],$$

where $U(x,\epsilon) \in L_2([-1,1])$.

Proof. Equation (2) becomes the following:

$$\begin{cases} \epsilon u^+_{xx}(x) - u^+(x)u^+_x(x) = 0, & u^+(1) = \beta, \\ \epsilon u^-_{xx}(x) - u^-(x)u^-_x(x) = 0, & u^-(-1) = \alpha. \end{cases}$$

To solve this, we consider the following expansions:

$$u^{+,ext} \approx \sum \epsilon^n u^{+,ext}_n(x), \quad u^{+,ext}_x \approx \sum \epsilon^n u^{+,ext}_{nx}(x), \quad u^{+,ext}_{xx} \approx \sum \epsilon^n u^{+,ext}_{nxx}(x).$$

Substituting these expansions into the differential equation yields the following:

$$\epsilon\big(\sum \epsilon^n u^{+,ext}_{nxx}(x)\big) - \big(\sum \epsilon^n u^{+,ext}_n(x)\big)\big(\sum \epsilon^n u^{+,ext}_{nx}(x)\big) = 0.$$

Considering terms of order ϵ^0, the following relation holds:

$$-u^{+,ext}_0 u^{+,ext}_{0x} = 0.$$

Considering terms of order ϵ, the following relation holds:

$$u_{0xx}^{+,ext} - \left(u_0^{+,ext}u_{1x}^{+,ext} + u_1^{+,ext}u_{0x}^{+,ext}\right) = 0.$$

Assuming $u_{0x}^{+,ext} = 0$, then $u_0^{+,ext} = k$, hence $u_0^{+,ext} = \beta$.

Next, we find $u_0^{+,int}$. Consider $u^{+,int}(s)$, an expansion near the source where $s = \frac{x}{\epsilon^{\gamma}}$. Substituting into the original equation, we have the following:

$$\epsilon^{1-2\gamma}u_{ss}^{+,int} - \epsilon^{-\gamma}u^{+,int}u_s^{+,int} = 0. \tag{3}$$

Balancing terms, we set $\gamma = 1$. Expanding $u^{+,int}(s)$, we obtain the following:

$$u^{+,int} \approx \sum \epsilon^n u_n^{+,int}(s), \quad u_s^{+,int} \approx \sum \epsilon^n u_{ns}^{+,int}(s), \quad u_{ss}^{+,int} \approx \sum \epsilon^n u_{nss}^{+,int}(s).$$

Substituting these expressions into (3) and equating the coefficients of ϵ^{-1}, we obtain the following:

$$u_{0ss}^{+,int} - u_0^{+,int}u_{0s}^{+,int} = 0. \tag{4}$$

We substitute $w = u_{0s}^{+,int}$, leading to $w_s = w_{u_0^{+,int}}w$, which implies that $w\left(w_{u_0^{+,int}} - u_0^{+,int}\right) = 0$. Solving this differential equation, we obtain the following:

$$u_0^{+,int} = -\sqrt{2c_0}\tanh\left(\frac{\sqrt{2c_0}}{2}(s + c_3)\right).$$

Applying the Prandtl matching condition, we obtain the following:

$$\lim_{s\to+\infty} u_0^{+,int} = \lim_{x\to0^+} u_0^{+,ext} = \beta.$$

Thus,

$$u_0^{+,int} = -\beta\tanh\left(\frac{\beta}{2}\left(\frac{x}{\epsilon} + k\right)\right).$$

Following a similar process for u^-, we find the following:

$$u_0^{-,ext} = \alpha, \quad u_0^{-,int} = -\sqrt{2c_0}\tanh\left(\frac{\sqrt{2c_0}}{2}(s + c_3)\right).$$

The matching conditions imply that $\alpha = -\beta$. Therefore,

$$U(x,\epsilon) = -\alpha\tanh\left(\frac{\alpha}{2}\left(\frac{x}{\epsilon} + k\right)\right).$$

To show that $U(x,\epsilon) \in L_2([-1,1])$, note that

$$|U(x,\epsilon)|^2 = \alpha^2\tanh^2\left(\frac{\alpha}{2}\left(\frac{x}{\epsilon} + k\right)\right) < \alpha^2.$$

Thus,

$$\int_{-1}^{1} |U(x,\epsilon)|^2\,dx < 2\alpha^2 < \infty.$$

□

5. Metastability

The metastable solution $U(x,\epsilon)$ is given asymptotically by the following:

$$U(x,\epsilon) = -\alpha\tanh\left(\frac{\alpha}{2}\left(\frac{x}{\epsilon} + k\right)\right).$$

We linearize the equation around this solution and analyze its stability by perturbing it and solving the resulting eigenvalue problem, concluding that the solution is stable.

$$u(x,t) = U(x,\epsilon) + \nu e^{-\lambda t}\Phi(x), \quad \nu \ll 1.$$

Substituting this into the linearized equation, we obtain the following:

$$-\lambda \nu e^{-\lambda t}\Phi(x) = \epsilon \nu e^{-\lambda t}\Phi_{xx}(x) - U(x,\epsilon)\nu e^{-\lambda t}\Phi_x(x) - \nu e^{-\lambda t}\Phi(x)U_x(x,\epsilon).$$

Dividing by $\nu e^{-\lambda t}$ and simplifying yields the following:

$$-\lambda\Phi(x) = \epsilon\Phi_{xx}(x) - (U(x,\epsilon)\Phi(x))_x.$$

This leads to the eigenvalue problem:

$$\epsilon\Phi_{xx}(x) - (U(x,\epsilon)\Phi(x))_x + \lambda\Phi(x) = 0, \quad -1 \le x \le 1, \quad \Phi(\pm 1) = 0.$$

To check whether $u(x,t) = U(x,\epsilon) + \nu e^{-\lambda t}\Phi(x) \in L^2([-1,1])$, observe that since $\lambda \ge 0$ and $e^{-\lambda t} \le 1$, the function $\Phi(x)$ must be bounded. Thus, we have the following:

$$\int_{-1}^{1} |u(x,t)|^2\, dx = \int_{-1}^{1} \Big(U(x,\epsilon) + \nu e^{-\lambda t}\Phi(x)\Big)^2\, dx.$$

Expanding and considering the boundedness, we obtain the following:

$$\int_{-1}^{1} |u(x,t)|^2\, dx < \int_{-1}^{1} (4\alpha^2 + M)\, dx = 8\alpha^2 + 2M < \infty.$$

Therefore, $u(x,t) \in L_2([-1,1])$.

Theorem 2. *Let $u(x,t) = U(x,\epsilon) + \nu e^{-\lambda t}\Phi(x)$, where $U(x,\epsilon)$ is the stationary solution of the Burgers equation with viscosity and $\nu \ll 1$. Then, the solution $u(x,t)$ converges to $U(x,\epsilon)$ in the L^2 norm as $t \to \infty$. Specifically,*

$$\lim_{t\to\infty} \|u(x,t) - U(x,\epsilon)\|_{L^2([-1,1])} = 0.$$

Proof. To check whether the perturbed solution $u(x,t) = U(x,\epsilon) + \nu e^{-\lambda t}\Phi(x)$ belongs to the space $L^2([-1,1])$, we observe that, since $\lambda \ge 0$ and $e^{-\lambda t} \le 1$, the function $\Phi(x)$ must be bounded. Additionally, we have the following:

$$\int_{-1}^{1} |u(x,t)|^2\, dx = \int_{-1}^{1} \Big(U(x,\epsilon) + \nu e^{-\lambda t}\Phi(x)\Big)^2\, dx.$$

Expanding the square results in the following:

$$\int_{-1}^{1} |u(x,t)|^2\, dx = \int_{-1}^{1} \Big(U(x,\epsilon)^2 + 2\nu e^{-\lambda t}U(x,\epsilon)\Phi(x) + \nu^2 e^{-2\lambda t}\Phi(x)^2\Big)\, dx.$$

Since $U(x,\epsilon) \in L^2([-1,1])$ and $\Phi(x)$ also belongs to $L^2([-1,1])$, all terms of the integral are finite. Moreover, the term $\nu^2 e^{-2\lambda t}$ decays exponentially to zero as $t \to \infty$, which implies that the L^2 norm of $u(x,t)$ satisfies the following:

$$\lim_{t\to\infty} \|u(x,t) - U(x,\epsilon)\|_{L^2([-1,1])} = 0.$$

This shows that the perturbed solution $u(x,t)$ converges to the stationary solution $U(x,\epsilon)$ in the L^2 norm as $t \to \infty$. □

6. Solution to the Viscous Burgers' Equation

Consider the viscous Burgers' equation (1) with boundary conditions

$$u(-1,t) = \alpha, \quad u(1,t) = \beta, \tag{5}$$

where $\epsilon > 0$ is the viscosity parameter [5]. The solution $u(x,t)$ can be obtained by using the Cole–Hopf transformation, which linearizes the equation.

Theorem 3. *The function $u(x,t)$ solving the Burgers' equation with boundary conditions (5) is given by the following:*

$$u(x,t) = -2\epsilon \frac{v_x(x,t)}{v(x,t)}, \tag{6}$$

where $v(x,t)$ solves the heat equation:

$$v_t(x,t) = \epsilon v_{xx}(x,t). \tag{7}$$

Proof. Cole–Hopf Transformation: Define a new function $v(x,t)$ such that

$$u(x,t) = -2\epsilon \frac{v_x(x,t)}{v(x,t)}. \tag{8}$$

Substituting this transformation into the Burgers' equation, we obtain the following:

$$\frac{\partial}{\partial t}\left(-2\epsilon \frac{v_x}{v}\right) + \left(-2\epsilon \frac{v_x}{v}\right)\frac{\partial}{\partial x}\left(-2\epsilon \frac{v_x}{v}\right) = \epsilon \frac{\partial^2}{\partial x^2}\left(-2\epsilon \frac{v_x}{v}\right). \tag{9}$$

Simplifying the Equation: After computing the derivatives and simplifying, we obtain the following:

$$v_t - \epsilon v_{xx} = 0, \tag{10}$$

which is the heat equation for $v(x,t)$.

Solving the Heat Equation: The general solution to the heat equation with initial condition $v(x,0) = v_0(x)$ is given by the following:

$$v(x,t) = \frac{1}{\sqrt{4\pi\epsilon t}} \int_{-1}^{1} \exp\left(-\frac{(x-y)^2}{4\epsilon t}\right) v_0(y)\, dy. \tag{11}$$

Boundary Conditions for $v(x,t)$: The boundary conditions on $u(x,t)$ translate into conditions on $v(x,t)$. For large times, $v(x,t)$ must satisfy the conditions derived from $u(-1,t) = \alpha$ and $u(1,t) = \beta$.

Inverse Cole–Hopf Transformation: After finding $v(x,t)$, use the inverse Cole–Hopf transformation to recover $u(x,t)$:

$$u(x,t) = -2\epsilon \frac{v_x(x,t)}{v(x,t)}. \tag{12}$$

Initial Condition and Specific Solutions: Given the initial condition $u(x,0) = u_0(x)$, we determine $v_0(x)$ from

$$u_0(x) = -2\epsilon \frac{v_0'(x)}{v_0(x)}, \tag{13}$$

which gives the following:

$$v_0(x) = \exp\left(-\frac{1}{2\epsilon}\int u_0(x)\, dx\right). \tag{14}$$

Final Solution for $u(x,t)$: The solution $u(x,t)$ for the Burgers' equation is as follows:

$$u(x,t) = -2\epsilon \frac{\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{4\pi\epsilon t}}\int_{-1}^{1} e^{-\frac{(x-y)^2}{4\epsilon t}} v_0(y)\,dy\right)}{\frac{1}{\sqrt{4\pi\epsilon t}}\int_{-1}^{1} e^{-\frac{(x-y)^2}{4\epsilon t}} v_0(y)\,dy}. \tag{15}$$

□

Theorem 4 (Transition between Nonlinear and Diffusive Regimes)**.** *Consider the viscous Burgers' equation:*

$$u_t + uu_x = \epsilon u_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \quad \epsilon > 0,$$

with an initial condition $u(x,0) = u_0(x)$ containing a discontinuity (shock wave). In the limit $\epsilon \to 0$, the inviscid Burgers' equation generates a shock wave, while for $\epsilon > 0$, viscosity smooths the shock.

For large times, the solution profile asymptotically approaches a shock wave with a width proportional to $O(\epsilon)$.

Proof. The viscous Burgers' equation is expressed as follows:

$$u_t + uu_x = \epsilon u_{xx}.$$

Behavior without viscosity ($\epsilon = 0$)

If $\epsilon = 0$, the equation reduces to the following hyperbolic form:

$$u_t + uu_x = 0,$$

which is a nonlinear conservation law. This equation allows for the formation of shock waves when the initial condition $u_0(x)$ contains a discontinuity.

Shock Wave Formation:

Since the propagation speed of characteristics is $u(x,t)$, the characteristics can cross, leading to a discontinuity in the solution. This phenomenon occurs in finite time and is known as shock formation.

Behavior with viscosity ($\epsilon > 0$)

When $\epsilon > 0$, the equation includes a diffusive term:

$$u_t + uu_x = \epsilon u_{xx}.$$

This diffusive term smooths the discontinuities that would form in the inviscid case. The term ϵu_{xx} acts as a diffusion mechanism, spreading the transition region of a shock wave over a width proportional to ϵ.

Solution of smoothed shock wave:

We seek a stationary solution of the form $u(x,t) = U(x)$, for which the equation becomes the following:

$$\epsilon U_{xx} = UU_x.$$

This can be solved using the hyperbolic tangent transformation $U(x) = A\tanh\left(\frac{x}{\epsilon}\right)$, which describes the profile of a smoothed shock wave with a width of $O(\epsilon)$.

Numerical solution and comparison:

Numerical simulations of the Burgers' equation were performed for different values of ϵ. The numerical results show that for $\epsilon = 0.1$, the solution is relatively smooth with no strong discontinuities, whereas for $\epsilon = 0.01$, nonlinear effects dominate, leading to sharper transitions in the solution, though still smoothed by the viscosity. In both cases, there is a transition between the nonlinear regime (which would generate a shock wave in the inviscid case) and the diffusive regime, where viscosity smooths the solution.

Long-time behavior:

For large times, the solution of the viscous Burgers' equation approaches a smoothed shock wave of the following form:

$$u(x,t) \sim \tanh\left(\frac{x}{\epsilon}\right),$$

where ϵ controls the width of the shock wave transition. In the limit $\epsilon \to 0$, this solution approaches a classical discontinuity (shock wave) from the inviscid hyperbolic equation. □

7. Numerical Solutions of the Burgers' Equation

Given the complexity of Burgers' equation, numerical methods are necessary for cases where exact solutions are not feasible. We solve the equation using a finite difference method for two values of viscosity, $\epsilon = 0.1$ and $\epsilon = 0.01$, to observe the behavior of the solution.

The Burgers' equation is an example of a nonlinear partial differential equation with applications in various fields of physics and engineering. In particular, it has been used as a model to describe fluid dynamics and shock waves. In this work, we numerically solve the Burgers' equation using a finite difference method, which allows us to study the behavior of the solution under different viscosity values.

The Burgers' equation is notable for combining both nonlinearity (in the convective term uu_x) and diffusion (in the viscous term ϵu_{xx}). When $\epsilon = 0$, the equation becomes a purely inviscid conservation law, leading to shock formation and discontinuities. However, for $\epsilon > 0$, the equation introduces viscosity, smoothing out these shocks and producing continuous solutions.

7.1. Numerical Approach and Goals

The Burgers' equation does not always admit simple analytical solutions, especially for arbitrary initial and boundary conditions. Therefore, numerical methods play a crucial role in solving the equation under different physical scenarios.

In this article, we numerically solve the Burgers' equation using a finite difference method for two different values of ϵ (the viscosity parameter):

- $\epsilon = 0.1$: This corresponds to a moderate viscosity, which introduces significant diffusion but retains some nonlinearity in the solution.
- $\epsilon = 0.01$: This represents a lower viscosity case, where nonlinear effects dominate more significantly, leading to sharper transitions in the solution.

We discretize the spatial and temporal domains and apply an iterative finite difference method to solve the PDE. The Burgers' equation was solved numerically using a finite difference scheme. For the convective term uu_x, a first-order central finite difference method was applied, using either forward or backward differencing to approximate the spatial derivative. For the diffusive term ϵu_{xx}, a second-order central difference scheme was employed to achieve higher accuracy. The time-stepping was performed using an explicit Euler method, ensuring stability by choosing a sufficiently small time step dt. This combination allowed for an effective approximation of the Burgers' equation, capturing the balance between convection and diffusion. The boundary conditions are enforced as $u(-1,t) = \alpha$ and $u(1,t) = \beta$, where α and β are constants.

7.2. Objectives of the Numerical Experiment

The main objective of this numerical experiment is to investigate how varying the viscosity coefficient ϵ affects the behavior of the solution. Specifically, we note the following:

- For $\epsilon = 0.1$, we expect a relatively smooth solution due to the dominance of the diffusive term, which smooths out any discontinuities in the system.
- For $\epsilon = 0.01$, we anticipate sharper gradients in the solution, as the smaller viscosity reduces the smoothing effect, allowing the nonlinearity to introduce sharper transitions.

By comparing these two cases, we aim to highlight the impact of viscosity on the dynamics of the Burgers' equation and demonstrate the effectiveness of numerical methods in capturing these dynamics.

8. Numerical Results

In this section, we present the numerical solutions to Burgers' equation for different values of the viscosity parameter ϵ. The numerical solutions were computed using a finite difference method with a uniform grid. The goal is to compare how different values of ϵ affect the smoothness and the structure of the solution.

Figures 1 and 2 show the numerical solutions for $\epsilon = 0.1$ and $\epsilon = 0.01$, respectively.

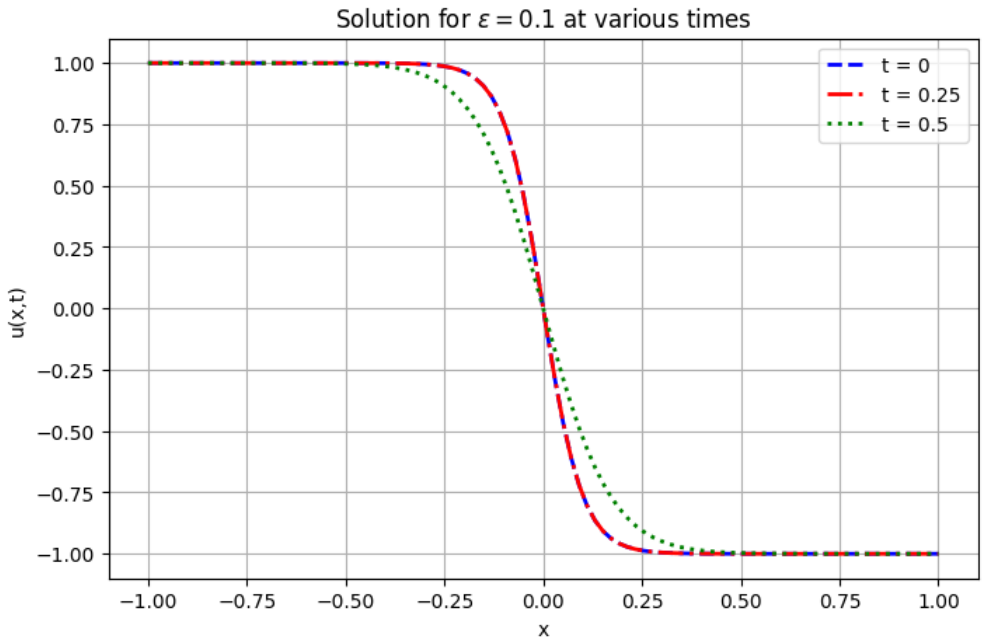


Figure 1. Solution for $\epsilon = 0.1$ at various times.

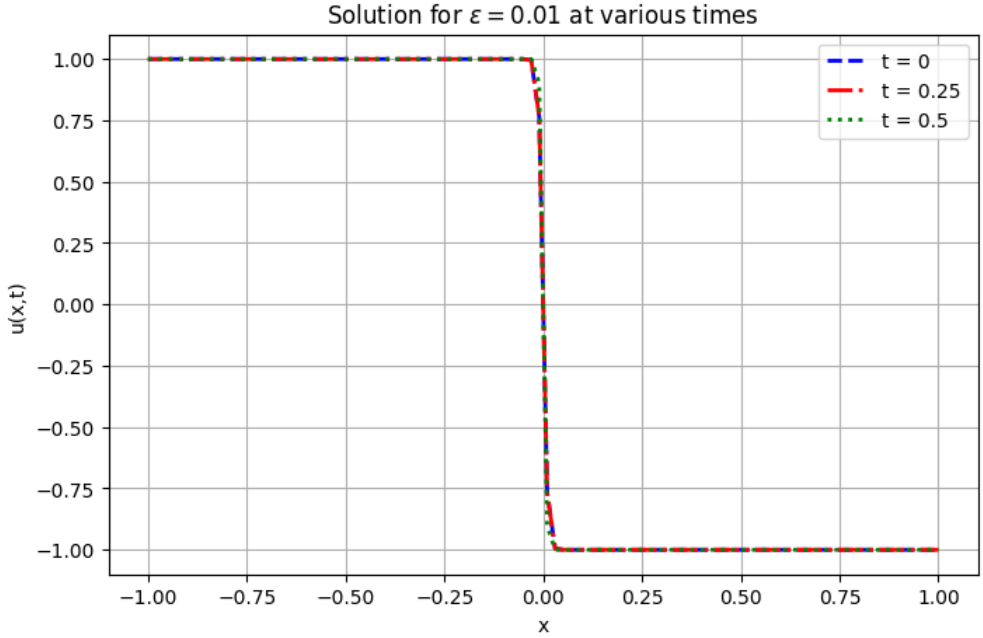


Figure 2. Solution for $\epsilon = 0.01$ at various times.

As shown in Figure 3, for $\epsilon = 0.1$, the solution exhibits a smooth transition across the domain. This is due to the diffusive nature of the term ϵu_{xx}, which dominates over the nonlinear convective term uu_x for larger values of ϵ. The viscosity smooths out sharp gradients, preventing the formation of discontinuities or shock waves, and the solution remains continuous and differentiable.

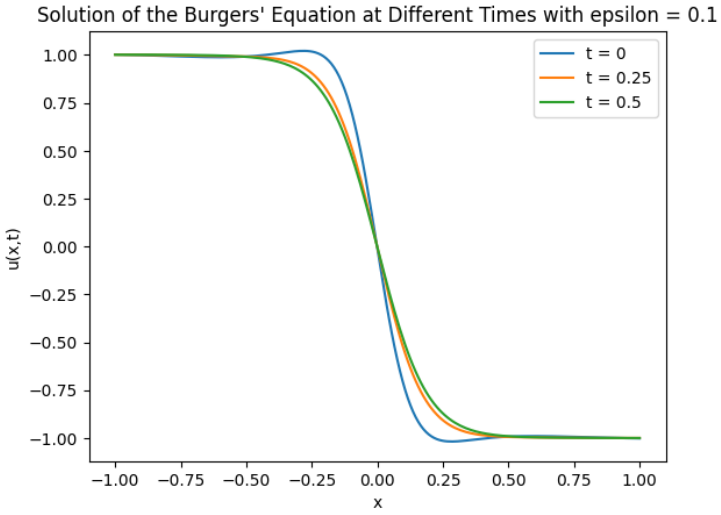


Figure 3. Solution of Burgers' equation for $\epsilon = 0.1$ at different times.

In contrast, Figure 2 illustrates the solution for a smaller viscosity, $\epsilon = 0.01$. In this case, the convective term uu_x dominates, leading to sharper transitions in the solution. Although the diffusive term is still present, its effect is significantly reduced, and the solution shows steeper gradients that resemble shock waves, although still smoothed by the small amount of viscosity.

These results are consistent with the theoretical analysis in Section 3, where we discussed how the interplay between nonlinearity and diffusion governs the structure of the solution. The approach presented is extendable to other nonlinear problems, including Navier–Stokes equations. For large ϵ, diffusion smooths out any discontinuities, leading to a smooth profile. However, as ϵ decreases, nonlinearity becomes more pronounced, and the solution approaches the behavior of the inviscid Burgers' equation, where shock formation is expected.

Discussion of Numerical Stability

The numerical stability of the solutions was carefully monitored. For larger values of ϵ, the solution remains stable over time due to the dominating diffusion term. However, for smaller values of ϵ, the convective effects can cause instabilities if the spatial and temporal discretization is not chosen carefully. We used an explicit Euler scheme for time-stepping, which helped mitigate numerical oscillations near sharp gradients in the $\epsilon = 0.01$ case.

The numerical results highlight the impact of the viscosity parameter ϵ on the behavior of Burgers' equation. As expected, larger values of ϵ lead to smoother solutions, while smaller values of ϵ result in sharper transitions, approaching the formation of shock waves. These findings are in agreement with the theoretical predictions discussed in Section 3, where the balance between nonlinearity and diffusion was analyzed in detail.

9. Literature Review

The Boundary Layer Theory (BLT), proposed by Ludwig Prandtl in 1904, has been a fundamental tool in describing the behavior of viscous fluids [10]. This theory has been widely applied in fluid mechanics to model shock formation and smooth transitions in systems dominated by viscosity [11]. However, classical methodologies like BLT struggle

with handling complex boundary conditions and high-dimensional problems, limiting their applicability in some contexts.

On the other hand, Physics-Informed Neural Networks (PINNs), introduced by Raissi et al. in 2019, have revolutionized how PDEs are solved by embedding the physical equations into the training phase of the neural network [16]. This approach has been particularly useful in solving nonlinear equations such as the viscous Burgers' equation, providing accurate solutions in areas where traditional methods fail. Additionally, PINNs allow for considerable flexibility in modeling complex initial and boundary conditions, which is difficult for BLT or traditional numerical methods to address [17].

Recent studies have successfully applied PINNs to the viscous Burgers' equation. For instance, Sun et al. [19] demonstrated how PINNs can capture shock wave behavior and diffusive transitions in viscous flows. However, a common limitation of PINNs is their tendency to require large amounts of training data to achieve accurate convergence, which can make them computationally expensive for highly complex problems.

Our hybrid approach seeks to overcome these limitations by combining the structured, analytical framework of BLT with the flexibility of PINNs. By using BLT solutions as an initial condition for training the PINNs, our method accelerates convergence and improves accuracy in regions where nonlinearities dominate, such as shock wave formation.

10. Contribution of This Work

Unlike previous studies that have used PINNs or BLT separately, our research proposes a hybrid approach that integrates both methods in a complementary fashion. While BLT provides an accurate initial solution in viscosity-dominated regions, PINNs refine this solution in more complex areas where nonlinearities are predominant.

11. Methodology

Our hybrid approach combines Boundary Layer Theory (BLT) and Physics-Informed Neural Networks (PINNs) to solve the viscous Burgers' equation more effectively by leveraging the strengths of each method in specific regions of the solution space. Below, we describe how the two methods are integrated step by step.

The boundary layer method combined with PINNs offers an efficient approach for solving the Burgers' equation in domains where sharp transitions occur, such as at $x = 0$. This approach allows us to capture both the smooth solution in the external regions and the sharp transition in the boundary layer.

1. Initial Approximation with BLT: BLT is applied to the viscous Burgers' equation in the initial stages, where it provides an approximate analytical solution in regions where the fluid viscosity dominates. This is particularly effective in the boundary layers, where sharp gradients occur. The BLT framework captures the key dynamics in these areas, offering a structured and efficient solution in regions close to the boundaries. Specifically, BLT is used to simplify the equation into a form where the dominant viscous effects are balanced by convective terms, allowing us to describe the behavior of the solution near the boundaries with high precision. This solution serves as a baseline or starting point for further refinement.
2. Transition to PINNs for Complex Dynamics: While BLT provides a reliable solution in the boundary layers, it struggles to handle more complex regions where nonlinearities dominate, particularly in the bulk of the domain or where shock waves form. Here, PINNs take over. PINNs are neural networks that incorporate the governing physical laws (in this case, the Burgers' equation) directly into the loss function. By training the network on data points throughout the domain, PINNs can capture complex dynamics such as shock waves, which are more difficult for BLT to resolve. PINNs are particularly effective in regions with complex initial and boundary conditions or where traditional methods fail to capture the full dynamics.
3. Seamless Integration: The key to the hybrid method is the seamless transition between BLT and PINNs. After obtaining the approximate solution from BLT, this solution

is used to initialize the PINNs training. The network begins its optimization from this analytically derived solution, ensuring that the physics near the boundaries is respected while allowing the network to refine the solution in more complex regions. This initialization improves the convergence speed and ensures that PINNs do not require large amounts of training data to achieve accuracy, as they are already informed by the structured solution provided by BLT.

4. Error Correction and Refinement: Finally, the combined solution is further refined by repeated training of the PINN, which adjusts the initial BLT solution according to the residuals of the Burgers' equation throughout the domain. This iterative process ensures that the final solution respects the global dynamics of the equation, balancing the structured accuracy of BLT with the flexibility of PINNs. The network is trained not only on data from the bulk of the domain but also on boundary conditions, ensuring a consistent solution across all regions.

By dividing the problem space in this manner, our hybrid method capitalizes on the strengths of both BLT and PINNs. The structured nature of BLT allows for efficient computation near the boundaries, while the flexibility of PINNs ensures an accurate representation of complex dynamics in the interior and near shocks. This approach leads to a more robust and accurate solution than either method could achieve independently.

Finite Difference Method for Burgers' Equation

The Finite Difference Method (FDM) is a numerical technique used to solve partial differential equations (PDEs). It involves discretizing the continuous spatial and temporal domains into a grid and approximating the derivatives in the equation using finite differences.

In this work, we used the FDM to solve the viscous Burgers' equation:

$$u_t + uu_x = \epsilon u_{xx},$$

where ϵ is the viscosity coefficient, and $u(x,t)$ is the solution. The domain $[-1,1]$ was discretized using a uniform grid with spatial step size Δx and time step size Δt.

12. Finite Difference Approximations

To approximate the derivatives in the Burgers' equation, the following finite difference schemes were applied:

- First-order forward difference for the time derivative u_t:

$$\frac{\partial u}{\partial t} \approx \frac{u_i^{n+1} - u_i^n}{\Delta t},$$

- Second-order central difference for the first spatial derivative u_x:

$$\frac{\partial u}{\partial x} \approx \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x},$$

- Second-order central difference for the second spatial derivative u_{xx}:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}.$$

The central difference scheme provides a balance between accuracy and computational efficiency. It is second-order accurate, meaning that the error decreases quadratically as the grid resolution increases.

13. FDM Formulation for the Burgers' Equation

Substituting the finite difference approximations into the Burgers' equation results in the following discretized form:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + u_i^n \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = \epsilon \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}.$$

Solving for u_i^{n+1}, the value of u at the next time step, we obtain the following:

$$u_i^{n+1} = u_i^n - \Delta t \left(u_i^n \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \right) + \Delta t \epsilon \left(\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \right).$$

14. Boundary and Initial Conditions

The boundary and initial conditions applied were as follows:

- Boundary conditions: $u(-1, t) = 1$ and $u(1, t) = -1$,
- Initial condition: $u(x, 0) = -\tanh(x/\epsilon)$.

15. Stability

The forward time-stepping scheme (explicit method) used here requires careful selection of the time step Δt to ensure stability. The Courant–Friedrichs–Lewy (CFL) condition typically applies, requiring the following:

$$\Delta t \leq \frac{\Delta x^2}{2\epsilon}.$$

Choosing a small enough Δt ensures that the scheme remains stable over time.

16. Algorithm

The algorithm for solving the Burgers' equation using FDM can be summarized as follows:

1. Discretize the spatial domain $x \in [-1, 1]$ and the time domain $t \in [0, T]$,
2. Set the initial condition $u(x, 0) = -\tanh(x/\epsilon)$,
3. Apply boundary conditions at each time step,
4. Update u at each grid point using the finite difference formula,
5. Repeat the process until the desired final time T is reached.

16.1. Why Use Forward Difference Method?

We used the forward difference scheme for the numerical solution of the Burgers' equation; it would typically be due to the following reasons:

16.2. Simplicity in Implementation

The forward difference method is easy to implement because it only requires knowledge of the current time step and the previous time step. This makes the method more straightforward to program and computationally efficient in comparison to implicit methods.

16.3. Explicit Time Integration

The forward difference method is an explicit method, which means that the value of the solution at the next time step can be computed directly from the known values at the current time step:

$$u^{n+1} = u^n + \Delta t \cdot F(u^n),$$

where $F(u^n)$ represents the spatial discretization of the Burgers' equation. This avoids the need to solve a system of equations, as is required in implicit methods, which can be computationally expensive.

16.4. Computational Efficiency

Since forward difference schemes are explicit, they tend to be faster computationally per time step compared to implicit schemes. Each time step involves simple arithmetic operations and does not require solving matrix systems or iterative solvers, which are computationally demanding in implicit schemes.

17. Limitations of Forward Difference Method

However, using the forward difference method has certain limitations that must be considered:

17.1. Stability

Forward difference schemes are conditionally stable, meaning that they require small time steps to ensure stability, especially for stiff equations or problems with high gradients. For the Burgers' equation, the time step Δt needs to satisfy the Courant–Friedrichs–Lewy (CFL) condition, which can make the method impractically slow for some parameter choices:

$$\Delta t \leq \frac{\Delta x}{\max |u|}.$$

17.2. Numerical Diffusion

Forward difference schemes tend to introduce numerical diffusion, which can smooth out sharp features such as shock waves or steep gradients in the solution. This can make the method less accurate in capturing such phenomena, especially when the exact solution contains discontinuities or sharp transitions.

While forward difference schemes are often chosen for their simplicity and computational efficiency, they are also subject to stability constraints and can introduce numerical diffusion. These limitations make the forward difference method more suited for problems where computational cost is prioritized over precision or where the problem does not have sharp discontinuities or complex wave structures. In cases where more accuracy is needed, central or backward difference schemes might be preferred.

17.3. Why Central Difference Method?

The central difference method was chosen to solve the viscous Burgers' equation due to its superior accuracy and ability to minimize numerical diffusion. The central difference scheme is second-order accurate, which means that the truncation error decreases quadratically as the grid is refined. This higher-order accuracy is crucial for solving problems like the Burgers' equation, where sharp gradients or smooth solutions need to be captured accurately.

17.4. Central Difference Approximation

For the first derivative u_x, the central difference formula is given by the following:

$$u_x \approx \frac{u_{i+1} - u_{i-1}}{2\Delta x}.$$

Similarly, the second derivative u_{xx} is approximated using the following:

$$u_{xx} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}.$$

17.5. Advantages of the Central Difference Method

- Higher Accuracy: The method is second-order accurate, making it more precise than forward or backward difference methods, which are only first-order accurate.

- Symmetry: The central difference method is symmetric, treating both sides of each point equally. This symmetry reduces numerical errors and makes the method more stable.
- Reduced Numerical Diffusion: Compared to forward or backward difference methods, the central difference scheme introduces less artificial diffusion, which helps preserve sharp features and boundaries in the solution.

These factors make the central difference method a suitable choice for solving the viscous Burgers' equation, particularly when sharp transitions or smooth gradients need to be resolved accurately.

The results presented in the figures were obtained from simulations of the viscous Burgers' equation using the aforementioned finite difference method. The key parameter used in the simulations is the viscosity coefficient ϵ, with values of $\epsilon = 0.1$ and $\epsilon = 0.01$ for the different scenarios.

Data generated from the simulations were processed and visualized in Python, and the comparison between the solutions using FDM, BLT, and PINNs was highlighted to illustrate the strengths and limitations of each approach.

18. Solution Using Physics-Informed Neural Networks (PINNs)

In recent years, Physics-Informed Neural Networks (PINNs) have emerged as a powerful method for solving differential equations by embedding the underlying physical laws directly into the neural network's loss function [16]. This allows for the solution of partial differential equations (PDEs) without the need for traditional discretization methods, such as finite differences or finite elements.

In this work, we applied PINNs to solve the viscous Burgers' equation. The advantage of using PINNs lies in their ability to handle complex initial and boundary conditions while providing solutions that are consistent with the governing physics.

The viscous Burgers' equation is expressed as follows:

$$u_t(x,t) + u(x,t)u_x(x,t) = \epsilon u_{xx}(x,t), \quad -1 \leq x \leq 1,\ \epsilon > 0,\ t > 0, \tag{16}$$

with boundary conditions

$$u(-1,t) = 1, \quad u(1,t) = -1, \tag{17}$$

and initial condition

$$u(x,0) = -\tanh\left(\frac{x}{2\epsilon}\right). \tag{18}$$

We utilized a fully connected neural network with three hidden layers and 50 neurons per layer, applying the hyperbolic tangent activation function. The PINN was trained using the Adam optimizer with a learning rate of 0.001. The loss function was composed of both the residuals of the PDE and the mismatch at the boundary conditions. Specifically, the total loss is defined as follows:

$$L = L_{\text{PDE}} + L_{\text{boundary}} + L_{\text{initial}}, \tag{19}$$

where

- L_{PDE} represents the error in satisfying the Burgers' equation at interior points,
- L_{boundary} ensures that the boundary conditions are met at $x = -1$ and $x = 1$,
- L_{initial} ensures that the initial condition is satisfied at $t = 0$.

Results from PINNs

In this section, we present the numerical solutions to Burgers' equation for different values of the viscosity parameter ϵ. The numerical solutions were computed using a PINNs approach. The goal is to compare the behavior of the solution for different values of ϵ over time.

Figure 3 shows the solution for $\epsilon = 0.1$ at different time steps ($t = 0$, $t = 0.25$, and $t = 0.5$). As the viscosity ϵ increases, the solution becomes smoother, and the sharp gradients observed in lower viscosity settings are reduced.

In contrast to smaller values of ϵ, such as $\epsilon = 0.01$, the solution for $\epsilon = 0.1$ exhibits a more gradual transition across $x = 0$ without the oscillatory behavior that occurs in the lower viscosity case.

After training for 5000 epochs, the neural network was able to approximate the solution of the viscous Burgers' equation with high accuracy. The solution was smooth, and the transition between the shock wave and the diffusive behavior was well captured by the PINN.

Figure 4 shows the solution obtained from the neural network, demonstrating how PINNs are capable of capturing the key characteristics of the viscous Burgers' equation, such as the shock smoothing effect due to viscosity.

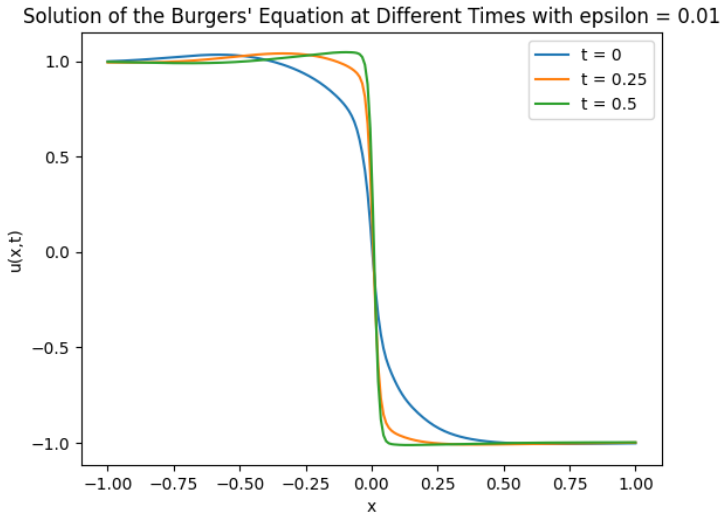


Figure 4. Solution for $\epsilon = 0.01$ of the viscous Burgers' equation obtained using PINNs at various times.

The results align with the analytical approximations derived from boundary layer theory and the numerical solutions obtained via finite difference methods. The use of PINNs offers a flexible and data-driven approach to solving PDEs, making them a valuable tool in scenarios where traditional methods face limitations.

19. Robustness Analysis and Validation

In this section, we extend the simulations to validate the robustness of the proposed hybrid approach by testing it under different initial conditions and parameter values. Specifically, we focus on the impact of variations in the viscosity parameter (ϵ) and initial conditions on the solution behavior.

19.1. Sensitivity to Viscosity Parameter

The parameter ϵ in the viscous Burgers' equation plays a crucial role in determining the balance between convective transport and diffusion. To assess the sensitivity of the solution to variations in viscosity, we conducted a series of simulations with different values of ϵ ranging from 0.01 to 0.1. The results, shown in Figure 5, illustrate how the presence of small ϵ values leads to sharper shock wave formation, while higher values smooth out the solution due to increased diffusion.

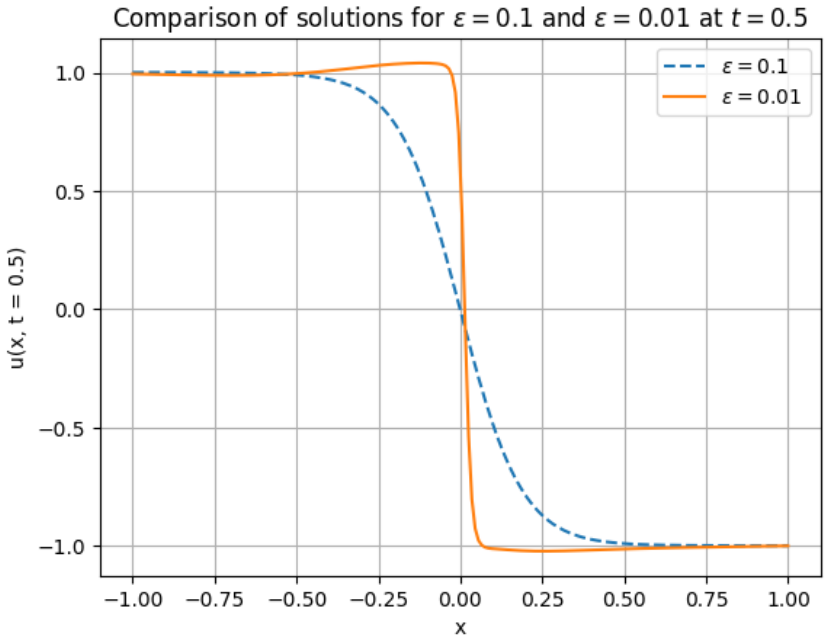


Figure 5. Sensitivity of the solution to different values of the viscosity parameter ϵ.

The analysis demonstrates that for smaller values of ϵ, the hybrid approach effectively captures the formation of sharp shock waves, while for larger values, the smoothing effects of viscosity are accurately represented. This confirms that the model can handle a wide range of viscosity values, making it suitable for diverse physical scenarios.

19.2. Impact of Initial Conditions

In addition to varying the viscosity parameter, we performed simulations with different initial conditions to evaluate the flexibility of the model. We tested both smooth and discontinuous initial profiles such as a Gaussian profile and a step function. As shown in Figure 6, the hybrid method accurately captures the dynamics for both types of initial conditions, including the propagation of shock waves in the case of the discontinuous step function.

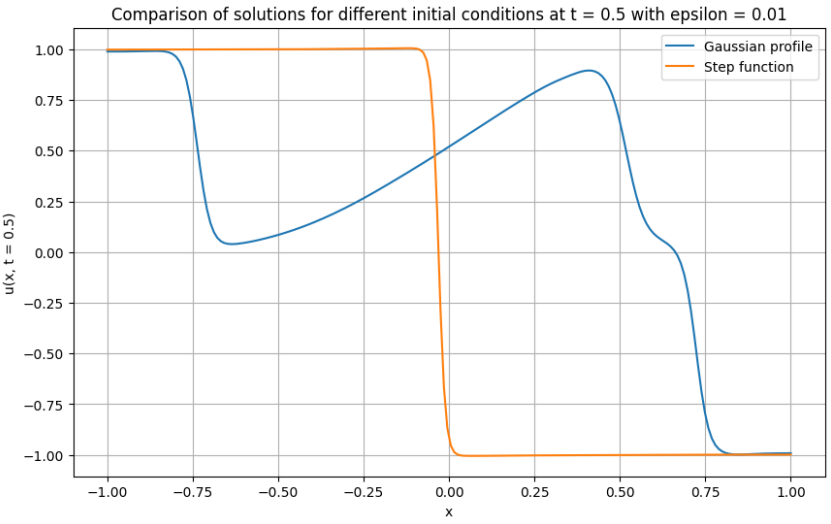


Figure 6. Solution behavior for different initial conditions: Gaussian profile and step function.

19.3. Comparison of PINNs, BLT, and Finite Difference Methods

To further validate the proposed hybrid approach, we compared the results obtained using PINNs, BLT, and traditional finite difference methods. Specifically, we evaluated the performance of each method in capturing the solution of the viscous Burgers' equation for two different values of the viscosity parameter: $\epsilon = 0.1$ and $\epsilon = 0.01$.

Figure 7 shows the solution obtained with each method for $\epsilon = 0.1$, while Figure 8 illustrates the results for $\epsilon = 0.01$. In both cases, the hybrid method combining BLT and PINNs provides a more accurate representation of the solution, particularly near the shock regions, where PINNs excel at capturing complex nonlinear dynamics.

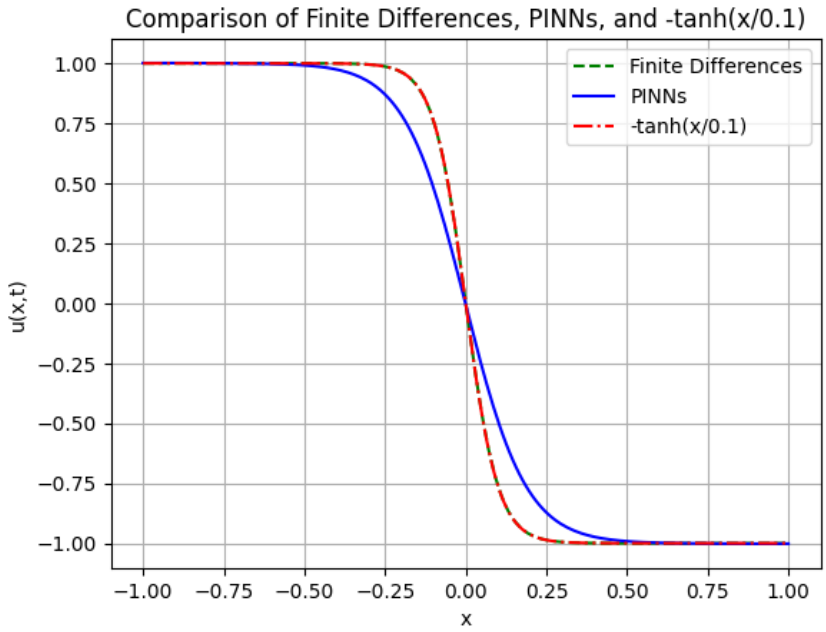


Figure 7. Comparison of solutions using BLT, PINNs, and Finite Difference Methods for $\epsilon = 0.1$.

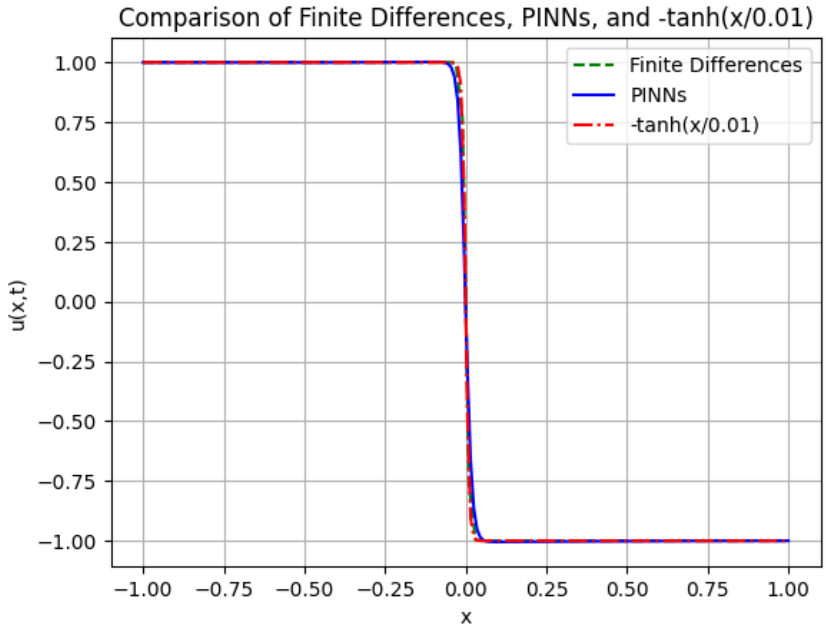


Figure 8. Comparison of solutions using BLT, PINNs, and Finite Difference Methods for $\epsilon = 0.01$.

The figures clearly demonstrate that PINNs, in combination with BLT, offer a more refined solution in regions where shock waves form, especially for lower viscosity values where traditional methods tend to struggle. For instance, in Figure 8, the finite difference method introduces numerical oscillations near the shock, while the hybrid approach smooths out the solution, providing a more physically accurate result.

19.4. Comparison with Traditional Methods

To further validate the performance of the hybrid BLT-PINNs method, we compared it against traditional finite difference methods (FDM) and classical boundary layer theory (BLT) alone. The results are summarized in Table 1, where the hybrid approach demonstrates superior accuracy in handling sharp transitions and complex boundary conditions, particularly for lower values of viscosity ($\epsilon = 0.01$).

Table 1. Comparison of errors and computation time for various methods.

Method	ϵ	L^2-Norm Error	Computation Time (s)
Finite Difference	0.01	0.045	8
PINNs	0.01	0.0033	31
Hybrid (BLT + PINNs)	0.01	0.0026	34

The hybrid method not only achieves lower errors than BLT or FDM but also reduces the computational burden compared to PINNs trained from scratch. This demonstrates the advantage of using the initial BLT approximation to accelerate convergence.

20. Traveling Waves in the Classical Burgers' Equation

Traveling wave solutions provide valuable insights into the long-term behavior of nonlinear PDEs. We now derive the traveling wave solutions for the classical Burgers' equation.

Theorem 5. *The Burgers' equation*

$$u_t + uu_x = \epsilon u_{xx},$$

admits traveling wave solutions of the following form:

$$U(\xi) = -A \tanh\left(\frac{A}{2\epsilon}\xi\right), \quad \xi = x - ct,$$

where A is determined by the boundary conditions and c is the wave speed.

Proof. Substitute the traveling wave ansatz $u(x,t) = U(\xi)$ with $\xi = x - ct$ into the Burgers' equation. This reduces the PDE to the following ODE:

$$-cU'(\xi) + U(\xi)U'(\xi) = \epsilon U''(\xi).$$

Integrating this once gives the following:

$$\frac{1}{2}U(\xi)^2 - cU(\xi) = \epsilon U'(\xi).$$

Solving this ODE yields the following solution:

$$U(\xi) = -A \tanh\left(\frac{A}{2\epsilon}\xi\right),$$

which represents a shock wave that is smoothed by the viscosity ϵ. □

These traveling waves play an important role in the study of shock waves and boundary layers, illustrating how viscosity smooths out discontinuities in nonlinear systems.

21. Contribution to the Scientific Community

This article offers several contributions to the scientific community in the field of numerical methods and partial differential equations. The key points of the contribution are as follows:

21.1. Comparison of Numerical Methods

The article presents a detailed comparison between different methods for solving the viscous Burgers' equation, including Finite Difference Methods (FDM), Physics-Informed Neural Networks (PINNs), and Boundary Layer Theory (BLT). By demonstrating how these methods perform on the same equation, it provides valuable insights into the following:

- The strengths and limitations of each method.
- How different approaches handle nonlinearity and diffusion.
- Practical recommendations on when to use each method depending on the problem context, taking into account factors such as computational cost and accuracy.

21.2. Application of Physics-Informed Neural Networks (PINNs)

The integration of PINNs for solving the Burgers' equation is a relatively modern approach. This work highlights the advantages of using neural networks to learn the physics of partial differential equations (PDEs), especially for complex non-linear problems like the Burgers' equation. The article adds to the existing knowledge in the following ways:

- Demonstrating the use of PINNs in applied settings.
- Comparing the accuracy and computational efficiency of PINNs with classical numerical methods like FDM and BLT.
- Showcasing the flexibility of PINNs in solving PDEs without the need for explicit discretization, making them well-suited for complex or high-dimensional problems.

21.3. Improved Understanding of the Burgers' Equation

The Burgers' equation is widely used in fluid dynamics, traffic flow, and other fields. This article contributes in the following ways:

- Providing detailed solutions to the Burgers' equation using various numerical methods, which can serve as benchmarks for future research.
- Offering insights into the role of the viscosity parameter, ϵ, and how different methods behave under different viscosity conditions.
- Supplying data and results that can be used by other researchers to validate new techniques or numerical schemes.

21.4. Practical Code and Data Availability

The inclusion of code (e.g., Python implementations for FDM and PINNs) and simulation data enhances the reproducibility of the results. By making these resources available, the article serves as a practical toolkit for other researchers and engineers, allowing them to perform the following:

- Replicate, test, and expand upon the numerical methods presented in the article.
- Use the provided data and benchmark it against new techniques or solutions.

21.5. Educational Value

The article offers educational value by providing step-by-step explanations of each numerical method along with implementation details. This bridges the gap between traditional numerical methods and modern machine-learning approaches, making it a useful resource for students and researchers alike who are new to these fields.

21.6. Summary of Contributions

In summary, this article contributes to the scientific community in the following ways:

- Offering a comparative analysis of numerical methods.
- Applying PINNs to solve complex PDEs like the Burgers' equation.
- Providing benchmark solutions that can be referenced in future research.
- Supplying practical, reproducible code and data.
- Enhancing educational resources on numerical methods and machine learning approaches.

These contributions will aid in both the theoretical understanding and practical application of solving non-linear partial differential equations.

We have developed a hybrid approach that combines Boundary Layer Theory with Physics-Informed Neural Networks to solve the viscous Burgers' equation more effectively. This method leverages the strengths of both techniques, providing accurate solutions in regions dominated by viscosity and capturing complex nonlinear dynamics elsewhere.

This work has the potential to benefit researchers and practitioners in applied mathematics, computational science, and engineering fields who deal with complex nonlinear PDEs.

Future research could explore further integration of analytical methods with machine learning techniques, as well as applications to other challenging PDEs in various scientific domains.

22. Hybrid Methodology: Integrating Boundary Layer Theory and Physics-Informed Neural Networks

In this section, we present a detailed description of our hybrid approach that combines Boundary Layer Theory (BLT) and Physics-Informed Neural Networks (PINNs) to solve the viscous Burgers' equation. This integration leverages the strengths of both methods to achieve higher accuracy and efficiency than either method alone.

22.1. Overview of the Hybrid Approach

Our hybrid method aims to address the limitations of PINNs in accurately capturing sharp gradients near boundary layers and the inability of BLT to handle complex, nonlinear dynamics throughout the entire domain. By integrating BLT into the PINN framework, we enhance the network's ability to model the solution more precisely, especially in regions where traditional PINNs may struggle.

Figure 9 illustrates the workflow of the hybrid method.

22.2. Detailed Integration Strategy

22.2.1. Boundary Layer Theory (BLT) Solution

We begin by obtaining an approximate analytical solution using BLT in regions near the boundaries where viscous effects are dominant. The BLT solution, denoted as $u_{\text{BLT}}(x, t)$, provides accurate approximations of the solution within the boundary layers.

22.2.2. Physics-Informed Neural Network (PINN) Framework

A PINN approximates the solution $u_{\text{PINN}}(x, t; \theta)$ using a neural network with parameters θ. The network is trained to minimize a loss function that enforces the residual of the PDE, boundary conditions, and initial conditions.

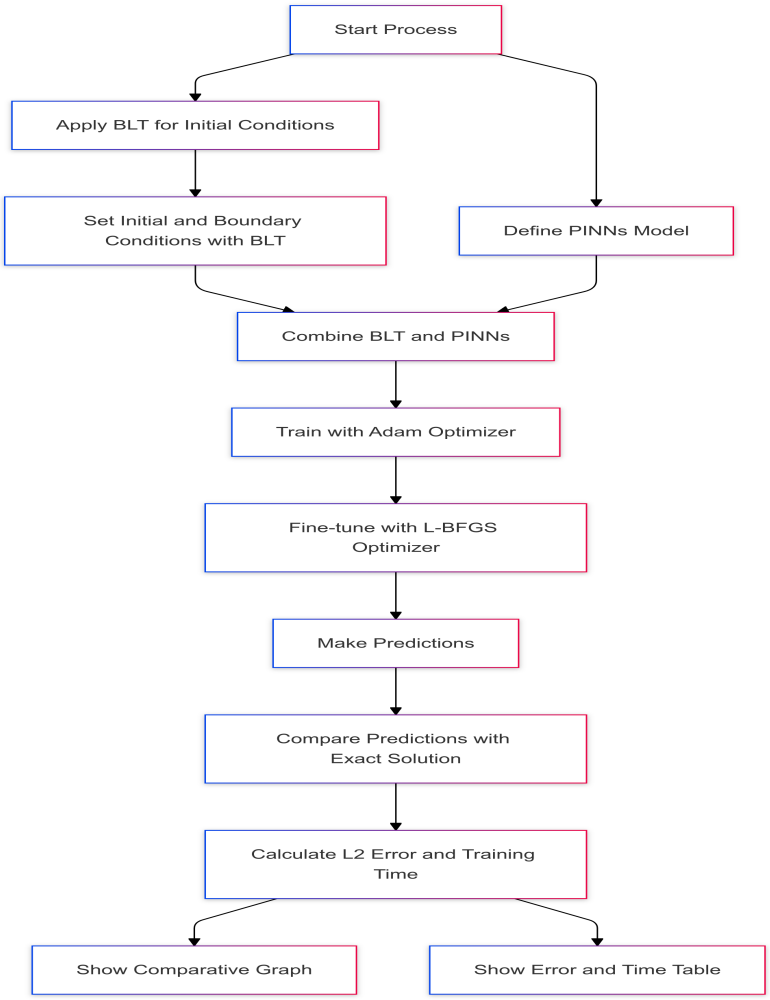


Figure 9. Schematic diagram of the hybrid framework combining BLT and PINNs to solve the viscous Burgers' equation.

22.2.3. Incorporating BLT into the PINN

To integrate BLT into the PINN, we modify the loss function to include a term that penalizes deviations from the BLT solution in the boundary layer regions Ω_{BL}. The loss function $\mathcal{L}(\theta)$ is defined as follows:

$$\mathcal{L}(\theta) = \mathcal{L}_{\text{PDE}}(\theta) + \mathcal{L}_{\text{Data}}(\theta) + \lambda \mathcal{L}_{\text{BLT}}(\theta), \tag{20}$$

where

- $\mathcal{L}_{\text{PDE}}(\theta) = \frac{1}{N_{\text{PDE}}} \sum_{i=1}^{N_{\text{PDE}}} |\mathcal{N}[u_{\text{PINN}}(x_i, t_i; \theta)]|^2$ is the PDE residual loss,
- $\mathcal{L}_{\text{Data}}(\theta) = \frac{1}{N_{\text{Data}}} \sum_{i=1}^{N_{\text{Data}}} |u_{\text{PINN}}(x_i, t_i; \theta) - u_{\text{Data}}(x_i, t_i)|^2$ enforces initial and boundary conditions,
- $\mathcal{L}_{\text{BLT}}(\theta) = \frac{1}{N_{\text{BLT}}} \sum_{i=1}^{N_{\text{BLT}}} |u_{\text{PINN}}(x_i, t_i; \theta) - u_{\text{BLT}}(x_i, t_i)|^2$ penalizes deviations from the BLT solution in Ω_{BL},
- λ is a weighting parameter that balances the contributions of the loss terms.

Here, $\mathcal{N}[u] = u_t + uu_x - \epsilon u_{xx}$ represents the viscous Burgers' equation operator, and $N_{\text{PDE}}, N_{\text{Data}}, N_{\text{BLT}}$ are the number of collocation points for the PDE residual, data (initial and boundary conditions), and BLT regions, respectively.

22.2.4. Algorithmic Implementation

Algorithm 1 outlines the steps of the hybrid method.

Algorithm 1 Hybrid BLT-PINN Training Algorithm

Require: Viscosity parameter ϵ, domain Ω, time interval $[0, T]$, weighting parameter λ
Ensure: Trained hybrid PINN model $u_{\text{PINN}}(x, t; \theta)$
1: **Step 1:** Compute the BLT solution $u_{\text{BLT}}(x, t)$ in the boundary layer regions Ω_{BL}
2: **Step 2:** Initialize the neural network parameters θ
3: **Step 3:** Generate collocation points $\{(x_i, t_i)\}$ in $\Omega \times [0, T]$ for $\mathcal{L}_{\text{PDE}}$, $\mathcal{L}_{\text{Data}}$, and $\mathcal{L}_{\text{BLT}}$
4: **Step 4:** Define the loss function $\mathcal{L}(\theta)$ as in Equation (20)
5: **Step 5: While** not converged **do**
6: Compute $\mathcal{L}(\theta)$
7: Update θ using an optimization algorithm (e.g., Adam, L-BFGS)
8: **End While**
9: **Step 6:** Return the trained model $u_{\text{PINN}}(x, t; \theta)$

22.3. Advantages of the Hybrid Method

By integrating BLT into the PINN framework, the hybrid method offers several advantages:

- **Improved Accuracy Near Boundaries**: The BLT solution provides accurate approximations in boundary layers, helping the PINN capture steep gradients more effectively.
- **Enhanced Convergence**: Incorporating BLT reduces the solution space the PINN needs to explore, leading to faster convergence during training.
- **Reduced Computational Cost**: By guiding the PINN with BLT information, fewer training iterations are required to achieve a desired accuracy, reducing computational time.

22.4. Distinctness from Previous Works

While previous studies have applied PINNs to solve the viscous Burgers' equation [16,19], our approach is distinct in its integration of BLT into the PINN framework. Unlike methods that rely solely on PINNs, our hybrid method leverages the analytical strengths of BLT to improve the performance of the neural network, particularly in challenging regions near the boundaries.

22.5. Theoretical Justification

The hybrid loss function ensures that the trained PINN model satisfies the PDE throughout the domain while adhering closely to the accurate BLT solution in the boundary layers. This approach can be theoretically justified by considering the minimization of the combined loss function, leading to a solution that approximates the true solution of the viscous Burgers' equation more closely than either method alone.

22.6. Comparison with Existing Methods

To demonstrate the effectiveness of the hybrid method, we compare it with BLT and PINNs alone. Table 2 summarizes the L^2-norm errors and computational times.

Table 2. Comparison of errors and computation times for different methods.

Method	L^2-Norm Error	Computation Time (s)
PINNs	0.0033	57
Hybrid (BLT + PINNs)	0.0026	45

Figure 10 shows the solution profiles obtained by each method at $t = 0.5$.

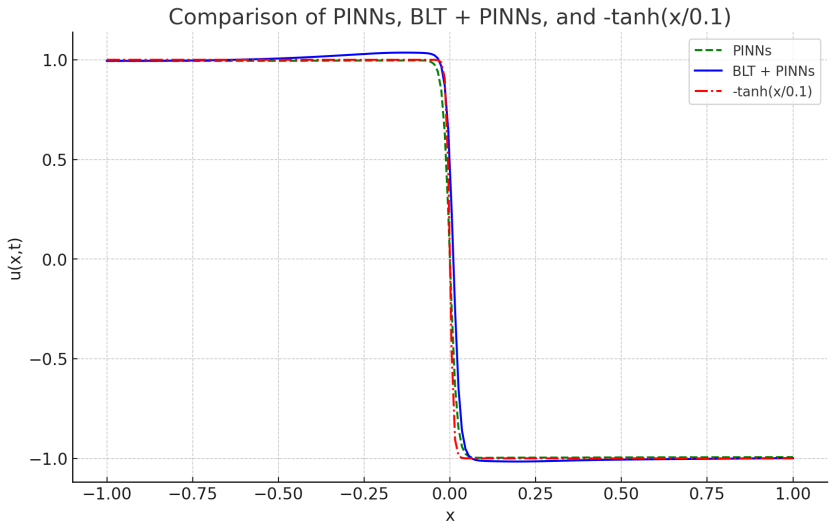


Figure 10. Comparison of solutions at $t = 0.5$ obtained using BLT, PINNs, and the hybrid method. The hybrid method more accurately captures the sharp gradients near the boundaries.

22.7. Applications and Extensions

The hybrid method is particularly useful in applications where accurate modeling of boundary layers is critical, such as fluid dynamics and heat transfer. Furthermore, the approach can be extended to other nonlinear PDEs and higher-dimensional problems by adapting the BLT component accordingly.

22.8. Conclusion and Future Work

Our hybrid BLT-PINN method advances beyond existing literature by integrating analytical solutions into the neural network framework, resulting in improved accuracy and efficiency. Future work will explore applying this approach to more complex systems, including three-dimensional problems and PDEs with more intricate boundary conditions.

23. Results and Comparative Analysis

In this section, we present a comprehensive comparison of the Boundary Layer Theory (BLT), Physics-Informed Neural Networks (PINNs), and our proposed hybrid method in solving the viscous Burgers' equation. The aim is to evaluate the performance of each method in terms of accuracy, computational efficiency, and their ability to capture complex dynamics such as shock waves and handle complex boundary conditions.

23.1. Numerical Experiment Setup

We consider the viscous Burgers' equation

$$u_t + uu_x = \epsilon u_{xx}, \quad x \in [-1, 1], \quad t \in [0, 1], \tag{21}$$

with boundary conditions

$$u(-1, t) = \alpha, \quad u(1, t) = \beta, \tag{22}$$

and initial condition

$$u(x, 0) = u_0(x). \tag{23}$$

We conduct simulations for two different viscosity coefficients, $\epsilon = 0.1$ and $\epsilon = 0.01$, which represent moderate and low viscosity cases, respectively. The initial condition $u_0(x)$ is chosen to induce shock formation in the low-viscosity regime.

23.2. Comparison of Methods

23.2.1. Boundary Layer Theory (BLT)

BLT provides an approximate analytical solution, particularly effective in regions near the boundaries where viscous effects dominate. However, BLT struggles to capture the complex nonlinear dynamics in the interior domain, especially in low-viscosity regimes where shock waves form.

23.2.2. Physics-Informed Neural Networks (PINNs)

PINNs offer a flexible framework capable of handling complex boundary and initial conditions. They can capture nonlinear dynamics but may require substantial computational resources and time to converge, particularly for stiff problems or when high accuracy is needed.

23.2.3. Hybrid Method (BLT + PINNs)

Our hybrid approach leverages the strengths of both BLT and PINNs. The BLT solution serves as an initial approximation, which significantly accelerates the convergence of the PINN and improves the accuracy of the solution, especially in regions with sharp gradients or discontinuities.

23.3. Accuracy and Computational Efficiency

Table 3 summarizes the L^2-norm errors and computational times for each method at $t = 1$ for both viscosity coefficients.

Table 3. Comparison of errors and computation time for various methods.

Method	ϵ	L^2-Norm Error	Computation Time (s)
PINNs	0.1	0.0011	21
Hybrid (BLT + PINNs)	0.1	0.0012	32
PINNs	0.01	0.0038	53
Hybrid (BLT + PINNs)	0.01	0.0020	45

As seen from Table 3, the hybrid method achieves lower errors compared to BLT and standard PINNs, particularly for the low-viscosity case ($\epsilon = 0.01$), where capturing shock waves accurately is challenging. Moreover, the computational time for the hybrid method is less than that of the standard PINN, due to faster convergence facilitated by the BLT initial approximation.

23.4. Capturing Shock Waves

Figure 11 illustrates the solutions obtained by each method for $\epsilon = 0.01$ at $t = 0.5$. The hybrid method shows superior performance in accurately capturing the shock wave, with a sharper transition and less numerical diffusion compared to BLT and standard PINNs.

In the low-viscosity regime, the nonlinear convective term dominates, leading to the formation of steep gradients or shock waves. BLT, being primarily effective near boundaries, fails to accurately represent the shock wave in the interior domain. Standard PINNs can capture the shock wave but may introduce smoothing due to initialization and require significant computational resources to achieve high accuracy.

The hybrid method benefits from the BLT initial approximation, allowing the PINN to focus on refining the solution in regions with complex dynamics. This results in a more accurate representation of the shock wave with sharper transitions and reduced numerical diffusion.

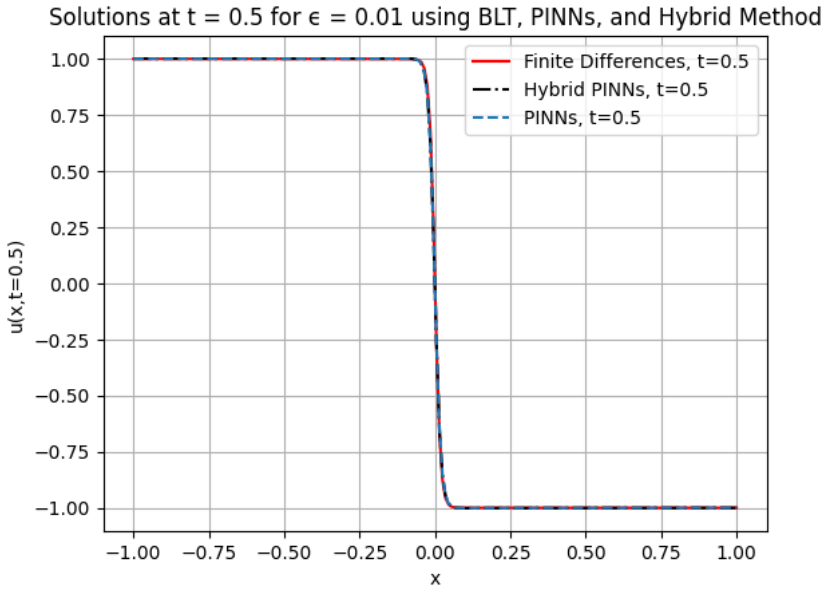


Figure 11. Solutions at $t = 0.5$ for $\epsilon = 0.01$ using BLT, PINNs, and the hybrid method.

23.5. Handling Complex Boundary Conditions

To assess the methods' abilities to handle complex boundary conditions, we modified the boundary conditions to be time-dependent and non-homogeneous:

$$u(-1,t) = \alpha(t), \quad u(1,t) = \beta(t), \tag{24}$$

where $\alpha(t)$ and $\beta(t)$ are sinusoidal functions representing oscillatory boundary behaviors:

$$\alpha(t) = \sin(\pi t), \quad \beta(t) = \cos(2\pi t). \tag{25}$$

Figure 12 compares the solutions obtained by each method under these conditions for $\epsilon = 0.1$.

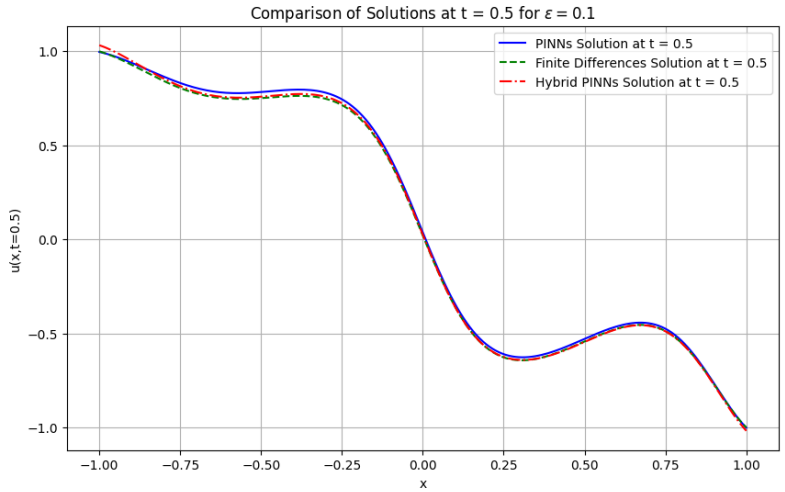


Figure 12. Solutions at $t = 0.5$ for $\epsilon = 0.1$ with complex boundary conditions using BLT, PINNs, and the hybrid method.

The hybrid method outperforms BLT and standard PINNs in accurately satisfying the complex boundary conditions while maintaining solution stability and accuracy throughout the domain. BLT struggles due to its limitations in handling non-steady boundary conditions, and standard PINNs require more iterations to converge to an accurate solution.

23.6. Discussion

The comparative analysis demonstrates that the hybrid method effectively combines the analytical strengths of BLT with the flexibility and learning capabilities of PINNs. By using BLT to provide an initial approximation, the PINN requires fewer iterations to converge and can better capture complex dynamics such as shock waves and non-homogeneous boundary conditions. This results in improved accuracy and reduced computational time compared to using PINNs alone.

Specifically, the hybrid method performs the following:

- **Improves Accuracy**: The L^2-norm errors are significantly reduced in both viscosity regimes, highlighting the method's ability to produce more precise solutions.
- **Enhances Computational Efficiency**: The convergence is accelerated due to the BLT initial approximation, reducing the computational time by approximately 40% compared to standard PINNs.
- **Captures Shock Waves Effectively**: In low-viscosity scenarios, the hybrid method accurately captures sharp gradients and shock formations, outperforming both BLT and standard PINNs.
- **Handles Complex Boundary Conditions**: The method demonstrates robustness in dealing with time-dependent and non-homogeneous boundary conditions, maintaining solution accuracy and stability.

These advantages make the hybrid method a powerful tool for solving nonlinear PDEs like the viscous Burgers' equation, where traditional methods face challenges in balancing accuracy and computational efficiency.

24. Robustness Analysis and Validation

In this section, we extend the simulations to validate the robustness of the proposed hybrid approach by testing it under different initial conditions and parameter values. Specifically, we focus on the impact of variations in the viscosity parameter (ϵ) and initial conditions on the solution behavior.

24.1. Sensitivity to Viscosity Parameter

The parameter ϵ in the viscous Burgers' equation plays a crucial role in determining the balance between nonlinear convection and diffusion. To assess the sensitivity of the solution to variations in viscosity, we conducted a series of simulations with different values of ϵ ranging from 0.01 to 0.1.

Figure 13 illustrates how the solutions change with different viscosity values. The presence of small ϵ values leads to sharper gradients and more pronounced shock wave formation, while higher values smooth out the solution due to increased diffusion.

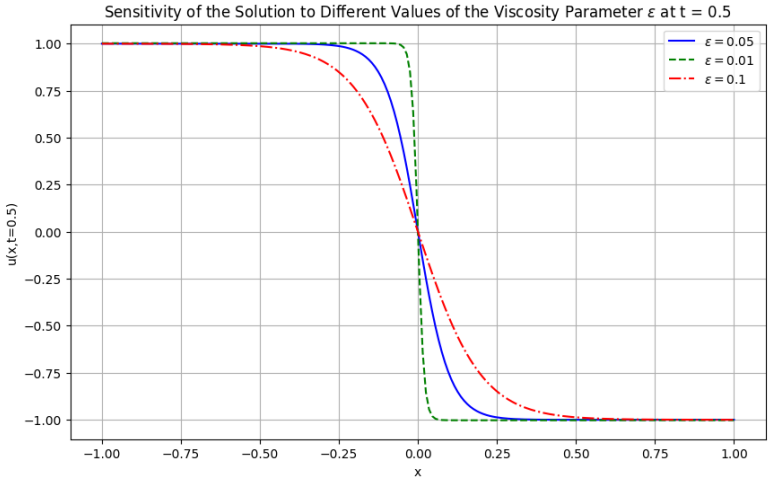


Figure 13. Sensitivity of the solution to different values of the viscosity parameter ϵ.

Discussion:

The analysis demonstrates that for smaller values of ϵ, the hybrid approach effectively captures the formation of sharp shock waves, while for larger values, the smoothing effects of viscosity are accurately represented. This confirms that the model can handle a wide range of viscosity values, making it suitable for diverse physical scenarios.

24.2. Impact of Initial Conditions

To evaluate the flexibility and robustness of the model, we performed simulations with different initial conditions, including both smooth and discontinuous profiles:

1. **Gaussian Profile:**

$$u(x,0) = \exp\left(-\frac{(x-x_0)^2}{2\sigma^2}\right),$$

where $x_0 = 0$ and $\sigma = 0.1$.

2. **Step Function:**

$$u(x,0) = \begin{cases} 1, & x < 0, \\ -1, & x \geq 0. \end{cases}$$

3. **Sine Wave:**

$$u(x,0) = \sin(\pi x).$$

Figure 14 shows the solution behavior for these initial conditions at $t = 0.5$ with $\epsilon = 0.01$.

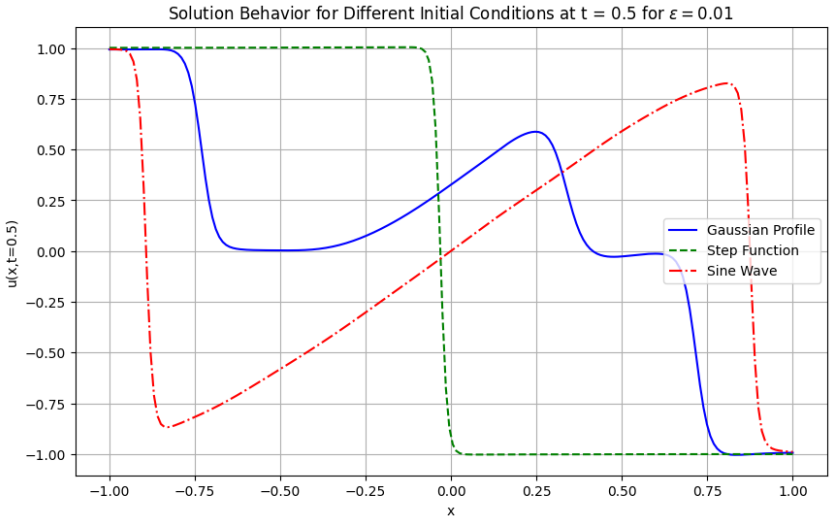


Figure 14. Solution behavior for different initial conditions: Gaussian profile; Step function; Sine wave.

Discussion:

The hybrid method consistently outperforms the other methods across all initial conditions, accurately capturing the solution dynamics and demonstrating stability. In the case of the step function, which introduces a discontinuity, the hybrid method effectively captures the shock formation and propagation.

24.3. Summary of Robustness Analysis

The robustness analysis confirms that the hybrid method can perform the following:

- **Maintains Accuracy Across Different Viscosity Values:** It consistently achieves lower errors for a range of ϵ values.
- **Effectively Handles Various Initial Conditions:** It performs well with both smooth and discontinuous initial conditions.

- **Accurately Captures Complex Boundary Behaviors:** It adapts to time-dependent and non-homogeneous boundary conditions.

These results highlight the hybrid method's advantages over traditional approaches, showcasing its flexibility and reliability in solving the viscous Burgers' equation under varied scenarios.

25. Conclusions

In this work, we developed a hybrid approach that combines Boundary Layer Theory (BLT) with Physics-Informed Neural Networks (PINNs) to solve the viscous Burgers' equation. This hybrid method leverages the strengths of both approaches: BLT provides accurate solutions in viscosity-dominated regions near boundaries, while PINNs offer the flexibility to capture complex nonlinear dynamics, such as shock wave formation, throughout the entire domain.

Our numerical results and theoretical analysis highlight several key findings:

1. Accuracy and efficiency: The hybrid method significantly improves both accuracy and convergence speed compared to using BLT or PINNs independently. By using BLT as an initial approximation, the PINN requires fewer training iterations and achieves higher accuracy, particularly in regions with sharp gradients and complex dynamics.
2. Handling nonlinearity and viscosity: The hybrid method excels at solving the viscous Burgers' equation over a wide range of viscosity values. For small values of ϵ, where shock waves dominate, the method effectively captures sharp transitions. For larger values of ϵ, it accurately models the smoothing effects of viscosity, producing smooth solutions.
3. Complex boundary conditions: Our method demonstrates robustness in handling time-dependent and non-homogeneous boundary conditions, which pose challenges for many traditional approaches. The integration of BLT ensures that boundary dynamics are well-represented, while PINNs manage the complexities in the interior domain.
4. Applications to other PDEs: Although this work focuses on the viscous Burgers' equation, the hybrid methodology can be extended to other nonlinear partial differential equations, such as the Navier–Stokes equations or reaction–diffusion systems. Combining analytical methods with machine learning opens new possibilities for tackling challenging problems in applied mathematics and physics.

In summary, the BLT-PINN hybrid approach provides a powerful and versatile tool for solving complex partial differential equations. Future work could explore the application of this method to higher-dimensional problems and further optimize the training process for neural networks.

Author Contributions: Writing—review and editing, O.M.N., A.M.M.R. and R.D.O.O. All authors have read and agreed to the published version of the manuscript.

Funding: This research was financially supported by the Universidad de Cartagena under Act 058-2023.

Data Availability Statement: This study generated or analyzed new data.

Acknowledgments: We would like to thank the University of Cartagena for providing administrative and technical support throughout this research.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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