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Ulam–Hyers and Generalized Ulam–Hyers Stability of Fractional Differential Equations with Deviating Arguments

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Abstract: In this paper, we study the initial value problem for the fractional differential equation with multiple deviating arguments. By using Krasnoselskii's fixed point theorem, the conditions of solvability of the problem are obtained. Furthermore, we establish Ulam–Hyers and generalized Ulam–Hyers stability of the fractional functional differential problem. Finally, two examples are presented to illustrate our results, one is with a pantograph-type equation and the other is numerical.

Keywords: Caputo derivative; Krasnoselskii's fixed point theorem; solvability; UH stability; GUH stability

MSC: 26A33; 34A08; 34K20

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Citation: Dilna, N.; Fekete, G.; Langerová, M.; Tóth, B. Ulam–Hyers and Generalized Ulam–Hyers Stability of Fractional Differential Equations with Deviating Arguments. *Mathematics* **2024**, *12*, 3418. https://doi.org/10.3390/math12213418

Academic Editor: Mieczysław Cichoń

Received: 10 October 2024
Revised: 25 October 2024
Accepted: 28 October 2024
Published: 31 October 2024

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1. Introduction

The aim of this paper is to establish Ulam–Hyers (UH) stability and generalized Ulam–Hyers (GUH) stability of the initial value problem for the fractional differential equation (FDE) with multiple deviating arguments such that the equation consists of linear and nonlinear parts. To achieve this goal, first, we have to prove that the problem is solvable. We obtain sufficient conditions for the existence of a solution of the problem by using Krasnoselskii's fixed point theorem (see [1]).

Differential equations of integer order with deviating arguments have been studied in [2–5]. Solvability conditions for fractional functional differential equations (FFDEs) have been investigated in many papers, e.g., there are some results on solvability for linear FFDEs that one can find in [6–8], and for general nonlinear FFDEs, results can be found in [9–13]. UH stability (firstly considered in [14,15]) or GUH stability of fractional differential equations are widely studied in modern investigation (see [16–18]). Ulam–Hyers–Rasias (UHR) stability was considered in [19–21], and regarding results for practical UHR stability for nonlinear equations, one can refer to [19,22]. UH and GUH stability conditions for a pantograph model (see [23]) were investigated in [24–27].

2. Problem Formulation

We consider FDE with multiple deviating arguments

$$D_0^q x(t) = \sum_{i=1}^{m} \Big(p_i(t)x(\tau_i(t)) - z_i(t)x(\nu_i(t)) \Big) + \mathcal{F}\big(t, x(\omega_1(t)), x(\omega_2(t)), \ldots, x(\omega_n(t))\big) + f(t), \quad t \in [0,T], \tag{1}$$

with the initial condition

$$x(0) = c, \tag{2}$$

where D_0^q is the Caputo fractional derivative of order q, $q \in (0,1)$, $c \in \mathbb{R}$, $0 < T < +\infty$, p_i, z_i, $f \in L^\infty([0,T],\mathbb{R})$, $i = 1,2,\dots,m$, and τ_i, ν_i, $\omega_j : [0,T] \to [0,T]$, $j = 1,2,\dots,n$, are measurable functions, and $\mathcal{F} : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is an L^∞ - Carathéodory function, $m, n \in \mathbb{N}$.

We understand a solution of FFDE in the sense that is traditional for the general theory of functional differential equations [28].

Definition 1. *The solution of the fractional functional differential problem (FFDP)* (1), (2) *is an absolutely continuous function* $x : [0,T] \to \mathbb{R}$ *that satisfies equality* (1) *for almost all* $t \in [0,T]$ *and possesses the property* (2).

We introduce notations, definitions and auxiliary propositions in Sections 3 and 4; the conditions for the solvability of FFDP are established in Section 5; the result with the theorem on UH and GUH stability is in Section 6; and in Section 7, two pantograph-type equations as examples are given to illustrate the mentioned results.

3. Notations and Definitions

Throughout the paper, we use the following notation:

- $\mathbb{R} = (-\infty, +\infty)$; $\mathbb{R}_+ = [0, +\infty)$; $\mathbb{N} = \{1,2,3,\dots\}$;
- $L^\infty([0,T],\mathbb{R})$ is the Banach space of all Lebesgue integrable functions $x : [0,T] \to \mathbb{R}$ with the norm

$$\|x\|_\infty = \operatorname*{ess\,sup}_{t\in[0,T]} |x(t)|;$$

- $\mathcal{W}^{1,\infty}([0,T],\mathbb{R})$ is the Sobolev space of all functions $x : [0,T] \to \mathbb{R}$ that are absolutely continuous, having essentially bounded first derivatives on $[0,T]$, with the norm

$$\|x\| = \|x\|_\infty + \|x'\|_\infty.$$

Definition 2 ([29])**.** *The function* $\mathcal{F}$ *satisfies* L^∞ *- Carathéodory conditions on* $[0,T] \times \mathbb{R}^n$ *if*

- $\mathcal{F}(t,\cdot,\dots,\cdot) : \mathbb{R}^n \to \mathbb{R}$ *is continuous for a. e.* $t \in [0,T]$;
- $\mathcal{F}(\cdot, x_1, x_2, \dots, x_n) : [0,T] \to \mathbb{R}$ *is measurable for all* $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$;
- *For each compact set* $\mathcal{K} \subset \mathbb{R}^n$, *there exists a function* $\eta \in L^\infty([0,T],\mathbb{R})$, $\eta \geq 0$, *such that*

$$|\mathcal{F}(t, x_1, x_2, \dots, x_n)| \leq \eta(t)$$

for all $(x_1, x_2, \dots, x_n) \in \mathcal{K}$ *and a. e.* $t \in [0,T]$.

Definition 3 (§2.1 and §2.4 in [30])**.**

- *For* $x(t) \in \mathcal{W}^{1,\infty}([0,T],\mathbb{R})$, *the Caputo fractional derivative* $D_0^q x(t)$ *of order* q, $q \in (0,1)$, *exists almost everywhere on* $[0,T]$ *and*

$$D_0^q x(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} x'(s) ds,$$

where $\Gamma(q) : [0,\infty) \to \mathbb{R}$ *is the Gamma function*

$$\Gamma(q) := \int_0^\infty t^{q-1} e^{-t} dt.$$

- *For* $x(t) \in L^\infty([0,T],\mathbb{R})$, *the fractional integral* $\mathcal{I}_{0t}^q$ *of order* q, $q \in (0,1)$, *is defined by*

$$\mathcal{I}_{0t}^q x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} x(s) ds, \quad t \in [0,T].$$

Definition 4 ([14,15]). *The problem* (1), (2) *is UH-stable if there exists a constant* $\kappa_{\mathcal{F}} > 0$ *such that for arbitrary* $\varepsilon > 0$, *and for each solution* $u \in \mathcal{W}^{1,\infty}([0,T],\mathbb{R})$ *of the inequality*

$$\left| D_0^q u(t) - \sum_{i=1}^{m} \Big(p_i(t)u(\tau_i(t)) - z_i(t)u(\nu_i(t)) \Big) \right.$$
$$\left. - \mathcal{F}\big(t, u(\omega_1(t)), u(\omega_2(t)), \ldots, u(\omega_n(t))\big) - f(t) \right| \le \varepsilon, \quad t \in [0,T], \tag{3}$$

there exists a solution $x \in \mathcal{W}^{1,\infty}([0,T],\mathbb{R})$ *of the problem* (1), (2) *that satisfies*

$$|u(t) - x(t)| \le \kappa_{\mathcal{F}} \varepsilon.$$

Additionally, if there exists a nondecreasing function $F : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $F(0) = 0$ *such that*

$$|u(t) - x(t)| \le \kappa_{\mathcal{F}} F(\varepsilon), \quad t \in [0,T],$$

then the problem (1), (2) *is GUH-stable.*

In other words, UH stability of a problem is its property of having an exact solution close to an approximate solution.

Definition 5 ([20]). *The problem* (1), (2) *is UHR-stable with respect to* $\phi \in \mathcal{W}^{1,\infty}([0,T],\mathbb{R}_+)$ *if there exists a constant* $\kappa_{\mathcal{F}} > 0$ *such that for arbitrary* $\varepsilon > 0$, *and for each solution* $u \in \mathcal{W}^{1,\infty}([0,T],\mathbb{R})$ *of the inequality*

$$\left| D_0^q u(t) - \sum_{i=1}^{m} \Big(p_i(t)u(\tau_i(t)) - z_i(t)u(\nu_i(t)) \Big) \right.$$
$$\left. + \mathcal{F}\big(t, u(\omega_1(t)), u(\omega_2(t)), \ldots, u(\omega_n(t))\big) - f(t) \right| \le \varepsilon\phi(t), \quad t \in [0,T],$$

there exists a solution $x \in \mathcal{W}^{1,\infty}([0,T],\mathbb{R})$ *of* (1), (2) *that satisfies*

$$|u(t) - x(t)| \le \kappa_{\mathcal{F}} \varepsilon \phi(t).$$

4. Auxiliary Propositions

In our investigation, we use the following statements.

Lemma 1 (Lemmas 2.21 and 2.22, [30]). *The next properties hold almost everywhere on* $[0,T]$:

(i) Suppose that $0 < q < 1$ *and* $x(t) \in L^\infty([0,T],\mathbb{R})$, *then*

$$D_0^q \mathcal{I}_{0t}^q x(t) = x(t);$$

(ii) Suppose that $0 < q < 1$ *and* $x(t) \in \mathcal{W}^{1,\infty}([0,T],\mathbb{R})$, *then*

$$\mathcal{I}_{0t}^q D_0^q x(t) = x(t) - x(0).$$

Lemma 2. *The nonhomogeneous initial value problem* (1), (2) *is equivalent to the fractional integral equation*

$$\begin{aligned} x(t) = c &+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \sum_{i=1}^{m} (p_i(s)x(\tau_i(s)) - z_i(s)x(\nu_i(s)))ds \\ &+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \mathcal{F}(s, x(\omega_1(s)), x(\omega_2(s)), \ldots, x(\omega_n(s)))ds \\ &+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds, \end{aligned} \tag{4}$$

where $t \in [0,T]$ *and* $x(t) \in \mathcal{W}^{1,\infty}([0,T],\mathbb{R})$.

Proof. Suppose that $x \in \mathcal{W}^{1,\infty}([0,T],\mathbb{R})$. We apply the fractional integral $\mathcal{I}_{0t}^q$ on both sides of Equation (1)

$$\begin{aligned} \mathcal{I}_{0t}^q D_0^q(x(t)) = \mathcal{I}_{0t}^q \Big(&\sum_{i=1}^{m} \big(p_i(t)x(\tau_i(t)) - z_i(t)x(\nu_i(t))\big) \\ &+ \mathcal{F}\big(t, x(\omega_1(t)), x(\omega_2(t)), \ldots, x(\omega_n(t))\big) + f(t)\Big). \end{aligned}$$

Using Lemma 1 (ii), we obtain

$$\begin{aligned} x(t) - x(0) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \Big(&\sum_{i=1}^{m} \big(p_i(s)x(\tau_i(s)) - z_i(t)x(\nu_i(s))\big) \\ &+ \mathcal{F}\big(s, x(\omega_1(s)), x(\omega_2(s)), \ldots, x(\omega_n(s))\big) + f(s)\Big)ds, \quad t \in [0,T]. \end{aligned}$$

On the other side, applying the fractional derivative D_0^q on both sides of (4) and taking Lemma 1 (i) into account, we obtain

$$\begin{aligned} D_0^q x(t) =& D_0^q \Bigg(c + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \sum_{i=1}^{m} (p_i(s)x(\tau_i(s)) - z_i(s)x(\nu_i(s)))ds \\ &+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \mathcal{F}(s, x(\omega_1(s)), x(\omega_2(s)), \ldots, x(\omega_n(s)))ds \\ &+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds \Bigg) \\ =& D_0^q \Big(\mathcal{I}_{0t}^q \Big(\sum_{i=1}^{m} \big(p_i(s)x(\tau_i(s)) - z_i(s)x(\nu_i(s))\big) \\ &+ \mathcal{F}(s, x(\omega_1(s)), x(\omega_2(s)), \ldots, x(\omega_n(s))) + f(s)\Big)\Big) \\ =& \sum_{i=1}^{m} \big(p_i(t)x(\tau_i(t)) - z_i(t)x(\nu_i(t))\big) \\ &+ \mathcal{F}(t, x(\omega_1(t)), x(\omega_2(t)), \ldots, x(\omega_n(t)) + f(t). \end{aligned}$$

□

Proposition 1 (Krasnoselskii's fixed point theorem, Chapter III, p. 148, [1])**.** *Assume that* S *is a closed, convex, bounded subset of a Banach space* E *and suppose that* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *are operators on* S *satisfying the following conditions:*

1. $\mathcal{B}_1(x) + \mathcal{B}_2(y) \in S$, *for all* $x, y \in S$;
2. $\mathcal{B}_1$ *is continuous and compact;*
3. $\mathcal{B}_2$ *is a contraction.*

Then, there exists $x \in S$ such that $\mathcal{B}_1(x) + \mathcal{B}_2(x) = x$.

5. Solvability Conditions

In order to establish UH and GUH stability of the problem (1), (2), first, we need to prove the existence of a solution of the problem. For this purpose, we use Krasnoselskii's fixed point theorem.

Let us set the operators $\mathcal{B}_1, \mathcal{B}_2 : \mathcal{W}^{1,\infty}([0,T],\mathbb{R}) \to \mathcal{W}^{1,\infty}([0,T],\mathbb{R})$ as

$$\mathcal{B}_1(x)(t) := \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left(\sum_{i=1}^{m} \left(p_i(s)x(\tau_i(s)) - z_i(s)x(\nu_i(s)) \right) + f(s) \right) ds + c, \quad (5)$$

$$\mathcal{B}_2(x)(t) := \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \mathcal{F}\big(s, x(\omega_1(s)), x(\omega_2(s)), \ldots, x(\omega_n(s))\big) ds, \quad (6)$$

where $t \in [0,T]$.

Remark 1. *In view of Lemma 2 and Formulas (5) and (6), we have that the problem (1), (2) is equivalent to the equation*

$$x(t) = \mathcal{B}_1(x)(t) + \mathcal{B}_2(x)(t), \quad t \in [0,T]. \quad (7)$$

Remark 2. *By applying Leibnitz's integral rule, the following relations are true for $t \in [0,T]$:*

$$\frac{d}{dt} \int_0^t (t-s)^{q-1} \phi(s) ds = (q-1) \int_0^t (t-s)^{q-2} \phi(s) ds \quad \phi \in L^\infty([0,T],\mathbb{R}); \quad (8)$$

$$\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds + \frac{1}{\Gamma(q)} \frac{d}{dt} \int_0^t (t-s)^{q-1} ds = \frac{t^q}{\Gamma(q+1)} + \frac{t^{q-1}}{\Gamma(q)} = \frac{t^{q-1}(t+q)}{\Gamma(q+1)}. \quad (9)$$

Theorem 1. *Assume that there exist constants $M_p < \infty$, $M_z < \infty$, $M_{\mathcal{F}} < \infty$, $M_f < \infty$, $0 < L_j < \infty$, $j = 1, \ldots n$, such that the conditions*

$$\sum_{i=1}^{m} \|p_i(t)\|_\infty = M_p; \quad \sum_{i=1}^{m} \|z_i(t)\|_\infty = M_z; \quad (10)$$

$$\|\mathcal{F}(t,0,0,\ldots,0)\|_\infty = M_{\mathcal{F}}; \quad (11)$$

$$\|f(t)\|_\infty = M_f; \quad (12)$$

$$\left| \mathcal{F}\Big(\cdot, x_1, x_2, \ldots, x_n\Big) - \mathcal{F}\Big(\cdot, y_1, y_2, \ldots, y_n\Big) \right| \le \sum_{j=1}^{n} L_j |x_j - y_j| \quad (13)$$

are fulfilled for any x_j and $y_j \in \mathcal{W}^{1,\infty}([0,T],\mathbb{R})$, $j = 1,2,\ldots,n$. Moreover, suppose that there exists $\kappa \in (0,1)$ such that

$$\kappa := \kappa_1 + \kappa_2, \quad \kappa_1 = \frac{T^{q-1}(T+q)}{\Gamma(q+1)}(M_p + M_z), \quad \kappa_2 = \frac{T^{q-1}(T+q)}{\Gamma(q+1)} \sum_{j=1}^{n} L_j. \quad (14)$$

Then, FFDP (1), (2) has a solution.

Proof. To prove Theorem 1, we consider the operators $\mathcal{B}_1, \mathcal{B}_2 : \mathcal{W}^{1,\infty}([0,T],\mathbb{R}) \to \mathcal{W}^{1,\infty}([0,T],\mathbb{R})$ defined by (5) and (6).

We define the set

$$X_r = \{x \in \mathcal{W}^{1,\infty}([0,T],\mathbb{R}) : \|x\| \le r\}$$

with $r > 0$ such that

$$\frac{|c| + \frac{T^{q-1}(T+q)}{\Gamma(q+1)}(M_{\mathcal{F}} + M_f)}{1-\kappa} \le r.$$

Obviously, X_r is nonempty, bounded, convex and closed.

Let us prove that the operators $\mathcal{B}_1$ and $\mathcal{B}_2$ satisfy the assumptions of Proposition 1. The proof consists of three steps.

1. Let us prove that $\mathcal{B}_1(x) + \mathcal{B}_2(y) \in X_r$ for all $x, y \in X_r$.
Using (8) –(14), for any $x, y \in X_r$ and $t \in [0, T]$, we have

$$\begin{aligned}
&\|\mathcal{B}_1(x)(t) + \mathcal{B}_2(y)(t)\| \\
&\quad = \Bigg\| c + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \Bigg(\sum_{i=1}^{m} \Big(p_i(s)x(\tau_i(s)) - z_i(s)x(\nu_i(s)) \Big) + f(s) \\
&\quad + \mathcal{F}\big(s, 0, 0, \ldots, 0\big) \\
&\quad + \mathcal{F}\big(s, y(\omega_1(s)), y(\omega_2(s)), \ldots, y(\omega_n(s))\big) - \mathcal{F}\big(s, 0, 0, \ldots, 0\big) \Bigg) ds \Bigg\|_\infty \\
&\quad + \Bigg\| \frac{1}{\Gamma(q)} \frac{d}{dt} \int_0^t (t-s)^{q-1} \Bigg(\sum_{i=1}^{m} \Big(p_i(s)x(\tau_i(s)) - z_i(s)x(\nu_i(s)) \Big) + f(s) \\
&\quad + \mathcal{F}\big(s, 0, 0, \ldots, 0\big) \\
&\quad + \mathcal{F}\big(s, y(\omega_1(s)), y(\omega_2(s)), \ldots, y(\omega_n(s))\big) - \mathcal{F}\big(s, 0, 0, \ldots, 0\big) \Bigg) ds \Bigg\|_\infty \\
&\quad \leq |c| + \Big((M_p + M_z)\|x\|_\infty + (M_f + M_{\mathcal{F}}) \\
&\quad + \sum_{j=1}^{n} L_j \|y\|_\infty \Big) \frac{1}{\Gamma(q)} \int_0^T (t-s)^{q-1} ds \\
&\quad + \Big((M_p + M_z)\|x\|_\infty + (M_f + M_{\mathcal{F}}) + \sum_{j=1}^{n} L_j \|y\|_\infty \Big) \frac{q-1}{\Gamma(q)} \int_0^T (t-s)^{q-2} ds \\
&\quad \leq |c| + \frac{T^{q-1}(T+q)}{\Gamma(q+1)} \Big(M_f + M_{\mathcal{F}} \Big) + \frac{T^{q-1}(T+q)}{\Gamma(q+1)} (M_p + M_z)\|x\| \\
&\quad + \frac{T^{q-1}(T+q)}{\Gamma(q+1)} \sum_{j=1}^{n} L_j \|y\| \\
&\quad \leq |c| + \frac{T^{q-1}(T+q)}{\Gamma(q+1)} \Big(M_f + M_{\mathcal{F}} \Big) + \frac{T^{q-1}(T+q)}{\Gamma(q+1)} \left(M_p + M_z + \sum_{j=1}^{n} L_j \right) r \\
&\quad \leq (1-\kappa)r + \kappa r = r.
\end{aligned}$$

This leads to $\|\mathcal{B}_1(x)(t) + \mathcal{B}_2(y)(t)\| \leq r$ for all $x, y \in X_r$. Thus,

$$\mathcal{B}_1(x) + \mathcal{B}_2(y) \in X_r.$$

2. Let us prove that $\mathcal{B}_1$ is continuous on X_r.
Let $\{x_k\}$ be a sequence such that $x_k \to x$ in X_r as $k \to \infty$. Then, for every $t \in [0, T]$, we have

$$
\begin{aligned}
&\|\mathcal{B}_1(x_k)(t) - \mathcal{B}_1(x)(t)\| \\
&\quad = \left\| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left(\sum_{i=1}^{m} \Big(p_i(t) x_k(\tau_i(s)) - z_i(s) x_k(\nu_i(s)) \Big) \right. \right. \\
&\quad \left. \left. - \sum_{i=1}^{m} \Big(p_i(t) x(\tau_i(s)) - z_i(s) x(\nu_i(s)) \Big) \right) ds \right\|_\infty \\
&\quad + \left\| \frac{1}{\Gamma(q)} \frac{d}{dt} \int_0^t (t-s)^{q-1} \left(\sum_{i=1}^{m} \Big(p_i(t) x_k(\tau_i(s)) - z_i(s) x_k(\nu_i(s)) \Big) \right. \right. \\
&\quad \left. \left. - \sum_{i=1}^{m} \Big(p_i(t) x(\tau_i(s)) - z_i(s) x(\nu_i(s)) \Big) \right) ds \right\|_\infty \\
&\quad \leq \frac{T^{q-1}(T+q)}{\Gamma(q+1)} (M_p + M_z) \|x_k - x\|_\infty \\
&\quad \leq \frac{T^{q-1}(T+q)}{\Gamma(q+1)} (M_p + M_z) \|x_k - x\|. \qquad (15)
\end{aligned}
$$

Thus, in view of (15),

$$\|\mathcal{B}_1(x_k)(t) - \mathcal{B}_1(x)(t)\| \to 0 \quad \text{as} \quad k \to \infty.$$

This means that $\mathcal{B}_1$ is continuous on X_r.

Next, let us prove $\mathcal{B}_1(X_r)$ is relatively compact.
$\mathcal{B}_1(X_r)$ is uniformly bounded due to Step 1:

$$\mathcal{B}_1(X_r) = \{\mathcal{B}_1(x) : x \in X_r\} \subset \mathcal{B}_1(X_r) + \mathcal{B}_2(X_r) \subseteq X_r.$$

Now, let us prove that $\mathcal{B}_1$ is equicontinuous on X_r. For $t_1, t_2 \in [0, T]$, $t_1 < t_2$, and $x \in X_r$, we have

$$
\begin{aligned}
&\Big\| \mathcal{B}_1(x)(t_2) - \mathcal{B}_1(x)(t_1) \Big\| \\
&\quad = \Big\| \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2 - s)^{q-1} \sum_{i=1}^{m} \Big(p_i(s) x(\tau_i(s)) - z_i(s) x(\nu_i(s)) \Big) ds \\
&\quad - \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} \sum_{i=1}^{m} \Big(p_i(s) x(\tau_i(s)) - z_i(s) x(\nu_i(s)) \Big) ds \Big\|_\infty \\
&\quad + \Big\| \frac{1}{\Gamma(q)} \frac{d}{dt} \int_0^{t_2} (t_2 - s)^{q-1} \sum_{i=1}^{m} \Big(p_i(s) x(\tau_i(s)) - z_i(s) x(\nu_i(s)) \Big) ds \\
&\quad - \frac{1}{\Gamma(q)} \frac{d}{dt} \int_0^{t_1} (t_1 - s)^{q-1} \sum_{i=1}^{m} \Big(p_i(s) x(\tau_i(s)) - z_i(s) x(\nu_i(s)) \Big) ds \Big\|_\infty \\
&\quad \leq (M_p + M_z) \|x\|_\infty \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} ds \\
&\quad + (M_p + M_z) \|x\|_\infty \frac{1}{\Gamma(q)} \int_0^{t_1} \Big((t_2 - s)^{q-1} - (t_1 - s)^{q-1} \Big) ds \\
&\quad + (M_p + M_z) \|x\|_\infty \frac{q-1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-2} ds \\
&\quad + (M_p + M_z) \|x\|_\infty \frac{q-1}{\Gamma(q)} \int_0^{t_1} \Big((t_2 - s)^{q-2} - (t_1 - s)^{q-2} \Big) ds
\end{aligned}
$$

$$\leq \frac{(M_p+M_z)\|x\|}{q\Gamma(q)}(t_2-t_1)^q + \frac{(M_p+M_z)\|x\|}{q\Gamma(q)}\Big((t_2-t_1)^q - t_2^q + t_1^q\Big)$$
$$+ \frac{(M_p+M_z)\|x\|}{\Gamma(q)}(t_2-t_1)^{q-1} + \frac{(M_p+M_z)\|x\|}{\Gamma(q)}\Big((t_2-t_1)^{q-1} - t_2^{q-1} + t_1^{q-1}\Big)$$
$$\leq \frac{2(M_p+M_z)\|x\|}{\Gamma(q+1)}|t_2-t_1|^q + \frac{2(M_p+M_z)\|x\|}{\Gamma(q)}|t_2-t_1|^{q-1}.$$

Hence, $\|\mathcal{B}_1(x)(t_2) - \mathcal{B}_1(x)(t_1)\| \to 0$ as $|t_2 - t_1| \to 0$. This implies that $\mathcal{B}_1$ is equicontinuous on X_r. Thus, $\mathcal{B}_1(X_r)$ is relatively compact. By Arzela–Ascoli's theorem, $\mathcal{B}_1$ is compact.

3. Let us prove that $\mathcal{B}_2$ is a contraction on X_r. Using (13), for any $x, y \in X_r$ and $t \in [0,T]$, we have

$$\begin{aligned}\|\mathcal{B}_2(x)(t) - \mathcal{B}_2(y)(t)\| &= \Big\|\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\Big(\mathcal{F}\big(s, x(\omega_1(s)), x(\omega_2(s)), \ldots, x(\omega_n(s))\big)\\ &\quad - \mathcal{F}\big(s, y(\omega_1(s)), y(\omega_2(s)), \ldots, y(\omega_n(s))\big)\Big)ds\Big\|_\infty\\ &\quad + \Big\|\frac{1}{\Gamma(q)}\frac{d}{dt}\int_0^t (t-s)^{q-1}\Big(\mathcal{F}\big(s, x(\omega_1(s)), x(\omega_2(s)), \ldots, x(\omega_n(s))\big)\\ &\quad - \mathcal{F}\big(s, y(\omega_1(s)), y(\omega_2(s)), \ldots, y(\omega_n(s))\big)\Big)ds\Big\|_\infty\\ &\leq \frac{T^q}{\Gamma(q+1)}\sum_{j=1}^n L_j\|y\|_\infty + \frac{T^{q-1}}{\Gamma(q)}\sum_{j=1}^n L_j\|y\|_\infty\\ &\leq \frac{T^{q-1}(T+q)}{\Gamma(q+1)}\sum_{j=1}^n L_j\|x-y\|_\infty \leq \kappa\|x-y\|.\end{aligned}$$

Accordingly, by (14), $\mathcal{B}_2$ is a contraction.

Hence, by Krasnoselskii's fixed point theorem, there exists a fixed point $x \in X_r$ such that $\mathcal{B}_1(x) + \mathcal{B}_2(x) = x$, which is a solution of FFDP (1), (2). □

6. Stability Result

The following assertion is essential to prove the main result.

Remark 3. *A function $u \in \mathcal{W}^{1,\infty}([0,T],\mathbb{R})$ is a solution of (3) if and only if there exists function $g \in \mathcal{W}^{1,\infty}([0,T],\mathbb{R})$ with the following properties:*
(i) $|g(t)| \leq \varepsilon$;
(ii) $D_0^q u(t) = \sum_{i=1}^m \Big(p_i(t)u(\tau_i(t)) - z_i(t)u(\nu_i(t))\Big) + \mathcal{F}\big(t, u(\omega_1(t)), u(\omega_2(t)), \ldots, u(\omega_n(t))\big) + f(t) + g(t), \quad t \in [0,T]$.

Lemma 3. *Assume that $u \in \mathcal{W}^{1,\infty}([0,T],\mathbb{R}^n)$ is a solution of (3). Then, for $t \in [0,T]$, the function u satisfies the inequality*

$$\begin{aligned}\Big\|u(t) - u(0) - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\Big(\sum_{i=1}^m (p_i(s)u(\tau_i(s)) - z_i(s)u(\nu_i(s)))\\ - \mathcal{F}\Big(s, u(\omega_1(s)), u(\omega_2(s)), \ldots, u(\omega_n(s))\Big) - f(s)\Big)ds\Big\|\\ \leq \frac{T^{q-1}(T+q)}{\Gamma(q+1)}\varepsilon.\end{aligned} \tag{16}$$

Proof. Let u be a solution of the inequality (3). It follows from Remark 3 (ii) that for $t \in [0, T]$,

$$D_0^q u(t) = \sum_{i=1}^{m}\Big(p_i(t)u(\tau_i(t)) - z_i(t)u(\nu_i(t))\Big) + \mathcal{F}\big(t, u(\omega_1(t)), u(\omega_2(t)), \ldots, u(\omega_n(t))\big) + f(t) + g(t), \quad t \in [0, T]. \tag{17}$$

The solution of (17) is

$$u(t) = u(0) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\Bigg(\sum_{i=1}^{m}\Big(p_i(s)u(\tau_i(s)) - z_i(s)u(\nu_i(s))\Big) + \mathcal{F}\big(s, u(\omega_1(s)), u(\omega_2(s)), \ldots, u(\omega_n(s))\big) + f(s) + g(s)\Bigg)ds.$$

Next,

$$u(t) = u(0) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\Bigg(\sum_{i=1}^{m}\Big(p_i(s)u(\tau_i(s)) - z_i(s)u(\nu_i(s))\Big) + \mathcal{F}\big(s, u(\omega_1(s)), u(\omega_2(s)), \ldots, u(\omega_n(s))\big) + f(s)\Bigg)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}g(s)ds.$$

Finally, for $t \in [0, T]$, we have

$$\begin{aligned}
&\Bigg\|u(t) - u(0) - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\Bigg(\sum_{i=1}^{m}\Big(p_i(s)u(\tau_i(s)) - z_i(s)u(\nu_i(s))\Big) + \mathcal{F}\big(s, u(\omega_1(s)), u(\omega_2(s)), \ldots, u(\omega_n(s))\big) + f(s)\Bigg)ds\Bigg\| \\
&= \left\|\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}g(s)ds\right\| \\
&= \left\|\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}g(s)\,ds\right\|_\infty + \left\|\frac{1}{\Gamma(q)}\frac{d}{dt}\int_0^t (t-s)^{q-1}g(s)\,ds\right\|_\infty \\
&\leq \frac{T^q}{\Gamma(q+1)}\varepsilon + \frac{T^{q-1}}{\Gamma(q)}\varepsilon = \frac{T^{q-1}(T+q)}{\Gamma(q+1)}\varepsilon.
\end{aligned}$$

□

Now, we are ready to formulate and prove our main result.

Theorem 2. *If the conditions* (3) *and* (10)–(14) *hold, then FFDP* (1), (2) *is UH-stable and GUH-stable.*

Proof. Assume that $u \in \mathcal{W}^{1,\infty}([0, T], \mathbb{R})$ is a solution of the inequality (3), and assume that $x \in \mathcal{W}^{1,\infty}([0, T], \mathbb{R})$ is a solution of the following problem

$$\begin{aligned}
D_0^q x(t) &= \sum_{i=1}^{m}\Big(p_i(t)x(\tau_i(t)) - z_i(t)x(\nu_i(t))\Big) \\
&\quad + \mathcal{F}\big(t, x(\omega_1(t)), x(\omega_2(t)), \ldots, x(\omega_n(t))\big) + f(t), \quad t \in [0, T], \\
x(0) &= u(0).
\end{aligned}$$

Taking into account Lemma 2, we have

$$x(t) = c + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left(\sum_{i=1}^{m} \Big(p_i(s)x(\tau_i(s)) - z_i(s)x(\nu_i(s)) \Big) + \mathcal{F}\big(s, x(\omega_1(s)), x(\omega_2(s)), \ldots, x(\omega_n(s))\big) + f(s) \right) ds,$$

where $c \in \mathbb{R}$. Next, if $x(0) = u(0)$, then $u(0) = c$. Therefore,

$$x(t) = u(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left(\sum_{i=1}^{m} \Big(p_i(s)x(\tau_i(s)) - z_i(s)x(\nu_i(s)) \Big) + \mathcal{F}\big(s, x(\omega_1(s)), x(\omega_2(s)), \ldots, x(\omega_n(s))\big) + f(s) \right) ds.$$

Then, in view of Lemma 2 and properties (10), (13), and (16), for $t \in [0, T]$, we have

$$\begin{aligned}
\Big\| u(t) - x(t) \Big\| &= \Big\| u(t) - x(0) \\
&\quad - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left(\sum_{i=1}^{m} \Big(p_i(s)x(\tau_i(s)) - z_i(s)x(\nu_i(s)) \Big) \right. \\
&\quad \left. - \mathcal{F}\big(s, x(\omega_1(s)), x(\omega_2(s)), \ldots, x(\omega_n(s))\big) + f(s) \right) ds \Big\| \\
&\leq \Big\| u(t) - u(0) \\
&\quad - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left(\sum_{i=1}^{m} \Big(p_i(s)u(\tau_i(s)) - z_i(s)u(\nu_i(s)) \Big) \right. \\
&\quad \left. - \mathcal{F}\big(s, u(\omega_1(s)), u(\omega_2(s)), \ldots, u(\omega_n(s))\big) + f(s) \right) ds \\
&\quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left(\sum_{i=1}^{m} \Big(p_i(s)u(\tau_i(s)) - z_i(s)u(\nu_i(s)) \Big) \right. \\
&\quad \left. - \sum_{i=1}^{m} \Big(p_i(s)x(\tau_i(s)) - z_i(s)x(\nu_i(s)) \Big) \right) ds \\
&\quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left(\mathcal{F}\big(s, u(\omega_1(s)), u(\omega_2(s)), \ldots, u(\omega_n(s))\big) \right. \\
&\quad \left. - \mathcal{F}\big(s, x(\omega_1(s)), x(\omega_2(s)), \ldots, x(\omega_n(s))\big) \right) ds \Big\| \\
&\leq \frac{T^{q-1}(T+q)}{\Gamma(q+1)} \varepsilon + \frac{T^{q-1}(T+q)}{\Gamma(q+1)} \left(M_p + M_z + \sum_{j=1}^{n} L_j \right) \|u - x\|_\infty \\
&\leq \frac{T^{q-1}(T+q)}{\Gamma(q+1)} \varepsilon + \frac{T^{q-1}(T+q)}{\Gamma(q+1)} \left(M_p + M_z + \sum_{j=1}^{n} L_j \right) \|u - x\|.
\end{aligned}$$

Thus, we obtain

$$\|u-x\| \leq \frac{\frac{T^{q-1}(T+q)}{\Gamma(q+1)}}{1-\frac{T^{q-1}(T+q)}{\Gamma(q+1)}\left(M_p+M_z+\sum_{j=1}^{n}L_j\right)}\,\varepsilon = \frac{T^{q-1}(T+q)}{\Gamma(q+1)(1-\kappa)}\varepsilon,$$

where $\kappa \in (0,1)$ is defined by (14). Therefore,

$$\|u-x\| \leq \kappa_{\mathcal{F}}\varepsilon \tag{18}$$

with

$$\kappa_{\mathcal{F}} := \frac{T^{q-1}(T+q)}{\Gamma(q+1)(1-\kappa)} > 0.$$

We established that FFDP (1), (2) is UH-stable in $\mathcal{W}^{1,\infty}([0,T],\mathbb{R})$.

Now, to obtain the GUH stability of FFDP (1), (2), we take a nondecreasing function $F:\mathbb{R}^+ \to \mathbb{R}^+$ such that $F(\varepsilon)=\varepsilon$ with $F(0)=0$. Then according to Definition 4, from (18) we have

$$\|u-x\| \leq F(\varepsilon)\kappa_{\mathcal{F}}.$$

Thus, FFDP (1), (2) is GUH-stable in $\mathcal{W}^{1,\infty}([0,T],\mathbb{R})$. □

7. Examples of an Application

Example 1. *Let us consider the nonlinear fractional differential equation with deviating arguments that is a nonlinear fractional pantograph-type equation (see [25–27]):*

$$\begin{aligned} D_0^q x(t) &= p_1(t)x(\tau_1 t) - z_1(t)x(\nu_1 t) + p_2(t)x(\tau_2 t) - z_2(t)x(\nu_2 t) \\ &\quad + \left(\alpha(t)\right)^2\sqrt{\beta(t)+\sin(x(\omega t))} + f(t), \quad t\in[0,1], \end{aligned} \tag{19}$$

with the initial condition (2), where functions $p_1, p_2, z_1, z_2, \alpha, \beta, f \in L^\infty([0,1],\mathbb{R})$, $\tau_1, \tau_2, \nu_1, \nu_2$, $\omega \in (0,1)$, *and*

$$\beta(t) > 1, \quad t\in[0,1]. \tag{20}$$

Equation (19) is a pantograph-type equation [23] used in electrodynamics, quantum mechanics, number theory, astrophysics, nonlinear dynamical systems. The pantograph is a device that can be used, for example, in electric trains to collect electric currents from the overload lines.

Equation (19) is a particular case of Equation (1) for $m=2$, *and* $n=1$ *with*

$$\mathcal{F}\left(t,x(\omega(t))\right) := \left(\alpha(t)\right)^2\sqrt{\beta(t)+\sin(x(\omega t))}. \tag{21}$$

First, we prove solvability conditions of the problem (19), (2). Let us assume that there exist $M_p<\infty$, $M_{\mathcal{F}}<\infty$, $M_z<\infty$, *and* $L_1<\infty$ *such that*

$$\|p_1(t)\|_\infty + \|p_2(t)\|_\infty = M_p; \quad \|z_1(t)\|_\infty + \|z_2(t)\|_\infty = M_z; \tag{22}$$

$$\left\|(\alpha(t))^2\sqrt{\beta(t)}\right\|_\infty = M_{\mathcal{F}}; \quad \left\|\frac{(\alpha(t))^2}{2\sqrt{\beta(t)-1}}\right\|_\infty = L_1. \tag{23}$$

Also, suppose that there exists $\kappa\in(0,1)$ *such that*

$$\kappa = \frac{1+q}{\Gamma(q+1)}\left(M_p+M_z+L_1\right). \tag{24}$$

Corollary 1. *If the conditions (12) and (22)–(24) hold, then FFDP (19), (2) has a solution.*

Proof. To prove Corollary 1, we use the results from Theorem 1.

Here, the operators $\mathcal{B}_1, \mathcal{B}_2 : \mathcal{W}^{1,\infty}([0,1],\mathbb{R}) \to \mathcal{W}^{1,\infty}([0,1],\mathbb{R})$ have the form

$$\begin{aligned}\mathcal{B}_1(x)(t) =& c + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\Big(p_1(s)x(\tau_1 s) - z_1(s)x(\nu_1 s)\\ &+ p_2(s)x(\tau_2 s) - z_2(s)x(\nu_2 s) + f(s)\Big)ds, \quad t \in [0,1],\\ \mathcal{B}_2(x)(t) =& \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\big(\alpha(s)\big)^2\sqrt{\beta(s)+\sin(x(\omega s))}ds, \quad t \in [0,1],\end{aligned} \tag{25}$$

and set X_r is

$$X_r = \{x \in \mathcal{W}^{1,\infty}([0,1],\mathbb{R}) : \|x\| \le r\}$$

with $r > 0$ such that

$$\frac{|c| + \frac{1+q}{\Gamma(q+1)}(M_{\mathcal{F}} + M_f)}{1-\kappa} \le r,$$

where κ is defined by (24).

Analogously, the proof consists of three steps. Obviously, steps 1–2 hold for $\mathcal{B}_1$ defined by (25). Let us prove step 3, that $\mathcal{B}_2$ is a contraction on X_r.

Using (23), (24) and Lagrange's theorem (also known as the mean value theorem), for any $x, y \in X_r$ and $t \in [0,T]$, we have

$$\begin{aligned}\big\|\mathcal{B}_2(x)(t) - \mathcal{B}_2(y)(t)\big\| =& \left\|\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\Big(\big(\alpha(s)\big)^2\sqrt{\beta(s)+\sin(x(\omega s))}\right.\\ &\left. - \big(\alpha(s)\big)^2\sqrt{\beta(s)+\sin(y(\omega s))}ds\Big)\right\|\\ \le& \left\|\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\sup_{\xi\in\mathbb{R}}\frac{(\alpha(s))^2|\cos\xi|}{2\sqrt{\beta(s)+\sin\xi}}(x(\omega s) - y(\omega s))ds\right\|_\infty\\ &+ \left\|\frac{1}{\Gamma(q)}\frac{d}{dt}\int_0^t (t-s)^{q-1}\sup_{\xi\in\mathbb{R}}\frac{(\alpha(s))^2|\cos\xi|}{2\sqrt{\beta(s)+\sin\xi}}(x(\omega s) - y(\omega s))\right\|_\infty\\ \le& \left\|\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\frac{(\alpha(s))^2}{2\sqrt{\beta(s)-1}}(x(\omega s) - y(\omega s))ds\right\|_\infty\\ &+ \left\|\frac{1}{\Gamma(q)}\frac{d}{dt}\int_0^t (t-s)^{q-1}\frac{(\alpha(s))^2}{2\sqrt{\beta(s)-1}}(x(\omega s) - y(\omega s))ds\right\|_\infty\\ \le& \frac{1+q}{\Gamma(q+1)}L_1\|x-y\|_\infty \le \kappa\|x-y\|.\end{aligned}$$

Then, by (24), $\mathcal{B}_2$ is a contraction. Applying Krasnoselskii's fixed point theorem, there exists $x \in X_r$ such that $\mathcal{B}_1(x) + \mathcal{B}_2(x) = x$, which is a solution of FFDP (19), (2). □

Now, we formulate the stability result.

Corollary 2. *Assume that the conditions* (3), (22)–(24) *are true. Then, FFDP* (19), (2) *is UH- and GUH-stable.*

Proof. To prove Corollary 2, we use the results from Theorem 2.

Let $u \in \mathcal{W}^{1,\infty}([0,1],\mathbb{R})$ be a solution of the inequality (3) and let $x \in \mathcal{W}^{1,\infty}([0,1],\mathbb{R})$ be a solution of FFDP (19), (2) with $u(0) = x(0)$. Using Lemma 3, (22), (23) and Lagrange's theorem, for all $t \in [0,1]$, we get

$$\begin{aligned}\left\|u(t) - x(t)\right\| &= \Big\|u(t) - c - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\Big(p_1(s)x(\tau_1 s) - z_1(s)x(\nu_1 s)\\ &\quad + p_2(s)x(\tau_2 s) - z_2(s)x(\nu_2 s)\Big)ds\\ &\quad - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\Big((\alpha(s))^2\sqrt{(\beta(s)+1)\sin(x(\omega s))} + f(s)\Big)ds\Big\|\\ &\leq \frac{1+q}{\Gamma(q+1)}\varepsilon + \frac{1+q}{\Gamma(q+1)}\left(M_p + M_z + L_1\right)\|u - x\|.\end{aligned}$$

Hence, it follows that

$$\|u - x\| \leq \frac{\frac{1+q}{\Gamma(q+1)}}{1 - \frac{1+q}{\Gamma(q+1)}\left(M_p + M_z + L_1\right)}\,\varepsilon = \frac{1+q}{\Gamma(q+1)(1-\kappa)}\varepsilon$$

with $\kappa \in (0,1)$ defined by (24). Thus, we have

$$\|u - x\| \leq \kappa_{\mathcal{F}}\varepsilon,$$

where $\kappa_{\mathcal{F}} = \frac{1+q}{\Gamma(q+1)(1-\kappa)} > 0$.
So, we established that FFDP (19), (2) is UH-stable in $\mathcal{W}^{1,\infty}([0,1],\mathbb{R})$.

If we take the nondecreasing function $F : \mathbb{R}^+ \to \mathbb{R}^+$ such that $F(\varepsilon) = \varepsilon$ with $F(0) = 0$, we get $\|u - x\| \leq F(\varepsilon)\kappa_{\mathcal{F}}$. Therefore, according to Definition 4, FFDP (19), (2) is GUH-stable in $\mathcal{W}^{1,\infty}([0,1],\mathbb{R})$. □

Example 2. *Here, we consider a numerical example of FFDP* (1), (2)*:*

$$\begin{aligned}D_0^{\frac{1}{2}}x(t) &= \frac{1}{40}x\left(\frac{1}{2}t\right) - \frac{1}{60}x\left(\frac{1}{3}t\right) + \frac{1}{80}x\left(\frac{1}{4}t\right) - \frac{1}{120}x\left(\frac{1}{5}t\right)\\ &\quad + \frac{1}{8}\sin\left(x\left(\frac{1}{2}t\right)\right) + \frac{1}{4}\cos\left(x\left(\frac{1}{6}t\right)\right) + 5, \quad t \in [0,1]; \qquad x(0) = 1. \qquad (26)\end{aligned}$$

The problem (26) *is a particular case of FDP with multiple deviating arguments* (1), (2) *for* $m = 2, n = 2$*, with*

$$\begin{aligned}&q = \frac{1}{2}, \quad p_1(t) = \frac{1}{40}, \quad p_2(t) = \frac{1}{80}, \quad z_1(t) = \frac{1}{60}, \quad z_2(t) = \frac{1}{120},\\ &\tau_1(t) = \frac{1}{2}t, \quad \tau_2(t) = \frac{1}{4}t, \quad \nu_1(t) = \frac{1}{3}t, \quad \nu_2(t) = \frac{1}{5}t,\\ &\omega_1(t) = \frac{1}{2}t, \quad \omega_2(t) = \frac{1}{6}t, \quad f(t) = 5, \quad c = 1\end{aligned}$$

and

$$\mathcal{F}(t, x(\omega_1(t)), x(\omega_2(t))) = \frac{1}{8}\sin\left(x\left(\frac{1}{2}t\right)\right) + \frac{1}{4}\cos\left(x\left(\frac{1}{6}t\right)\right).$$

The operators $\mathcal{B}_1, \mathcal{B}_2 : \mathcal{W}^{1,\infty}([0,1],\mathbb{R}) \to \mathcal{W}^{1,\infty}([0,1],\mathbb{R})$ *are*

$$\begin{aligned}\mathcal{B}_1(x)(t) &= 1 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left(\frac{1}{40}x\left(\frac{1}{2}s\right) - \frac{1}{60}x\left(\frac{1}{3}s\right) + \frac{1}{80}x\left(\frac{1}{4}s\right) - \frac{1}{120}x\left(\frac{1}{5}s\right) + 5\right)ds,\\ \mathcal{B}_2(x)(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left(\frac{1}{8}\sin\left(x\left(\frac{1}{2}s\right)\right) + \frac{1}{4}\cos\left(x\left(\frac{1}{6}s\right)\right)\right)ds.\end{aligned}$$

In this case, we have

$$M_p = \frac{3}{80}, \quad M_z = \frac{1}{40}, \quad M_{\mathcal{F}} = \frac{1}{6}, \quad L_1 = \frac{1}{8}, \quad L_2 = \frac{1}{3},$$
$$\kappa_1 = \frac{3}{16\sqrt{\pi}}, \quad \kappa_2 = \frac{9}{8\sqrt{\pi}}, \quad \kappa = \frac{21}{16\sqrt{\pi}} < 1.$$

Thus, the assumptions of Theorem 1 are fulfilled, and then the problem (26) is solvable.

Because the assumptions of Theorem 2 also hold, the problem (26) is UH- and GUH-stable. According to Definition 4, we have $\kappa_{\mathcal{F}} = \frac{48}{16\sqrt{\pi}-21} > 0$ *such that for any* $\varepsilon > 0$ *and for each solution* $u \in \mathcal{W}^{1,\infty}([0,1],\mathbb{R})$ *of the inequality*

$$\left| D_0^{\frac{1}{2}} u(t) - \frac{1}{40} u\left(\frac{1}{2}t\right) + \frac{1}{60} u\left(\frac{1}{3}t\right) - \frac{1}{80} u\left(\frac{1}{4}t\right) + \frac{1}{120} u\left(\frac{1}{5}t\right) - \frac{1}{8}\sin\left(u\left(\frac{1}{2}t\right)\right) - \frac{1}{4}\cos\left(u\left(\frac{1}{6}t\right)\right) - 5 \right| \leq \varepsilon$$

there exists a solution x of (26) that satisfies $\|u - x\| \leq \kappa_{\mathcal{F}}\varepsilon$.

8. Conclusions

In this paper, we formulated UH and GUH stability results for initial value problems for fractional differential equations with multiple deviating arguments. Krasnoselskii's fixed point theorem was used to establish the existence conditions for the problem. In our examples, these results were applied on pantograph-type equations. The presented results are novel and extend some previous research.

Author Contributions: Conceptualization, N.D.; methodology, N.D. and M.L.; formal analysis, N.D. and M.L.; investigation, N.D. and M.L.; writing – original draft, N.D. and M.L.; writing – review and editing, N.D., M.L., G.F. and B.T.; project administration, N.D., M.L., G.F. and B.T.; funding acquisition, N.D., G.F. and B.T. All authors have read and agreed to the published version of the manuscript.

Funding: This work is partially supported by the Slovak Grant Agency VEGA No. 2/0062/24 (N. Dilna, M. Langerová), the Jànos Bolyai Research Scholarship of the Hungarian Academy of Sciences (BO/00047/21/6), the National Scholarship Programme of the Slovak Republic (ID 46598) (G. Fekete), and the National Research, Development and Innovation Office (NKFIH) under grant No. FK 147030 (B. Tóth).

Data Availability Statement: The original contributions presented in the study are included in the article, further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflict of interest.

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