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Existence of a Global Attractor for the Reaction–Diffusion Equation with Memory and Lower Regularity Terms

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Abstract: This paper investigates the large time behavior for the reaction–diffusion equation with memory and the forcing term $g \in H^{-1}(\Omega)$. We prove the existence of a global attractor in $L^2(\Omega) \times L^2_\mu(\mathbb{R}; H^1_0(\Omega))$. Due to the lower regularity of g, one can hardly use the traditional energy estimates to derive the existence of a bounded absorbing set in the higher regularity space and then the compactness of the semigroup. Here, we utilize the contractive function method to establish the asymptotic smoothness of the semigroup.

Keywords: global attractor; reaction–diffusion equation; memory term; lower regularity; asymptotically smooth

MSC: 35B41; 35K65; 37L30

1. Introduction

We investigate the existence of a global attractor for the following reaction–diffusion equation with memory:

$$\begin{cases} u_t - \Delta u - \int_0^\infty k(s)\Delta u(t-s)ds + f(u) = g, & \text{in } \Omega, t > 0, \\ u(t) = u_0(t), & \text{in } \Omega, t \leqslant 0, \\ u|_{\partial\Omega} = 0, & t \in \mathbb{R}, \end{cases} \tag{1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n (n \geq 3)$, where $g \in H^{-1}(\Omega)$, and memory kernel function $k(s)$ is assumed to satisfy the posterior adaptation hypothesis, which will be detailed in the subsequent sections.

It is well established that numerous physical phenomena can be described more accurately by incorporating terms into the governing equations that account for the system's historical behavior. In particular, models pertaining to high-viscosity fluids at low temperatures or the thermomechanical behavior of polymers (refer to [1,2] and the associated references) demonstrate the significant influence of historical effects. In certain mathematical formulations of physical equations, the presence of the memory term is particularly salient, as it is reasonable to assert that the evolution of a system is contingent upon its entire historical trajectory rather than solely its present state.

The long-time behavior of solutions of Equation (1) with memory has been studied extensively. In order to generate a dynamical system in an appropriate phase space, Dafermos [3] introduced an additional variable η^t, the (integrated) past history of u,

$$\eta^t = \eta^t(x,s) = \int_0^s u(x, t-\tau)d\tau,$$

of which the evolution is ruled by a first-order hyperbolic equation. Thus, the original problem (1) can be transformed into a dynamical system on a phase space with two components

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$$\begin{cases} u_t - \Delta u - \int_0^\infty \mu(s)\Delta\eta^t(s)ds + f(u) = g, \\ \eta_t^t = -\eta_s^t + u, \end{cases}$$

Following the Dafermos' idea, the long-time behavior of solutions to the reaction–diffusion equations with memory have been studied largely. In [4], the existence of global attractors for reaction–diffusion systems with finite delay was obtained, while in [5], the authors studied the existence of trajectory attractors for reaction–diffusion equations with an infinite-delay memory term. In [6], the existence of global attractors in $L^2(\Omega) \times L^2_\mu(\mathbb{R}; H^1_0(\Omega))$ was obtained for Equation (1) with critical nonlinearity. In [7], when the nonlinearity is subcritical, the existence of a bounded absorbing set in $H^1_0(\Omega) \times L^2_\mu(\mathbb{R}; H^2(\Omega) \cup H^1_0(\Omega))$ is established. For more related works on the long-time behavior of the equation with memory, we refer readers to [8–13].

In general, the aforementioned results presuppose that $g \in L^2(\Omega)$ in order to obtain a bounded absorbing set or an energy estimate within a higher regularity space (such as $H^1_0(\Omega) \times L^2_\mu(\mathbb{R}^+, H^2(\Omega) \cap H^1_0)$). This assumption is pivotal for establishing compactness or asymptotic compactness in a lower regularity space (the original phase space of the semigroup), which is essential for proving the existence of attractors. In this paper, we shall investigate the existence of a global attractor for Equation (1) under the weaker assumption that $g \in H^{-1}(\Omega)$. We recall that when $g \in H^{-1}(\Omega)$, there exists a unique weak solution to Equation (1) in $L^2(\Omega) \times L^2_\mu(\mathbb{R}; H^1_0(\Omega))$, where the solution generates a semigroup as the standard reaction–diffusion equation. It is natural to ask whether the global attractor exists in such a space. Note that when $g \in H^{-1}(\Omega)$, the standard energy estimates derived by using $-\Delta u$ or u_t as test functions are not in hand. For standard reaction–diffusion equations, to overcome this difficulty, the authors of [14,15] (see also [16,17]) developed the ω–limit compactness method to investigate the existence of a global attractor. Yet, such a method is not applicable for our problem with memory. Actually, it only ensures that the co-dimension of the first component u is sufficiently small after orthogonal decomposition, but does not address the second component η^t. To overcome this difficulty, we adapt the ideas of Chueshov in [18] to use the contractive function method to establish asymptotic smoothness. First, we prove that the semigroup of solutions possesses a bounded absorbing set in the phase space. Then, we show that the elements of this absorbing set satisfy the compression function condition, and thereby obtain the asymptotic compactness of the semigroup, and the existence of the attractor.

2. Some Preliminaries

Assume that the nonlinear term f in Equation (1) satisfies the following conditions:

$$\alpha_1|s|^p - k_1 \leqslant f(s)s \leqslant \alpha_2|s|^p + k_2, \quad p \geqslant 2, \tag{2}$$

where $\alpha_1, \alpha_2,\ k_1,\ k_2, l$ and C are positive constants satisfying the dissipative conditions

$$f'(s) \geqslant -l, \quad \forall s \in \mathbb{R}, \tag{3}$$

$$\langle f(s_1) - f(s_2), s_1 - s_2\rangle \leqslant C.$$

Let $F(s) = \int_0^s f(\tau)d\tau$. It is not difficult to see that there exist positive constants $\alpha_3, \alpha_4,\ k_3$, and δ_1, such that

$$\alpha_3|s|^p - \delta_1|s|^2 \leqslant F(s)s \leqslant \alpha_4|s|^p + k_3|s|^2. \tag{4}$$

Concerning the memory kernel function $k(s)$, let $\mu(s) = -k'(s)$. Hypothesize that

$$\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \quad \mu(s) \geqslant 0, \quad \mu'(s) \leqslant 0, \quad \forall s \in \mathbb{R}^+, \tag{5}$$

and there are constants $\delta,\ \gamma > 0$, such that

$$\mu'(s) + \delta\mu(s) \leqslant 0, \tag{6}$$

$$\mu(\infty) = \lim_{s\to\infty} \mu(s) = 0, \tag{7}$$

$$m_0 = \int_0^\infty \mu(s)ds.$$

For the past history of u, denote

$$\eta^t = \eta^t(x,s) := \int_0^s u(x,t-\tau)d\tau, \quad \forall s \in \mathbb{R}^+, \tag{8}$$

and

$$\eta_t^t = \frac{\partial}{\partial t}\eta^t,$$
$$\eta_s^t = \frac{\partial}{\partial s}\eta^t.$$

The historical variable $u_0(\cdot, -s)$ of u satisfies the following conditions: there exist two positive N and $\sigma \leq \delta$ such that

$$\int_0^\infty e^{-\delta s}||u_0(-s)||_0^2 ds \leqslant N. \tag{9}$$

From Equations (7) and (8), and utilizing properties of the Lebesgue integral and partial integrals, we get

$$\int_0^\infty k(s)\Delta u(t-s)ds = \int_0^\infty \mu(s)\Delta\eta^t(s)ds.$$

Therefore, transforming the memory term in Equation (1) yields the following system

$$\begin{cases} u_t - \Delta u - \int_0^\infty \mu(s)\Delta\eta^t(s)ds + f(u) = g, \\ \eta_t^t = -\eta_s^t + u, \end{cases} \tag{10}$$

with the initial boundary conditions

$$\begin{cases} u(x,t)|_{\partial\Omega} = 0, \quad \eta^t(x,s)|_{\partial\Omega\times\mathbb{R}^+} = 0, \quad t \geq 0, \\ u(x,0) = u_0(x), \quad \eta^0(x,s) = \int_0^s u_0(x,-\tau)d\tau, \quad (x,s) \in \Omega \times \mathbb{R}^+. \end{cases} \tag{11}$$

In this manuscript, let Ω be a fixed domain in $\mathbb{R}^n (n \geqslant 3)$. $L^2(\Omega)$ is denoted by a Hilbert space H, $H_0^1(\Omega)$ is denoted by V. The norms in H and V will be denoted by $|\cdot|$ and $\|\cdot\|$, respectively. Let X be a Hilbert space, and $L_\mu^2(\mathbb{R}^+; X)$ be a Hilbert space of functions $\varphi : \mathbb{R} \to X$ endowed with the inner product

$$\langle\varphi_1, \varphi_2\rangle_{\mu,X} = \int_0^\infty \mu(s)\langle\varphi_1(s), \varphi_2(s)\rangle_X ds.$$

Let $\|\cdot\|_{\mu,X}$ denote the corresponding norm

$$||\varphi||_{\mu,X}^2 = \int_0^\infty \mu(s)||\varphi(s)||_X^2 ds.$$

In particular, in the case that $X = V$, $X = V \cap H^2(\Omega)$, we denote the norm as $\|\varphi\|_{\mu,1}$ and $\|\varphi\|_{\mu,2}$, respectively.

Moreover, let

$$\mathcal{L}_1 = H \times L_\mu^2(\mathbb{R}^+, V)$$

and

$$\mathcal{L}_2 = V \times L^2_\mu(\mathbb{R}^+; V \cap H^2(\Omega)).$$

with norm

$$\|z_1\|^2_{\mathcal{L}_1} = \|(u_1, \varphi_1)\|^2_{\mathcal{L}_1} = |u_1| + \|\varphi_1\|^2_{\mu,1},$$

$$\|z_2\|^2_{\mathcal{L}_2} = \|(u_2, \varphi_2)\|^2_{\mathcal{L}_2} = \|u_2\| + \|\varphi_2\|^2_{\mu,2}$$

for any $z_1 = (u_1, \varphi_1) \in \mathcal{L}_1$, and $z_2 = (u_2, \varphi_2) \in \mathcal{L}_2$.

Lemma 1 ([19]). *Let X be a Hilbert space, $I = [0, T]$, $\forall T \geq 0$. Assume the memory kernel $\mu(s)$ satisfies (4) and (5). Then, for any $\eta^t \in C(I, L^2_\mu(\mathbb{R}^+, X))$, we have*

$$\langle \eta^t, \eta^t_s \rangle_{\mu,X} \geqslant \frac{\delta}{2} ||\eta^t||^2_{\mu,X},$$

where δ is from (6).

Proof. Integrating by parts in s and using (6), it yields

$$\begin{aligned}
\langle \eta^t, \eta^t_s \rangle_{\mu,X} &= \int_0^\infty \mu d\left(\frac{1}{2}\int_\Omega ||\nabla \eta^t(s)||^2 dx\right) \\
&= \left[\frac{1}{2}\mu(s)||\nabla \eta^t(s)||^2\right]_0^\infty - \frac{1}{2}\int_0^\infty \mu'(s)||\nabla \eta^t(s)||^2 ds \\
&= -\frac{1}{2}\int_0^\infty \mu'(s)||\nabla \eta^t(s)||^2 ds \\
&\geqslant \frac{\delta}{2}||\eta^t(s)||^2_{\mu,X}.
\end{aligned}$$

□

We now recall the definitions of the asymptotic smoothness of the semigroup and the existence theorem of attractors, which will be used later.

Definition 1 ([18], Definition 2.2.1). *Let $\{S(t)\}_{t\geq 0}$ be a semigroup on a complete metric space X. $\{S(t)\}_{t\geq 0}$ is said to be asymptotically compact if the following Ladyzhenskaya condition holds: for any bounded set B in X such that the tail $\gamma^\tau(B) := \cup_{t\geqslant\tau} S(t)B$ is bounded for some $\tau \geqslant 0$, we have that any sequence of the form $\{S(t_n)x_n\}$ with $x_n \in B$ and $t_n \to \infty$ is relatively compact.*

Definition 2 ([18], Definition 2.2.1). *Let $\{S(t)\}_{t\geq 0}$ be a semigroup on a complete metric space X, and let (X, S_t) be the corresponding dynamical system. A closed set $B \subset X$ is said to be absorbing for $\{S(t)\}_{t\geq 0}$ if for any bounded set $D \subset X$ there exists $t_0(D)$ such that $S(t)D \subset B$ for all $t \geqslant t_0(D)$.*

Definition 3 ([18], Definition 2.2.1). *Let $\{S(t)\}_{t\geq 0}$ be a semigroup on a complete metric space X. $\{S(t)\}_{t\geq 0}$ is said to be asymptotically smooth if the following Hale condition is valid: for every bounded set D such that $S(t)D \subset D$ for $t > 0$ there exists a compact set K in the closure $\overline{D}$ of D, such that $S(t)D$ converges uniformly to K in the sense that*

$$\lim_{t\to+\infty} d_X\{S(t)D|K\} = 0, \quad \text{where} \quad d_X\{A|B\} = \sup_{x\in A} dist_X(x, B).$$

Lemma 2 ([18], Proposition 2.2.4). *A semigoup $\{S(t)\}_{t\geq 0}$ in some metric space X is asymptotically compact if and only if it is asymptotically smooth.*

Lemma 3 ([18], Theorem 2.3.5)**.** *Let $(X, S(t))$ be a dissipative asymptotically compact dynamical system on a complete metric space X. Then, $S(t)$ possesses a unique compact global attractor E such that*

$$E = \omega(B_0) = \bigcap_{t>0} \overline{\bigcup_{\tau \geq t_1} S(\tau)B_0},$$

for every bounded absorbing set B_0 and

$$\lim_{t \to +\infty} (d_X\{S(t)B_0|E\} + d_X\{E|S(t)B_0\}) = 0.$$

where, as above, $d_X\{A|B\} = \sup_{x \in A} dist_X(x, B)$. Moreover, if there exists a connected absorbing bounded set, then E is connected.

Thanks to Lemmas 2 and 3, the existence of the attractor can be achieved through asymptotic smoothness. The next lemma provides a sufficient condition for verifying the asymptotic smoothness using the contractive functions method.

Lemma 4 ([18], Theorem 2.2.17)**.** *Let $\{S(t)\}_{t \geq 0}$ be a semigroup on a complete metric space X. Assume that for any bounded forward invariant set B in X and for any $\varepsilon > 0$ there exists $T \equiv T(\varepsilon, B)$ such that*

$$dist(S(T)y_1, S(T)y_2) \leqslant \varepsilon + \Psi_{\varepsilon,B,T}(y_1, y_2),\ y_i \in B,$$

where $\Psi_{\varepsilon,B,T}(y_1, y_2)$ is a functional defined on $B \times B$ such that

$$\liminf_{m \longrightarrow \infty} \liminf_{n \longrightarrow \infty} \Psi_{\varepsilon,B,T}(y_n, y_m) = 0, \quad for\ \ every\ \ sequence \quad y_n \subset B.$$

Then, $\{S(t)\}_{t \geq 0}$ is asymptotically smooth.

A function $\Psi_{\varepsilon,B,T}(y_1, y_2)$ satisfying (2.23) is said to be a contractive function on B, and $Cont(B)$ denotes the set of all contractive functions on B.

Additionally, the following lemma is useful in subsequent disccusions.

Lemma 5 ([20], Aubin–Dubinskii–Lions Lemma)**.** *Assume that $X \subset Y \subset Z$ is a triple of Banach spaces such that $X \hookrightarrow\hookrightarrow Y \hookrightarrow Z$.*

(1) Let $W_1 = \{u \in L^p(a, b; X)|u_t \in L^q(a, b; Z)\}$ for some $1 \leq p \leq \infty$ and $q \geq 1$. Here, u_t denotes the derivative of u in the distributional sense. Then, $W_1 \hookrightarrow\hookrightarrow L^p(a, b; Y)$. If $q > 1$, then $W_1 \hookrightarrow\hookrightarrow C(a, b; Z)$;

(2) Let $W_2 = \{u \in L^\infty(a, b; X)|u_t \in L^r(a, b; Z)\}$ for some $r > 1$, then $W_2 \hookrightarrow\hookrightarrow C(a, b; Y)$.

3. Asymptotical Smoothness and Global Attractors

For convenience, denote g as $h + D_i g^i (= h + \Sigma_{i=1}^n D_i g^i)$, where $h,\ g^i \in L^2(\Omega)$. As presented in the previous section, the Dafermos transformation allows us to reformulate the original problem (1) as a dynamical system

$$\begin{cases} u_t - \Delta u - \int_0^\infty \mu(s)\Delta\eta^t(s)ds + f(u) = h + D_i g^i, \\ \eta_t^t = -\eta_s^t + u, \end{cases} \tag{12}$$

with initial boundary conditions

$$\begin{cases} u(x,t)|_{\partial\Omega} = 0, \quad \eta^t(x,s)|_{\partial\Omega\times\mathbb{R}^+} = 0, \quad t \geq 0, \\ u(x,0) = u_0(x), \quad \eta^0(x,s) = \int_0^s u_0(x,-\tau)d\tau, \quad (x,s) \in \Omega \times \mathbb{R}^+. \end{cases} \tag{13}$$

This section addresses the existence of a global attractor for the Equation (12) under the assumptions (2)–(9). By [21] (see also [7]), we have the existence and uniqueness of weak solutions.

Lemma 6. *Assume that (2)–(9) hold and $h, g^i \in L^2(\Omega)$. Then, for any $T > 0$ and $z_0 = (u_0, \eta^0) \in \mathcal{L}_1$, there exists a unique weak solution $z = (u, \eta^t)$ for the Equation (12) with the initial conditions (13), satisfying*

$$u \in L^\infty(0,T;H) \cap L^2(0,T;V) \cap L^P(0,T;L^P(\Omega)),$$
$$\eta^t \in L^\infty(0,T;L^2_\mu(\mathbb{R}^+;V),$$
$$u_t \in L^2(0,T;H^{-1}(\Omega)).$$

Furthermore,

$$z \in C(0,T;H), \quad \forall T > 0.$$

The above lemma insures the existence of a solution semigroup $\{S(t)\}_{t\geqslant 0}$ in $\mathcal{L}_1$

$$S(t) : \mathbb{R}^+ \times \mathcal{L}_1 \to \mathcal{L}_1,$$
$$S(t)z_0 = z(t).$$

Remark 1. *Actually, it suffices for the above Lemma 6 to hold when $h, g^i \in L^2(\Omega)$ (i.e., $g = h + D_i g^i \in H^{-1}(\Omega)$). It is only necessary for $g \in L^2(\Omega)$ when proving that the solutions have higher regularity, such as if $z_0 \in \mathcal{L}_2$, then*

$$u \in L^\infty(0,T;V) \cap L^2(0,T;V \cap H^2(\Omega)),$$
$$\eta^t \in L^\infty(0,T;L^2_\mu(\mathbb{R}^+;V \cap H^2(\Omega)), \quad \text{and} \quad z \in C(0,T;V).$$

The following lemma provides the existence of a bounded absorbing set in $\mathcal{L}_1 = H \times L^2_\mu(\mathbb{R}^+, V)$.

Lemma 7. *Assume that the conditions in Lemma 6 hold. Then, there exist positive constants C and $t_0 = t_0(||z_0||_{\mathcal{L}_1})$ such that,*

$$||S(t)z_0||_{\mathcal{L}_1} \leq C, \quad \forall\, t \geq t_0,$$

Therefore,

$$E_0 = \{z \in \mathcal{L}_1 : ||z||_{\mathcal{L}_1} \leq C\}$$

is a bounded absorbing set for the semigroup $S(t)_{t\geq t_0}$ in $\mathcal{L}_1$.

Proof. Multiplying Equation (12) by u and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}|u|^2 + ||u||^2 + \int_0^\infty \mu(s)\langle -\Delta\eta^t(s), u(t)\rangle ds + \langle f(u), u\rangle = \langle h, u\rangle + \langle D_i g^i, u\rangle. \tag{14}$$

For the third term on the left-hand side, we use (8), where $u = \eta^t_t + \eta^t_s$ to deduce that

$$\begin{aligned}\int_0^\infty \mu(s)\langle -\Delta\eta^t(s), u(t)\rangle ds &= \int_0^\infty \mu(s)\langle -\Delta\eta^t(s), \eta^t_t\rangle ds + \int_0^\infty \mu(s)\langle -\Delta\eta^t(s), \eta^t_s\rangle ds \\ &= \frac{1}{2}\frac{d}{dt}\int_0^\infty \mu(s)||\eta^t(s)||^2 ds + \int_0^\infty \mu(s)\langle \nabla\eta^t(s), \nabla\eta^t_s(s)\rangle ds \\ &= \frac{1}{2}\frac{d}{dt}||\eta^t||^2_{\mu,1} + \langle \eta^t, \eta^t_s\rangle_{\mu,1}.\end{aligned} \tag{15}$$

From Lemma 1, $\langle \eta^t, \eta_s^t \rangle_{\mu,1} \geqslant \frac{\delta}{2}||\eta^t||^2_{\mu,1}$. We therefore obtain that

$$\int_0^\infty \mu(s)\langle -\Delta\eta^t(s), u(t)\rangle ds \geqslant \frac{1}{2}\frac{d}{dt}||\eta^t||_{\mu,1} + \frac{\delta}{2}||\eta^t||_{\mu,1}. \tag{16}$$

On the other hand, note that

$$\langle f(u), u)\rangle \geqslant \alpha|u|_p^p - k_1|\Omega|^2, \tag{17}$$

$$\langle h, u\rangle \leqslant \frac{1}{\lambda}|h|^2 + \frac{\lambda}{4}|u|^2, \tag{18}$$

where λ is the Sobolev constant, which satisfies the condition $||u||^2 \geqslant \lambda|u|^2$. Moreover,

$$\langle D_i g^i, u\rangle = -\int \widetilde{g}\cdot\nabla u \leqslant |\widetilde{g}|^2 + \frac{1}{4}||u||^2. \tag{19}$$

where $\widetilde{g} = (g^1, g^2, \cdots, g^n)$, $|\widetilde{g}|^2 = \sum_{i=1}^n |g^i|^2$. By taking (15)–(19) into (14), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|u|^2 + ||u||^2 + \frac{1}{2}\frac{d}{dt}||\eta^t||^2_{\mu,1} + \frac{\delta}{2}||\eta^t||^2_{\mu,1} + \alpha|u|_p^p\\ \leqslant& k_1|\Omega|^2 + \frac{1}{\lambda}|h|^2 + \frac{\lambda}{4}||u||_2^2 + |\widetilde{g}|^2 + \frac{1}{4}||u||^2.\end{aligned}$$

By substituting $||u||^2 \geqslant \lambda|u|_2^2$ into the formula and rearranging, it follows that

$$\frac{1}{2}\frac{d}{dt}(|u|^2 + ||\eta^t||^2_{\mu,1}) + (||u||^2 - \frac{\lambda}{2}|u|^2 + \frac{\delta}{2}||\eta^t||^2_{\mu,1}) \leqslant \frac{\widetilde{C}(|h|, |\widetilde{g}|, \lambda, |\Omega|)}{2}.$$

Taking $\gamma = min\{\lambda, \delta\}$, we have

$$\frac{d}{dt}(|u|^2 + ||\eta^t||^2_{\mu,1}) + \gamma(|u|^2 + ||\eta^t||^2_{\mu,1}) \leqslant \widetilde{C}(|h|, |\widetilde{g}|, \lambda, |\Omega|),$$

for which Gronwall's lemma implies that

$$|u(t)|^2 + ||\eta^t(s)||^2_{\mu,1} \leqslant (|u_0(x)|^2 + ||\eta^0(s)||^2_{\mu,1})e^{-\gamma t} + \frac{1}{\gamma}\widetilde{C}.$$

Therefore,

$$||z(t)||_{\mathcal{L}_1} \leqslant ||z_0||_{\mathcal{L}_1}e^{-\gamma t} + \frac{\widetilde{C}}{\gamma}.$$

Taking $t_0 = \frac{1}{\gamma}ln\frac{\gamma}{\widetilde{C}}$, $C^2 = \frac{2}{\gamma}\widetilde{C}$, we have

$$||z(t)||_{\mathcal{L}_1} \leqslant C, \quad \forall\, t \geqslant t_0.$$

□

Next, we will prove that the semigroup generated by the weak solution of Equation (12) satisfies the conditions of Lemma 4, thereby obtaining the asymptotic smoothness of the semigroup.

Lemma 8. *Assume that the conditions in Lemma 6 hold. Then, $\{S(t)\}_{t\geq 0}$ is asymptotically smooth.*

Proof. It suffices to prove that the statement in Lemma 4 holds true for the absorbing set E_0 constructed in Lemma 7. To this end, take any $(u_0^1, \eta_1^0), (u_0^1, \eta_1^0) \in \mathcal{L}_1$ and set $(u_p(t), \eta_p^t) = S_t(u_0^p, \eta_p^0),\ p = 1, 2$. Then, we have

$$(u_p)_t - \Delta u_p - \int_0^\infty \mu(s)\Delta\eta_p^t(s)ds + f(u_p) = h + D_i g^i, \quad (p = 1, 2).$$

Let $w = u_1 - u_2$, $\eta = \eta_1^t - \eta_2^t$; it is obvious that

$$w_t - \Delta w - \int_0^\infty \mu(\Delta s)\Delta\eta^t(s)ds + f(u_1) - f(u_2) = 0. \tag{20}$$

Multiplying the Equation (20) by w, we deduce that

$$\frac{1}{2}\frac{d}{dt}|w|^2 + |\nabla w|^2 + \frac{1}{2}\frac{d}{dt}||\eta^t||_{\mu,1}^2 + \frac{\delta}{2}||\eta^t||_{\mu,1}^2 \leqslant \langle f(u_2) - f(u_1), u_1 - u_2\rangle.$$

From (3), it follows that

$$\langle f(u_2) - f(u_1), u_1 - u_2\rangle \leq l|u_1 - u_2|^2 = l|w|^2.$$

Note that

$$|w|^2 \leqslant \frac{1}{\lambda_1}|\nabla w|^2.$$

Therefore

$$\frac{d}{dt}(|w|^2 + \|\eta^t\|_{\mu,1}^2) + (2\lambda_1 + \delta)(|w|^2 + \|\eta^t\|_{\mu,1}^2) \leqslant l|w|^2.$$

By Gronwall's lemma, we get for all $t \geq 0$ that

$$\begin{aligned} |w(t)|^2 + \|\eta^t\|_{\mu,1}^2 &\leqslant e^{-(2\lambda_1+\delta)t}(|w(0)|^2 + \|\eta^0\|_{\mu,1}^2) + l\int_0^t e^{-(2\lambda_1+\delta)(t-s)}|w(s)|^2 ds \\ &\leqslant e^{-(2\lambda_1+\delta)t}\big(|u_0^1 - u_0^2|^2 + \|\eta_1^0 - \eta_2^0\|_{\mu,1}^2\big) + l\int_0^t |w(s)|^2 ds, \end{aligned} \tag{21}$$

where $(u_0^1, \eta_1^0), (u_0^2, \eta_2^0) \in B_0$ are the initial data of solutions u_1, u_2, and B_0 is the absorbing set. For any $\varepsilon > 0$, we choose $T = T(\varepsilon, B_0) > 0$ such that

$$e^{-\frac{(2\lambda_1+\delta)}{2}T}\big(|u_0^1 - u_0^2|^2 + \|\eta_1^0 - \eta_2^0\|_{\mu,1}^2\big)^{\frac{1}{2}} \leqslant C_{B_0}e^{-\frac{(2\lambda_1+\delta)}{2}T} < \varepsilon.$$

From (21), it follows that

$$\|S(T)(u_0^1, \eta_1^0) - S(T)(u_0^2, \eta_2^0)\| \leqslant \varepsilon + \Psi_T\big((u_0^1, \eta_1^0), (u_0^2, \eta_2^0)\big),$$

where

$$\Psi_T\big((u_0^1, \eta_1^0), (u_0^2, \eta_2^0)\big) = \sqrt{l}\left(\int_0^T |u_1(s) - u_2(s)|^2 ds\right)^{\frac{1}{2}}.$$

Next, we show that $\Psi_T(\cdot, \cdot) \in \mathrm{Cont}(B_0)$.

By (2.1), we know that

$$|f(s)| \leqslant c_0(1 + |s|^{p-1}), \ \forall s \in \mathbb{R}.$$

As a result, $u \in L^p([0, T] \times \Omega)$ implies that $f(u) \in L^q([0, T] \times \Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Specifically,

$$\begin{aligned} \|f(u)\|_{L^q([0,T]\times\Omega)}^q &\leqslant c_0\int_0^T\int_\Omega (1 + |u|^{p-1})^q dxdt \\ &\leqslant C\int_0^T\int_\Omega (1 + |u|^{q(p-1)})dxdt \\ &\leqslant CT|\Omega| + C\|u\|_{L^p([0,T]\times\Omega)}^p. \end{aligned}$$

From Lemma 6, it follows that $u_t \in L^2(0,T;H^{-1}) \cap L^q(0,T;L^q)$. Note that both $L^2(0,T;H^{-1})$ and $L^q(0,T;L^q)$ are continuous and included in $L^q(0,T;H^{-s})$ (with $q<2$ since $p>2$ and s are chosen such that $H^s \hookrightarrow L^p$). Thus, for any sequence $\{(u_0^n,\eta_n^0)\}_{n\geqslant 1} \subset B_0$, setting $(u^n(t),\eta_n^t) = S(t)(u_0^n,\eta_n^0)$, it follows that

$$u^n \in L^2(0,T;H_0^1), \quad u_t^n \in L^q(0,T;H^{-s}).$$

Since

$$H_0^1 \hookrightarrow\hookrightarrow L^2 \hookrightarrow H^{-s}.$$

By the Aubin–Lions Lemma 5, there exists a subsequence $\{(u_0^{n_k},\eta_{n_k}^0)\}_{k\geqslant 1} \subset \{(u_0^n,\eta_n^0)\}_{n\geqslant 1}$ such that

$$\lim_{i\to\infty}\lim_{j\to\infty} \Psi_T\big((u_0^{n_i},\eta_{n_i}^0),(u_0^{n_j},\eta_{n_j}^0)\big) = \lim_{i\to\infty}\lim_{j\to\infty} \sqrt{l}\left(\int_0^T |u_{n_i}(s)-u_{n_j}(s)|^2 ds\right)^{\frac{1}{2}} = 0.$$

This implies that $\Psi_T(\cdot,\cdot) \in Cont(B_0)$ and the proof is complete by Lemma 4. □

Based on Lemmas 2, 3 and 8, we are ready to derive the main result regarding the existence of the global attractor immediately.

Theorem 1. *Assume that* (2)–(9) *hold and* $h, g^i \in L^2(\Omega)$. *The semigroup* $\{S(t)\}_{t\geq 0}$ *generated by Equation* (12) *possesses a global attractor* $\mathcal{A}$ *in* $\mathcal{L}_1 = H \times L_\mu^2(\mathbb{R}^+,V)$.

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References

1. Fabrizio, M.; Morro, A. *Mathematical Problems in Linear Viscoelasticity*; SIAM Studies in Applied Mathematics; SIAM: Philadelphia, PA, USA, 1992; Volume 12 .
2. Renardy, M.; Hrusa, W.; Nohel, J. *Mathematical Problems in Viscoelasticity, Longman, Harlow*; John Wiley: New York, NY, USA, 1987.
3. Dafermos, C. Asymptotic stability in viscoelasticity. *Arch. Ration. Mech. Anal.* **1970**, *37*, 297–308. [CrossRef]
4. Wang, L.; Xu, D. Asymptotic behavior of a class of reaction–diffusion equations with delay. *J. Math. Anal. Appl.* **2003**, *281*, 439–453.
5. Chepyzhov, V.; Gatti, S.; Grasselli, M.; Miranville, A.; Patta, V. Trajectory and global attractors for evolutions equations with memory. *Appl. Math. Lett.* **2006**, *19*, 87–96.
6. Gatti, S.; Grasselli, M.; Pata, V. Lyapunov functionals for reaction–diffusion equations with memory. *Math. Methods Appl. Sci.* **2005**, *28*, 1725–1735. [CrossRef]
7. Giorgi, C.; Pata, V.; Marzocchi, A. Asymptotic behavior of a semilinear problem in heat conduction with memory. *NoDEA Nonlinear Differ. Equ. Appl.* **1998**, *5*, 333–354. [CrossRef]
8. Caraballo, T.; Garrido-Atienza, M.; Schmalfuß, B.; Valero, J. Global attractor for a non-autonomous integro-differential equation in materials with memory. *Nonlinear Anal.* **2010**, *73*, 183–201. [CrossRef]
9. Kloeden, P.; Real, J.; Sun, C. Robust exponential attractors for non–autonomous equations with memory. *Commun. Pure Appl. Anal.* **2011**, *10*, 885–915. [CrossRef]
10. Li, Y.; Wang, Y. The existence and asymptotic behavior of solutions to fractional stochastic evolution equations with infinite delay. *J. Differ. Equ.* **2019**, *266*, 3514–3558. [CrossRef]
11. Tao, C.; Zhe, C.; Tang, Y. Finite dimensionality of global attractors for a non-classical reaction–diffusion equation with memory. *Appl. Math. Lett.* **2012**, *25*, 357–362.

12. Wang, J.; Ma, Q.; Zhou, W. Attractor of the nonclassical diffusion equation with memory on time-dependent space. *AIMS Math.* **2023**, *8*, 14820–14841. [CrossRef]
13. Zhang, J.; Xie, Z.; Xie, Y. Asymptotic behavior of solutions to nonclassical diffusion equations with degenerate memory and a time-dependent perturbed parameter. *Electron. J. Differ. Equ.* **2024**, *2024*, 1–27. [CrossRef]
14. Ma, Q.; Wang, S.; Zhong, C. Necessary and sufficient conditions for the existence of global attractors for semigroups and applications. *Indiana Univ. Math. J.* **2002**, *51*, 1541–1559. [CrossRef]
15. Zhong, C.; Sun, C.; Niu, M. On the existence of global attractor for a class of infinite dimensional dissipative nonlinear dynamical systems. *Chin. Ann. Math. Ser. B* **2005**, *26*, 393–400. [CrossRef]
16. Sun, C.; Zhong, C. Attractors for the semilinear reaction–diffusion equation with distribution derivatives in unbounded domains. *Nonlinear Anal.* **2005**, *63*, 49–65. [CrossRef]
17. Zhang, J.; Zhong, C. The existence of global attractors for a class of reaction–diffusion equations with distribution derivatives terms in R^n. *J. Math. Anal. Appl.* **2015**, *427*, 365–376. [CrossRef]
18. Chueshov, I. *Dynamics of Quasi-Stable Dissipative Systems*; Springer: Cham , Switzerland, 2015.
19. Xie, Y.; Li, Y.; Zeng, Y. Uniform Attractors for Nonclassical Diffusion Equations with Memory. *J. Funct. Spaces* **2016**, *2016*, 5340489. [CrossRef]
20. Lions, J. *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*; Dunod: Paris, France, 1969.
21. Xu, J.; Caraballo, T.; Valero, J. Asymptotic behavior of a semilinear problem in heat conduction with long time memory and non-local diffusion. *J. Differ. Equ.* **2022**, *327*, 418–447. [CrossRef]

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