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Order-Bounded Difference in Weighted Composition Operators Between Fock Spaces

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Abstract: There are two aims in this paper. The first aim is to characterize the order-bounded weighted composition operators between Fock spaces, and the second is to further characterize the order-bounded difference in weighted composition operators between Fock spaces. At the same time, six examples are given to illustrate the relations between boundedness and ordered boundedness. Moreover, an interesting result is found that differences in weighted composition operators defined by some special weighted functions and symbol functions are order-bounded between Fock spaces if and only if each weighted composition operator is compact between Fock spaces. Finally, two open questions are also put forward for converting larger Fock spaces into smaller ones.

Keywords: weighted composition operator; Fock space; order-boundedness; compactness

MSC: 47B38; 47B33; 47B37; 30H05

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1. Introduction

Let $\mathbb{C}$ denote the complex plane and $H(\mathbb{C})$ the space of all holomorphic functions on $\mathbb{C}$ with the usual compact open topology. For two given functions φ, $u \in H(\mathbb{C})$, the weighted composition operator on (or between) the subspaces of $H(\mathbb{C})$ is defined by

$$W_{u,\varphi}f(z) = u(z)f(\varphi(z)).$$

If $u \equiv 1$, then $W_{u,\varphi}$ is the composition operator usually denoted by C_φ. If $\varphi(z) = z$, then $W_{u,\varphi}$ is the multiplication operator usually denoted by M_u. For various investigations of these operators acting on holomorphic functions, see [1–9].

Let φ_j, $u_j \in H(\mathbb{C})$ for $j = 1, 2$. The difference in the operators W_{u_1,φ_1} and W_{u_2,φ_2} on (or between) the subspaces of $H(\mathbb{C})$ is defined by

$$(W_{u_1,\varphi_1} - W_{u_2,\varphi_2})f(z) = u_1(z)f(\varphi_1(z)) - u_2(z)f(\varphi_2(z)).$$

While some basic properties of weighted composition operators have been studied widely and deeply, the topological structure of the composition operators and weighted composition operators has been paid more attention. To this end, Shapiro and Sundberg in [10] studied whether two composition operators belong to the same connected component when their difference is compact. Influenced by this inspiring question, differences in composition operators or weighted composition operators have been studied and have become a very active topic. For example, Hosokawa in [11] characterized the bounded and compact difference in weighted composition operators in Bloch spaces. Moorhouse in [12] gave a characterization of a compact difference for composition operators in weighted Bergman spaces. Some interesting phenomena were also found in the progress of characterizing difference in weighted composition operators. Here, we will just list a few to

illustrate this and inspire you. For instance, Choe et al. in [13] characterized compact difference in composition operators in weighted Bergman space of the upper half-plane, but Matache in [14] proved that the compact composition operators in this space do not exist. Moorhouse in [12] proved that the difference in noncompact composition operators can be compact in the weighted Bergman space of the unit disk. All of these have aroused the interest of experts and scholars in this research topic (see [15–18] and the references therein).

Next, let us turn our attention to the order-boundedness of operators in holomorphic function spaces. Let μ be a positive measure on $\mathbb{D}$ and X be a Banach space of holomorphic functions. Let $0 < p < \infty$, $(\Omega, \mathcal{A}, \mu)$ be a measure space and

$$L^p(\Omega, \mathcal{A}, \mu) = \Big\{ f | f : \Omega \to \mathbb{C} \text{ is measurable and } \int_\Omega |f|^p d\mu < \infty \Big\}.$$

We say that an operator $T : X \to L^p(\Omega, \mathcal{A}, \mu)$ is order-bounded if T maps the close unit ball B_X of X into an order interval of $L^p(\Omega, \mathcal{A}, \mu)$. From this, we give the following definition.

Definition 1. *Let $0 < p < \infty$. An operator $T : X \to L^p(\Omega, \mathcal{A}, \mu)$ is said to be order-bounded if there exists a nonnegative function $h \in L^p(\Omega, \mathcal{A}, \mu)$ such that for all $f \in X$ with $\|f\|_X \leq 1$, it holds that*

$$|(Tf)(x)| \leq h(x), \ \ a.e.\ [\mu].$$

Hunziker and Jarchow in [19] introduced this definition, and they also proved that if the composition operator is order-bounded from Hardy space to weighted Bergman space, then it is also compact. Ueki in [20] characterized the order-boundedness of weighted composition operators from weighted-type space to Bergman space, and proved the equivalent relations between boundedness and ordered boundedness. A sufficient condition for the order-boundedness of weighted composition operators in Dirichlet spaces was given in [21]. Subsequently, the necessary and sufficient conditions of order-bounded weighted composition operators in Dirichlet spaces was obtained in [22]. Wolf in [23] studied the order-bounded weighted composition operators from weighted Bergman spaces with the weights to $L^q(\mu)$. For some other studies, also see [24–28].

Inspired by these studies of order-boundedness, it is natural to consider how to characterize the order-bounded weighted composition operators and difference in weighted composition operators between Fock spaces. Indeed, Tien and Khoi in [29] characterized the boundedness and compactness of differences in weighted composition operators between Fock spaces. This reinforces the need to study the order-bounded difference in weighted composition operators between Fock spaces. Hence, one of the aims of this paper is to characterize order-bounded difference in weighted composition operators between Fock spaces.

In this paper, positive numbers are denoted by C, and they may vary in different situations. The notation $a \lesssim b$ (resp. $a \gtrsim b$) means that there is a normal number C such that $a \leq Cb$ (resp. $a \geq Cb$).

2. Preliminaries

For $0 < q < \infty$, $L^q(\mathbb{C}, e^{\frac{-q|z|^2}{2}} dA)$ is the space of all f in $H(\mathbb{C})$ for which

$$\|f\|_{L^q(\mathbb{C}, e^{\frac{-q|z|^2}{2}} dA)} = \Big(\int_{\mathbb{C}} |f(z)|^q e^{-\frac{q|z|^2}{2}} dA(z) \Big)^{\frac{1}{q}} < \infty,$$

where dA is the usual Lebesgue area measure on $\mathbb{C}$. For $0 < p < \infty$, the Fock space $\mathcal{F}^p$ is the set of all functions $f \in H(\mathbb{C})$ such that

$$\|f\|_p = \Big(\frac{p}{2\pi}\int_{\mathbb{C}} |f(z)|^p e^{-\frac{p|z|^2}{2}} dA(z)\Big)^{\frac{1}{p}} < \infty.$$

If $p = \infty$, then the space $\mathcal{F}^\infty$ is the set of all functions $f \in H(\mathbb{C})$ such that

$$\|f\|_\infty = \sup_{z\in\mathbb{C}} |f(z)|e^{-\frac{|z|^2}{2}} < \infty.$$

If $p = 2$, it is a Hilbert space with the inner product

$$\langle f, g\rangle = \frac{1}{\pi}\int_{\mathbb{C}} f(z)\overline{g(z)}e^{-|z|^2} dA(z).$$

By [30], $\mathcal{F}^p$ is a Banach space for $1 \le p \le \infty$. $\mathcal{F}^p$ is a complete metric space with the distance $d(f, g) = \|f - g\|_p^p$ when $0 < p < 1$.

Let $w \in \mathbb{C}$. On $\mathbb{C}$, the function is defined:

$$K_w(z) = e^{\overline{w}z}.$$

Then, $\|K_w\|_p = e^{\frac{|w|^2}{2}}$ for $0 < p \le \infty$. Indeed, K_w is the reproducing kernel of $\mathcal{F}^2$, which means that $\langle f, K_z\rangle = f(z)$ for $f \in \mathcal{F}^2$ and all $z \in \mathbb{C}$. Let k_w be the normalization of K_w, that is,

$$k_w(z) = e^{\overline{w}z - \frac{|w|^2}{2}}.$$

From the calculations, it follows that $\|k_w\|_p = 1$ for $0 < p \le \infty$, and $k_w \to 0$ as $|w| \to \infty$.

Here, we would also like to mention some applications of the space $\mathcal{F}^2$ in physics. For instance, $\mathcal{F}^2$ is used to legitimately state quantum that lack definite particle numbers like the states of quantum harmonic oscillators. For various aspects of the theory of weighted composition operators acting on Fock spaces, see [31–36].

The following evaluation can be found (see [30]).

Lemma 1. *For each $f \in \mathcal{F}^p$ and $z \in \mathbb{C}$, it holds that*

$$|f(z)| \le \begin{cases} \big(\frac{2\pi}{p}\big)^{1/p} e^{\frac{|z|^2}{2}} \|f\|_p, & 0 < p < \infty \\ e^{\frac{|z|^2}{2}} \|f\|_p, & p = \infty. \end{cases}$$

Remark 1. *Regardless of whether $0 < p < \infty$ or $p = \infty$, for each $f \in \mathcal{F}^p$ and $z \in \mathbb{C}$, it holds that*

$$|f(z)| \le Ce^{\frac{|z|^2}{2}} \|f\|_p.$$

where the positive constant C can be taken as $\max\{1, (\frac{2\pi}{p})^{1/p}\}$.

Next, a sufficient condition for the boundedness of the operator $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ was given as follows.

Lemma 2. *Let $0 < p \le \infty$ and $0 < q < \infty$. The operator $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ is bounded if u and φ satisfy the condition*

$$\int_{\mathbb{C}} |u(z)|^q e^{\frac{q(|\varphi(z)|^2 - |z|^2)}{2}} dA(z) < \infty. \tag{1}$$

Proof. For $f \in \mathcal{F}^p$ and $z \in \mathbb{C}$, it follows from Remark 1 that

$$|W_{u,\varphi}f(z)| = |u(z)f(\varphi(z))| \leq C|u(z)|e^{\frac{|\varphi(z)|^2}{2}}\|f\|_p.$$

Thus, we have

$$\begin{aligned}\|W_{u,\varphi}f\|_q^q &= \frac{q}{2\pi}\int_{\mathbb{C}} |u(z)f(\varphi(z))|^q e^{-\frac{q|z|^2}{2}} dA(z)\\ &\leq \frac{q}{2\pi}C^q\int_{\mathbb{C}} |u(z)|^q e^{\frac{q(|\varphi(z)|^2-|z|^2)}{2}}\|f\|_p^q dA(z)\\ &= \frac{q}{2\pi}C^q\int_{\mathbb{C}} |u(z)|^q e^{\frac{q(|\varphi(z)|^2-|z|^2)}{2}} dA(z)\|f\|_p^q. \end{aligned} \tag{2}$$

From (2), we obtain

$$\|W_{u,\varphi}f\|_q \leq C\Big(\frac{q}{2\pi}\int_{\mathbb{C}} |u(z)|^q e^{\frac{q(|\varphi(z)|^2-|z|^2)}{2}} dA(z)\Big)^{\frac{1}{q}}\|f\|_p.$$

By (1), the operator $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ is bounded. The proof is completed. □

The following result was obtained in [31].

Lemma 3. *Let $0 < p \leq q < \infty$ and $\varphi(z) = az + b$ with $0 < |a| \leq 1$ and u a nonzero function in $\mathcal{F}^q$. Then, the following statements hold.*

(i) The operator $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ is bounded if and only if

$$m(u,\varphi) = \sup_{z\in\mathbb{C}} |u(z)|e^{\frac{|\varphi(z)|^2-|z|^2}{2}} < \infty. \tag{3}$$

(ii) The operator $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ is compact if and only if

$$\lim_{|z|\to\infty} |u(z)|e^{\frac{|\varphi(z)|^2-|z|^2}{2}} = 0.$$

The case $a = 0$ is not considered in Lemma 3. Indeed, we have the following result for this case.

Lemma 4. *Let $0 < p, q \leq \infty$, $u(z) = e^{\bar{c}z}$ and $\varphi(z) = b$. Then, the operator $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ is bounded.*

Proof. For each $f \in \mathcal{F}^p$, from Remark 1, we have

$$\|W_{u,\varphi}f\|_q = |f(b)|\|u\|_q \leq Ce^{\frac{|b|^2}{2}}\|u\|_q\|f\|_p = Ce^{\frac{|b|^2+|c|^2}{2}}\|f\|_p,$$

which shows that the operator $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ is bounded. The proof is completed. □

In [33], the author proved that if u is a nonzero function in $\mathcal{F}^q$ and $m(u,\varphi)$ is finite, then $\varphi(z) = az + b$ with $|a| \leq 1$. From this, we therefore assume that $\varphi(z) = az + b$ with $|a| \leq 1$ throughout this paper. From Lemmas 3 and 4, we have the next result, which extends the case $p = q$ (see [37]) to the case $0 < p \leq q < \infty$.

For $z \in \mathbb{C}$, we write $z = x + iy$ and $\bar{c} + a\bar{b} = l + i\beta$ for convenience.

Lemma 5. *Let $0 < p \leq q < \infty$, $u(z) = e^{\bar{c}z}$ and $\varphi(z) = az + b$ with $|a| \leq 1$. Then, the operator $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ is bounded if and only if either $|a| < 1$ or $|a| = 1$ and $c + \bar{a}b = 0$.*

Proof. If $|a| < 1$, then

$$\begin{aligned}\sup_{z\in\mathbb{C}} |u(z)|e^{\frac{|\varphi(z)|^2-|z|^2}{2}} &= \sup_{x,y\in\mathbb{R}} e^{\frac{|b|^2}{2}+\frac{l^2+\beta^2}{2(1-|a|^2)}} e^{-\frac{1-|a|^2}{2}(x-\frac{l}{1-|a|^2})^2-\frac{1-|a|^2}{2}(y+\frac{\beta}{1-|a|^2})^2} \\ &= e^{\frac{|b|^2}{2}+\frac{l^2+\beta^2}{2(1-|a|^2)}} \sup_{x,y\in\mathbb{R}} e^{-\frac{1-|a|^2}{2}(x-\frac{l}{1-|a|^2})^2-\frac{1-|a|^2}{2}(y+\frac{\beta}{1-|a|^2})^2} \\ &= e^{\frac{|b|^2}{2}+\frac{|\bar{c}+a\bar{b}|^2}{2(1-|a|^2)}} \\ &< \infty. \end{aligned} \tag{4}$$

This means that $m(u,\varphi)$ is finite. From Lemmas 3 and 4, we obtain that the operator $W_{u,\varphi}$ is bounded.

If $|a| = 1$ and $c + \bar{a}b = 0$, then

$$\begin{aligned} m(u,\varphi) = \sup_{z\in\mathbb{C}} |u(z)|e^{\frac{|\varphi(z)|^2-|z|^2}{2}} &= \sup_{x,y\in\mathbb{R}} e^{\frac{|b|^2}{2}} e^{\frac{-(1-|a|^2)x^2}{2}+lx} e^{\frac{-(1-|a|^2)y^2}{2}-\beta y} \\ = e^{\frac{|b|^2}{2}} \sup_{x,y\in\mathbb{R}} e^{lx-\beta y} &= e^{\frac{|b|^2}{2}}, \end{aligned}$$

which shows that $m(u,\varphi)$ is finite. By Lemma 3, we find the operator $W_{u,\varphi}$ is bounded.

Conversely, let us assume that $W_{u,\varphi}$ is bounded. Then, from Proposition 3.1 in [31], we obtain that $m(u,\varphi)$ is finite and $|a| \le 1$. By the above proofs, we have

$$m(u,\varphi) = \begin{cases} e^{\frac{|b|^2}{2}+\frac{|\bar{c}+a\bar{b}|^2}{2(1-|a|^2)}}, & |a| < 1 \\ e^{\frac{|b|^2}{2}} \sup\limits_{x,y\in\mathbb{R}} e^{lx-\beta y}, & |a| = 1. \end{cases} \tag{5}$$

For the case $|a| = 1$, form (5), we see that $W_{u,\varphi}$ is bounded if and only if $c + \bar{a}b = 0$. The proof is completed. □

The next results are about the characterizations of compactness. The first was obtained in [31].

Lemma 6. *Let $0 < p, q < \infty$, $\varphi(z) = b$ and $u(z) = e^{\bar{c}z}$. Then, the operator $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ is compact.*

By Lemmas 3 and 6, we have

Lemma 7. *Let $0 < p \le q < \infty$, $\varphi(z) = az + b$ with $|a| \le 1$ and $u(z) = e^{\bar{c}z}$. Then, the operator $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ is compact if and only if $|a| < 1$.*

Proof. Assume that $|a| < 1$. By Lemma 6, we can further assume that $0 < |a| < 1$. From the Cauchy–Schwarz inequality, we obtain

$$lx - \beta y \le |z||\bar{c} + a\bar{b}| \le |z|(|c| + |ab|).$$

Then,

$$|u(z)|e^{\frac{|\varphi(z)|^2-|z|^2}{2}} \le e^{\frac{|b|^2}{2}} e^{\frac{-(1-|a|^2)|z|^2}{2}+(|ab|+|c|)|z|},$$

which shows

$$\lim_{|z|\to\infty} |u(z)|e^{\frac{|\varphi(z)|^2-|z|^2}{2}} = 0.$$

By Lemma 3, the operator $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ is compact.

Conversely, let us assume that the operator $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ is compact. If $|a| = 1$, then from the proof of Lemma 5 we have

$$\lim_{|z|\to\infty} |u(z)| e^{\frac{|\varphi(z)|^2-|z|^2}{2}} = e^{\frac{|b|^2}{2}} \lim_{x^2+y^2\to\infty} e^{lx-\beta y}.$$

This limit does not exist. Therefore, we obtain $|a| < 1$. The proof is completed. □

Here, we cite the following two results (see [29]) in order to study the order-boundedness of difference in weighted composition operators between Fock spaces.

Lemma 8. *Let $0 < p, q < \infty$ and $u_1, u_2, \varphi_1, \varphi_2$ be holomorphic functions on $\mathbb{C}$ with $u_1 \not\equiv 0$, $u_2 \not\equiv 0$ and $\varphi_1 \not\equiv \varphi_2$. If the operator $W_{u_1,\varphi_1} - W_{u_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ is bounded, then*

$$m(u_j, \varphi_j) = \sup_{z\in\mathbb{C}} |u_j(z)|^2 e^{|\varphi_j(z)|^2-|z|^2} < \infty.$$

In this case, $\varphi_j = a_j z + b_j$ with $|a_j| \le 1$ and $j = 1, 2$.

Lemma 9. *Let $0 < p \le q < \infty$ and $u_1, u_2, \varphi_1, \varphi_2$ be holomorphic functions on $\mathbb{C}$ with $u_1 \not\equiv 0$, $u_2 \not\equiv 0$ and $\varphi_1 \not\equiv \varphi_2$. Then, the operator $W_{u_1,\varphi_1} - W_{u_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ is compact if and only if both W_{u_1,φ_1} and $W_{u_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ are compact.*

Next, we have an integral estimate.

Lemma 10. *Let $0 < q < \infty$. For $d \in \mathbb{C}$, it holds that*

$$\int_{\mathbb{C}} |e^{dz}|^q dA(z) = \infty.$$

Proof. Let $z = x + iy$ and $d = a + ib$. Then,

$$\begin{aligned}
\int_{\mathbb{C}} |e^{dz}|^q dA(z) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |e^{(a+ib)(x+iy)}|^q dxdy \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |e^{(ax-by)+i(bx+ay)}|^q dxdy \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{qax-qby} dxdy \\
&= \int_{-\infty}^{+\infty} e^{qax} dx \int_{-\infty}^{+\infty} e^{-qby} dy \\
&= \infty.
\end{aligned}$$

The proof is completed. □

3. Order-Bounded Weighted Composition Operators

We begin to study the order-bounded weighted composition operators between Fock spaces. For the special symbol functions, we characterize the order-bounded weighted composition operators between Fock spaces and prove that order-boundedness is equivalent to compactness.

First, let us recall that the operator $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ is order-bounded if and only if there exists a nonnegative function $g \in L^q(\mathbb{C}, e^{\frac{-q|z|^2}{2}} dA)$ such that for all $f \in \mathcal{F}^p$ with $\|f\|_p \le 1$, it holds that

$$|W_{u,\varphi} f(z)| \le g(z), \ \ a.e.\ [e^{\frac{-q|z|^2}{2}} dA].$$

Theorem 1. *Let $0 < p \leq \infty$ and $0 < q < \infty$. Then, the operator $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ is order-bounded if and only if u and φ satisfy the condition*

$$\int_{\mathbb{C}} |u(z)|^q e^{\frac{q(|\varphi(z)|^2-|z|^2)}{2}} dA(z) < \infty. \tag{6}$$

Proof. Assume that (6) holds. Let $f \in \mathcal{F}^p$ with $\|f\|_p \leq 1$. From Remark 1 , we obtain

$$|W_{u,\varphi}f(z)| = |u(z)f(\varphi(z))| \leq C|u(z)|e^{\frac{|\varphi(z)|^2}{2}}. \tag{7}$$

From (6), we know that the function

$$h(z) = |u(z)|e^{\frac{|\varphi(z)|^2}{2}}, \ z \in \mathbb{C},$$

belongs to $L^q(\mathbb{C}, e^{\frac{-q|z|^2}{2}} dA)$, and it follows from (7) that $|W_{u,\varphi}f(z)| \lesssim h(z)$. This shows that the operator $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ is order-bounded.

Now, assume that the operator $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ is order-bounded. Then, there exists a nonnegative function $g \in L^q(\mathbb{C}, e^{\frac{-q|z|^2}{2}} dA)$ such that for $f \in \mathcal{F}^p$ with $\|f\|_p \leq 1$, it follows that

$$|W_{u,\varphi}f(z)| \leq g(z), \ \ a.e. \ [e^{\frac{-q|z|^2}{2}} dA]. \tag{8}$$

For $w \in \mathbb{C}$, from Preliminaries, we know that the function

$$k_w(z) = e^{\overline{w}z - \frac{|w|^2}{2}}$$

and $\|k_w\|_p = 1$. Then, replacing f by k_w in (8), we obtain

$$g(z) \geq |W_{u,\varphi}k_w(z)| = |u(z)k_w(\varphi(z))| = |u(z)e^{\overline{w}\varphi(z) - \frac{|w|^2}{2}}|, \ \ a.e. \ [e^{\frac{-q|z|^2}{2}} dA]. \tag{9}$$

Letting $w = \varphi(z)$ in (9), we have

$$g(z) \geq |u(z)||e^{\frac{|\varphi(z)|^2}{2}}| = |u(z)|e^{\frac{|\varphi(z)|^2}{2}}, \ \ a.e. \ [e^{\frac{-q|z|^2}{2}} dA],$$

which implies

$$\int_{\mathbb{C}} |u(z)|^q e^{\frac{q(|\varphi(z)|^2-|z|^2)}{2}} dA(z) < \infty.$$

That is, (6) holds, and the proof is completed. □

Example 1. *Let $0 < p \leq \infty$, $0 < q < \infty$, $u(z) = e^{(1-i)z}$ and $\varphi(z) = (\frac{1}{2} - \frac{1}{2}i)z + 1 + i$. Then, the operator $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ is order-bounded.*

Proof. Let $z = x + iy$. From Theorem 1, we have

$$\begin{aligned} |u(z)|^2 e^{|\varphi(z)|^2-|z|^2} &= |e^{2(1-i)z}|e^{|(\frac{1}{2}-\frac{1}{2}i)z+1+i|^2-|z|^2} \\ &= |e^{2(1-i)(x+iy)}|e^{|(\frac{1}{2}-\frac{1}{2}i)(x+iy)+1+i|^2-|x+iy|^2} \\ &= e^{2(x+y)}e^{-\frac{1}{2}x^2-\frac{1}{2}y^2+2y+2} \\ &= e^{-\frac{1}{2}(x-2)^2-\frac{1}{2}(y-4)^2+12}. \end{aligned}$$

Then,

$$\begin{aligned}\int_{\mathbb{C}} |u(z)|^q e^{\frac{q(|\varphi(z)|^2-|z|^2)}{2}} dA(z) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{q}{4}(x-2)^2-\frac{q}{4}(y-4)^2+6q} dxdy \\ &= e^{6q}\int_{-\infty}^{+\infty} e^{-\frac{q}{4}(x-2)^2} dx \int_{-\infty}^{+\infty} e^{-\frac{q}{4}(y-4)^2} dy \\ &< \infty.\end{aligned}$$

which implies

$$\int_{\mathbb{C}} |u(z)|^q e^{\frac{q(|\varphi(z)|^2-|z|^2)}{2}} dA(z) < \infty.$$

Hence, the operator $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ is order-bounded. The proof is completed. □

Example 2. *Let* $0 < p \leq \infty$, $0 < q < \infty$, $\varphi(z) = (\frac{1}{2} + \frac{\sqrt{3}}{2}i)z + i$ *and* $u(z) = e^{(-\frac{\sqrt{3}}{2}+\frac{1}{2}i)z}$. *Then, the operator* $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ *is not order-bounded.*

Proof. Let $z = x + iy$. From Theorem 1, we have

$$\begin{aligned}|u(z)|e^{\frac{|\varphi(z)|^2-|z|^2}{2}} &= |e^{(-\frac{\sqrt{3}}{2}+\frac{1}{2}i)(x+iy)}|e^{\frac{|i(\frac{\sqrt{3}}{2}x+\frac{1}{2}y+1)+\frac{1}{2}x-\frac{\sqrt{3}}{2}y|^2-|x+iy|^2}{2}} \\ &= |e^{i(\frac{1}{2}x-\frac{\sqrt{3}}{2}y)}|e^{-\frac{1}{2}y-\frac{\sqrt{3}}{2}x}e^{\frac{1}{2}y+\frac{\sqrt{3}}{2}x+1} = e.\end{aligned}$$

Then,

$$\int_{\mathbb{C}} |u(z)|^q e^{\frac{q(|\varphi(z)|^2-|z|^2)}{2}} dA(z) = \infty.$$

Hence, the operator $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ is not order-bounded. The proof is completed. □

From the next example, we see that there exists some functions u and φ such the operators $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ are bounded but not order-bounded.

Example 3. *Let* $0 < p \leq q < \infty$, $u(z) = e^{(1+i)z}$ *and* $\varphi(z) = iz - i - 1$. *Then, the operator* $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ *is bounded but not order-bounded.*

Proof. Let $z = x + iy$. From Lemma 3, we have

$$\begin{aligned}|u(z)|e^{\frac{|\varphi(z)|^2-|z|^2}{2}} &= |e^{(1+i)(x+iy)}|e^{\frac{|i(x-1)-(y+1)|^2-|x+iy|^2}{2}} \\ &= |e^{i(x+y)}|e^{x-y}e^{-x+y+1} = e,\end{aligned}$$

which implies

$$m(u,\varphi) = \sup_{z\in\mathbb{C}} e = e < \infty.$$

Hence, the operator $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ is bounded.

On the other hand, it follows from the calculations that

$$\int_{\mathbb{C}} |u(z)|^p e^{\frac{p(|\varphi(z)|^2-|z|^2)}{2}} dA(z) = \infty.$$

By Theorem 1, the operator $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ is not order-bounded. The proof is completed. □

Example 3 shows that we need to further study the relations between the order-boundedness and some properties such as boundedness or compactness of weighted composition operators. For such a problem, we will provide some answers in the sequel.

Since $W_{1,\varphi} = C_\varphi$, the following result was obtained.

Corollary 1. *Let $0 < q < \infty$ and $0 < p \leq \infty$. Then, the operator $C_\varphi : \mathcal{F}^p \to \mathcal{F}^q$ is order-bounded if and only if u and φ satisfy the condition*

$$\int_{\mathbb{C}} e^{\frac{q(|\varphi(z)|^2-|z|^2)}{2}} dA(z) < \infty.$$

We see that if the operator $T : \mathcal{F}^p \to \mathcal{F}^q$ is order-bounded, then it is bounded. We give the proof as follows.

If the operator $T : \mathcal{F}^p \to \mathcal{F}^q$ is order-bounded, then there exists a nonnegative function $h \in L^q(\mathbb{C}, e^{\frac{-q|z|^2}{2}} dA)$ such that for $f \in \mathcal{F}^p$ with $\|f\|_p \leq 1$, it holds that

$$|Tf(z)| \leq h(z), \ \ a.e.\ [e^{\frac{-q|z|^2}{2}} dA].$$

Thus, for each $f \in \mathcal{F}^p$ with $\|f\|_p \leq 1$, it holds that

$$\|Tf\|_q^q = \int_{\mathbb{C}} |Tf(z)|^q e^{\frac{-q|z|^2}{2}} dA(z) \leq \int_{\mathbb{C}} |h(z)|^q e^{\frac{-q|z|^2}{2}} dA(z),$$

which shows that the operator $T : \mathcal{F}^p \to \mathcal{F}^q$ is bounded.

Theorem 2. *Let $0 < p \leq q < \infty$, $\varphi(z) = az + b$ with $|a| \leq 1$ and $u(z) = e^{\bar{c}z}$. Then, the operator $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ is order-bounded if and only if $|a| < 1$.*

Proof. Assume that $|a| < 1$. Now, we prove that u and φ satisfy the condition (6). The proof is divided into the following two cases.

Case 1. Assume that $|a| = 0$, that is, $a = 0$. We notice that $\|u\|_q^q = \|K_c\|_q^q = e^{\frac{q|c|^2}{2}}$. Then, we have

$$\begin{aligned}
\int_{\mathbb{C}} |u(z)|^q e^{\frac{q(|\varphi(z)|^2-|z|^2)}{2}} dA(z) &= \int_{\mathbb{C}} |e^{\bar{c}z}|^q e^{\frac{q(|b|^2-|z|^2)}{2}} dA(z) \\
&= \int_{\mathbb{C}} |e^{\bar{c}z}|^q e^{\frac{q(|b|^2-|z|^2)}{2}} dA(z) \\
&= e^{\frac{q|b|^2}{2}} \int_{\mathbb{C}} |e^{\bar{c}z}|^q e^{\frac{-q|z|^2}{2}} dA(z) \\
&= e^{\frac{q|b|^2}{2}} \|K_c\|_q^q = e^{\frac{q(|b|^2+|c|^2)}{2}} \\
&< \infty,
\end{aligned}$$

which shows that (6) holds for this case.

Case 2. Assume that $0 < |a| < 1$. We have

$$\begin{aligned}
\int_{\mathbb{C}} |u(z)|^q e^{\frac{q(|\varphi(z)|^2-|z|^2)}{2}} dA(z) &= \int_{\mathbb{C}} |e^{\bar{c}z}|^q e^{\frac{q(|az+b|^2-|z|^2)}{2}} dA(z) \\
&\leq \int_{\mathbb{C}} |e^{\bar{c}z}|^q e^{\frac{q(|a|^2|z|^2+|b|^2+2|a||b||z|-|z|^2)}{2}} dA(z) \\
&= e^{\frac{q|b|^2}{2}} \int_{\mathbb{C}} |e^{\bar{c}z}|^q e^{\frac{q[(|a|^2-1)|z|^2+2|a||b||z|]}{2}} dA(z) \\
&\leq e^{\frac{q|b|^2}{2}} \int_{\mathbb{C}} e^{q|z|(|\bar{c}|+|a||b|)+q(|a|^2-1)|z|^2} dA(z).
\end{aligned}$$

Let $k_1 = q(|c| + |a||b|)$ and $k_2 = q(1 - |a|^2)$. We obtain that $k_1 \geq 0$ and $k_2 \in (0, 1)$. At this moment, we have

$$\int_{\mathbb{C}} e^{q|z|(|c|+|a||b|)+q(|a|^2-1)|z|^2} dA(z) = \int_{\mathbb{C}} e^{k_1|z|-k_2|z|^2} dA(z).$$

Let $z = x + iy$, $x = \rho\cos\theta$ and $y = \rho\sin\theta$. From changing the variables in the integral, we obtain

$$\begin{aligned}
\int_{\mathbb{C}} e^{k_1|z|-k_2|z|^2} dA(z) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{k_1\sqrt{x^2+y^2}-k_2(x^2+y^2)} dxdy \\
&= \int_0^{2\pi}\int_0^{+\infty} \rho e^{k_1\rho-k_2\rho^2} d\rho d\theta \\
&= 2\pi\int_0^{+\infty} \rho e^{k_1\rho-k_2\rho^2} d\rho \\
&= 2\pi\int_0^{+\infty} -\frac{1}{2k_2} e^{k_1\rho} de^{-k_2\rho^2} \\
&= 2\pi\Big(\frac{-1}{2k_2} e^{k_1\rho-k_2\rho^2}\Big|_0^{+\infty} + \int_0^{+\infty} \frac{k_1}{2k_2} e^{k_1\rho-k_2\rho^2} d\rho\Big) \\
&= \frac{\pi}{k_2}\Big(-e^{k_1\rho-k_2\rho^2}\Big|_0^{+\infty} + k_1\int_0^{+\infty} e^{k_1\rho-k_2\rho^2} d\rho\Big) \\
&= \frac{\pi}{k_2}(I_1(\rho) + k_1 I_2(\rho)).
\end{aligned}$$

We have

$$\begin{aligned}
I_2(\rho) &= \int_0^{+\infty} e^{k_1\rho-k_2\rho^2} d\rho = \int_0^{+\infty} e^{-k_2(\rho-\frac{k_1}{2k_2})^2+\frac{k_1^2}{4k_2}} d\rho \\
&= e^{\frac{k_1^2}{4k_2}}\int_0^{+\infty} e^{-k_2(\rho-\frac{k_1}{2k_2})^2} d\rho < \infty,
\end{aligned}$$

and

$$I_1(\rho) = -e^{k_1\rho-k_2\rho^2}\Big|_0^{+\infty} < \infty,$$

which implies

$$\int_{\mathbb{C}} e^{k_1|z|-k_2|z|^2} dA(z) < \infty,$$

that is,

$$\int_{\mathbb{C}} |u(z)|^q e^{\frac{q(|\varphi(z)|^2-|z|^2)}{2}} dA(z) < \infty,$$

which also shows that (6) holds for this case.

Now, assume that the operator $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ is order-bounded. From the Definition 1, there exists a nonnegative function $h \in L^p(\mathbb{C}, e^{\frac{-q|z|^2}{2}} dA)$ such that for all $f \in \mathcal{F}^p$ with $\|f\|_p \leq 1$, it holds that

$$|W_{u,\varphi} f(z)| = |u(z) f(\varphi(z))| = |e^{\bar{c}z} f(az + b)| \leq h(z), \ \ a.e.\ [e^{\frac{-p|z|^2}{2}} dA]. \tag{10}$$

Replacing f by k_w in (10), we have

$$|W_{u,\varphi} k_w(z)| = |e^{\bar{c}z} e^{\overline{w}|az+b|-\frac{|w|^2}{2}}| \leq h(z), \ \ a.e.\ [e^{\frac{-p|z|^2}{2}} dA].$$

Letting $w = \varphi(z)$, we obtain

$$|e^{\bar{c}z} e^{\frac{|az+b|^2}{2}}| \leq h(z), \ \ a.e.\ [e^{\frac{-p|z|^2}{2}} dA],$$

which shows

$$\int_{\mathbb{C}} \left|e^{\bar{c}z + \frac{|az+b|^2}{2}}\right|^p e^{\frac{-p|z|^2}{2}} dA(z) < \infty. \tag{11}$$

If $|a| = 1$, then by Lemma 10 and the fact $e^{\Re z} = |e^z|$, we have

$$\begin{aligned}
\int_{\mathbb{C}} \left|e^{\bar{c}z + \frac{|az+b|^2}{2}}\right|^q e^{\frac{-q|z|^2}{2}} dA(z) &= \int_{\mathbb{C}} |e^{\bar{c}z}|^q e^{\frac{q}{2}(|az+b|^2 - |z|^2)} dA(z) \\
&= e^{\frac{q|b|^2}{2}} \int_{\mathbb{C}} |e^{\bar{c}z}|^q e^{q\Re(a\bar{b}z)} dA(z) \\
&= e^{\frac{q|b|^2}{2}} \int_{\mathbb{C}} |e^{\bar{c}z}|^q |e^{a\bar{b}z}|^q dA(z) \\
&= e^{\frac{q|b|^2}{2}} \int_{\mathbb{C}} |e^{(\bar{c}+a\bar{b})z}|^q dA(z) \\
&= \infty,
\end{aligned}$$

which contradicts (11). So, we have $|a| < 1$. The proof is completed. □

From Lemma 7 and Theorem 2, we obtain the following result.

Corollary 2. *Let $0 < p \leq q < \infty$, $\varphi(z) = az + b$ with $|a| \leq 1$ and $u(z) = e^{\bar{c}z}$. Then, the operator $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ is compact if and only if the operator $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ is order-bounded.*

Since $C_\varphi = W_{K_0,\varphi}$, we also have

Corollary 3. *Let $0 < p \leq q < \infty$. Then, the operator $C_\varphi : \mathcal{F}^p \to \mathcal{F}^q$ is compact if and only if the operator $C_\varphi : \mathcal{F}^p \to \mathcal{F}^q$ is order-bounded.*

We obtain Theorem 2 only for the case $0 < p \leq q < \infty$. The case $0 < q < p < \infty$ is more complicated, so we put forward the following:

Question 1. *Let $0 < q < p < \infty$, $u(z) = e^{\bar{c}z}$ and $\varphi(z) = az + b$ with $|a| \leq 1$. What is the necessary and sufficient condition for the operator $W_{u,\varphi} : \mathcal{F}^p \to \mathcal{F}^q$ to be order-bounded?*

4. Order-Bounded Difference in Weighted Composition Operators

Now, we characterize the order-boundedness of difference in weighted composition operators between Fock spaces. We have successfully applied improved methods and techniques from [29] to the research of the following Theorem 3.

Theorem 3. *Let $0 < p \leq \infty$, $0 < q < \infty$ and $u_1, u_2, \varphi_1, \varphi_2$ be holomorphic functions with $u_1 \not\equiv 0$, $u_2 \not\equiv 0$ and $\varphi_1 \not\equiv \varphi_2$ on $\mathbb{C}$. Then, the operator $W_{u_1,\varphi_1} - W_{u_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ is order-bounded if and only if both W_{u_1,φ_1} and $W_{u_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ are order-bounded.*

Proof. Assume that the operators W_{u_1,φ_1} and $W_{u_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ are order-bounded. By Theorem 1,

$$\int_{\mathbb{C}} |u_j(z)|^q e^{\frac{q(|\varphi_j(z)|^2 - |z|^2)}{2}} dA(z) < \infty$$

for $j = 1, 2$.

For $f \in \mathcal{F}^p$ with $\|f\|_p \leq 1$, it follows from Lemma 1 that

$$\begin{aligned}|(W_{u_1,\varphi_1}-W_{u_2,\varphi_2})(f)(z)| &= |u_1(z)f(\varphi_1(z))-u_2(z)f(\varphi_2(z))|\\ &\le |u_1(z)f(\varphi_1(z))|+|u_2(z)f(\varphi_2(z))|\\ &\lesssim |u_1(z)|e^{\frac{|\varphi_1(z)|^2}{2}}+|u_2(z)|e^{\frac{|\varphi_2(z)|^2}{2}}.\end{aligned}$$

Choose the function

$$h(z)=|u_1(z)|e^{\frac{|\varphi_1(z)|^2}{2}}+|u_2(z)|e^{\frac{|\varphi_2(z)|^2}{2}}.$$

Then,

$$\begin{aligned}h^q(z) &= \left(|u_1(z)|e^{\frac{|\varphi_1(z)|^2}{2}}+|u_2(z)|e^{\frac{|\varphi_2(z)|^2}{2}}\right)^q\\ &\le \left(2\max\left\{|u_1(z)|e^{\frac{|\varphi_1(z)|^2}{2}},|u_2(z)|e^{\frac{|\varphi_2(z)|^2}{2}}\right\}\right)^q\\ &\le 2^q\left(|u_1(z)|^q e^{\frac{q|\varphi_1(z)|^2}{2}}+|u_2(z)|^q e^{\frac{q|\varphi_2(z)|^2}{2}}\right),\end{aligned}$$

which shows

$$\int_{\mathbb{C}} h^q(z)e^{\frac{-q|z|^2}{2}}dA(z)\le 2^q\int_{\mathbb{C}}\left(|u_1(z)|^q e^{\frac{q(|\varphi_1(z)|^2-|z|^2)}{2}}+|u_2(z)|^q e^{\frac{q(|\varphi_2(z)|^2-|z|^2)}{2}}\right)dA(z)<\infty,$$

that is, $h\in L^q(\mathbb{C},e^{\frac{-q|z|^2}{2}}dA)$. Therefore, the operator $W_{u_1,\varphi_1}-W_{u_2,\varphi_2}:\mathcal{F}^p\to\mathcal{F}^q$ is order-bounded.

Conversely, assume that the operator $W_{u_1,\varphi_1}-W_{u_2,\varphi_2}:\mathcal{F}^p\to\mathcal{F}^q$ is order-bounded. By Definition 1, there exists a nonnegative function $g\in L^q(\mathbb{C},e^{\frac{-q|z|^2}{2}}dA)$ such that for all $f\in\mathcal{F}^p$ with $\|f\|_p\le 1$, it holds that

$$|(W_{u_1,\varphi_1}-W_{u_2,\varphi_2})f(z)|\le g(z),\ \ a.e.\ [e^{\frac{-q|z|^2}{2}}dA]. \tag{12}$$

Consider the function $f=k_w$ in (12). Then, we have

$$\begin{aligned}g(z) &\ge |(W_{u_1,\varphi_1}-W_{u_2,\varphi_2})k_w(z)|\\ &= |u_1(z)k_w(\varphi_1(z))-u_2(z)k_w(\varphi_2(z))|\\ &= |u_1(z)e^{\overline{w}\varphi_1(z)-\frac{|w|^2}{2}}-u_2(z)e^{\overline{w}\varphi_2(z)-\frac{|w|^2}{2}}|\\ &\ge |u_1(z)e^{\overline{w}\varphi_1(z)-\frac{|w|^2}{2}}|-|u_2(z)e^{\overline{w}\varphi_2(z)-\frac{|w|^2}{2}}|,\ \ a.e.\ [e^{\frac{-q|z|^2}{2}}dA].\end{aligned} \tag{13}$$

Letting $w=\varphi_1(z)$ in (13), we obtain

$$\begin{aligned}g(z) &\ge |u_1(z)e^{\frac{|\varphi_1(z)|^2}{2}}|-|u_2(z)e^{\overline{\varphi_1(z)}\varphi_2(z)-\frac{|\varphi_1(z)|^2}{2}}|\\ &= |u_1(z)|e^{\frac{|\varphi_1(z)|^2}{2}}-|u_2(z)|e^{\frac{|\varphi_2(z)|^2-|\varphi_1(z)-\varphi_2(z)|^2}{2}},\ \ a.e.\ [e^{\frac{-q|z|^2}{2}}dA].\end{aligned} \tag{14}$$

Also, letting $w=\varphi_2(z)$ in (13), we have

$$g(z)\ge |u_2(z)|e^{\frac{|\varphi_2(z)|^2}{2}}-|u_1(z)|e^{\frac{|\varphi_1(z)|^2-|\varphi_1(z)-\varphi_2(z)|^2}{2}},\ \ a.e.\ [e^{\frac{-q|z|^2}{2}}dA]. \tag{15}$$

Consequently, combining (14) and (15) leads to

$$2g(z)\ge\left(|u_1(z)|e^{\frac{|\varphi_1(z)|^2}{2}}+|u_2(z)|e^{\frac{|\varphi_2(z)|^2}{2}}\right)\left(1-e^{-\frac{|\varphi_1(z)-\varphi_2(z)|^2}{2}}\right),\ \ a.e.\ [e^{\frac{-q|z|^2}{2}}dA].$$

Since $1 - e^x \geq \frac{x}{1+x}$ for all $x \geq 0$, we obtain

$$1 - e^{-\frac{|\varphi_1(z)-\varphi_2(z)|^2}{2}} \geq \frac{|\varphi_1(z) - \varphi_2(z)|^2}{2 + |\varphi_1(z) - \varphi_2(z)|^2}.$$

We therefore have

$$g(z) \geq \frac{|\varphi_1(z) - \varphi_2(z)|^2}{2(2 + |\varphi_1(z) - \varphi_2(z)|^2)} |u_1(z)| e^{\frac{|\varphi_1(z)|^2}{2}}, \ \ a.e.\ [e^{\frac{-q|z|^2}{2}} dA], \tag{16}$$

and

$$g(z) \geq \frac{|\varphi_1(z) - \varphi_2(z)|^2}{2(2 + |\varphi_1(z) - \varphi_2(z)|^2)} |u_2(z)| e^{\frac{|\varphi_2(z)|^2}{2}}, \ \ a.e.\ [e^{\frac{-q|z|^2}{2}} dA]. \tag{17}$$

From Lemma 8, for some $a, b \in \mathbb{C}$, we conclude that $\varphi_1(z) - \varphi_2(z) = az + b$. Therefore, we see that if $a \neq 0$, then

$$\lim_{|z|\to\infty} \frac{|\varphi_1(z) - \varphi_2(z)|^2}{2 + |\varphi_1(z) - \varphi_2(z)|^2} = 1; \tag{18}$$

if $a = 0$, then

$$\lim_{|z|\to\infty} \frac{|\varphi_1(z) - \varphi_2(z)|^2}{2 + |\varphi_1(z) - \varphi_2(z)|^2} = \frac{|b|^2}{2 + |b|^2} > 0, \tag{19}$$

where we have used the assumption of $\varphi_1 \not\equiv \varphi_2$. Next, the proof is divided into the following two cases.

Case 1. Assume that $a \neq 0$. By (18), for $\varepsilon = \frac{1}{2}$, there exists a positive constant $r_1 > 0$ such that

$$\Big| \frac{|\varphi_1(z) - \varphi_2(z)|^2}{2 + |\varphi_1(z) - \varphi_2(z)|^2} - 1 \Big| < \frac{1}{2} \tag{20}$$

for all $|z| > r_1$. It is clear that (20) is equivalent to

$$\frac{1}{2} < \frac{|\varphi_1(z) - \varphi_2(z)|^2}{2 + |\varphi_1(z) - \varphi_2(z)|^2} < \frac{3}{2} \tag{21}$$

for all $|z| > r_1$.

It follows from (16) and (21) that there exists a positive constant C_1 such that

$$|u_1(z)| e^{\frac{|\varphi_1(z)|^2}{2}} \leq C_1 g(z), \ \ a.e.\ [e^{\frac{-q|z|^2}{2}} dA] \tag{22}$$

for all $|z| > r_1$. By the continuousness, there exists a positive constant C_2 such that

$$|u_1(z)| e^{\frac{|\varphi_1(z)|^2}{2}} \leq C_2 \tag{23}$$

for all $|z| \leq r_1$. Consequently, from (22) and (23) we obtain

$$|u_1(z)| e^{\frac{|\varphi_1(z)|^2}{2}} \leq C_1 g(z) + C_2, \ \ a.e.\ [e^{\frac{-q|z|^2}{2}} dA].$$

Then

$$\int_{\mathbb{C}} |u_1(z)|^q e^{\frac{q(|\varphi_1(z)|^2 - |z|^2)}{2}} dA(z) < \infty.$$

Similarly, we also can obtain

$$\int_{\mathbb{C}} |u_2(z)|^q e^{\frac{q(|\varphi_2(z)|^2-|z|^2)}{2}} dA(z) < \infty.$$

By Theorem 1, both W_{u_1,φ_1} and $W_{u_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ are order-bounded for this case.

Case 2. Assume that $a = 0$. By (19), there exists a positive constant r_2 such that

$$\Big|\frac{|\varphi_1(z)-\varphi_2(z)|^2}{2+|\varphi_1(z)-\varphi_2(z)|^2} - \frac{|b|^2}{2+|b|^2}\Big| < \frac{|b|^2}{2(2+|b|^2)} \tag{24}$$

for all $|z| > r_2$. (24) implies that

$$\frac{|b|^2}{2(2+|b|^2)} < \frac{|\varphi_1(z)-\varphi_2(z)|^2}{2+|\varphi_1(z)-\varphi_2(z)|^2} < \frac{3|b|^2}{2(2+|b|^2)} \tag{25}$$

for all $|z| > r_2$. By (17) and (25), there exists a positive constant C_3 such that

$$|u_1(z)|e^{\frac{|\varphi_1(z)|^2}{2}} \le C_3 g(z) \ \ a.e.\ [e^{\frac{-q|z|^2}{2}} dA]$$

for $|z| > r_2$. On the other hand, there exists a positive constant C_4 such that

$$|u_1(z)|e^{\frac{|\varphi_1(z)|^2}{2}} \le C_4$$

for all $|z| \le r_2$.

Therefore, we have

$$|u_1(z)|e^{\frac{|\varphi_1(z)|^2}{2}} \le C_3 g(z) + C_4, \ \ a.e.\ [e^{\frac{-q|z|^2}{2}} dA].$$

From this, we deduce that

$$\int_{\mathbb{C}} |u_1(z)|^q e^{\frac{q(|\varphi_1(z)|^2-|z|^2)}{2}} dA(z) < \infty.$$

We also can similarly obtain

$$\int_{\mathbb{C}} |u_2(z)|^q e^{\frac{q(|\varphi_2(z)|^2-|z|^2)}{2}} dA(z) < \infty.$$

Also by Theorem 1, both W_{u_1,φ_1} and $W_{u_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ are order-bounded. The proof is completed. □

Remark 2. *In the proof of Theorem 4, we have used the two obvious facts* $1 - e^x \ge \frac{x}{1+x}$ *for all* $x \ge 0$, *and* $|e^{\bar{z}w-\frac{|z|^2}{2}}| = e^{\frac{|w|^2-|z-w|^2}{2}}$ *for all* $w, z \in \mathbb{C}$.

Example 4. *Let* $0 < p \le \infty$, $0 < q < \infty$, $\varphi_1(z) = \frac{i}{2}z - 1 + i$, $\varphi_2(z) = (\frac{1}{3} - \frac{2}{3}i)z + i$, $u_1(z) = e^z$ *and* $u_2(z) = e^{(-1+3i)z}$. *Then the operator* $W_{u_1,\varphi_1} - W_{u_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ *is order-bounded.*

Proof. Let $z = x + iy$. From Theorem 1, we have

$$\begin{aligned}|u_1(z)|^2 e^{|\varphi_1(z)|^2-|z|^2} &= |e^{2z}|e^{|\frac{i}{2}z-1+i|^2-|z|^2}\\ &\le e^{2|z|}e^{|\frac{i}{2}|^2|z|^2+|-1+i|^2+2|-1+i||\frac{i}{2}||z|-|z|^2}\\ &= e^{-\frac{3}{4}|z|^2+(2+\sqrt{2})|z|+2}\\ &= e^{-\frac{3}{4}(|z|-\frac{4}{3}-\frac{2\sqrt{2}}{3})^2+4+\frac{4\sqrt{2}}{3}},\end{aligned}$$

which implies

$$\int_{\mathbb{C}} |u_1(z)|^q e^{\frac{q(|\varphi_1(z)|^2-|z|^2)}{2}} dA(z) = \int_{\mathbb{C}} e^{-\frac{3q}{8}(|z|-\frac{4}{3}-\frac{2\sqrt{2}}{3})^2+2q+\frac{2\sqrt{2}q}{3}} dA(z) < \infty.$$

Hence, the operator $W_{u_1,\varphi_1} : \mathcal{F}^p \to \mathcal{F}^q$ is order-bounded.

Similarly, we have

$$\begin{aligned}|u_2(z)|^2 e^{|\varphi_2(z)|^2-|z|^2} &= |e^{2(-1+3i)z}|e^{|(\frac{1}{3}-\frac{2}{3}i)z+i|^2-|z|^2}\\ &\le e^{2|-1+3i||z|}e^{|\frac{1}{3}-\frac{2}{3}i|^2|z|^2+|i|^2+2|\frac{1}{3}-\frac{2}{3}i||i||z|-|z|^2}\\ &= e^{-\frac{4}{9}|z|^2+(\frac{2\sqrt{5}}{3}+2\sqrt{10})|z|+1}\\ &= e^{-\frac{4}{9}(|z|-\frac{3\sqrt{5}}{4}-\frac{9\sqrt{10}}{4})^2+\frac{99}{4}+\frac{15\sqrt{5}}{2}}.\end{aligned}$$

Obviously,

$$\int_{\mathbb{C}} |u_2(z)|^q e^{\frac{q(|\varphi_2(z)|^2-|z|^2)}{2}} dA(z) = \int_{\mathbb{C}} e^{-\frac{2q}{9}(|z|-\frac{3\sqrt{5}}{4}-\frac{9\sqrt{10}}{4})^2+\frac{99q}{8}+\frac{15\sqrt{5}q}{4}} dA(z) < \infty.$$

From Theorem 1, the operator $W_{u_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ is order-bounded. Then, the operator $W_{u_1,\varphi_1} - W_{u_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ is order-bounded by Theorem 3. The proof is completed. □

Example 5. *$0 < p \le \infty$, $0 < q < \infty$, $\varphi_2(z) = (1+i)z + 1 - i$, $u_1(z) = e^{(3+4i)z}$ and $u_2(z) = e^{(1-2i)z}$. Then the operator $W_{u_1,\varphi_1} - W_{u_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ is not order-bounded.*

Proof. Let $z = x + iy$. From Theorem 1, we have

$$\begin{aligned}|u_1(z)|^2 e^{|\varphi_1(z)|^2-|z|^2} &= |e^{2z(3+4i)}|e^{|1-i|^2-|z|^2}\\ &\le e^{2|3+4i||z|}e^{2-|z|^2}\\ &= e^{-(|z|-5)^2+27},\end{aligned}$$

which implies

$$\int_{\mathbb{C}} |u_1(z)|^q e^{\frac{q(|\varphi_1(z)|^2-|z|^2)}{2}} dA(z) = \int_{\mathbb{C}} e^{-\frac{q}{2}(|z|-5)^2+\frac{27q}{2}} dA(z) < \infty.$$

Hence, the operator $W_{u_1,\varphi_1} : \mathcal{F}^p \to \mathcal{F}^q$ is order-bounded.

Similarly, we have

$$\begin{aligned}|u_2(z)|^2 e^{|\varphi_2(z)|^2-|z|^2} &= |e^{2(1-2i)z}|e^{|(1+i)z+1-i|^2-|z|^2}\\ &\le e^{2|1-2i||z|}e^{|1+i|^2|z|^2+|1-i|^2+2|1+i||1-i||z|-|z|^2}\\ &= e^{|z|^2+(4+2\sqrt{5})|z|+2}\\ &= e^{(|z|+2+\sqrt{5})^2+11+4\sqrt{5}}.\end{aligned}$$

Obviously,

$$\int_{\mathbb{C}} |u_2(z)|^q e^{\frac{q(|\varphi_2(z)|^2-|z|^2)}{2}} dA(z) = \int_{\mathbb{C}} e^{\frac{q}{2}(|z|+2+\sqrt{5})^2+\frac{q}{2}(11+4\sqrt{5})} dA(z) = \infty.$$

From Theorem 1, the operator $W_{u_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ is not order-bounded. That is, the operator $W_{u_1,\varphi_1} - W_{u_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ is not order-bounded. The proof is completed. □

From the next example, we see that there exists some symbols u and φ such the operators $W_{u_1,\varphi_1} - W_{u_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ are bounded but not order-bounded.

Example 6. *Let* $0 < p \le q < \infty$, $\varphi_1(z) = (\frac{1}{2} - \frac{1}{2}i)z + 1 - i$, $u_1(z) = e^{(1+\sqrt{3}i)z}$, $\varphi_2(z) = -iz + i - 1$ *and* $u_2(z) = e^{(1-i)z}$. *Then, the operator* $W_{u_1,\varphi_1} - W_{u_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ *is bounded but not order-bounded.*

Proof. By Theorem 1, we have

$$\begin{aligned} |u_1(z)|^2 e^{|\varphi_1(z)|^2-|z|^2} &\le e^{2|1+\sqrt{3}i||z|} e^{|\frac{1}{2}-\frac{1}{2}i|^2|z|^2+2|\frac{1}{2}-\frac{1}{2}i||1-i||z|+|1-i|^2-|z|^2} \\ &= e^{6|z|-\frac{|z|^2}{2}+2} = e^{-\frac{1}{2}(|z|-6)^2+20}, \end{aligned}$$

that is,

$$\int_{\mathbb{C}} e^{\frac{p}{2}[-\frac{1}{2}(|z|-6)^2+20]} dA(z) < \infty,$$

which shows that the operator $W_{u_1,\varphi_1} : \mathcal{F}^p \to \mathcal{F}^q$ is order-bounded. At the same time,

$$m(u_1, \varphi_1) = e^{20} < \infty.$$

So, from Lemma 3, the operator $W_{u_1,\varphi_1} : \mathcal{F}^p \to \mathcal{F}^q$ is bounded.

Let $z = x + iy$. From Lemma 3, we have

$$\begin{aligned} |u_2(z)|^2 e^{|\varphi_2(z)|^2-|z|^2} &= |e^{(1-i)(x+iy)}|^2 e^{|(y-1)-i(x-1)|^2-|x+iy|^2} \\ &= |e^{i(y-x)}|^2 e^{2x+2y} e^{-2x-2y+2} = e^2, \end{aligned}$$

which implies that

$$m(u_2, \varphi_2) = \sup_{z\in\mathbb{C}} e^2 < \infty.$$

Hence, the operator $W_{u_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ is bounded.

On the other hand, it follows from the calculations that

$$\int_{\mathbb{C}} |u_2(z)|^p e^{\frac{p(|\varphi_2(z)|^2-|z|^2)}{2}} dA(z) = \infty.$$

From Theorem 1, the operator $W_{u_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ is not order-bounded. Then the operator $W_{u_1,\varphi_1} - W_{u_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ is bounded but not order-bounded. The proof is completed. □

We have the following result, which can be obtained by Theorems 2 and 3.

Corollary 4. *Let* $0 < p \le q < \infty$, $u_j(z) = e^{\bar{c}_j z}$, $\varphi_j(z) = a_j z + b_j$ *with* $|a_j| \le 1$, $c_1 \ne c_2$ *and* $a_1 \ne a_2$ *or* $b_1 \ne b_2$ *for* $j = 1, 2$. *Then the operator* $W_{u_1,\varphi_1} - W_{u_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ *is order-bounded if and only if* $|a_j| < 1$ *for* $j = 1, 2$.

To sum up, we obtain

Theorem 4. *Let $0 < p \le q < \infty$, $u_j(z) = e^{\bar{c}_j z}$ and $\varphi_j(z) = a_j z + b_j$ with $|a_j| \le 1$, $c_1 \ne c_2$ and $a_1 \ne a_2$ or $b_1 \ne b_2$ for $j = 1, 2$. Then the following statements are equivalent:*
(a) $W_{u_1,\varphi_1} - W_{u_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ is order-bounded;
(b) $W_{u_1,\varphi_1} - W_{u_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ is compact;
(c) $|a_j| < 1$, for $j = 1, 2$;
(d) both $W_{u_1,\varphi_1} : \mathcal{F}^p \to \mathcal{F}^q$ and $W_{u_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ are order-bounded;
(e) both $W_{u_1,\varphi_1} : \mathcal{F}^p \to \mathcal{F}^q$ and $W_{u_2,\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ are compact.

Proof. From Theorem 2 and Corollary 2, it follows that $(c) \Leftrightarrow (d)$ and $(d) \Leftrightarrow (e)$. Obviously, $(a) \Leftrightarrow (d)$ from Theorem 3. From Lemma 9, $(b) \Leftrightarrow (e)$ is true. The proof is completed. □

Notice that $C_{\varphi_1} - C_{\varphi_2} = W_{K_0,\varphi_1} - W_{K_0,\varphi_2}$. Therefore, the following result was obtained.

Corollary 5. *Let $0 < p \le q < \infty$, $\varphi_j(z) = a_j z + b_j$ with $|a_j| \le 1$, and $a_1 \ne a_2$ or $b_1 \ne b_2$ for $j = 1, 2$. Then the following statements are equivalent:*
(a) $C_{\varphi_1} - C_{\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ is order-bounded;
(b) $C_{\varphi_1} - C_{\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ is compact;
(c) $|a_j| < 1$ for $j = 1, 2$;
(d) both $C_{\varphi_1} : \mathcal{F}^p \to \mathcal{F}^q$ and $C_{\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ are order-bounded;
(e) both $C_{\varphi_1} : \mathcal{F}^p \to \mathcal{F}^q$ and $C_{\varphi_2} : \mathcal{F}^p \to \mathcal{F}^q$ are compact.

We finish the section by posing the following

Question 2. *What will be the results in this section for the case $0 < q < p < \infty$?*

5. Conclusions

In this paper, we completely characterize the order-bounded weighted composition operators $W_{u,\varphi}$ from $\mathcal{F}^p$ to $\mathcal{F}^q$ for $0 < p \le \infty$ and $0 < q < \infty$. We also completely characterize the order-bounded difference in these operators. When $0 < p \le q < \infty$, we successfully characterize the order-boundedness of the operators $W_{e^{\bar{c}z},az+b}$ from $\mathcal{F}^p$ to $\mathcal{F}^q$ by using the compactness. But, we find that there are still some interesting unresolved questions such as Questions 1 and 2. Therefore, we hope that the study can attract people's more attention to this topic.

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