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Twistor and Reflector Spaces for Paraquaternionic Contact Manifolds

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Abstract: We consider certain fiber bundles over paraquaternionic contact manifolds, called twistor and reflector spaces. We show that the twistor space carries an integrable CR structure (Cauchy–Riemann structure) and the reflector space is an integrable para-CR structure, both with neutral signatures.

Keywords: paraquaternionic; twistor; reflector; contact

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1. Introduction

The geometry of paraquaternionic contact structures is essentially a tool to study a special type of co-dimension three distribution on $(4n+3)$ manifolds with properties closely related to the algebra of paraquaternions, known also as split-quaternions [1], quaternions of the second kind [2], and complex product structures [3]. The paraquaternionic contact structure, introduced in [4], can be considered a generalization of the para three-Sasakian geometry developed in [1,5]. In many ways, paraquaternionic contact structures resemble the geometry of quaternionic contact manifolds, introduced by O. Biquard [6], which has been very useful in relation to the quaternionic contact Yamabe problem and the determination of extremals and the best constant in the L^2 Folland–Stein inequality on the quaternionic Heisenberg group [7–11]. Despite the similarities between these two types of geometry, there are also some major differences determined mainly by the fact that in the paraquaternionic contact setting, one is often forced to consider sub-hyperbolic PDEs instead of sub-elliptic PDEs.

As shown in [6], the study of quaternionic contact structures leads back in a natural way to the study of a particular class of integrable CR manifolds (which are never pseudo-convex), called twistor spaces, which appear as certain sphere bundles over the base quaternionic contact manifold (see also [12]). This is a generalization of the concept of a twistor space of a quaternionic Kähler manifold [13]. In the paraquaternionic contact case, we have two different types of bundles: the twistor space $\mathcal{Z}$ and the reflector space $\mathcal{R}$. The situation is very similar to the discussion in [14]. The fibers of $\mathcal{Z}$ are diffeomorphic to the two-sheeted hyperboloid $x^2+y^2-z^2=-1$ in $\mathbb{R}^3$, whereas the fibers of $\mathcal{R}$ are diffeomorphic to the one-sheeted hyperboloid $x^2+y^2-z^2=1$ (see Section 3 below for the details). The purpose of this paper is to demonstrate the following:

Theorem 1. *If (M,H) is any paraquaternionic contact manifold with twistor space $\mathcal{Z}$ and reflector space $\mathcal{R}$, then we have a natural integrable CR structure on $\mathcal{Z}$ and a natural integrable para-CR structure on $\mathcal{R}$. The Levi form for each of these structures is of signature $(2n+2,2n+2)$.*

The proof of this theorem is divided into several steps throughout the paper and follows the results obtained in Propositions 1, 2, 3, and 5.

Conventions. *In this paper we use the following general conventions:*

(a) *Indices s and t usually run from 1 to 3 (when nothing else specified).*

(b) *Indices i, j, k always represent a positive (cyclic) permutation of* $1, 2, 3$.

(c) *The summation symbol* $\sum_{(ijk)}$ *indicates summation over all positive permutations* (ijk) *of* $1, 2, 3$*; that is,*

$$\sum_{(ijk)} X_{ijk} \;=\; X_{123} + X_{231} + X_{312}.$$

(d) *We fix the following signs:* $\epsilon_1 = -1$, $\epsilon_2 = -1$, *and* $\epsilon_3 = 1$.

2. Preliminaries

2.1. CR and Para-CR Structures on Manifolds

A CR structure (or a Cauchy–Riemann structure) on a differentiable manifold is a type of geometric structure that models the geometry of a real hypersurface in a complex manifold. Formally, a CR manifold is a differentiable manifold N of odd dimension, say $2n+1$, endowed with a complex subbundle K of the complexified tangent bundle $\mathbb{C}TN = TN \otimes_{\mathbb{R}} \mathbb{C}$, so that the fibers of K are of complex dimension n; $[K, K] \subset K$ (i.e., K is formally integrable), and $K \cap \overline{K} = \{0\}$.

If we set D to be the real component of $K \oplus \overline{K}$, then D is a 2n-dimensional (real) distribution on N. There is a natural field J of endomorphisms of the distribution D with the following properties: $J^2 = -\mathrm{Id}_D$; the fibers of K and $\overline{K}$ are eigenspaces of J with eigenvalues of $\sqrt{-1}$ and $-\sqrt{-1}$, respectively. The Levi form of the CR structure (D, J) is a vector-valued hermitian 2-form L, defined on D, where the values in the line bundle TN/D. L is given by the following formula:

$$L(x, y) = [x, Jy] \qquad mod\ D, \qquad x, y \in D. \tag{1}$$

For a more detailed discussion on this topic, see [15].

Similarly, a para-CR structure on a $2n+1$-dimensional differentiable manifold N can be defined as a pair (D, J) of a co-dimension distribution D on N and a field of endomorphisms J of D with the following properties: $J^2 = \mathrm{Id}_D$ and $J \neq \pm\mathrm{Id}_D$; $[K, K] \subset K$ and $[\tilde{K}, \tilde{K}] \subset \tilde{K}$, where K and $\tilde{K}$ are now the 1 and -1 eigenspaces of J. The Levi form, in this case, is a vector-valued symmetric 2-form L, defined on D, with values in the line bundle TN/D, which are given again by Formula (1). See, for example, refs. [16] or [17] and the references contained therein for a more detailed discussion on para-CR manifolds and their applications.

2.2. The Algebra of Split-Quaternions

Both the quaternions and the split-quaternions are real Clifford algebras generated by a two-dimensional non-degenerate quadratic form. In the negative-definite case, we obtain the algebra of quaternions, whereas in the other two cases, i.e., of a positive-definite or indefinite quadratic form, we get the same (up to an isomorphism) Clifford algebra, which is denoted here by $\mathbb{B}$ and is called the algebra of the split-quaternions (or paraquaternions) (see, e.g., [1]). The elements of $\mathbb{B}$ are generally represented in the following form:

$$a \;=\; a_0 + a_1\, j_1 + a_2\, j_2 + a_3\, j_3,$$

where a_s are real numbers, and j_s are basic split-quaternions; that is, some fixed elements of $\mathbb{B}$ satisfy the following identities:

$$j_1^2 = j_2^2 = 1, \qquad j_1\, j_2 = -j_2\, j_1 = j_3.$$

The remaining multiplication rules for $\mathbb{B}$ are easily derived from the following:

$$j_3^2 = -1, \qquad j_2\, j_3 = -j_3\, j_2 = -j_1, \qquad j_3\, j_1 = -j_1\, j_3 = -j_2.$$

The conjugate to a is defined by $\overline{a} \;=\; a_0 - a_1\, j_1 - a_2\, j_2 - a_3\, j_3$. We obtain the typical identity $\overline{a\,b} = \overline{b}\,\overline{a}$. The real and imaginary parts of a split-quaternion are given by $Re(a) = a_0$ and $Im(a) = a_1\, j_1 + a_2\, j_2 + a_3\, j_3$. There is a natural inner product on $Im(\mathbb{B}) = \mathbb{R}^3$,

$$\langle a, b\rangle \;=\; -\; Re(a\,b) \;=\; -a_1\, b_1 - a_2\, b_2 + a_3\, b_3, \tag{2}$$

and a cross product "$\times$",

$$a \times b \;=\; \sum_{\substack{s,t=1\\ s\neq t}}^{3} (a_s\, b_t)\, j_s\, j_t, \tag{3}$$

so that

$$\langle a \times b\, ,\, c\rangle \;=\; \det\begin{pmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{pmatrix}, \qquad a, b, c \in Im(\mathbb{B}).$$

We observe that $\mathbb{B}$ is isomorphic to the algebra $M_2(\mathbb{R})$ of all 2×2 matrices with real entries under the identification

$$j_1 \;=\; \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}, \qquad j_2 \;=\; \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}, \qquad j_3 \;=\; \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}.$$

Let $SO(1,2)$ be the group of all 3×3 real matrices of determinant 1 that preserve the inner product (2). We need the following basic lemma, which is easily derived from the multiplication rules of $\mathbb{B}$.

Lemma 1. *Three split-quaternions, γ_1, γ_2, and γ_3, satisfy the identities*

$$\gamma_1^2 = \gamma_2^2 = 1, \qquad \gamma_1\, \gamma_2 = -\gamma_2\, \gamma_1 = \gamma_3, \tag{4}$$

if and only if there exists a matrix $A = (a_{st}) \in SO(1,2)$ so that $\gamma_s = \sum_t a_{st}\, j_t$, $s = 1, 2, 3$.

If we regard the vector space $\mathbb{B}^n$ (the elements of $\mathbb{B}^n$ are thought of as column vectors) as a right $\mathbb{B}$ module, the multiplication from the left with $n \times n$ matrices with entries in $\mathbb{B}$ represents the space of all $\mathbb{B}$-linear endomorphisms of $\mathbb{B}^n$. We define $Sp(n,\mathbb{B})$ to be the group of all $\mathbb{B}$-linear transformations that preserve the inner product $\langle x, y\rangle = Re(\overline{x}^T\, y)$, $x, y \in \mathbb{B}^n$,

$$Sp(n,\mathbb{B}) = \Big\{ A \in M_n(\mathbb{B}) \;:\; \overline{A}^T A = 1 \Big\}.$$

In particular, $Sp(1,\mathbb{B})$ is the group of the unite split-quaternions,

$$Sp(1,\mathbb{B}) = \Big\{ z = z_0 + z_1\, j_1 + z_2\, j_2 + z_3\, j_3 \;:\; z_0^2 - z_1^2 - z_2^2 + z_3^2 = 1 \Big\}.$$

Consider the action of the direct product $Sp(n,\mathbb{B}) \times Sp(1,\mathbb{B})$ on the vector space $\mathbb{B}^n$, defined by

$$(A, z) \cdot x \;=\; A\, x\, \overline{z}, \tag{5}$$

and let us fix (once and for all) identification $\mathbb{B}^n = \mathbb{R}^{4n}$. Since the induced inner product is of signature $(2n, 2n)$, we obtain an embedding of the quotient group

$$\frac{Sp(n,\mathbb{B}) \times Sp(1,\mathbb{B})}{\{\pm(1,1)\}}$$

into the matrix group $SO(2n, 2n)$. The image of this embedding is denoted by $Sp(n,\mathbb{B})Sp(1,\mathbb{B})$ and consists of all elements of $SO(2n, 2n)$ that preserve the three-dimensional subspace $\mathcal{Q} \subset End(\mathbb{R}^{4n})$ generated by the right action of $Im(\mathbb{B})$ on $\mathbb{B}^n$.

2.3. Paraquaternionic Contact Structures

Consider a $4n$-dimensional smooth distribution H on a $(4n+3)$-dimensional manifold M. Suppose that at each point p in an open subset $U \subset M$, we are given a triple (η_1, η_2, η_3) of 1-forms on T_pM, a triple (I_1, I_2, I_3) of endomorphisms of $H_p \subset T_pM$, and a non-degenerate quadratic form g on H_p, all depending smoothly on point p. The list (η_s, I_s, g) is called a (local) paraquaternionic contact (shortly: pqc) structure for H on U if the following three conditions are satisfied at each $p \in U$:

(i) $H_p = \Big\{A \in T_pM \ : \ \eta_1(A) = \eta_2(A) = \eta_3(A) = 0\Big\}$;

(ii) $d\eta_s(X,Y) = 2g(I_sX,Y), \quad \forall X,Y \in H_p, \quad$ s = 1, 2, 3;

(iii) $I_1^2 = I_2^2 = \mathrm{id}, \quad I_1I_2 = -I_2I_1 = I_3.$

Clearly, for every local pqc-structure (η_s, I_s, g) for H, the quadratic form g must be of signature (2n,2n). The pair (M,H) is called a paraquaternionic contact manifold if, around each point of M, there exists at least one local pqc-structure for H. Here arises the natural question: to what extent are the different local pqc-structures determined by distribution H? The answer is given by the following.

Lemma 2. *Suppose that (M,H) is a pqc manifold. If (η_s, I_s, g) and (η'_s, I'_s, g') are two pqc-structures for H on an open set $U \subset M$, then*

$$(\eta'_1, \eta'_2, \eta'_3) = f(\eta_1, \eta_2, \eta_3)\mathcal{S}, \quad (I'_1, I'_2, I'_3) = (I_1, I_2, I_3)\mathcal{S}, \quad g' = f\,g,$$

for some non-vanishing real valued smooth function f on U and some matrix-valued smooth function $\mathcal{S} = (a_{ij}) : U \to SO(1,2)$.

Proof. By assumption $H = \cap_{s=1}^{3} Ker(\eta_s) = \cap_{s=1}^{3} Ker(\eta'_s)$, there exists a matrix-valued function $\mathcal{A} = (a_{st}) : U \to GL(3)$ so that $\eta'_s = \sum_{t=1}^{3} a_{st}\eta_t$, $s = 1,2,3$. Applying the exterior derivative to both sides of this equation and taking the restriction of the resulting two forms to distribution H, we obtain

$$d\eta'_s|_H = \sum_t a_{st}(d\eta_t|_H). \tag{6}$$

If G' is a field of endomorphisms of H defined by the equation $g'(X,Y) = g(G'X,Y)$, $X,Y \in H$, then $d\eta'_s(X,Y) = g(G'I'_sX,Y)$, and using (6),

$$G'I'_s = \sum_t a_{st}I_t.$$

This yields

$$I'_1 = (I'_2)^{-1}\,I'_3 = (G'I'_2)^{-1}\,(G'I'_3) = (\sum_s a_{2s}I_s)^{-1}(\sum_t a_{3t}I_t) \ \in \ \mathrm{span}_{\mathbb{R}}\,\{id_H, I_1, I_2, I_3\},$$

similarly to I'_2 and I'_3. Let us observe that $\mathrm{span}_{\mathbb{R}}\,\{id_H, I_1, I_2, I_3\} \subset End(H)$ is an algebra with respect to the usual composition of endomorphisms, which is isomorphic to the algebra of split-quaternions. Therefore, using Lemma 1, we have

$$\mathrm{span}_{\mathbb{R}}\,\{I_1, I_2, I_3\} \ = \ \mathrm{span}_{\mathbb{R}}\,\{I'_1, I'_2, I'_3\}.$$

In particular, this yields that I'_1, I'_2, I'_3 are skew-symmetric with respect to both g and g'. Furthermore, we calculate the following:

$$\begin{aligned} g\Big((G'I'_1I'_2 + I'_2G'I'_1)X, Y\Big) &= g(G'I'_3X,Y) - g(G'I'_1X, I'_2Y) \\ &= g'(I'_3X,Y) - g'(I'_1X, I'_2Y) = 0, \end{aligned}$$

i.e., $G'I'_1$ anti-commutes with I'_2, similarly to I'_3. Therefore, $G'I'_1$ must be proportional to I'_1, i.e., G' is proportional to the identity. This means

$$g' = f\,g$$

for some appropriate non-vanishing real-valued function f. The rest follows from Lemma 1. □

An important consequence of the above lemma is that for each pqc manifold (M, H), we can associate a canonical line bundle $\mathcal{G}(M) \to M$ so that if (η_s, I_s, g) is a local pqc structure for H, then g is a local section of $\mathcal{G}(M)$. Furthermore, the vector bundle $\pi : \mathcal{Q}(M) \to M$ with fiber (over p)

$$\mathcal{Q}_p = \mathrm{span}\{I_1, I_2, I_3\}, \tag{7}$$

is also globally defined. It has a canonical inner product,

$$\langle I_s, I_t\rangle = \begin{cases} \epsilon_s, \text{ if } s = t \\ 0, \text{ otherwise} \end{cases}, \qquad \epsilon_1 = \epsilon_2 = -\epsilon_3 = -1 \tag{8}$$

of signature $(-, -, +)$ and an orientation defined by the ordering of I_1, I_2 and I_3.

2.4. Invariant Tensor Decomposition

Let (M, H) be a pqc manifold and consider some local pqc-structure (η_s, I_s, g) for H, defined around a fixed $p \in M$. Each endomorphism $\Psi \in End(H_p)$ can be decomposed uniquely into a sum of four components, $\Psi = \Psi^{+++} + \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}$, where Ψ^{+++} commutes with I_1, I_2, and I_3 and Ψ^{+--} commutes with I_1 and anti-commutes with I_2 and I_3, etc. Explicitly,

$$4\Psi^{+++} = \Psi + I_1\Psi I_1 + I_2\Psi I_2 - I_3\Psi I_3; \quad 4\Psi^{+--} = \Psi + I_1\Psi I_1 - I_2\Psi I_2 + I_3\Psi I_3;$$
$$4\Psi^{-+-} = \Psi - I_1\Psi I_1 + I_2\Psi I_2 + I_3\Psi I_3; \quad 4\Psi^{--+} = \Psi - I_1\Psi I_1 - I_2\Psi J_2 - I_3\Psi I_3.$$

Clearly, this decomposition depends on the particular choice of a pqc structure. To obtain invariant decomposition, we shall consider the action of the Casimir operator † on $End(H_p)$, given by

$$\dagger(\Psi) = I_1\Psi I_1 + I_2\Psi I_2 - I_3\Psi I_3.$$

The leading signs $(+, +, -)$ in the above summation are opposite to the signature of the invariant inner product on $\mathcal{Q}_p$ (cf. (8)); therefore, † must be invariant too. It is easily seen that this Casimir operator has eigenvalues 3 and -1, and that, if $\Psi = \Psi_{[3]} + \Psi_{[-1]}$ is the induced decomposition of $\Psi \in End(H_p)$ into a sum of eigenvectors, then

$$\Psi_{[3]} = \Psi^{+++} \qquad \text{and} \qquad \Psi_{[-1]} = \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}.$$

2.5. The Canonical Connection

In general, a pqc manifold (M, H) is a parabolic type of geometry that cannot be characterized by a linear connection on the tangent bundle of M; it requires more complicated construction involving a certain Cartan connection, which we shall not deal with here. Instead, we shall use an auxiliary assumption. We require that the naturally induced line bundle $\mathcal{G}(M) \to M$ (cf. Section 2.3) admits a global non-vanishing section g; that is, there is a globally defined g on M so that around each point, one can find at least one local pqc structure for H of the form (η_s, I_s, g) (with last entry the same g).

The triple (M, H, g) is already a much simpler type of geometry that can be characterized by a unique linear connection ∇ on the tangent bundle of M (as shown in [4]) called the canonical connection of the triple. We shall summarize all the relevant properties of this connection below. Let us first observe that the differential invariants produced by ∇ depend strongly on the choice of g. If we are interested only in the geometry defined by

(M, H), we need to consider those differential invariants that remain unchanged after an arbitrary multiplication of g by a non-vanishing function (cf. Lemma 2). The relationship between (M, H, g) and (M, H) is similar to that between the Riemannian and conformal Riemannian geometry.

In [4], it is shown (with a slightly different notation) that if the dimension of M is at least 11, to each choice of (a global) g, there exists a unique complementary (vertical) distribution $V \subset TM$ on M,

$$TM = H \oplus V. \tag{9}$$

If we pick any local pqc structure (η_s, I_s, g) for H, then V is the real span of local vector fields ξ_1, ξ_2 and ξ_3 on M, called Reeb vector fields, which are defined by the following equations:

$$\begin{aligned} &(i) \quad \eta_s(\xi_t) = \begin{cases} \epsilon_s, \text{ if } s = t \\ 0, \text{ otherwise} \end{cases}, \qquad \epsilon_1 = \epsilon_2 = -\epsilon_3 = -1; \\ &(ii) \quad d\eta_s(\xi_t, X) + d\eta_t(\xi_s, X) = 0, \quad \forall X \in H, \quad s,t = 1,2,3. \end{aligned} \tag{10}$$

Remark 1. *In (the lowest) dimension 7, the existence of Reeb vector fields is an additional condition on the structure, which we shall assume is always satisfied.*

At each $p \in M$, the vector space H_p is isomorphic as a $Sp(n, \mathbb{B})Sp(1, \mathbb{B})$ module to $\mathbb{B}^n$ $\big($with the action (5)$\big)$, and the set of all isomorphisms from H_p to $\mathbb{B}^n$ constitutes a fiber over p of a certain principle bundle $\mathcal{P}(M) \to M$ with a structure group $Sp(n, \mathbb{B})Sp(1, \mathbb{B})$. The Reeb vector fields (10) allow us to extend the action of $Sp(n, \mathbb{B})Sp(1, \mathbb{B})$ on H_p to an action on the whole tangent space at p, $T_pM = H_p \oplus V_p$ by declaring that $Sp(n, \mathbb{B})Sp(1, \mathbb{B})$ acts on the Reeb vector fields ξ_s in the same way as it acts on the endomorphisms $I_s \in End(H_p)$. It is easily verified (using Lemma 2) that this action remains unchanged if we replace the initial pqc structure (η_s, I_s, g) with another (of course, the Rieb vector fields must undergo a respective transformation as well) as long as the g-entry remains the same; that is, the choice of g allows us to consider T_pM as a $Sp(n, \mathbb{B})Sp(1, \mathbb{B})$ module isomorphic to $\mathbb{B}^n \oplus Im(\mathbb{B})$, and the set of all isomorphisms is a principle fiber bundle $\mathcal{P}(M) \to M$ with a structure group $Sp(n, \mathbb{B})Sp(1, \mathbb{B})$.

The canonical connection ∇ is a principle $Sp(n, \mathbb{B})Sp(1, \mathbb{B})$-connection on $\mathcal{P}$, whose torsion tensor

$$T(A, B) = \nabla_A B - \nabla_B A - [A, B], \qquad A, B \in TM, \tag{11}$$

can be described as follows.

We define three (local) two-forms, ω_1, ω_2, and ω_3, on M by setting

$$\omega_s(A, B) \;=\; g\big(I_s(A_H), B_H\big), \qquad s = 1,2,3, \tag{12}$$

where by subscript H we mean projection onto H w.r.t. the decomposition (9). There exists a (unique) triple $(Scal, \tau, \mu)$, where $Scal$ is a (global) function on M; τ and μ are globally defined as traceless symmetric sections of the endomorphism bundle $End(H) \to M$, satisfying $\tau = \tau_{[-1]}$, $\mu = \mu_{[3]}$ (cf. Section 2.4), so that the following is true:

$$\begin{aligned} &(i) \quad T(X, Y) \;=\; -2\omega_1(X, Y)\xi_1 - 2\omega_2(X, Y)\xi_2 + 2\omega_3(X, Y)\xi_3, \\ &(ii) \quad T(\xi_s, X) \;=\; \Big(\frac{1}{4}\left(I_s\,\tau \;-\; \tau\, I_s\right) \;+\; I_s\,\mu\Big)X, \\ &(iii) \quad T(\xi_s, \xi_t) \;=\; -\frac{Scal}{8n(n+1)}\,\xi_s \times \xi_t - \big[\xi_s, \xi_t\big]_H \qquad cf, (15) \end{aligned} \tag{13}$$

for any $X, Y \in H,\ s, t = 1, 2, 3$. Notice also that the vertical distribution V has an induced inner product $\langle , \rangle$ of signature $(-, -, +)$, so that

$$\langle \xi_s, \xi_t \rangle = \begin{cases} \epsilon_s, \text{ if } s = t \\ 0, \text{ otherwise.} \end{cases} \tag{14}$$

On V, we also have a natural orientation and a cross product "$\times$" (cf. (3)):

$$\xi_1 \times \xi_2 = \xi_3, \qquad \xi_2 \times \xi_3 = -\xi_1, \qquad \xi_3 \times \xi_1 = -\xi_2. \tag{15}$$

Clearly, the two vector bundles $V \to M$ and $\pi : \mathcal{Q}(M) \to M$ over M (cf. (7)) are isomorphic, and the cross product on V_p corresponds to the half-commutator on $\mathcal{Q}_p$:

$$\frac{1}{2}[I, J] \;=\; \frac{1}{2}(IJ - JI), \qquad I, J \in \mathcal{Q}_p.$$

Since both $\langle , \rangle$ and "$\times$" are ∇-parallel, locally, on the same domain where the considered local pqc-structure (η_s, I_s, g) is defined, we can find certain one-forms, i.e., α_1, α_2 and α_3 (called connection 1-forms), so that

$$\nabla_A I_s \;=\; \frac{1}{2}\Big[\sum_t \alpha_t(A) I_t,\ I_s\Big], \tag{16}$$

or equivalently,

$$\nabla_A \xi_s \;=\; \Big(\sum_t \alpha_t(A)\xi_t\Big) \times \xi_s, \tag{17}$$

for all $A \in T_pM$ and $s = 1, 2, 3$. As shown in [4], the connection one-forms are completely determined by the exterior derivatives of the three one-forms η_s and the function $Scal$,

$$\begin{aligned} \alpha_i(X) &= d\eta_k(\xi_j, X) = -d\eta_j(\xi_k, X), \\ \alpha_i(\xi_s) &= d\eta_s(\xi_j, \xi_k) - \delta_{is}\Bigg(\frac{Scal}{16n(n+2)}, \\ &\qquad + \frac{1}{2}(d\eta_1(\xi_2, \xi_3) + d\eta_2(\xi_3, \xi_1) + d\eta_3(\xi_1, \xi_2))\Bigg), \end{aligned} \tag{18}$$

for all $X \in H$ and $s = 1, 2, 3$, where δ_{is} is the Kronecker delta, and (ijk) is any positive permutation of $1, 2, 3$.

2.6. Curvature

It turns out that not only the torsion (cf. (13)) but also many of the contractions of the curvature tensor,

$$R(A, B) = [\nabla_A, \nabla_B] \;-\; \nabla_{[A,B]}, \qquad A, B \in TM,$$

are completely determined by the triple $(Scal, \tau, \mu)$. Consider a local frame $e_a \in H, 1 \le a \le 4n$ for H, and let $e_a^* \in H$ be its dual; that is, the frame defined by the following equations:

$$g(e_a, e_b^*) \;=\; \begin{cases} 1, & \text{if } a = b \\ 0, & \text{otherwise} \end{cases}, \quad \forall a, b = 1, \ldots, 4n. \tag{19}$$

The Ricci curvature, Ric, is defined by

$$Ric(A, B) = \sum_a g(R(e_a, A)B, e_a^*), \qquad A, B \in TM.$$

According to [4], we have

$$Ric(X,Y) \;=\; g\Big(\frac{Scal}{4n}X \;+\; (2n+2)\tau(X) \;+\; (4n+10)\mu(X),\, Y\Big),$$

for all $X, Y \in H$. In particular, $Scal = \sum_a Ric(e_a, e_a^*)$, i.e., $Scal$ is indeed the scalar curvature of ∇.

Since, by design, ∇ is a principle $Sp(n,\mathbb{B})Sp(1,\mathbb{B})$ connection, its curvature splits into a sum of two components, $sp(n,\mathbb{B}) \oplus sp(1,\mathbb{B})$. We shall use the Ricci two-forms ρ_s to represent the $sp(1,\mathbb{B})$ component of the curvature:

$$\big[R(A,B), I_s\big] \;=\sum_{t=1}^{3}\rho_t(A,B)\big[I_t, I_s\big], \qquad \forall A,B \in TM, \quad s = 1,2,3, \tag{20}$$

or equivalently,

$$\rho_s(A,B) \;=\; \frac{\epsilon_s}{4n}\sum_a g\big(R(A,B)e_a, I_s e_a^*\big).$$

Using the result in [4],

$$\begin{aligned}
\rho_s(X,Y) \;&=\; -\,\epsilon_s\, g\bigg(\frac{1}{2}\big(\tau\, I_s + I_s\,\tau\big)X + 2\Big(\mu + \frac{Scal}{16n(n+2)}\Big)I_s X,\, Y\bigg),\\
\rho_i(X,\xi_i) \;&=\; -\,\frac{d\,Scal(X)}{32n(n+2)} \;+\; \frac{1}{2}g\Big(I_i\big[\xi_j,\xi_k\big]_H + I_j\big[\xi_k,\xi_i\big]_H + I_k\big[\xi_i,\xi_j\big]_H,\, X\Big),\\
\rho_i(X,\xi_s) \;&=\; g\Big(I_s\big[\xi_j,\xi_k\big]_H,\, X\Big), \qquad i \neq s,
\end{aligned} \tag{21}$$

for all $X, Y \in H$ and $s = 1,2,3$, where (ijk) is any positive permutation of 1,2,3 ($d\,Scal$ is the differential of $Scal$). For the values of the three Ricci two-forms on a pair of vertical vector fields, we have the identity

$$\rho_i(\xi_i,\xi_j) + \rho_k(\xi_k,\xi_j) \;=\; \frac{d\,Scal(\xi_j)}{16n(n+2)}. \tag{22}$$

3. Twistor and Reflector Spaces

The twistor space $\mathcal{Z}$ and the reflector space $\mathcal{R}$ of a pqc manifold (M,H) are defined as subbundles of the canonical vector bundle $\pi : \mathcal{Q}(M) \to M$ (cf. (7)). The corresponding fibers over a point $p \in M$ are

$$\mathcal{Z}_p \;=\; \Big\{I \in \mathcal{Q}_p(M) \;:\; I^2 = -\mathrm{id}\Big\} \qquad \text{and} \qquad \mathcal{R}_p \;=\; \Big\{I \in \mathcal{Q}_p(M) \;:\; I^2 = \mathrm{id}\Big\}.$$

The purpose of this section is to prove the two following propositions.

Proposition 1. *On the twistor space $\mathcal{Z}$, there exists a natural co-dimension one distribution $\mathcal{K} \subset T\mathcal{Z}$ and a smooth field J of endomorphisms of $\mathcal{K}$, which satisfies $J^2 = -id$ (such a pair $(\mathcal{K}, J)$ is called an almost CR structure).*

Furthermore, if η is any local one-form on $\mathcal{Z}$ with $\mathcal{K} = \ker(\eta)$, then at each $I \in \mathcal{Z}$, $d\eta(J.,.)$ there is a non-degenerate symmetric two-tensor on $\mathcal{K}_I$ of signature $(2n+2, 2n+2)$, $\dim(M) = 4n+3$; that is, the Levi form of the almost CR structure on $\mathcal{Z}$ is of signature $(2n+2, 2n+2)$.

Proposition 2. *On the reflector space $\mathcal{R}$, there exists a natural co-dimension one distribution $\mathcal{K} \subset T\mathcal{R}$ and a smooth field J of endomorphisms of $\mathcal{K}$, which satisfies $J^2 = id$ (such a pair $(\mathcal{K}, J)$ is called an almost para-CR structure).*

Furthermore, if η is any local one-form on $\mathcal{R}$ with $\mathcal{K} = \ker(\eta)$, then at each $I \in \mathcal{R}$, $d\eta(J.,.)$ is a non-degenerate symmetric two-tensor on $\mathcal{K}_I$ of signature $(2n+2, 2n+2)$; that is, the Levi form of the almost para-CR structure is of signature $(2n+2, 2n+2)$.

Later in this paper (Section 4), we will show that both the almost CR structure on $\mathcal{Z}$ and the almost para-CR structure on $\mathcal{R}$ are in fact integrable.

3.1. The Induced Structure on $\mathcal{Q}$

To begin with, let us fix an arbitrary non-vanishing section g of the line bundle $\mathcal{G}(M) \to M$ (cf. Section 2.3) and consider the corresponding canonical connection ∇ on TM. We shall use ∇ to induce a certain structure on the tangent space of the vector bundle $\mathcal{Q} = \mathcal{Q}(M)$. Indeed, since ∇ preserves the vector bundle $\mathcal{Q} \subset End(TM)$, it defines a horizontal distribution $\mathcal{D} \subset T\mathcal{Q}$ so that the horizontal lift A^h (w.r.t. ∇) of any vector field A on M is a vector field on $\mathcal{Q}$ tangent to $\mathcal{D}$. On the other hand, there is a distribution $\mathcal{F} = \ker(\pi_*) \subset T\mathcal{Q}$ that consists of all vectors that are tangent to the fibers of the bundle $\pi : \mathcal{Q} \to M$. We have the following direct sum decomposition:

$$T\mathcal{Q} \;=\; \mathcal{D} \oplus \mathcal{F}.$$

The differential π_* of the projection map $\pi : \mathcal{Q} \to M$ at any $I \in \mathcal{Q}$ is an isomorphism between $\mathcal{D}_I$ and T_pM, where $p = \pi(I)$. There is also a natural isomorphism $\mathcal{F}_I \cong \mathcal{Q}_p$ that identifies the tangent vector to a curve $t \mapsto I(t) \in \mathcal{Q}_p$ at $I(0) = I$ (that is, any element of $\mathcal{F}_I$) with the respective derivative $\frac{d\,I(t)}{dt}_{|t=0}$ (which is as an element of the fiber $\mathcal{Q}_p$).

Let us consider a (small enough) domain U of local coordinates u_α, $1 \le \alpha \le 4n+3$ on M. For each $I \in \pi^{-1}(U) \subset \mathcal{Q}$, we know that $I = x_1 I_1 + x_2 I_2 + x_3 I_3$; thus, we may consider the functions

$$u_\alpha \circ \pi,\; x_1,\; x_2,\; x_3, \qquad 1 \le \alpha \le 4n+3, \tag{23}$$

as local coordinates on $\mathcal{Q}$ (we shall abbreviate $u_\alpha \circ \pi$ to u_α). In this coordinate chart, the isomorphism between $\mathcal{F}_I$ and $\mathcal{Q}_p$ identifies $\frac{\partial}{\partial x_s}$ with I_s for $s = 1,2,3$.

Lemma 3. *Within the coordinate chart (23), the horizontal lift A^h of a vector field*

$$A \;=\; \sum_{a=1}^{4n+3} A_s \frac{\partial}{\partial u_a}$$

on M, at $I = \sum_s x_s I_s \in \mathcal{Q}$, is given by

$$\begin{aligned} A^h_I \;&=\; \sum_{\alpha=1}^{4n+3} A_\alpha \frac{\partial}{\partial u_\alpha} \;-\; \sum_{s,t=1}^{3} x_s \Big\langle \nabla_A I_s, \epsilon_t I_t \Big\rangle \frac{\partial}{\partial x_t} \\ &=\; \sum_{\alpha=1}^{4n+3} A_\alpha \frac{\partial}{\partial u_\alpha} \;+\; \sum_{(ijk)} \epsilon_i \Big(x_j\, \alpha_k(A) - x_k \alpha_j(A) \Big) \frac{\partial}{\partial x_i}, \end{aligned} \tag{24}$$

where A^h_I denotes the value of A^h at I, and α_s are the connection one-forms of ∇ (cf. (16)).

Proof. Consider a curve $t \;\mapsto\; \Big(u_\alpha(t),\, x_s(t)\Big)$ within the coordinate chart (23), passing through a fixed $I \in \mathcal{Q}$ at a time $t = 0$. Suppose that the tangent vector to this curve at $t = 0$ is A^h_I. Then,

$$0 \;=\; \nabla_A \Big(\sum_s x_s(t) I_s \Big) \;=\; \sum_s \Big(\dot{x}_s(0) I_s \;+\; x_s(0) \nabla_A I_s \Big);$$

therefore, since $x_s(0)I_s = I$,

$$\dot{x}_s(0) = -\sum_t x_t \left\langle \nabla_A I_t, \epsilon_s I_s \right\rangle; \tag{25}$$

that is, for the horizontal lift A^h_I we have

$$A^h_I = \sum_{\alpha=1}^{4n+3} A_\alpha \frac{\partial}{\partial u_\alpha} + \sum_{s=1}^{3} \dot{x}_s(0) \frac{\partial}{\partial x_s},$$

where $\dot{x}_s(0)$ are given by (25). Applying (16) to the latter yields the result. □

Lemma 4. *For any two vector fields A and B on M, within a coordinate chart like (23), the commutator of their respective horizontal lifts A^h and B^h at any $I = \sum_s x_s I_s \in \mathcal{Q}$ is given by*

$$\left[A^h, B^h\right]_I = \left[A, B\right]^h_I + \sum_{(ijk)} 2\epsilon_i \Big(x_j \rho_k(A,B) - x_k \rho_j(A,B) \Big) \frac{\partial}{\partial x_i},$$

where ρ_s are the corresponding Ricci two-forms (cf. (20)).

Proof. Using (24), we calculate

$$\begin{aligned} \left[A^h, B^h\right]_I &= \sum_{\alpha=1}^{4n+3} \left[A, B\right]_\alpha \frac{\partial}{\partial u_\alpha} - \sum_{s,t=1}^{3} x_s \Big\langle \nabla_A(\nabla_B I_s) - \nabla_B(\nabla_A I_s), \epsilon_t I_t \Big\rangle \frac{\partial}{\partial x_t} \\ &= \left[A, B\right]^h_I - \sum_{s,t=1}^{3} x_s \Big\langle \left[R(A,B), I_s\right], \epsilon_t I_t \Big\rangle \frac{\partial}{\partial x_t} \end{aligned}.$$

The result follows from (20). □

Next, we consider two naturally defined (global) vector fields, χ and $\mathcal{N}$, on $\mathcal{Q}$. At any $I = \sum_s x_s I_s \in \mathcal{Q}$, we set, with respect to the coordinate chart (23),

$$\chi = \sum_s x_s \xi^h_s \qquad \text{and} \qquad \mathcal{N} = \sum_s x_s \frac{\partial}{\partial x_s}. \tag{26}$$

Clearly, $\mathcal{N}$ is a section of the vertical distribution $\mathcal{F} \subset T\mathcal{Q}$. On the other hand, the splitting of $TM = H \oplus V$ (cf. (9)) defines the splitting of the horizontal distribution, $\mathcal{D} = \mathcal{H} \oplus \mathcal{V}$, and the vector field χ is tangent everywhere to $\mathcal{V}$.

Suppose that $I \in \mathcal{Q}$, considered as an endomorphism of the vector space $H_p \subset T_pM$, does not square to 0, $I^2 \neq 0$. Letting $\mathcal{W}_I$ be the orthogonal complement of $\mathcal{N}$ in $\mathcal{F}_I$, and $\mathcal{U}_I$ the orthogonal complement of χ in $\mathcal{V}_I$ (the orthogonality is with respect to (14) and (8)), we obtain the splitting

$$T_I\mathcal{Q} = \overbrace{\mathcal{H}_I \oplus \underbrace{\mathcal{U}_I \oplus \mathbb{R}\cdot\chi_I}_{\mathcal{V}_I}}^{\mathcal{D}_I} \oplus \underbrace{\mathcal{W}_I \oplus \mathbb{R}\cdot\mathcal{N}_I}_{\mathcal{F}_I}. \tag{27}$$

We now consider a canonical one-form η on $\mathcal{Q}$, defined at any $I = x_s I_s \in \mathcal{Q}$, by

$$\eta = \sum_s x_s \, \pi^*(\eta_s), \tag{28}$$

where $\pi^*(\eta_s)$ is the pullback of η_s via $\pi : \mathcal{Q} \to M$. In order to calculate the exterior derivative of η, we introduce three local one-forms ϕ_1, ϕ_2 and ϕ_3 on $\mathcal{Q}$ using the following formula

$$\phi_i = \epsilon_i \, dx_i - x_j \, \pi^*(\alpha_k) - x_k \, \pi^*(\alpha_j) \tag{29}$$

for any positive permutation (ijk) of $1,2,3$. Clearly, the forms ϕ_s are only defined within the coordinate chart (23). According to Lemma 3, each ϕ_s vanishes on the horizontal distribution $\mathcal{D}$, and we have

$$\phi_s\Big(\frac{\partial}{\partial x_t}\Big) \;=\; \begin{cases} \epsilon_s, \text{ if } s=t \\ 0, \text{ otherwise.} \end{cases} \tag{30}$$

For any $A \in T_I\mathcal{Q}$, we have

$$A = \Big(\big(\pi_* A\big)_H\Big)^h \;+\; \sum_s \epsilon_s \Big(\eta_s(A)\,\xi_s^h \;+\; \phi_s(A)\,\frac{\partial}{\partial x_s}\Big). \tag{31}$$

By subscript H we mean projection onto H w.r.t. the decomposition (9).

Lemma 5. *The exterior derivative of the canonical one-form η on $\mathcal{Q}$ is given (within the coordinate chart (23)) by*

$$d\eta \;=\; \sum_{(ijk)} \Big(2x_i\,\pi^*(\omega_i) \;+\; \epsilon_i\,\phi_i \wedge \pi^*(\eta_i) \;-\; \frac{Scal}{8n(n+2)}\,\epsilon_i\,x_i\,\pi^*(\eta_j\wedge\eta_k)\Big).$$

The two-forms ω_s are as in (12); for the wedge product, we use the formula $\phi_i \wedge \pi^*(\eta_i)\,(A,B) = \phi_i(A)\,\eta_i(\pi_* B) - \phi_i(B)\,\eta_i(\pi_* A)$.

Proof. Differentiating (28) yields

$$d\eta \;= \sum_s \Big(dx_s \wedge \pi^*(\eta_s) \;+\; x_s\,\pi^*(d\eta_s)\Big).$$

We calculate:

$$d\eta(X^h, \tilde X^h) \;=\; \sum_s x_s d\eta_s(X, \tilde X) \;=\; 2\sum_s x_s g(I_s X, \tilde X);$$

$$\begin{aligned} d\eta\big(X^h,\, \xi_i^h\big) \;&=\; \epsilon_i\,dx_i(X^h) \;+\; \sum_s x_s d\eta_s(X, \xi_i) \\ &\overset{cf.(24)}{=}\; x_j\alpha_k(X) - x_k\alpha_j(X) \;+\; \sum_s x_s d\eta_s(X,\xi_i) \;\overset{cf.(18)}{=}\; 0; \end{aligned}$$

$$\begin{aligned} d\eta\big(\xi_i^h,\, \xi_j^h\big) \;&=\; \epsilon_j\,dx_j(\xi_i^h) - \epsilon_i\,dx_i(\xi_j^h) + \sum_s x_s d\eta_s(\xi_i,\xi_j) \\ &\overset{cf.(24)}{=}\; x_k\alpha_i(\xi_i) - x_i\alpha_k(\xi_i) - x_j\alpha_k(\xi_j) + x_k\alpha_j(\xi_j) + \sum_s x_s d\eta_s(\xi_i,\xi_j) \\ &\overset{cf.(18)}{=}\; -\frac{Scal}{8n(n+2)}\,x_k; \end{aligned}$$

$$d\eta\Big(\xi_s^h,\, \frac{\partial}{\partial x_t}\Big) \;=\; -\eta_t(\xi_s).$$

□

As a consequence of the previous lemma, we obtain the following.

Corollary 1. *At any $I = \sum_s x_s\,I_s \in \mathcal{Q}$, the canonical one-form η and the vector field χ (cf. (26)) satisfy*

$$\eta(\chi) \;=\; \sum_s \epsilon_s\,x_s^2 \qquad \textit{and} \qquad \chi \lrcorner d\eta \;=\; -\sum_s \epsilon_s\,x_s\,dx_s,$$

where $\chi \lrcorner d\eta(A) = d\eta(\chi, A)$.

Let $\mathcal{Q}^o \subset \mathcal{Q}$ be the open subset consisting of all $I \in \mathcal{Q}$ with $I^2 \neq 0$. Clearly, the twistor and the reflector spaces $\mathcal{Z}$ and $\mathcal{R}$ are submanifolds in $\mathcal{Q}^o$. On the manifold $\mathcal{Q}^o$, we have the distribution

$$\mathcal{K} = \mathcal{H} \oplus \mathcal{U} \oplus \mathcal{W} \subset T\mathcal{Q}^o. \tag{32}$$

Using local coordinates (23) and one-forms ϕ_s (cf. (30)), $\mathcal{K}$ can be described with the equations

$$\sum_s x_s\, \pi^*(\eta_s) \;=\; 0 \qquad \text{and} \qquad \sum_s x_s\, \phi_s \;=\; 0.$$

We introduce a natural field J of endomorphisms of the distribution $\mathcal{K}$ that satisfies $J^2 = -\langle I, I\rangle\, \mathrm{id}$ by setting

$$J(X + U + W) \;=\; \Big(I(\pi_* X)\Big)^h_I \;+\; \chi_I \times U \;+\; \mathcal{N}_I \times W, \tag{33}$$

where $X \in \mathcal{H}_I$, $U \in \mathcal{U}_I$ and $W \in \mathcal{W}_I$. For any $A \in \mathcal{K}_I$ within the coordinate chart (23), we have (cf. (31))

$$\begin{aligned} J(A) \;=\; \sum_s x_s \Big(I_s \pi_*(A)_H\Big)^h \;+\; \sum_{(ijk)} \Big(\epsilon_j\, x_j\, \eta_k(\pi_* A) - \epsilon_k\, x_k\, \eta_j(\pi_* A)\Big)\xi_i^h \\ + \sum_{(ijk)} \Big(\epsilon_j\, x_j\, \phi_k(A) - \epsilon_k\, x_k\, \phi_j(A)\Big)\frac{\partial}{\partial x_i}. \end{aligned} \tag{34}$$

Let us denote by G the bilinear form

$$G(A,B) = -\frac{1}{2\langle I, I\rangle} d\eta(JA, B), \qquad A, B \in \mathcal{K}_I.$$

Since $J^2 = -\langle I, I\rangle\, \mathrm{id}$, we have

$$d\eta(A,B) \;=\; 2G(JA,B), \qquad A, B \in \mathcal{K}_I.$$

Lemma 6. *At any $I \in \mathcal{Q}^o$, G is a symmetric two-form on $\mathcal{K}_I \subset T_I\mathcal{Q}^o$ (cf. (32)) of signature $(2n+2, 2n+2)$, which satisfies the relation*

$$G(JA,B) = -G(A,JB)$$

for $A, B \in \mathcal{K}_I$. Explicitly, within the coordinate chart (23), we have that

$$\begin{aligned} G(A,B) \;&= g\Big((\pi_* A)_H, (\pi_* B)_H\Big), \\ &- \frac{Scal}{16n(n+2)} \sum_s \epsilon_s\, \eta_s(\pi_* A)\, \eta_s(\pi_* B) \\ &- \frac{1}{2\langle I, I\rangle} \sum_{(ijk)} \epsilon_i\, x_i \Big(\phi_j(A)\, \eta_k(\pi_* B) \;+\; \eta_k(\pi_* A)\, \phi_j(B), \\ &\qquad - \phi_k(A)\, \eta_j(\pi_* B) \;-\; \eta_j(\pi_* A)\, \phi_k(B)\Big). \end{aligned} \tag{35}$$

Proof. Formula (35) is a straightforward application of Lemma 5. To calculate the signature of G on $\mathcal{K}_I$, we first observe that the two subspaces $\mathcal{H}_I$ and $\mathcal{U}_I + \mathcal{W}_I$ are G-orthogonal, and the restriction of G to $\mathcal{H}_I$ has the same signature as g. Therefore, we only need to show that the restriction of G to $\mathcal{U}_I + \mathcal{W}_I$ is of signature $(2, 2)$.

For any fixed $I \in \mathcal{Q}^o$, we can pick a local pqc structure (η_s, I_s, g) in such a way so that either $I = \lambda I_3$ or $I = \lambda I_1$, $\lambda \in \mathbb{R}$. In the first case, the restriction of G to $\mathcal{U}_I + \mathcal{W}_I$ is given, w.r.t. the frame $\{\xi_1^h, \xi_2^h\}$ of $\mathcal{U}_I$ and the frame $\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right\}$ of $\mathcal{W}_I$, by the matrix

$$\begin{pmatrix} h & 0 & 0 & l \\ 0 & h & -l & 0 \\ 0 & -l & 0 & 0 \\ l & 0 & 0 & 0 \end{pmatrix}, \tag{36}$$

where $h = \frac{Scal}{16n(n+2)}$ and $l = \frac{1}{2\lambda}$. This matrix has two eigenvalues, each with multiplications of two: $\frac{1}{2}\left(h \pm \sqrt{h^2 + 4l^2}\right)$. Therefore, the restriction of G to $\mathcal{U}_I + \mathcal{W}_I$ has signature (2,2).

Similarly, in the second case (when $I = \lambda I_1$), the restriction of G to $\mathcal{U}_I + \mathcal{W}_I$ is given, w.r.t. the frame $\{\xi_2^h, \xi_3^h\}$ of $\mathcal{U}_I$ and the frame $\left\{\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right\}$ of $\mathcal{W}_I$, by

$$\begin{pmatrix} h & 0 & 0 & l \\ 0 & -h & -l & 0 \\ 0 & -l & 0 & 0 \\ l & 0 & 0 & 0 \end{pmatrix}.$$

Given matrix has two positive and two negative eigenvalues:

$$\frac{1}{2}\left(\pm h + \sqrt{h^2 + 4l^2}\right) \quad \text{and} \quad \frac{1}{2}\left(\pm h - \sqrt{h^2 + 4l^2}\right);$$

thus, the signature is again $(2, 2)$. □

3.2. Invariance

For the definition of the distribution $\mathcal{K} \subset T\mathcal{Q}^o$ and the respective field J (cf. (33)), we have used, as an essential tool, the concept of a horizontal lift of a vector fields w.r.t. ∇. Since ∇ is the canonical connection determined by a choice of a section g of the canonical line bundle $\mathcal{G} \to M$ (cf. Section 2.3), the whole construction depends on that choice as well. Our purpose here is to show that this dependence is only formal and, in fact, if we replace g with

$$\bar{g} = \frac{1}{2f} g, \tag{37}$$

where f is any smooth and non-vanishing function on M, then both $\mathcal{K}$ and J remain unchanged.

If A is a vector field on M with a horizontal lift A^h to $\mathcal{Q}$ w.r.t. g and ∇, we shall denote $A^{\bar{h}}$ as the respective horizontal lift of A to $\mathcal{Q}$ w.r.t. $\bar{g}$ and its canonical connection $\overline{\nabla}$. Clearly, if (η_s, I_s, g) is any local pqc structure for H, then so is $(\overline{\eta}_s, I_s, \bar{g})$, where $\overline{\eta}_s = \frac{1}{2f}\eta_s$. More generally, we shall use the bar on objects related to the pqc structure (η_s, I_s, g) to indicate the respective objects related to $(\overline{\eta}_s, I_s, \bar{g})$, e.g., $\overline{\xi}_s$ will denote the Reeb vector fields (cf. (10)), defined by

$$(i) \quad \overline{\eta}_s(\overline{\xi}_t) = \begin{cases} \epsilon_s, \text{ if } s = t \\ 0, \text{ otherwise} \end{cases}$$
$$(ii) \quad d\overline{\eta}_s(\overline{\xi}_t, X) + d\overline{\eta}_t(\overline{\xi}_s, X) = 0, \quad \forall X \in H.$$

One can easily derive from the above that

$$\overline{\xi}_s = 2f\,\xi_s + I_s \nabla f,$$

where ∇f is the horizontal gradient of the function f; that is, the unique section of the distribution H, which satisfies $g(\nabla f, X) = df(X)$ for all $X \in H$. According to [18] (here we are using slightly different sign conventions), we have the following formulas concerning the connection one-forms $\overline{\alpha}_s$ $\big($cf. (16)$\big)$ of $\overline{\nabla}$:

$$
\begin{aligned}
\overline{\alpha}_s(X) &= \alpha_s(X) + \frac{\epsilon_s}{f}\, df(I_s\, X), \qquad s = 1,2,3, \qquad \forall X \in H, \\
\overline{\alpha}_i(\overline{\xi}_i) &= 2f\, \alpha_i(\xi_i) - \frac{\Delta f}{2n} + \frac{g(\nabla f, \nabla f)}{n\, f}, \\
\overline{\alpha}_j(\overline{\xi}_i) &= 2f\, \alpha_j(\xi_i) + \alpha_j(I_i \nabla f) - 2\epsilon_i\, df(\xi_k), \\
\overline{\alpha}_k(\overline{\xi}_i) &= 2f\, \alpha_k(\xi_i) + \alpha_k(I_i \nabla f) + 2\epsilon_i\, df(\xi_j),
\end{aligned} \tag{38}
$$

where (ijk) is any positive permutation of $1,2,3$, and $\Delta f = \sum_a \nabla df(e_a, e_a^*)$ (cf. (19)).

Lemma 7. *Within the coordinate chart (23) on $\mathcal{Q}$, we have the following formulas for the horizontal lift of a vector fields from M to $\mathcal{Q}$:*

$$
\begin{aligned}
X^{\bar{h}} &= X^h + \mathcal{N} \times \sum_t \epsilon_t\, df(I_t X)\, \frac{\partial}{\partial x_t}, \\
\overline{\xi}_s^{\;\bar{h}} &= 2\, f\, \xi_s^h + \big(I_s\, \nabla f\big)^h + 2\, df\big(\pi_* \chi\big) \frac{\partial}{\partial x_s} \\
&\quad - 2\epsilon_s\, x_s \sum_t \epsilon_t\, df(\xi_t)\, \frac{\partial}{\partial x_t} + \Big(- \frac{\Delta f}{2n} + \frac{g(\nabla f, \nabla f)}{n\, f} \Big)\, \mathcal{N} \times \frac{\partial}{\partial x_s},
\end{aligned} \tag{39}
$$

where X is any section of H, and $s = 1,2,3$.

Proof. The proof is obtained by a straightforward calculation using (38) and (24). □

Let us observe that the vector field $\mathcal{N}$ on $\mathcal{Q}$ $\big($cf. (26)$\big)$ does not depend on the choice of g and ∇, whereas the field χ is changed as follows:

$$
\begin{aligned}
\overline{\chi} &= \sum_s x_s\, \overline{\xi}_s^{\;\bar{h}} \\
&= \chi + 2\, df(\pi_* \chi)\, \mathcal{N} + J\big(\nabla f\big)^h - 2\, \langle I, I \rangle \sum_t \epsilon_t\, df(\xi_t)\, \frac{\partial}{\partial x_t}.
\end{aligned}
$$

Proposition 3. *The distribution $\mathcal{K}$ on $\mathcal{Q}^o$ $\big($defined by (32)$\big)$ and the field J of endomorphisms of $\mathcal{K}$ $\big($defined by (33)$\big)$ do not depend on the choice of g and ∇.*

Proof. Let us begin by constructing a distribution $\overline{\mathcal{K}}$ on $\mathcal{Q}^o$ as in (32) using (37) in place of g and $\overline{\nabla}$ in place of ∇. Within the coordinate chart (23), we have an orthogonal decomposition

$$
\operatorname{span}\Big\{ \overline{\xi}_1^{\;\bar{h}}, \overline{\xi}_2^{\;\bar{h}}, \overline{\xi}_3^{\;\bar{h}} \Big\} = \overline{\mathcal{U}} \oplus \mathbb{R}.\overline{\chi}
$$

that defines a distribution $\overline{\mathcal{U}}$. Then,

$$
\overline{\mathcal{K}} = \overline{\mathcal{H}} \oplus \overline{\mathcal{U}} \oplus \mathcal{W},
$$

where the distribution $\overline{\mathcal{H}}$ is defined by the requirement that its sections are precisely the horizontal lifts, w.r.t. the connection $\overline{\nabla}$, of vector fields on M tangent to the distribution $H \subset TM$, and $\mathcal{W}$ is as in (27).

If $A = \sum_s a_s \overline{\xi}_s^{\,\bar{h}}$ is any element of $\overline{\mathcal{U}}_I$, $I = \sum_s x_s I_s$, then $\sum_s \epsilon_s\, a_s\, x_s = 0$. Using (39), we calculate

$$A \;=\; \sum_s a_s \overline{\xi}_s^{\,\bar{h}} \;=\; \sum_s \Big(2\, f\, a_s\, \xi_s^h \;+\; \big(a_s\, I_s\, \nabla f\big)^h \;+\; 2\, df(\pi_* \chi)\, a_s \,\frac{\partial}{\partial x_s} \\ +\; \Big(-\,\frac{\Delta f}{2n} \;+\; \frac{g(\nabla f, \nabla f)}{n\, f} \Big)\, \mathcal{N} \times \Big(a_s \,\frac{\partial}{\partial x_s} \Big) \Big),$$

which yields that $A \in \mathcal{H} \oplus \mathcal{U} \oplus \mathcal{W}$ and $\overline{\mathcal{U}} \subset \mathcal{K}$. Similarly, if X is any section of H, then, by (39), $X^{\bar{h}} \in \mathcal{H} \oplus \mathcal{W}$; thus, $\overline{\mathcal{H}} \subset K$. Therefore, we get that $\overline{\mathcal{K}} = \mathcal{K}$, and $\mathcal{K}$ does not depend on the choice of g and ∇. The invariance of J is shown similarly. □

3.3. *Proof of Propositions 1 and 2*

The restriction of the onr-form η to the twistor space $\mathcal{Z} \subset \mathcal{Q}^o$ and the reflector space $\mathcal{R} \subset \mathcal{Q}^o$, respectively, satisfies

$$\eta \wedge d\eta^{2n} \neq 0; \tag{40}$$

therefore, it defines a contact structure on both $\mathcal{Z}$ and $\mathcal{R}$. The tangent bundles $T\mathcal{Z}$ and $T\mathcal{R}$, considered as subbundles in $T\mathcal{Q}^o$, are described by the equation

$$\sum_s x_s\, \phi_s \;=\; 0 \qquad \text{cf. } (30). \tag{41}$$

The vector field χ $\big($cf. (26)$\big)$ is tangent to $\mathcal{Z}$ ($\mathcal{R}$), and if we restrict to the tangent space of $\mathcal{Z}$ (resp. $\mathcal{R}$), we obtain that (cf. Corollary 1)

$$\eta(\chi) \;=\; \langle I, I \rangle \qquad \text{and} \qquad \chi \lrcorner d\eta \;=0;$$

that is, χ is a Reeb vector field for the contact form η on $\mathcal{Z}$ ($\mathcal{R}$).

At each $I \in \mathcal{Z}$ ($I \in \mathcal{R}$), the kernel of η $\big($cf. (32)$\big)$ is given by the subspace $\mathcal{K}_I \subset T_I\mathcal{Z}$ ($\mathcal{K}_I \subset T_I\mathcal{R}$) and the endomorphism J $\big($cf. (34)$\big)$ of $\mathcal{K}_I$ satisfies $J^2 = -id$ ($J^2 = id$). The pair $(\mathcal{K}, J)$ defines an almost CR structure on the twistor space $\mathcal{Z}$ and an almost para-CR structure on the reflector space $\mathcal{R}$. The signature of $d\eta(J.,.)$ is given by Lemma 6. By Proposition 3, the pair $(\mathcal{K}, J)$ is uniquely determined by the pqc distribution $H \subset TM$, which does not depend on the particular choice of the local pqc structure (η_s, I_s, g) for H.

4. Integrability

In this section, we consider the integrability question for the previously introduced (Section 3) almost CR structure $(\mathcal{K}, J)$ on the twistor space $\mathcal{Z}$, and for the respective almost para-CR structure on the reflector space $\mathcal{R}$.

Observe that using Lemma 6, if A and B are any two sections of $\mathcal{K}$, then

$$[JA, B] + [A, JB]$$

is also a section of $\mathcal{K}$. Therefore, the integrability of the almost CR structure $(\mathcal{K}, J)$ on $\mathcal{Z}$ is equivalent to the equation $N^{\mathcal{Z}}(A, B) = 0$, where $N^{\mathcal{Z}}$ is the so called Nijenhuis tensor, defined by

$$N^{\mathcal{Z}}(A, B) \;=\; -[A, B] \;+\; [JA, JB] \;-\; J\Big([JA, B] + [A, JB]\Big), \tag{42}$$

for any two vector fields A and B on $\mathcal{Z}$ that are tangent to the distribution $\mathcal{K} \subset T\mathcal{Z}$.

The complexified distribution $\mathcal{K}^c = \mathcal{K} \otimes_{\mathbb{R}} \mathbb{C}$ on $\mathcal{Z}$ splits as

$$\mathcal{K}^c = \mathcal{K}_{\sqrt{-1}} \,\oplus\, \mathcal{K}_{-\sqrt{-1}},$$

where $\mathcal{K}_{\sqrt{-1}}$ and $\mathcal{K}_{-\sqrt{-1}}$ are the eigenspaces of J with eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$. The vanishing of the Nijenhuis tensor $N^{\mathcal{Z}}$ is equivalent to the formal integrability of the complex distributions $\mathcal{K}_{\sqrt{-1}}$ and $\mathcal{K}_{-\sqrt{-1}}$; that is, to any of the following two conditions

$$\left[\mathcal{K}_{\sqrt{-1}}, \mathcal{K}_{\sqrt{-1}}\right] \subset \mathcal{K}_{\sqrt{-1}} \qquad \text{and} \qquad \left[\mathcal{K}_{-\sqrt{-1}}, \mathcal{K}_{-\sqrt{-1}}\right] \subset \mathcal{K}_{-\sqrt{-1}}.$$

Similarly, the almost para-CR structure $(\mathcal{K}, J)$ on the reflector space $\mathcal{R}$ is integrable if $N^{\mathcal{R}}(A, B) = 0$ for any two sections A and B of the distribution $\mathcal{K} \subset T\mathcal{R}$, where

$$N^{\mathcal{R}}(A, B) \;=\; [A, B] \;+\; [JA, JB] \;-\; J\Big([JA, B] + [A, JB]\Big). \tag{43}$$

Here, the complexified distribution splits as $\mathcal{K}^c = \mathcal{K}_{+1} \oplus \mathcal{K}_{-1}$, where $\mathcal{K}_{+1}$ and $\mathcal{K}_{-1}$ are the ± 1 eigenspaces of J. The vanishing of $N^{\mathcal{R}}$ is equivalent to the formal integrability of $\mathcal{K}_{+1}$ and $\mathcal{K}_{-1}$, i.e., to the following conditions:

$$\left[\mathcal{K}_{+1}, \mathcal{K}_{+1}\right] \subset \mathcal{K}_{+1} \qquad \text{and} \qquad \left[\mathcal{K}_{-1}, \mathcal{K}_{-1}\right] \subset \mathcal{K}_{-1}.$$

The following result is obtained as a straightforward application of Proposition 5 below.

Proposition 4. *The almost CR structure $(\mathcal{K}, J)$ on the twistor space $\mathcal{Z}$ and the respective almost para-CR structure on the reflector space $\mathcal{R}$ are integrable.*

Integrability on the Ambient Space $\mathcal{Q}^o$

The distribution $\mathcal{K}$ (cf. (32)) can be considered as a vector bundle over the manifold $\mathcal{Q}^o$. We introduce a Nijenhuis-like tensor field N defined for any two vector fields A and B on $\mathcal{Q}^o$ that are tangent to the distribution $\mathcal{K}$ by the following formula:

$$N(A, B) \;=\; -\langle I, I\rangle [A, B] \;+\; [JA, JB] \;-\; J\Big([JA, B] + [A, JB]\Big), \tag{44}$$

where N is indeed a tensor field, meaning that the value of $N(A, B)$ at any given $I \in \mathcal{Q}^o$ depends only on the values of A and B at I, due to the obvious property $N(fA, hB) = fhN(A, B)$ for any functions f and h on $\mathcal{Q}^o$. Notice that the expression on the right hand side in (44) also makes sense, since, by Lemma 5, the vector field $[JA, B] + [A, JB]$ is tangent to the distribution $\mathcal{K}$, and the action of J is well defined there (by definition J is a field of endomorphisms of $\mathcal{K}$, cf. (33)). Furthermore, applying Lemma 5 one more time, we observe that $N(A, B)$ is always a section of $\mathcal{K}$; thus, $N : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$. Clearly, if restricted to the twistor space $\mathcal{Z} \subset \mathcal{Q}^o$, N coincides with the Nijenhuis tensor $N^{\mathcal{Z}}$ (cf. (42)), and, similarly, on $\mathcal{R} \subset \mathcal{Q}^o$, it coincides with $N^{\mathcal{R}}$ (cf. (43)).

Proposition 5. *On $\mathcal{Q}^o$, we have that*

$$N(A, B) \;=\; 0,$$

for any two sections A and B of $\mathcal{K}$.

Proof. To begin with, we fix an arbitrary non-vanishing section g of the line bundle $\mathcal{G}(M) \to M$ (cf. Section 2.3) and consider the corresponding canonical connection ∇ on TM. Using Lemma 2, for any fixed $I \in \mathcal{Q}^o$, we can pick a local pqc structure (η_s, I_s, g) in such a way so that either $I = \lambda I_1$ or $I = \lambda I_3$, $\lambda \in \mathbb{R}$. Let us assume that $I = \lambda I_1$ (in the other case the proof is similar). Using the corresponding Reeb vector fields ξ_s, we construct a coordinate chart as in (23) around the fixed point $I = \lambda I_1 \in \mathcal{Q}^o$.

Following the structure (32) of $\mathcal{K}$ and observing that $N(A,B) = -N(B,A)$, we see that there are six different cases to consider in the proof: (I) $A, B \in \mathcal{H}$; (II) $A \in \mathcal{H}, B \in \mathcal{U}$; (III) $A \in \mathcal{H}, B \in \mathcal{W}$; (IV) $A, B \in \mathcal{U}$; (V) $A \in \mathcal{U}, B \in \mathcal{W}$; (VI) $A, B \in \mathcal{W}$.

Case (I) $A, B \in \mathcal{H}$:

Without loss of generality, in this case, we may assume that $A = X^h$ and $B = Y^h$ for some vector fields X and Y on M that are tangent to the distribution H. We calculate the following:

$$
\begin{aligned}
N(X^h, Y^h)_{\big| I=\lambda I_1} &= \\
&= \lambda^2 [X^h, Y^h] + \sum_{s,t} \Big[x_s (I_s X)^h,\, x_t (I_t Y)^h \Big]_{\big| I=\lambda I_1} \\
&\qquad - \lambda\, I_1 \sum_s \Big(\Big[x_s (I_s X)^h,\, Y^h \Big] + \Big[X^h,\, x_s (I_s Y)^h \Big] \Big)_{\big| I=\lambda I_1}, \\
&= \lambda^2 \Big([X,Y] + [I_1 X, I_1 Y] - I_1 \Big([I_1 X, Y] + [X,\, I_1 Y] \Big) \Big)^h_{\big| I=\lambda I_1} \\
&\quad + \sum_s \Big(dx_s (J X^h)(I_s Y)^h - dx_s (J Y^h)(I_s X)^h \\
&\qquad - dx_s (X^h) J (I_s Y)^h + dx_s (Y^h) J (I_s X)^h \Big)_{\big| I=\lambda I_1}. \\
&\quad + 2\lambda^3 \Big(\rho_3(X,Y) + \rho_3(I_1 X, I_1 Y) - \rho_2(I_1 X, Y) - \rho_2(X, I_1 Y) \Big) \frac{\partial}{\partial x_2} \\
&\quad + 2\lambda^3 \Big(\rho_2(X,Y) + \rho_2(I_1 X, I_1 Y) - \rho_3(I_1 X, Y) - \rho_3(X, I_1 Y) \Big) \frac{\partial}{\partial x_2}
\end{aligned}
$$

We observe that the last two lines in the above expression vanish as a consequence of (21). We may represent the remaining part of the expression as

$$
\lambda^2 \left(\Sigma_1 \right)^h_{\big| I=\lambda I_1} + \lambda^2 \left(\Sigma_2 \right)^h_{\big| I=\lambda I_1}, \tag{45}
$$

where

$$
\begin{aligned}
\Sigma_1 &= [X,Y] + [I_1 X, I_1 Y] - I_1 \Big([I_1 X, Y] + [X,\, I_1 Y] \Big), \\
\Sigma_2 &= \frac{1}{\lambda} \sum_s \Big(dx_s \big((I_1 X)^h \big)(I_s Y) - dx_s \big((I_1 Y)^h \big)(I_s X) \\
&\qquad - dx_s (X^h) I_1 (I_s Y) + dx_s (Y^h) I_1 (I_s X) \Big).
\end{aligned}
$$

Using the canonical connection ∇ on M and its torsion T (cf. (11)), we calculate

$$
\begin{aligned}
\Sigma_1 &= \nabla_X Y - \nabla_Y X - T(X,Y) + \nabla_{I_1 X}(I_1 Y) - \nabla_{I_1 Y}(I_1 X) - T(I_1 X, I_1 Y) \\
&\quad - I_1 \Big(\nabla_{I_1 X} Y - \nabla_Y (I_1 X) - T(I_1 X, Y) + \nabla_X (I_1 Y) - \nabla_{I_1 Y} X - T(X, I_1 Y)' \\
&= -I_1 (\nabla_X I_1) Y + I_1 (\nabla_Y I_1) X + (\nabla_{I_1 X} I_1) Y - (\nabla_{I_1 Y} I_1) X \\
&\qquad - T(X,Y) - T(I_1 X, I_1 Y) + I_1 \Big(T(I_1 X, Y) + T(X, I_1 Y) \Big)' \\
&= \Big(\alpha_2(X) + \alpha_3(I_1 X) \Big) I_2 Y - \Big(\alpha_2(Y) + \alpha_3(I_1 Y) \Big) I_2 X \\
&\qquad + \Big(\alpha_3(X) - \alpha_2(I_1 X) \Big) I_3 Y - \Big(\alpha_3(Y) - \alpha_2(I_1 Y) \Big) I_3 X,'
\end{aligned}
$$

where the last equality follows from (16) and (13).

Applying (24) to the expression Σ_2 gives

$$\begin{aligned}\Sigma_2 \;=\; -\Big(\alpha_2(X)+\alpha_3(I_1X)\Big)I_2Y \;+\; \Big(\alpha_2(Y)+\alpha_3(I_1Y)\Big)I_2X \\ -\;\Big(\alpha_3(X)-\alpha_2(I_1X)\Big)I_3Y \;+\; \Big(\alpha_3(Y)-\alpha_2(I_1Y)\Big)I_3X\end{aligned}.$$

Therefore, using (45), we get $N(X^h, Y^h) = 0$.

Case (II) $A \in \mathcal{H}$, $B \in \mathcal{U}$:

Here, we may assume that $A = X^h$ and $B = \mu_2\,\xi_2^h + \mu_3\,\xi_3^h$, where μ_2 and μ_3 are any real numbers, and X is a section of $H \subset TM$. We obtain that

$$N(A,B)_{\big|I=\lambda I_1} \;=\; \mu_2\,N(X^h,\xi_2^h)_{\big|I=\lambda I_1} \;+\; \mu_3\,N(X^h,\xi_3^h)_{\big|I=\lambda I_1}. \tag{46}$$

In order to calculate the quantity $N(X^h,\xi_2^h)_{\big|I=\lambda I_1}$, consider the vector field

$$\xi_2^h + \frac{x_2}{\langle I, I\rangle}\chi. \tag{47}$$

Clearly, (47) is a vector filed tangent to the distribution $\mathcal{U} \subset \mathcal{K} \subset T\mathcal{Q}$ (cf. (32)) that is defined in a neighborhood of the fixed point $I = \lambda I_1$, so that its value at this point coincides with the value of ξ_2^h. Therefore, we obtain that (cf. (44) and (33))

$$\begin{aligned}N(X^h,\xi_2^h)_{\big|I=\lambda I_1} \;&=\; N\Big(X^h,\,\xi_2^h + \frac{x_2}{\langle I,I\rangle}\chi\Big)_{\big|I=\lambda I_1}\\ &=\; \lambda^2\big[X^h,\xi_2^h + \frac{x_2}{\langle I,I\rangle}\chi\big]_{\big|I=\lambda I_1} \;+\; \sum_{s,t}\Big[x_s(I_sX)^h,\,x_t(\xi_t\times\xi_2)^h\Big]_{\big|I=\lambda I_1},\\ &\quad -\;\lambda\sum_s J\Big(\big[x_s(I_sX)^h,\,\xi_2^h + \frac{x_2}{\langle I,I\rangle}\chi\big] + \big[X^h,\,x_s(\xi_s\times\xi_2)^h\big]\Big)_{\big|I=\lambda I_1}\end{aligned}$$

$$\begin{aligned} =\; &\lambda^2[X,\xi_2]^h \;+\; \lambda^2[I_1X,\,\xi_3]^h \;-\; \lambda\,J\Big([I_1X,\,\xi_2] + [X,\,\xi_3]\Big)^h\\ &-\;\lambda^2\bigg(\Big(-\alpha_2(\xi_2)+\alpha_3(\xi_3)\Big)I_2X \;+\; \Big(\alpha_2(\xi_3)-\alpha_3(\xi_2)\Big)I_3X\\ &\qquad +\;\Big(\alpha_3(X)-\alpha_2(I_1X)\Big)\xi_1\bigg)^h\\ &+\;2\lambda^3\Big(\rho_3(X,\xi_2)+\rho_3(I_1X,\xi_3)-\rho_2(I_1X,\xi_2)-\rho_2(X,\xi_3)\Big)\frac{\partial}{\partial x_2}\\ &+\;2\lambda^3\Big(\rho_2(X,\xi_2)+\rho_2(I_1X,\xi_3)-\rho_3(I_1X,\xi_2)-\rho_3(X,\xi_3)\Big)\frac{\partial}{\partial x_3}\end{aligned}. \tag{48}$$

By the properties (21) of $\rho(X,\xi_s)$, the last four lines in the above expression vanish. Using the canonical connection ∇ on M and its torsion T (cf. (11)), we calculate that

$$\begin{aligned}&\lambda^2[X,\xi_2]^h \;+\; \lambda^2[I_1X,\,\xi_3]^h \;-\; \lambda\,J\Big([I_1X,\,\xi_2] + [X,\,\xi_3]\Big)^h\\ &\quad=\; \lambda^2\bigg(I_1(\nabla_{\xi_2}I_1)X - (\nabla_{\xi_3}I_1)X - T(X,\xi_2) + I_1T(I_1X,\xi_2)\\ &\qquad -\;T(I_1X,\xi_3) + I_1T(X,\xi_3) + \nabla_X\xi_2 + \nabla_{I_1X}\xi_3\\ &\qquad -\;\xi_1\times\Big(\nabla_{(I_1X)}\xi_2 + \nabla_X\xi_3\Big)\bigg)^h\end{aligned},$$

$$= \lambda^2 \Big(\Big(- \alpha_2(\xi_2) + \alpha_3(\xi_3) \Big) I_2 X + \Big(\alpha_2(\xi_3) - \alpha_3(\xi_2) \Big) I_3 X + \Big(\alpha_3(X) - \alpha_2(I_1 X) \Big) \xi_1 \Big)^{h\,\prime}, \quad (49)$$

where for the last identity, we have use Formulas (16), (17), and (13). Substituting (49) into (48), we get

$$N(X^h, \xi_2^h)_{|I=\lambda I_1} = 0.$$

Similarly, one can also show that

$$N(X^h, \xi_3^h)_{|I=\lambda I_1} = 0;$$

therefore, we obtain that, in this case, $N(A, B) = 0$.

Case (III) $A \in \mathcal{H}, B \in \mathcal{W}$:

We may assume here that $A = X^h$ and $B = \mu_2 \frac{\partial}{\partial x_2} + \mu_3 \frac{\partial}{\partial x_3}$, where μ_2 and μ_3 are any real numbers, and X is a section of $H \subset TM$. Then,

$$N(A,B)_{|I=\lambda I_1} = \mu_2 N\Big(X^h, \frac{\partial}{\partial x_2}\Big)_{|I=\lambda I_1} + \mu_3 N\Big(X^h, \frac{\partial}{\partial x_3}\Big)_{|I=\lambda I_1}. \quad (50)$$

In order to show that $N\Big(X^h, \frac{\partial}{\partial x_3}\Big)_{|I=\lambda I_1}$ vanishes (the vanishing of the other summand is shown similarly), we consider the vector field

$$\frac{\partial}{\partial x_3} - \frac{x_3}{\langle I, I \rangle} \mathcal{N}.$$

Clearly, this is a vector filed tangent to the distribution $\mathcal{W} \subset \mathcal{K} \subset TQ$ (cf. (32)), which is defined in a neighborhood of the fixed point $I = \lambda I_1$, so that its value at this point coincides with the value of $\frac{\partial}{\partial x_3}$. Therefore, using (44) and (33) we get

$$\begin{aligned}
N\Big(X^h, \frac{\partial}{\partial x_3}\Big)_{|I=\lambda I_1} &= N\Big(X^h, \frac{\partial}{\partial x_3} - \frac{x_3}{\langle I, I \rangle}\mathcal{N}\Big)_{|I=\lambda I_1} \\
&= \lambda^2 \Big[X^h, \frac{\partial}{\partial x_3} - \frac{x_3}{\langle I, I \rangle}\mathcal{N}\Big]_{|I=\lambda I_1} + \sum_{s,t} \Big[x_s (I_s X)^h, x_t \frac{\partial}{\partial x_t} \times \frac{\partial}{\partial x_2}\Big]_{|I=\lambda I_1}, \\
&\quad - \lambda \sum_s J\Big(\Big[x_s (I_s X)^h, \frac{\partial}{\partial x_3} - \frac{x_3}{\langle I, I \rangle}\mathcal{N}\Big] + \Big[X^h, x_t \frac{\partial}{\partial x_t} \times \frac{\partial}{\partial x_2}\Big]\Big)_{|I=\lambda I_1} \\
&= \lambda^2 \Big[X^h, \frac{\partial}{\partial x_3}\Big] + \lambda^2 \Big[(I_1 X)^h, \frac{\partial}{\partial x_2}\Big] \\
&\quad - \lambda J\Big(\Big[(I_1 X)^h, \frac{\partial}{\partial x_3}\Big] + \Big[X^h, \frac{\partial}{\partial x_2}\Big]\Big). \\
&\quad + \lambda^2 \Big(\alpha_2(X) - \alpha_3(I_1 X)\Big) \frac{\partial}{\partial x_1} = 0.
\end{aligned}$$

Case (IV) $A, B \in \mathcal{U}$:
It suffices to assume $A = \xi_2^h$, $B = \xi_3^h$. Using (44) and (24), we calculate

$$
\begin{aligned}
N(\xi_2^h, \xi_3^h)_{|I=\lambda I_1} &= N\Big(\xi_2^h + \frac{x_2}{\langle I, I\rangle}\succ, \xi_3^h - \frac{x_3}{\langle I, I\rangle}\succ\Big)_{|I=\lambda I_1} \\
&= \lambda^2\Big[\xi_2^h + \frac{x_2}{\langle I, I\rangle}\succ, \xi_3^h - \frac{x_3}{\langle I, I\rangle}\succ\Big]_{|I=\lambda I_1} \\
&\qquad + \sum_{s,t}\Big[x_s(\xi_s \times \xi_2)^h, x_t(\xi_t \times \xi_3)^h\Big]_{|I=\lambda I_1} \\
&\quad - \lambda\sum_s J\Big(\Big[x_s(\xi_s \times \xi_2)^h, \xi_3^h - \frac{x_3}{\langle I, I\rangle}\succ\Big] \\
&\qquad + \Big[\xi_2^h + \frac{x_2}{\langle I, I\rangle}\succ, x_s(\xi_s \times \xi_3)^h\Big]\Big)_{|I=\lambda I_1} = 0.
\end{aligned}
$$

Case (V) $A \in \mathcal{U}, B \in \mathcal{W}$:
Here, we need to consider the following assumptions: $A = \xi_s^h$ and $B = \frac{\partial}{\partial x_t}$ for $s, t = 2, 3$. We shall consider only the case where $s = 2$ and $t = 3$; the remaining three possibilities are entirely analogous.

$$
\begin{aligned}
N\Big(\xi_2^h, \frac{\partial}{\partial x_3}\Big)_{|I=\lambda I_1} &= N\Big(\xi_2^h + \frac{x_2}{\langle I, I\rangle}\succ, \frac{\partial}{\partial x_3} - \frac{x_3}{\langle I, I\rangle}\mathcal{N}\Big)_{|I=\lambda I_1} \\
&= \lambda^2\Big[\xi_2^h + \frac{x_2}{\langle I, I\rangle}\succ, \frac{\partial}{\partial x_3} - \frac{x_3}{\langle I, I\rangle}\mathcal{N}\Big]_{|I=\lambda I_1} \\
&\qquad + \sum_{s,t}\Big[x_s(\xi_s \times \xi_2)^h, x_t\frac{\partial}{\partial x_t} \times \frac{\partial}{\partial x_2}\Big]_{|I=\lambda I_1} \\
&\quad - \lambda\sum_s J\Big(\Big[x_s(\xi_s \times \xi_2)^h, \frac{\partial}{\partial x_3} - \frac{x_3}{\langle I, I\rangle}\mathcal{N}\Big] \\
&\qquad + \Big[\xi_2^h + \frac{x_2}{\langle I, I\rangle}\succ, x_t\frac{\partial}{\partial x_t} \times \frac{\partial}{\partial x_3}\Big]\Big)_{|I=\lambda I_1} = 0.
\end{aligned}
$$

Case (VI) $A, B \in \mathcal{W}$:
It suffices to consider only the case $A = \frac{\partial}{\partial x_2}$, $B = \frac{\partial}{\partial x_3}$.

$$
\begin{aligned}
N\Big(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\Big)_{|I=\lambda I_1} &= N\Big(\frac{\partial}{\partial x_2} + \frac{x_2}{\langle I, I\rangle}\mathcal{N}, \frac{\partial}{\partial x_3} - \frac{x_3}{\langle I, I\rangle}\mathcal{N}\Big)_{|I=\lambda I_1} \\
&= \lambda^2\Big[\frac{\partial}{\partial x_2} + \frac{x_2}{\langle I, I\rangle}\mathcal{N}, \frac{\partial}{\partial x_3} - \frac{x_3}{\langle I, I\rangle}\mathcal{N}\Big]_{|I=\lambda I_1} \\
&\qquad + \sum_{s,t}\Big[x_s\frac{\partial}{\partial x_s} \times \frac{\partial}{\partial x_2}, x_t\frac{\partial}{\partial x_t} \times \frac{\partial}{\partial x_3}\Big]_{|I=\lambda I_1} \\
&\quad - \lambda\sum_s J\Big(\Big[x_s\frac{\partial}{\partial x_s} \times \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} - \frac{x_3}{\langle I, I\rangle}\mathcal{N}\Big] \\
&\qquad + \Big[\frac{\partial}{\partial x_2} + \frac{x_2}{\langle I, I\rangle}\mathcal{N}, x_s\frac{\partial}{\partial x_s} \times \frac{\partial}{\partial x_3}\Big]\Big)_{|I=\lambda I_1} = 0.
\end{aligned}
$$

□

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