mathematics

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The Category $\mathfrak{G}$-*GrR-Mod* and Group Factorization

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Abstract: In this work, we use the concept of $\mathfrak{G}$-weak graded rings and $\mathfrak{G}$-weak graded modules, which are based on grading by a set $\mathfrak{G}$ of left coset representatives for the left action of a subgroup $\mathfrak{H}$ of a finite $\mathfrak{X}$ on $\mathfrak{X}$, to define the conjugation action of the set $\mathfrak{G}$ and to generalize and prove some results from the literature. In particular, we prove that a $\mathfrak{G}$-weak graded ring R is strongly graded if and only if each $\mathfrak{G}$-weak graded R-module V is induced by an $R_{\mathfrak{e}_\mathfrak{G}}$-module. Moreover, we prove that the additive induction functor $(-)^R$ and the restriction functor $(-)_{\mathfrak{e}_\mathfrak{G}}$ form an equivalence between the categories $\mathfrak{G}$-*GrR-Mod* and $R_{\mathfrak{e}_\mathfrak{G}}$-*Mod* when R is strongly $\mathfrak{G}$-weak graded. Furthermore, some related results and illustrative examples of $\mathfrak{G}$-weak graded R-modules and their morphisms are provided.

Keywords: left coset representatives; weak graded rings; weak graded module; weak graded homomorphisms; equivalence between categories

MSC: 16W50; 16W20; 16W22; 13A02

1. Introduction

Recall that, for a group $\mathfrak{X}$ and an $\mathfrak{X}$-graded ring R, a left R-module M is called an $\mathfrak{X}$-graded module if it can be written as a direct sum decomposition $M = \bigoplus_{\mathfrak{x}\in\mathfrak{X}} M_\mathfrak{x}$ such that, for all $\mathfrak{x}, \mathfrak{y} \in \mathfrak{X}$, we have $R_\mathfrak{x} M_\mathfrak{y} \subseteq M_{\mathfrak{x}\mathfrak{y}}$, where $M_\mathfrak{y}$ is an $R_\mathfrak{e}$-submodule for each $\mathfrak{y} \in \mathfrak{X}$ and $\mathfrak{e}$ is the identity element of $\mathfrak{X}$. If the condition $R_\mathfrak{x} M_\mathfrak{y} \subseteq M_{\mathfrak{x}\mathfrak{y}}$ is replaced by $R_\mathfrak{x} M_\mathfrak{y} = M_{\mathfrak{x}\mathfrak{y}}$ for all $\mathfrak{x}, \mathfrak{y} \in \mathfrak{X}$, then M is called a strongly $\mathfrak{X}$-graded module. Group-graded rings and modules have been extensively studied, either alone or in connection with different areas of mathematics; see ref. [1]. Many mathematicians have generalized the concept of group-graded rings and modules by using monoids or semigroups for grading; see, for example, refs. [2,3]. The concept of semi-graded rings and modules was introduced as a different means to generalize group-graded rings and modules; see ref. [4].

The properties of graded rings and modules have been investigated using various methods, such as duality theorems [5] and categorical methods, which are used to study separable functors; see refs. [6,7].

In ref. [8], an algebraic structure consisting of a set $\mathfrak{G}$ of left coset representatives and a binary operation " $*$ " was constructed. This structure led to interesting categories; see refs. [8,9]. Additionally, it was used to introduce the new concept of $\mathfrak{G}$-weak graded rings and modules, which generalizes the concept of group-graded rings and modules; see ref. [10].

This work is a continuation of [11], which generalized the concepts of group-graded rings and group-graded modules by associating the grading with the factorization of a given finite group and using a set of left coset representatives for grading, rather than groups. In this work, we generalize and prove some important results given in the literature concerning the category $\mathfrak{G}$-*GrR-Mod* and its objects of $\mathfrak{G}$-weak graded left R-modules. Specifically, we prove that a $\mathfrak{G}$-weak graded ring R is strongly graded if and only if each $\mathfrak{G}$-weak graded R-module V is induced from an $R_{\mathfrak{e}_\mathfrak{G}}$-module. We also prove that the additive induction functor $(-)^R$, which takes any left $R_{\mathfrak{e}_\mathfrak{G}}$-module K to the tensor product $R \otimes_{R_{\mathfrak{e}_\mathfrak{G}}} K$, and the restriction functor $(-)_{\mathfrak{e}_\mathfrak{G}}$ form an equivalence between the categories

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Citation: Al-Omari, R.; Al-Shomrani, M. The Category $\mathfrak{G}$-*GrR-Mod* and Group Factorization. *Mathematics* **2024**, *12*, 3344. https://doi.org/10.3390/math12213344

Academic Editors: Xiaomin Tang and Christos G. Massouros

Received: 30 August 2024
Revised: 15 October 2024
Accepted: 24 October 2024
Published: 25 October 2024

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$\mathfrak{G}$-*GrR-Mod* and $R_{\mathfrak{e}_\mathfrak{G}}$-*Mod* if R is a strongly $\mathfrak{G}$-weak graded ring. Some other important related results are also proven. Finally, some illustrative examples of $\mathfrak{G}$-weak graded R-modules are provided.

The significance of this work lies in the fact that it presents a generalization of the important concepts of group-graded rings and group-graded modules, which play a vital rule in abstract algebra. Consequently, the exploration of $\mathfrak{G}$-weak graded rings and modules and their properties remains an active field for interested researchers. Furthermore, this work may lead to further generalization to the quantum case, particularly to the bicrossproduct Hopf algebras associated with the factorization of a finite group $\mathfrak{X} = \mathfrak{H}\mathfrak{G}$; see refs. [12,13].

Throughout this research, we assume, unless otherwise stated, that all groups are finite, all rings are associative with unity, and all modules are unital.

2. Preliminaries

In this section, we include fundamental definitions and results that are essential to prove our results.

Recall that the category of $\mathfrak{G}$-weak graded left R-modules is denoted by $\mathfrak{G}$-*GrR-Mod* [11]. If a morphism $\varphi : V \longrightarrow W$ in $\mathfrak{G}$-*GrR-Mod*, then $\varphi(v) \in W_\mathfrak{p}$ for all $v \in V_\mathfrak{p}$ and $\mathfrak{p} \in \mathfrak{G}$, where V and W are $\mathfrak{G}$-weak graded left R-modules. We denote the class of these morphisms by $\mathfrak{G}$-$GrHom_R(V, W)$ [11]. Based on this, we denote the category of strongly $\mathfrak{G}$-weak graded R-modules by $\mathfrak{G}$-*StGrR-Mod*, with $\mathfrak{G}$-*Gr-R* as the category of $\mathfrak{G}$-weak graded rings and $\mathfrak{G}$-*StGr-R* as the category of strongly $\mathfrak{G}$-weak graded rings.

Definition 1 ([8]). *Consider a group $\mathfrak{X}$ and a subgroup $\mathfrak{H}$. The set $\mathfrak{G} \subset \mathfrak{X}$ is called a set of left coset representatives if there exists a unique $\mathfrak{p} \in \mathfrak{G}$ for each $\mathfrak{x} \in \mathfrak{X}$ such that $\mathfrak{x} \in \mathfrak{H}\mathfrak{p}$. We define a binary operation $*$ on $\mathfrak{G}$ that satisfies the right division property with a unique left identity $\mathfrak{e}_\mathfrak{G} \in \mathfrak{G}$.*

Definition 2 ([8]). *For $\mathfrak{p}, \mathfrak{q} \in \mathfrak{G}$, $f(\mathfrak{p}, \mathfrak{q}) \in \mathfrak{H}$ and $\mathfrak{p} * \mathfrak{q} \in \mathfrak{G}$ are defined by the unique factorization $\mathfrak{p}\mathfrak{q} = g(\mathfrak{p}, \mathfrak{q})(\mathfrak{p} * \mathfrak{q})$ in $\mathfrak{X}$, where g is the cocycle map satisfying $\mathfrak{o} \triangleleft g(\mathfrak{p}, \mathfrak{q}) * (\mathfrak{p} * \mathfrak{q}) = (\mathfrak{o} * \mathfrak{p}) * \mathfrak{q}$, provided that $\triangleright : \mathfrak{G} \times \mathfrak{H} \to \mathfrak{H}$ and $\triangleleft : \mathfrak{G} \times \mathfrak{H} \to \mathfrak{G}$ are defined by $\mathfrak{p}\mathfrak{h} = (\mathfrak{p} \triangleright \mathfrak{h})(\mathfrak{p} \triangleleft \mathfrak{h})$, for $\mathfrak{h} \in \mathfrak{H}$, which is also unique.*

Definition 3 ([10]). *A $\mathfrak{G}$-weak graded ring R is a ring that satisfies*

$$R = \bigoplus_{\mathfrak{p} \in \mathfrak{G}} R_\mathfrak{p}, \tag{1}$$

and

$$R_\mathfrak{p} R_\mathfrak{q} \subseteq R_{\mathfrak{p}*\mathfrak{q}} \quad \textit{for all} \quad \mathfrak{p}, \mathfrak{q} \in \mathfrak{G}, \tag{2}$$

where $R_\mathfrak{p}$ is an additive subgroup for each $\mathfrak{p} \in \mathfrak{G}$. If, instead of (2), we have

$$R_\mathfrak{p} R_\mathfrak{q} = R_{\mathfrak{p}*\mathfrak{q}}, \tag{3}$$

for all $\mathfrak{p}, \mathfrak{q} \in \mathfrak{G}$, then R is called a strongly $\mathfrak{G}$-weak graded ring.

Definition 4 ([10]). *A $\mathfrak{G}$-weak graded R-module V is a left R-module that satisfies*

$$V = \bigoplus_{\mathfrak{p} \in \mathfrak{G}} V_\mathfrak{p} \qquad (\textit{ in } R_{\mathfrak{e}_\mathfrak{G}}\textit{-Mod}) \tag{4}$$

and

$$R_\mathfrak{p} V_\mathfrak{q} \subseteq V_{\mathfrak{p}*\mathfrak{q}} \quad \textit{for all} \quad \mathfrak{p}, \mathfrak{q} \in \mathfrak{G}, \tag{5}$$

where $R \in \mathfrak{G}$-Gr-R. If, instead of (5), we have

$$R_\mathfrak{p} V_\mathfrak{q} = V_{\mathfrak{p}*\mathfrak{q}}, \tag{6}$$

for all $\mathfrak{p}, \mathfrak{q} \in \mathfrak{G}$, *then* V *is called a strongly* $\mathfrak{G}$*-weak graded* R*-module.*

Definition 5 ([11]). *For* $V, W \in \mathfrak{G}$*-GrR-Mod, the additive subgroup* $\mathfrak{G}\text{-}GrHom_R(V,W)_\mathfrak{q}$ *of* $\mathfrak{G}\text{-}GrHom_R(V,W)$ *for* $\mathfrak{q} \in \mathfrak{G}$ *is defined by*

$$\mathfrak{G}\text{-}GrHom_R(V,W)_\mathfrak{q} = \{\varphi \in \mathfrak{G}\text{-}GrHom_R(V,W) \ : \ \varphi(V_\mathfrak{p}) \subseteq W_{\mathfrak{p}*\mathfrak{q}}, \text{ for all } \mathfrak{p} \in \mathfrak{G}\}.$$

Definition 6 ([10]). *Let* K *be a left* $R_{e_\mathfrak{G}}$*-module. The tensor product* $R \otimes_{R_{e_\mathfrak{G}}} K$ *is a left* R*-module with*

$$(r_1(r_2 \otimes k) = (r_1 r_2) \otimes k, \ \text{ for all } r_1, r_2 \in R \text{ and } k \in K. \tag{7}$$

Theorem 1 ([10]). *For* $R \in \mathfrak{G}$*-Gr-R, the composite functor*

$$\beta_{R_{e_\mathfrak{G}},-} = (R \otimes_{R_{e_\mathfrak{G}}} -)_{e_\mathfrak{G}} = (R_{e_\mathfrak{G}} \otimes_{R_{e_\mathfrak{G}}} -) : R_{e_\mathfrak{G}}\text{-}Mod \to \mathfrak{G}\text{-}GrR\text{-}Mod,$$

which is defined by

$$\beta_{R_{e_\mathfrak{G}},K} = (R \otimes_{R_{e_\mathfrak{G}}} K)_{e_\mathfrak{G}} = R_{e_\mathfrak{G}} \otimes_{R_{e_\mathfrak{G}}} K \cong K,$$

for any left $R_{e_\mathfrak{G}}$*-module* K*, forms the following natural isomorphisms:*

$$((-)^R)_{e_\mathfrak{G}} \approx I_{R_{e_\mathfrak{G}}\text{-}Mod}, \tag{8}$$

where $(-)^R = (R \otimes_{R_{e_\mathfrak{G}}} -)$ *and* $I_{R_{e_\mathfrak{G}}\text{-}Mod}$ *is the identity functor on the category* $R_{e_\mathfrak{G}}$*-Mod.*

Proposition 1 ([10]). *If a right inverse* $\mathfrak{p}^R$ *exists for any* $\mathfrak{p} \in \mathfrak{G}$*, and if the ring* R *is strongly* $\mathfrak{G}$*-weak graded, then every* $\mathfrak{G}$*-weak graded* R*-module* V *is strongly graded.*

3. The Category of $\mathfrak{G}$-Weak Graded Modules

Definition 7. *For all* $\mathfrak{p}, \mathfrak{q} \in \mathfrak{G}$*, and all modules* V, W *and morphisms* $\varphi : V \to W$ *in* $\mathfrak{G}$*-GrR-Mod, a conjugation action of the set* $\mathfrak{G}$ *as automorphisms of* $\mathfrak{G}$*-GrR-Mod is defined by* $V^\mathfrak{p} = V$ *in* R*-Mod,* $(V^\mathfrak{p})_\mathfrak{q} = V_{\mathfrak{q}*\mathfrak{p}}$ *in* $R_{e_\mathfrak{G}}$*-Mod, and* $\varphi^\mathfrak{p} = \varphi$ *in* $Hom_R(V,W) = Hom_R(V^\mathfrak{p}, W^\mathfrak{p})$*, where* $e_\mathfrak{G}$ *is a two-sided identity.*

It can be noted that the $\mathfrak{G}$-weak grading is the only effect of this action for each module.

Theorem 2. *For any strongly* $\mathfrak{G}$*-weak graded* R*-module* V*, the natural map*

$$\beta_{R,V_{e_\mathfrak{G}}} : R \otimes_{R_{e_\mathfrak{G}}} V_{e_\mathfrak{G}} \to V,$$

which sends $r \otimes v_{e_\mathfrak{G}}$ *into* $rv_{e_\mathfrak{G}} \in V$ *for all* $r \in R$ *and* $v_{e_\mathfrak{G}} \in V_{e_\mathfrak{G}}$*, is an isomorphism in* $\mathfrak{G}$*-GrR-Mod.*

Proof. Let $V \in \mathfrak{G}$*-StGrR-Mod*; then, $V = \bigoplus_{\mathfrak{p} \in \mathfrak{G}} V_\mathfrak{p}$ in $R_{e_\mathfrak{G}}$*-Mod* and $V_\mathfrak{p} R_\mathfrak{q} = V_{\mathfrak{p}*\mathfrak{q}}$ for all $\mathfrak{p}, \mathfrak{q} \in \mathfrak{G}$. Since $V = \bigoplus_{\mathfrak{p} \in \mathfrak{G}} V_\mathfrak{p}$, we have

$$\beta_{R,V_{e_\mathfrak{G}}} : R \otimes_{R_{e_\mathfrak{G}}} V_{e_\mathfrak{G}} \to V,$$

which is an epimorphism in $\mathfrak{G}$*-GrR-Mod* for any $V \in \mathfrak{G}$*-StGrR-Mod*.

The kernel W of $\beta_{R,V_{e_\mathfrak{G}}}$ is a $\mathfrak{G}$-weak graded R-submodule of $R \otimes_{R_{e_\mathfrak{G}}} V_{e_\mathfrak{G}}$. Since, for any left $R_{e_\mathfrak{G}}$-module K, we have

$$(K^R)_\mathfrak{p} = R_\mathfrak{p} \otimes_{R_{e_\mathfrak{G}}} K, \quad (\text{ in } R_{e_\mathfrak{G}}\text{-}Mod) \text{ for all } \mathfrak{p} \in \mathfrak{G},$$

and

$$R_{e_\mathfrak{G}} \otimes_{e_\mathfrak{G}} V_{e_\mathfrak{G}} \cong V_{e_\mathfrak{G}}.$$

Then, the kernel of

$$\beta_{R_{\mathfrak{e}_{\mathfrak{G}}},V_{\mathfrak{e}_{\mathfrak{G}}}} : R_{\mathfrak{e}_{\mathfrak{G}}} \otimes_{R_{\mathfrak{e}_{\mathfrak{G}}}} V_{\mathfrak{e}_{\mathfrak{G}}} \xrightarrow{onto} V_{\mathfrak{e}_{\mathfrak{G}}}$$

is the $\mathfrak{e}_{\mathfrak{G}}$-component of W

$$W_{\mathfrak{e}_{\mathfrak{G}}} = W \cap (R_{\mathfrak{e}_{\mathfrak{G}}} \otimes_{R_{\mathfrak{e}_{\mathfrak{G}}}} V_{\mathfrak{e}_{\mathfrak{G}}}).$$

Hence, $W_{\mathfrak{e}_{\mathfrak{G}}} = 0$. Moreover, as $W \in \mathfrak{G}$-*StGrR-Mod*, we have, for all $\mathfrak{p}$ in $\mathfrak{G}$,

$$W_{\mathfrak{p}} = R_{\mathfrak{p}} W_{\mathfrak{e}_{\mathfrak{G}}} = 0.$$

Hence,

$$W = \bigoplus_{\mathfrak{p} \in \mathfrak{G}} W_{\mathfrak{p}} = 0.$$

Thus, $\beta_{R,V_{\mathfrak{e}_{\mathfrak{G}}}}$ is also a monomorphism and therefore is an isomorphism map as required. □

Theorem 3. *Let $R \in \mathfrak{G}$-StGr-R. Then, $\beta_{R,V_{\mathfrak{e}_{\mathfrak{G}}}}$ forms a natural isomorphism of the composite functor $((-)_{\mathfrak{e}_{\mathfrak{G}}})^R$ with the identity functor of $\mathfrak{G}$-GrR-Mod, i.e.,*

$$((-)_{\mathfrak{e}_{\mathfrak{G}}})^R \approx I_{\mathfrak{G}\text{-}GrR\text{-}Mod} \tag{9}$$

for all $\mathfrak{G}$-weak graded R-modules V in $\mathfrak{G}$-GrR-Mod.

Proof. The additive induction functor

$$(-)^R = R \otimes_{R_{e_{\mathfrak{G}}}} - : R_{\mathfrak{e}_{\mathfrak{G}}}\text{-}Mod \to \mathfrak{G}\text{-}GrR\text{-}Mod \tag{10}$$

is given, for any left $R_{\mathfrak{e}_{\mathfrak{G}}}$-module K, by

$$(K)^R = R \otimes_{R_{\mathfrak{e}_{\mathfrak{G}}}} K. \tag{11}$$

The $\mathfrak{p}$-component of K^R in $R_{\mathfrak{e}_{\mathfrak{G}}}$-*Mod* is given by

$$(K^R)_{\mathfrak{p}} = R_{\mathfrak{p}} \otimes_{R_{\mathfrak{e}_{\mathfrak{G}}}} K, \tag{12}$$

for all $\mathfrak{p} \in \mathfrak{G}$. For any map $\varphi : K \to L$ in $R_{\mathfrak{e}_{\mathfrak{G}}}$-*Mod*, we have

$$\varphi^R = I_R \otimes_{R_{\mathfrak{e}_{\mathfrak{G}}}} \varphi : R \otimes_{R_{\mathfrak{e}_{\mathfrak{G}}}} K \to R \otimes_{R_{\mathfrak{e}_{\mathfrak{G}}}} L, \tag{13}$$

where I_R is the identity map of R onto itself. In addition, the restriction functor

$$(-)_{\mathfrak{e}_{\mathfrak{G}}} : \mathfrak{G}\text{-}GrR\text{-}Mod \to R_{\mathfrak{e}_{\mathfrak{G}}}\text{-}Mod \tag{14}$$

sends any $V \in \mathfrak{G}$-*GrR-Mod* into its $R_{\mathfrak{e}_{\mathfrak{G}}}$-component $V_{\mathfrak{e}_{\mathfrak{G}}}$ and any morphism

$$\varphi : V \to W \quad \text{in } \mathfrak{G}\text{-}GrR\text{-}Mod$$

into its restriction

$$\varphi_{\mathfrak{e}_{\mathfrak{G}}} : V_{\mathfrak{e}_{\mathfrak{G}}} \to W_{\mathfrak{e}_{\mathfrak{G}}} \text{ in } R_{\mathfrak{e}_{\mathfrak{G}}}\text{-}Mod.$$

Combining the additive induction functor (10) and the restriction functor (14) gives the following composite functor:

$$((-)_{\mathfrak{e}_{\mathfrak{G}}})^R : \mathfrak{G}\text{-}GrR\text{-}Mod \to R_{\mathfrak{e}_{\mathfrak{G}}}\text{-}Mod \to \mathfrak{G}\text{-}GrR\text{-}Mod,$$

which is defined by

$$R \otimes_{R_{\mathfrak{e}_{\mathfrak{G}}}} V_{\mathfrak{e}_{\mathfrak{G}}} \to R_{\mathfrak{e}_{\mathfrak{G}}} \otimes_{R_{\mathfrak{e}_{\mathfrak{G}}}} V_{\mathfrak{e}_{\mathfrak{G}}} \to R \otimes_{R_{\mathfrak{e}_{\mathfrak{G}}}} V_{\mathfrak{e}_{\mathfrak{G}}},$$

for any $V_{e_{\mathfrak{G}}} \in R_{e_{\mathfrak{G}}}$-*Mod*. Since $R_{e_{\mathfrak{G}}} \otimes_{R_{e_{\mathfrak{G}}}} V_{e_{\mathfrak{G}}} \cong V_{e_{\mathfrak{G}}}$, we have

$$R \otimes_{R_{e_{\mathfrak{G}}}} V_{e_{\mathfrak{G}}} \to V_{e_{\mathfrak{G}}} \to R \otimes_{R_{e_{\mathfrak{G}}}} V_{e_{\mathfrak{G}}}.$$

□

Note that, for any $V \in \mathfrak{G}$-*GrR-Mod*, the map $\beta_{R,V_{e_{\mathfrak{G}}}}$ always forms a natural transformation of $((-)_{e_{\mathfrak{G}}})^R$ into the identity functor on $\mathfrak{G}$-*GrR-Mod*.

Corollary 1. *Let $R \in \mathfrak{G}$-StGr-R. Then, the additive induction functor $(-)^R$ and the restriction functor $(-)_{e_{\mathfrak{G}}}$ form an equivalence between the categories $\mathfrak{G}$-GrR-Mod and $R_{e_{\mathfrak{G}}}$-Mod.*

Proof. The proof is derived directly from Theorems 1 and 3. □

Theorem 4. *Let $e_{\mathfrak{G}}$ be a two-sided identity in $\mathfrak{G}$ and $R \in \mathfrak{G}$-Gr-R. Then, $R \in \mathfrak{G}$-StGr-R if and only if each $\mathfrak{G}$-weak graded R-module V in $\mathfrak{G}$-GrR-Mod is isomorphic in $\mathfrak{G}$-GrR-Mod to a module K^R induced from some $R_{e_{\mathfrak{G}}}$-module K.*

Proof. ($\Longrightarrow$) Assume that $R \in \mathfrak{G}$-*StGr-R*. Then, each $V \in \mathfrak{G}$-*GrR-Mod* is strongly graded according to Proposition 1. Thus, V is isomorpmic in $\mathfrak{G}$-*GrR-Mod* to a module K^R induced from some $R_{e_{\mathfrak{G}}}$-module K via Theorem 2 and Corollary 1.

($\Longleftarrow$) Here, assume that each $V \in \mathfrak{G}$-*GrR-Mod* is isomorphic in $\mathfrak{G}$-*GrR-Mod* to a module K^R induced from some $R_{e_{\mathfrak{G}}}$-module K. Then, according to Corollary 3.3 and Proposition 3.5 in [10], we have

$$R_{\mathfrak{q}} R_{e_{\mathfrak{G}}} = R_{\mathfrak{q}},$$

for any $q \in \mathfrak{G}$. Thus, from the tensor product's definition, and since $(K^R)_{\mathfrak{p}} = R_{\mathfrak{p}} \otimes_{R_{e_{\mathfrak{G}}}} K$, for all $\mathfrak{p} \in \mathfrak{G}$, we have

$$R_{\mathfrak{q}}(K^R)_{e_{\mathfrak{G}}} = R_{\mathfrak{q}}(R_{e_{\mathfrak{G}}} \otimes_{R_{e_{\mathfrak{G}}}} K) = (R_{\mathfrak{q}} R_{e_{\mathfrak{G}}}) \otimes_{R_{e_{\mathfrak{G}}}} K = R_{\mathfrak{q}} \otimes_{R_{e_{\mathfrak{G}}}} K = (K^R)_{\mathfrak{q}} \tag{15}$$

for any $R_{e_{\mathfrak{G}}}$-module K. Since each $V \in \mathfrak{G}$-*GrR-Mod* is isomorphic in $\mathfrak{G}$-*GrR-Mod* to a module K^R, we conclude that

$$R_{\mathfrak{q}} V_{e_{\mathfrak{G}}} = V_{\mathfrak{q} * e_{\mathfrak{G}}} = V_{\mathfrak{q}},$$

for any $V \in \mathfrak{G}$-*GrR-Mod*. The regular R-module R with its $\mathfrak{G}$-weak grading $R = \bigoplus_{\mathfrak{p} \in \mathfrak{G}} R_{\mathfrak{p}}$ is in $\mathfrak{G}$-*GrR-Mod* by $R_{\mathfrak{q}} R_{\mathfrak{p}} \subseteq R_{\mathfrak{q} * \mathfrak{p}}$. Let V be the conjugate module $R^{\mathfrak{p}}$ for some $\mathfrak{p} \in \mathfrak{G}$, and we conclude that

$$R_{\mathfrak{q}} R_{\mathfrak{p}} = R_{\mathfrak{q}}(R_{e_{\mathfrak{G}} * \mathfrak{p}}) = R_{\mathfrak{q}}(R^{\mathfrak{p}})_{e_{\mathfrak{G}}} = (R^{\mathfrak{p}})_{\mathfrak{q}} = R_{\mathfrak{q} * \mathfrak{p}}$$

according to Definition 7 and relation (15). Therefore, $R \in \mathfrak{G}$-*StGr-R*. □

Corollary 2. *Let $e_{\mathfrak{G}}$ be a two-sided identity in $\mathfrak{G}$ and R be a strongly $\mathfrak{G}$-graded ring; let $\varphi : V \to W$ be a morphism in $\mathfrak{G}$-GrR-Mod; and let $\mathfrak{p}$ be an element of $\mathfrak{G}$. Then, φ is a monomorphism in $\mathfrak{G}$-GrR-Mod if and only if its restriction $\varphi_{\mathfrak{p}} : V_{\mathfrak{p}} \to W_{\mathfrak{p}}$ is a monomorphism in $R_{e_{\mathfrak{G}}}$-Mod.*

Proof. From Definition 7, we have $\varphi^{\mathfrak{p}} = \varphi$ in $Hom_R(V, W)$, which is equal to $Hom_R(V^{\mathfrak{p}}, W^{\mathfrak{p}})$. The map φ is also a morphism $\varphi^{\mathfrak{p}}$ from $V^{\mathfrak{p}}$ to $W^{\mathfrak{p}}$ in $\mathfrak{G}$-*GrR-Mod*. From Corollary 1, $\varphi^{\mathfrak{p}}$ is a monomorphism if and only if its restriction $(\varphi^{\mathfrak{p}})_{e_{\mathfrak{G}}}$ is a monomorphism. Definition 7 implies that $(\varphi^{\mathfrak{p}})_{e_{\mathfrak{G}}}$ is precisely the map $\varphi_{\mathfrak{p}}$ of $(V^{\mathfrak{p}})_{e_{\mathfrak{G}}} = V_{e_{\mathfrak{G}} * \mathfrak{p}} = V_{\mathfrak{p}}$ into $(W^{\mathfrak{p}})_{e_{\mathfrak{G}}} = W_{e_{\mathfrak{G}} * \mathfrak{p}} = W_{\mathfrak{p}}$. Hence, the corollary holds. □

Corollary 3. *Let e_G be a two-sided identity in G and R be a strongly G-graded ring; let $\varphi : V \to W$ be a morphism in G-GrR-Mod; and let p be an element of G. Then, φ is an epimorphism in G-GrR-Mod if and only if its restriction $\varphi_p : V_p \to W_p$ is an epimorphism in R_{e_G}-Mod.*

Proof. Similarly to the proof of Corollary 2, the map φ is also a morphism $\varphi^p : V^p \to W^p$ in G-GrR-Mod. From Corollary 1, the map φ^p is an epimorphism if and only if its restriction $(\varphi^p)_{e_G}$ is an epimorphism. Moreover, we find that $(\varphi^p)_{e_G}$ is precisely the map φ_p of $(V^p)_{e_G} = V_{e_G * p} = V_p$ into $(W^p)_{e_G} = W_{e_G * p} = W_p$ via Definition 7. □

Corollary 4. *Let e_G be a two-sided identity in G and R be a strongly G-graded ring; let $\varphi : V \to W$ be a morphism in G-GrR-Mod; and let p be an element of G. Then, φ is an isomorphism in G-GrR-Mod if and only if its restriction $\varphi_p : V_p \to W_p$ is an isomorphism in R_{e_G}-Mod.*

Proof. The proof follows via Corollaries 2 and 3. □

Next, we use Definition 7 and relations (11)–(13) to obtain the following definition.

Definition 8. *For a two-sided identity e_G in G, $R \in G$-Gr-R and p in G, we define an additive functor $(-)^p : G$-GrR-Mod $\to G$-GrR-Mod as*

$$K^p = (K^R)_p = R_p \otimes_{R_{e_G}} K. \tag{16}$$

Equivalently, we write the composite additive functor $(-)^p$ as follows:

$$(-)^p = (((-)^R)^p)_{e_G} : R_{e_G}\text{-Mod} \to R_{e_G}\text{-Mod}. \tag{17}$$

Moreover, we define $\varphi^p : R_p \otimes_{R_{e_G}} K \to R_p \otimes_{R_{e_G}} L$ as

$$\varphi^p = I_p \otimes_{R_{e_G}} \varphi, \tag{18}$$

for any R_{e_G}-modules K and L and any R_{e_G}-homomorphism $\varphi : K \to L$, where I_p is the identity map of R_p onto itself.

Theorem 5. *Let e_G be a two-sided identity in G and $R \in G$-Gr-R. Then, for any p, q in G and for any left R_{e_G}-module K, the map*

$$\beta_{R_q, K^p} = R_q \otimes_{R_{e_G}} (R_p \otimes_{R_{e_G}} K) \to R_{q*p} \otimes_{R_{e_G}} K, \tag{19}$$

which sends $r_q \otimes (r_p \otimes k)$ into $(r_q r_p) \otimes k$ for any $k \in K, r_p \in R_p$, and $r_q \in R_q$, forms the following natural transformation:

$$((-)^p)^q \text{ into } (-)^{q*p} : R_{e_G}\text{-Mod} \to R_{e_G}\text{-Mod}.$$

Proof. From Definition 8, we have the composite additive functor

$$(-)^p = (((-)^R)^p)_{e_G} : R_{e_G}\text{-Mod} \to G\text{-GrR-Mod} \to G\text{-GrR-Mod} \to R_{e_G}\text{-Mod}.$$

From Theorem 1, we have

$$(-)^{e_G} = (((-)^R)^{e_G})_{e_G} = ((-)^R)_{e_G} \approx I_{R_{e_G}\text{-Mod}}.$$

Thus, in light of the relation (16) of Definition 8, we conclude that

$$\beta_{R_q, K^p} = R_q \otimes_{R_{e_G}} K^p = R_q \otimes_{R_{e_G}} (R_p \otimes_{R_{e_G}} K) \to R_{q*p} \otimes_{R_{e_G}} K,$$

which sends $r_q \otimes (r_p \otimes k)$ into $(r_q r_p) \otimes k$ for any $k \in K, r_p \in R_p$, and $r_q \in R_q$ forms a natural transformation of $((-)^p)^q$ into $(-)^{q*p} : R_{e_G}$-Mod $\to R_{e_G}$-Mod. □

Corollary 5. *Let $\mathfrak{e}_{\mathfrak{G}}$ be a two-sided identity in $\mathfrak{G}$ and $R \in \mathfrak{G}$-StGr-R. Then, the map $\beta_{R_{\mathfrak{q}},K^{\mathfrak{p}}}$ is an $R_{\mathfrak{e}_{\mathfrak{G}}}$-isomorphism of $(K^{\mathfrak{p}})^{\mathfrak{q}}$ onto $K^{\mathfrak{q}*\mathfrak{p}}$ for any $R_{\mathfrak{e}_{\mathfrak{G}}}$-module K.*

Proof. From Theorem 5, there is a natural transformation between $((-)^{\mathfrak{p}})^{\mathfrak{q}}$ and $(-)^{\mathfrak{q}*\mathfrak{p}}$: $\mathfrak{G}$-*GrR-Mod* $\to$ $\mathfrak{G}$-*GrR-Mod*, for any $\mathfrak{p}, \mathfrak{q} \in \mathfrak{G}$. Thus, Theorem 3 and relation (17) of Definition 8 yield that $\beta_{R_{\mathfrak{q}},K^{\mathfrak{p}}}$ is a natural isomorphism of $((-)^{\mathfrak{p}})^{\mathfrak{q}}$ onto $(-)^{\mathfrak{q}*\mathfrak{p}}$: $R_{\mathfrak{e}_{\mathfrak{G}}}$-*Mod* $\to$ $R_{\mathfrak{e}_{\mathfrak{G}}}$-*Mod*. □

Example 1. *Consider the Morita ring*

$$T = \left\{ \begin{pmatrix} r & v \\ w & s \end{pmatrix} : r \in R, v \in V, w \in W \text{ and } s \in S \right\},$$

with a Morita contex $(R, S, {}_R V_S, {}_S W_R, \varphi, \psi)$ such that the bimodule homomorphisms

$$\varphi : V \otimes_S W \to R \quad \text{and} \quad \psi : W \otimes_R V \to S$$

satisfy $(vw)v' = v(wv')$ as $\varphi(v, w) = vw$ and $\psi(w, v) = wv$, i.e., $\varphi(v \otimes w)v' = v\psi(w \otimes v')$ and $\psi(w \otimes v)w' = w\varphi(v \otimes w')$ for all $v, v' \in V$ and $w, w' \in W$. It is well known that T with the usual matrix addition and multiplication forms a ring. Here, let $\mathfrak{X}$ be the permutation group S_3 and $\mathfrak{H}$ be the non-normal subgroup $\{\mathfrak{e}, (23)\}$. We choose $\mathfrak{G} = \{(\mathfrak{e}, (132), (13)\}$ to be the set of left coset representatives. Then, the operation $*$ is as given in the following table (Table 1).

Table 1. The binary operation $*$.

$*$	$\mathfrak{e}$	(132)	(13)
$\mathfrak{e}$	$\mathfrak{e}$	(132)	(13)
(132)	(132)	(13)	$\mathfrak{e}$
(13)	(13)	$\mathfrak{e}$	(132)

Thus, $T = T_{\mathfrak{e}} \oplus T_{(132)} \oplus T_{(13)}$, where

$$T_{\mathfrak{e}} = \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} = \left\{ \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} : r \in R \text{ and } s \in S \right\},$$

$$T_{(132)} = \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} : v \in V \right\} \text{ and}$$

$$T_{(13)} = \begin{pmatrix} 0 & 0 \\ W & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix} : w \in W \right\}.$$

Next, we check the property $R_{\mathfrak{p}}R_{\mathfrak{q}} \subseteq R_{\mathfrak{p}*\mathfrak{q}}$ for all $\mathfrak{p}, \mathfrak{q} \in \mathfrak{G}$ as follows:

1. $T_{\mathfrak{e}}T_{\mathfrak{e}} \subseteq T_{\mathfrak{e}*\mathfrak{e}}$, *as, for all* $\begin{pmatrix} r_1 & 0 \\ 0 & s_1 \end{pmatrix}, \begin{pmatrix} r_2 & 0 \\ 0 & s_2 \end{pmatrix} \in T_e$, *we have*
$$\begin{pmatrix} r_1 & 0 \\ 0 & s_1 \end{pmatrix}\begin{pmatrix} r_2 & 0 \\ 0 & s_2 \end{pmatrix} = \begin{pmatrix} r_1r_2 & 0 \\ 0 & s_1s_2 \end{pmatrix} \in T_e = T_{e*e}.$$
2. $T_{\mathfrak{e}}T_{(132)} \subseteq T_{\mathfrak{e}*(132)}$, *as, for all* $\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \in T_{\mathfrak{e}}, \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \in T_{(132)}$, *we have*
$\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & rv \\ 0 & 0 \end{pmatrix} \in T_{(132)} = T_{e*(132)}$, *since $rv \in V$ as V is a left R-module.*
3. $T_{\mathfrak{e}}T_{(13)} \subseteq T_{\mathfrak{e}*(13)}$, *as, for all* $\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \in T_{\mathfrak{e}}, \begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix} \in T_{(13)}$, *we have*
$\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}\begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ sw & 0 \end{pmatrix} \in T_{(13)} = T_{2*(13)}$, *since $sw \in W$ as W is a left S-module.*

4. $T_{(132)}T_{\mathfrak{e}} \subseteq T_{(132)*\mathfrak{e}}$, *as, for all* $\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \in T_{(132)}$, $\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \in T_{\mathfrak{e}}$, *we have* $\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} = \begin{pmatrix} 0 & vs \\ 0 & 0 \end{pmatrix} \in T_{(132)} = T_{(132)*\mathfrak{e}}$.
5. $T_{(132)}T_{(132)} \subseteq T_{(132)*(132)}$, *as, for all* $\begin{pmatrix} 0 & v_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & v_2 \\ 0 & 0 \end{pmatrix} \in T_{(132)}$, *we have* $\begin{pmatrix} 0 & v_1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & v_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in T_{(13)} = T_{(132)*(132)}$.
6. $T_{(132)}T_{(13)} \subseteq T_{(132)*(13)}$, *as, for all* $\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} in T_{(132)}$, $\begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix} \in T_{(13)}$, *we have* $\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix} = \begin{pmatrix} vw & 0 \\ 0 & 0 \end{pmatrix} \in T_{\mathfrak{e}} = T_{(132)*(13)}$.
7. $T_{(13)}T_{\mathfrak{e}} \subseteq T_{(13)*\mathfrak{e}}$, *as, for all* $\begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix} \in T_{(13)}$, $\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \in T_{\mathfrak{e}}$, *we have* $\begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ wr & 0 \end{pmatrix} \in T_{(13)} = T_{(13)*\mathfrak{e}}$.
8. $T_{(13)}T_{(132)} \subseteq T_{(13)*(132)}$, *as, for all* $\begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix} \in T_{(13)}$, $\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \in T_{(132)}$, *we have* $\begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix}\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & wv \end{pmatrix} \in T_{\mathfrak{e}} = T_{(13)*(132)}$.
9. $T_{(13)}T_{(13)} \subseteq T_{(13)*(13)}$, *as, for all* $\begin{pmatrix} 0 & 0 \\ w_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ w_2 & 0 \end{pmatrix} \in T_{(13)}$, *we have* $\begin{pmatrix} 0 & 0 \\ w_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ w_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in T_{(132)} = T_{(13)*(13)}$.

Therefore, T is a $\mathfrak{G}$*-weak graded ring. However, T is not strongly graded. For example,* $T_{(13)}T_{(13)} \neq T_{(13)*(13)}$ *as* $T_{(132)} = T_{(13)*(13)} \nsubseteq T_{(13)}T_{(13)}$.

Example 2. *Let* $R = M_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$ *and let* $\mathfrak{X} = \mathbb{Z}_2 \times \mathbb{Z}_3 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\}$ *under addition with a subgroup* $\mathfrak{H} = \{(0,0), (1,0)\}$. *Choose* $\mathfrak{G} = \{(1,0), (0,1), (1,2)\}$. *Then, the* $*$ *operation is as given in the following table (Table 2).*

Table 2. The binary operation $*$.

$*$	$(1,0)$	$(0,1)$	$(1,2)$
$(1,0)$	$(1,0)$	$(0,1)$	$(1,2)$
$(0,1)$	$(0,1)$	$(1,2)$	$(1,0)$
$(1,2)$	$(1,2)$	$(1,0)$	$(0,1)$

Hence, we have $R = R_{(1,0)} \oplus R_{(0,1)} \oplus R_{(1,2)}$, *where* $R_{(1,0)} = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a, d \in \mathbb{R} \right\}$, $R_{(0,1)} = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in \mathbb{R} \right\}$, *and* $R_{(1,2)} = \left\{ \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} : c \in \mathbb{R} \right\}$. *Here, we check the property* $R_{\mathfrak{p}}R_{\mathfrak{q}} \subseteq R_{\mathfrak{p}*\mathfrak{q}}$ *for all* $\mathfrak{p}, \mathfrak{q} \in \mathfrak{G}$ *as follows:*

1. $R_{(1,0)}R_{(1,0)} \subseteq R_{(1,0)*(1,0)} = R_{(1,0)}$, *as, for all* $\begin{pmatrix} a_1 & 0 \\ 0 & d_1 \end{pmatrix}$ *and* $\begin{pmatrix} a_2 & 0 \\ 0 & d_2 \end{pmatrix} \in R_{(1,0)}$, *we have*

$$\begin{pmatrix} a_1 & 0 \\ 0 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & 0 \\ 0 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 & 0 \\ 0 & d_1d_2 \end{pmatrix} \in R_{(1,0)} = R_{(1,0)*(1,0)}.$$

2. $R_{(1,0)}R_{(0,1)} \subseteq R_{(1,0)*(0,1)} = R_{(0,1)}$, *as, for all* $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in R_{(1,0)}$ *and* $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in R_{(0,1)}$, *we have*
$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ab \\ 0 & 0 \end{pmatrix} \in R_{(0,1)} = R_{(1,0)*(0,1)}.$$

3. $R_{(1,0)}R_{(1,2)} \subseteq R_{(1,0)*(1,2)} = R_{(1,2)}$, *as, for all* $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in R_{(1,0)}$ *and* $\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \in R_{(1,2)}$, *we have*
$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ dc & 0 \end{pmatrix} \in R_{(1,2)} = R_{(1,0)*(1,2)}.$$

4. $R_{(0,1)}R_{(1,0)} \subseteq R_{(0,1)*(1,0)} = R_{(0,1)}$, *as, for all* $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in R_{(0,1)}$ *and* $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in R_{(1,0)}$, *we have*
$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & ab \\ 0 & 0 \end{pmatrix} \in R_{(0,1)} = R_{(0,1)*(1,0)}.$$

5. $R_{(0,1)}R_{(0,1)} \subseteq R_{(0,1)*(0,1)} = R_{(1,2)}$, *as, for all* $\begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & b_2 \\ 0 & 0 \end{pmatrix} \in R_{(0,1)}$, *we have*
$$\begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & b_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in R_{(1,2)} = R_{(0,1)*(0,1)}.$$

6. $R_{(0,1)}R_{(1,2)} \subseteq R_{(0,1)*(1,2)} = R_{(1,0)}$, *as, for all* $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in R_{(0,1)}$ *and* $\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \in R_{(1,2)}$, *we have*
$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} = \begin{pmatrix} bc & 0 \\ 0 & 0 \end{pmatrix} \in R_{(1,0)} = R_{(0,1)*(1,2)}.$$

7. $R_{(1,2)}R_{(1,0)} \subseteq R_{(1,2)*(1,0)} = R_{(1,2)}$, *as, for all* $\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \in R_{(1,2)}$ *and* $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in R_{(1,0)}$, *we have*
$$\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ ac & 0 \end{pmatrix} \in R_{(1,2)} = R_{(1,2)*(1,0)}.$$

8. $R_{(1,2)}R_{(0,1)} \subseteq R_{(1,2)*(0,1)} = R_{(1,0)}$, *as, for all* $\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \in R_{(1,2)}$ *and* $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in R_{(1,0)}$, *we have*
$$\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & bc \end{pmatrix} \in R_{(1,0)} = R_{(1,2)*(0,1)}.$$

9. $R_{(1,2)}R_{(1,2)} \subseteq R_{(1,2)*(1,2)} = R_{(0,1)}$, *as, for all* $\begin{pmatrix} 0 & 0 \\ c_1 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & 0 \\ c_2 & 0 \end{pmatrix} \in R_{(1,2)}$, *we have*
$$\begin{pmatrix} 0 & 0 \\ c_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ c_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in R_{(0,1)} = R_{(1,2)*(1,2)}.$$

Thus, R is a $\mathfrak{G}$*-weak graded ring. However, it is not strongly graded—for instance,* $R_{(1,2)*(1,2)} \neq R_{(1,2)}R_{(1,2)}$ *as* $R_{(0,1)} = R_{(1,2)*(1,2)} \nsubseteq R_{(1,2)}R_{(1,2)}$.

Next, if we define $V = M_{2\times 3}(\mathbb{R})$, *then V is a* $\mathfrak{G}$*-graded R-module with* $V = V_{(1,0)} \oplus V_{(0,1)} \oplus V_{(1,2)}$, *where* $V_{(0,1)} = \begin{pmatrix} \mathbb{R} & 0 & 0 \\ 0 & \mathbb{R} & 0 \end{pmatrix}$, $V_{(1,0)} = \begin{pmatrix} 0 & 0 & \mathbb{R} \\ \mathbb{R} & 0 & 0 \end{pmatrix}$ *and* $V_{(1,2)} = \begin{pmatrix} 0 & \mathbb{R} & 0 \\ 0 & 0 & \mathbb{R} \end{pmatrix}$.

Here, we show that the inclusion property $R_{\mathfrak{p}}V_{\mathfrak{q}} \subseteq V_{\mathfrak{p}*\mathfrak{q}}$ *is satisfied for all* $\mathfrak{p}, \mathfrak{q} \in \mathfrak{G}$ *as follows:*

1. $R_{(1,0)}V_{(0,1)} \subseteq V_{(0,1)*(0,1)}$, *as, for all* $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in R_{(1,0)}$ *and* $\begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_5 & 0 \end{pmatrix} \in V_{(0,1)}$, *we have* $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_5 & 0 \end{pmatrix} = \begin{pmatrix} ar_1 & 0 & 0 \\ 0 & dr_5 & 0 \end{pmatrix} \in V_{(0,1)} = V_{(1,0)*(0,1)}$.
2. $R_{(1,0)}V_{(1,0)} \subseteq V_{(1,0)*(1,0)}$, *as, for all* $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in R_{(1,0)}$ *and* $\begin{pmatrix} 0 & 0 & r_3 \\ r_4 & 0 & 0 \end{pmatrix} \in V_{(1,0)}$, *we have* $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & 0 & r_3 \\ r_4 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & ar_3 \\ r_4 & 0 & 0 \end{pmatrix} \in V_{(1,0)} = V_{(1,0)*(1,0)}$.
3. $R_{(1,0)}V_{(1,2)} \subseteq V_{(1,0)*(1,2)}$, *as, for all* $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in R_{(1,0)}$ *and* $\begin{pmatrix} 0 & r_2 & 0 \\ 0 & 0 & r_6 \end{pmatrix} \in V_{(1,2)}$, *we have* $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & r_2 & 0 \\ 0 & 0 & r_6 \end{pmatrix} = \begin{pmatrix} 0 & ar_2 & 0 \\ 0 & 0 & dr_6 \end{pmatrix} \in V_{(1,2)} = V_{(1,0)*(1,2)}$.
4. $R_{(0,1)}V_{(0,1)} \subseteq V_{(0,1)*(0,1)}$, *as, for all* $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in R_{(0,1)}$ *and* $\begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_5 & 0 \end{pmatrix} \in V_{(0,1)}$, *we have* $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_5 & 0 \end{pmatrix} = \begin{pmatrix} 0 & br_5 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in V_{(1,2)} = V_{(0,1)*(0,1)}$.
5. $R_{(0,1)}V_{(1,0)} \subseteq V_{(0,1)*(1,0)}$, *as, for all* $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in R_{(0,1)}$ *and* $\begin{pmatrix} 0 & 0 & r_3 \\ r_4 & 0 & 0 \end{pmatrix} \in V_{(1,0)}$, *we have* $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & r_3 \\ r_4 & 0 & 0 \end{pmatrix} = \begin{pmatrix} br_4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in V_{(0,1)} = V_{(0,1)*(1,0)}$.
6. $R_{(0,1)}V_{(1,2)} \subseteq V_{(0,1)*(1,2)}$, *as, for all* $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in R_{(0,1)}$ *and* $\begin{pmatrix} 0 & r_2 & 0 \\ 0 & 0 & r_6 \end{pmatrix} \in V_{(1,0)}$, *we have* $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & r_2 & 0 \\ 0 & 0 & r_6 \end{pmatrix} = \begin{pmatrix} 0 & 0 & br_6 \\ 0 & 0 & 0 \end{pmatrix} \in V_{(1,0)} = V_{(0,1)*(1,2)}$.
7. $R_{(1,2)}V_{(0,1)} \subseteq V_{(1,2)*(0,1)}$, *as, for all* $\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \in R_{(1,2)}$ *and* $\begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_5 & 0 \end{pmatrix} \in V_{(0,1)}$, *we have* $\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}\begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_5 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ cr_1 & 0 & 0 \end{pmatrix} \in V_{(1,0)} = V_{(1,2)*(0,1)}$.
8. $R_{(1,2)}V_{(1,0)} \subseteq V_{(1,2)*(1,0)}$, *as, for all* $\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \in R_{(1,2)}$ *and* $\begin{pmatrix} 0 & 0 & r_3 \\ r_4 & 0 & 0 \end{pmatrix} \in V_{(1,0)}$, *we have* $\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & r_3 \\ r_4 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & cr_3 \end{pmatrix} \in V_{(1,2)} = V_{(1,2)*(1,0)}$.
9. $R_{(1,2)}V_{(1,2)} \subseteq V_{(1,2)*(1,2)}$, *as, for all* $\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \in R_{(1,2)}$ *and* $\begin{pmatrix} 0 & r_2 & 0 \\ 0 & 0 & r_6 \end{pmatrix} \in V_{(1,2)}$, *we have* $\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}\begin{pmatrix} 0 & r_2 & 0 \\ 0 & 0 & r_6 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & cr_2 & 0 \end{pmatrix} \in V_{(0,1)} = V_{(1,2)*(1,2)}$. *Therefore, V is a* $\mathfrak{G}$*-weak graded R-module. It can be noted that V is not strongly graded as* $R_{(0,1)}V_{(0,1)} \subseteq V_{(1,2)}$ *but* $V_{(1,2)} = V_{(0,1)*(0,1)} \nsubseteq R_{(0,1)}V_{(0,1)}$.

4. $\mathfrak{G}$-Weak Homomorphism Groups

In ref. [11], it was proven that $\mathfrak{G}$-$GrHom_R(V,W)$ is an additive subgroup of $Hom_R(V,W)$ and that $\mathfrak{G}$-$GrHom_R(V,W) = \bigoplus_{\mathfrak{p}\in\mathfrak{G}} \mathfrak{G}$-$GrHom_R(V,W)_\mathfrak{p}$ as an additive subgroup. We can now prove the following.

Proposition 2. *Let* $\mathfrak{e}_\mathfrak{G}$ *be a two-sided identity in* $\mathfrak{G}$ *and* $V, W \in \mathfrak{G}$*-GrR-Mod. Then, for any* $\mathfrak{p}$, $\mathfrak{q}$ *in* $\mathfrak{G}$, $\mathfrak{G}$-$GrHom_R(V,W)_\mathfrak{q}$ *is the additive subgroup* $\mathfrak{G}$-$GrHom_R(V^\mathfrak{p}, W^{\mathfrak{p}*\mathfrak{q}})$ *of* $Hom_R(V,W)$.

Proof. Since $\mathfrak{G}$-$GrHom_R(V,W)_\mathfrak{q}$ is an additive subgroup of $\mathfrak{G}$-$GrHom_R(V,W)$ of $Hom_R(V,W)$, then Definition 7 yields the required result for the conjugation action. □

Proposition 3. *For the* $\mathfrak{G}$*-weak graded R-modules L, V, and W in* $\mathfrak{G}$*-GrR-Mod, we have*

$$\mathfrak{G}\text{-}GrHom_R(V,W)_\mathfrak{p}\,\mathfrak{G}\text{-}GrHom_R(L,V)_\mathfrak{q} \subseteq \mathfrak{G}\text{-}GrHom_R(L,W)_{\mathfrak{p}*\mathfrak{q}},\ \forall\,\mathfrak{p},\,\mathfrak{q} \in \mathfrak{G}. \tag{20}$$

Proof. Let φ in $\mathfrak{G}$-$GrHom_R(V,W)_{\mathfrak{p}}$ and ψ in $\mathfrak{G}$-$GrHom_R(L,V)_{\mathfrak{q}}$; then, for all $\mathfrak{p}, \mathfrak{q} \in \mathfrak{G}$, we have

$$\varphi(V_{\mathfrak{p}}) \subseteq W_{\mathfrak{p}*\mathfrak{q}},$$

and

$$\psi(L_{\mathfrak{q}}) \subseteq V_{\mathfrak{q}*\mathfrak{p}}.$$

From [8], Proposition 2.4, we have

$$\varphi(\psi(L_{\mathfrak{q}})) \subseteq \varphi(V_{\mathfrak{q}*\mathfrak{p}}) \subseteq W_{(\mathfrak{q}*\mathfrak{p})*\mathfrak{q}} = W_{\left(\mathfrak{q}\triangleleft g(\mathfrak{p},\mathfrak{q})\right)*(\mathfrak{p}*\mathfrak{q})}.$$

As $\mathfrak{q} \triangleleft g(\mathfrak{p},\mathfrak{q})$ is also in $\mathfrak{G}$, we have

$$(\psi \circ \varphi) \in \mathfrak{G}\text{-}GrHom_R(L,W)_{\mathfrak{p}*\mathfrak{q}}.$$

□

Lemma 1. *Let V and W be $\mathfrak{G}$-weak graded R-modules. Then, for any φ in $\mathfrak{G}$-$GrHom_R(V,W)$ and for all $\mathfrak{p}$ in $\mathfrak{G}$, we have*

$$\varphi(V_{\mathfrak{p}}) \subseteq \sum_{\mathfrak{q}\in\mathfrak{G}} W_{\mathfrak{p}*\mathfrak{q}}. \tag{21}$$

Proof. The proof follows directly via Definition 5 and [11], Theorem 1. □

Theorem 6. *For any $\mathfrak{G}$-weak graded R-modules V and W in $\mathfrak{G}$-GrR-Mod, the equality*

$$\mathfrak{G}\text{-}GrHom_R(V,W) = Hom_R(V,W)$$

is satisfied.

Proof. Since $\mathfrak{G}$-$GrHom_R(V,W)$ is an additive subgroup of $Hom_R(V,W)$, we have

$$\mathfrak{G}\text{-}GrHom_R(V,W) \subseteq Hom_R(V,W).$$

Here, let φ in $Hom_R(V,W)$, $\mathfrak{p} \in \mathfrak{G}$ and $v \in V$. Since $\mathfrak{G}$ is a finite set, we can write $v_{\mathfrak{p}}$ as the finite sum of its homogeneous components as follows: $v = \sum_{\mathfrak{p}\in\mathfrak{G}} v_{\mathfrak{p}}$, for all $\mathfrak{p} \in \mathfrak{G}$ and $v_{\mathfrak{p}} \in V_{\mathfrak{p}}$. Since $\mathfrak{G}$ is closed under the $*$ operation and W is a $\mathfrak{G}$-weak graded R-module, the direct sum $W = \bigoplus_{\mathfrak{p}\in\mathfrak{G}} W_{\mathfrak{p}}$ is equivalent to

$$W = \bigoplus_{\mathfrak{q}\in\mathfrak{G}} W_{\mathfrak{p}*\mathfrak{q}},$$

for any $\mathfrak{p} \in \mathfrak{G}$. Thus, there exists $\varphi_{\mathfrak{p}*\mathfrak{q}} \in Hom_{R_e}(V_{\mathfrak{p}}, W_{\mathfrak{p}*\mathfrak{q}})$ for $\mathfrak{p}, \mathfrak{q} \in \mathfrak{G}$, such that

$$\varphi(v_{\mathfrak{p}}) = \sum_{\mathfrak{q}\in\mathfrak{G}} \varphi_{\mathfrak{p}*\mathfrak{q}}(v_{\mathfrak{p}}) \text{ for all } \mathfrak{p} \in \mathfrak{G} \text{ and } v_{\mathfrak{p}} \in V_{\mathfrak{p}},$$

where $\varphi_{\mathfrak{p}*\mathfrak{q}}(v_{\mathfrak{p}})$ is the homogeneous component of $\varphi(v_{\mathfrak{p}})$ that lies in $W_{\mathfrak{p}*\mathfrak{q}}$. Here, for any $\mathfrak{q} \in \mathfrak{G}$, we define a map $\varphi_{\mathfrak{q}} \in \mathfrak{G}$-$GrHom_R(V,W)$ as

$$\varphi_{\mathfrak{q}}\Big(\sum_{\mathfrak{p}\in\mathfrak{G}} v_{\mathfrak{p}}\Big) = \sum_{\mathfrak{p}\in\mathfrak{G}} \varphi_{\mathfrak{p}*\mathfrak{q}}(v_{\mathfrak{p}}) = \varphi_{\mathfrak{p}*\mathfrak{q}}\Big(\sum_{\mathfrak{p}\in\mathfrak{G}} v_p\Big). \tag{22}$$

Since $\mathfrak{G}$ is a finite set, this sum is finite and well defined. Moreover, for all $\mathfrak{p} \in \mathfrak{G}$ and all $v_{\mathfrak{p}} \in V_{\mathfrak{p}}$, we know that $\varphi(v_{\mathfrak{p}}) \in W$ because φ is an R-homomorphism. Hence,

$\varphi_{\mathfrak{q}}(V_{\mathfrak{p}}) = \sum_{\mathfrak{q}\in\mathfrak{G}} \varphi_{\mathfrak{p}*\mathfrak{q}}(V_{\mathfrak{p}}) = \varphi_{\mathfrak{p}*\mathfrak{q}}(\sum_{\mathfrak{q}\in\mathfrak{G}} V_{\mathfrak{p}}) \subseteq W_{\mathfrak{p}*\mathfrak{q}}$. Thus, from Definition 5, each $\varphi_{\mathfrak{q}} \in \mathfrak{G}\text{-}GrHom_R(V,W)_{\mathfrak{q}}$. From (22), we have

$$\varphi(v_{\mathfrak{p}}) = \sum_{\mathfrak{q}\in\mathfrak{G}} \varphi_{\mathfrak{p}*\mathfrak{q}}(v_{\mathfrak{p}}) = \sum_{\mathfrak{q}\in\mathfrak{G}} \varphi_{\mathfrak{q}}(v_{\mathfrak{p}}),$$

which implies that $\varphi = \sum_{\mathfrak{q}\in\mathfrak{G}} \varphi_{\mathfrak{q}}$. This sum is finite and well defined since $\mathfrak{G}$ is a finite set. Therefore, $\varphi \in \mathfrak{G}\text{-}GrHom_R(V,W)$. □

Proposition 4. *If $\mathfrak{e}_{\mathfrak{G}}$ is a two-sided identity in $\mathfrak{G}$, R is a strongly $\mathfrak{G}$-graded ring, and K and L are $R_{\mathfrak{e}_{\mathfrak{G}}}$-modules. Then, the restriction to $K^{\mathfrak{p}} = (K^R)_{\mathfrak{p}}$ induces*

$$Hom_R(K^R, L^R)_{\mathfrak{q}} \cong Hom_{R_{\mathfrak{e}_{\mathfrak{G}}}}(K^{\mathfrak{p}}, L^{\mathfrak{p}*\mathfrak{q}}) \quad \text{for all } \mathfrak{p},\mathfrak{q} \text{ in } \mathfrak{G}.$$

Proof. From Definition 8, the functor $(-)_{\mathfrak{e}_{\mathfrak{G}}}$ sends $((K)^R)^{\mathfrak{p}}$ into $K^{\mathfrak{p}}$, for any $R_{\mathfrak{e}_{\mathfrak{G}}}$-module K. We know from Proposition 2 that $Hom_R(V,W)_{\mathfrak{q}}$ is the additive subgroup $\mathfrak{G}\text{-}GrR\text{-}Hom_R(V^{\mathfrak{p}}, W^{\mathfrak{p}*\mathfrak{q}})$ of $Hom_R(V,W)$, for any $\mathfrak{p},\mathfrak{q}$ in $\mathfrak{G}$. Moreover, we know from Corollary 1 that the functors $(-)_{\mathfrak{e}_{\mathfrak{G}}}$ and $(-)^R$ form $\mathfrak{G}\text{-}GrR\text{-}Mod \approx R_{\mathfrak{e}_{\mathfrak{G}}}\text{-}Mod$. Thus, we conclude that the restriction to $K^{\mathfrak{p}} = (K^R)_{\mathfrak{p}}$ is an isomorphism of the additive group $Hom_R(K^R, L^R)_{\mathfrak{q}}$ onto $Hom_{R_{\mathfrak{e}_{\mathfrak{G}}}}(K^{\mathfrak{p}}, L^{\mathfrak{p}*\mathfrak{q}})$ for all $\mathfrak{p},\mathfrak{q}$ in $\mathfrak{G}$. □

Example 3. *Let $\mathfrak{X} = D_6 = \{1, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\}$ and*

$$\mathfrak{H} = \{1, a^2, a^4, b, a^2b, a^4b\}$$

be a subgroup of $\mathfrak{X}$. Choose the set of left coset representatives to be $\mathfrak{G} = \{1, a\}$. Then, the $*$ is as given in the following table (Table 3).

Table 3. The binary operation $*$.

$*$	1	a
1	1	a
a	a	1

Consider the ring of polynomials with integer coefficients $R = \mathbb{Z}[x]$ to be graded by $\mathfrak{G} = \{1, a\}$. In this case, the grading is determined by the degree of the polynomial. The subgroups R_1 and R_a correspond to polynomials with even and odd degrees, respectively:

$$R_1 = \{c_0 + c_2x^2 + c_4x^4 + \ldots | c_i \in \mathbb{Z} \text{ for all even } i\},$$
$$R_a = \{c_1x + c_3x^3 + c_5x^5 + \ldots | c_i \in \mathbb{Z} \text{ for all odd } i\}.$$

Hence, R is a $\mathfrak{G}$-weak graded ring. Here, let V be defined as the module of polynomials; thus, $V = V_1 \oplus V_a$, where V_1 is defined as the module of polynomials with even degrees:

$$V_1 = \{c_0 + c_2x^2 + c_4x^4 + \ldots | c_i \in \mathbb{Z} \text{ for all even } i\}$$

and V_a is defined as the module of polynomials with odd degrees:

$$V_a = \{c_1x + c_3x^3 + c_5x^5 + \ldots | c_i \in \mathbb{Z} \text{ for all odd } i\}.$$

Hence, $V \in \mathfrak{G}\text{-}GrR\text{-}Mod$. Similarly, let W be the module of polynomials; thus, $W = W_1 \oplus W_a$, where W_1 is defined as the module of polynomials with even degrees:

$$W_1 = \{d_0 + d_2x^2 + d_4x^4 + \ldots | d_i \in \mathbb{Z} \text{ for all even } i\}$$

and W_a is defined as the module of polynomials with odd degrees:

$$W_a = \{d_1x + d_3x^3 + d_5x^5 + \dots | d_i \in \mathbb{Z} \text{ for all odd } i\}.$$

Then, $W \in \mathfrak{G}$-GrR-Mod. Here, we can define a morphism $\varphi : V \to W$ as follows:

$$\varphi(c_0 + c_2x^2 + c_4x^4 + \dots) = d_2x^2 + d_4x^4 + \dots \subseteq W_1,$$

and

$$\varphi(c_1x + c_3x^3 + c_5x^4 + \dots) = d_3x^3 + d_5x^5 + \dots \subseteq W_a,$$

where $c_i = d_{i+2}$ for all i. Hence, $\varphi \in \mathfrak{G}$-GrHom$_R(V, W)$ since $\varphi(V_1) \subseteq W_1$ and $\varphi(V_a) \subseteq W_a$. Thus, $\varphi(V_\mathfrak{g}) \subseteq W_\mathfrak{g}$ for all $v \in V_\mathfrak{g}$ and $\mathfrak{g} \in \mathfrak{G}$. Note that, since $\mathfrak{G}$ is a finite set, according to Theorem 6, we have $\mathfrak{G}$-GrHom$_R(V, W) =$ Hom$_R(V, W)$.

5. Conclusions

In this work, it was shown that many results in the literature concerning group-graded rings and group-graded modules can be generalized and proven using the new concepts of $\mathfrak{G}$-weak graded rings and $\mathfrak{G}$-weak graded modules. Moreover, this generalization may form a bridge between the classical group theory and the theory of quantum groups. Using these new concepts, interested readers can study several properties of group- or semigroup-graded rings and modules in the literature, such as simplicity and semi-simplicity.

Author Contributions: Conceptualization, M.A.-S.; Software, R.A.-O.; Validation, M.A.-S.; Formal analysis, R.A.-O.; Investigation, R.A.-O.; Resources, R.A.-O.; Data curation, R.A.-O.; Writing—original draft, R.A.-O.; Writing—review & editing, M.A.-S.; Visualization, M.A.-S.; Supervision, M.A.-S.; Funding acquisition, R.A.-O. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The original contributions presented in the study are included in the article, further inquiries can be directed to the corresponding author.

Acknowledgments: The authors would like to express their gratitude to the academic editor and the anonymous reviewers for their valuable remarks and suggestions that helped to improve this work.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Dade, E.C. Group graded rings and modules. *Math. Z.* **1980**, *174*, 241–262. [CrossRef]
2. Dascalescu, S.; Kelarev, A.V.; Van Wyk, L. Semigroup gradings of full matrix rings. *Comm. Algebra* **2001**, *29*, 5023–5031. [CrossRef]
3. Nystedt, P.; Oinert, J. Simple semigroup graded rings. *J. Algebra Appl.* **2015**, *14*, 1–10. [CrossRef]
4. Lezama, O.; Latorre, E. Non-commutative algebraic geometry of semigraded rings. *Int. J. Algebra Comput.* **2017**, *27*, 361–389. [CrossRef]
5. Cohen, M.; Montgomery, S. Group-graded rings, smash product and group actions. *Trans. Amer. Math. Soc.* **1984**, *282*, 237–258. [CrossRef]
6. Năstăsescu, C.; Bergh, M.; Oystaeyen, F. Separable functor, applications to graded rings and modules. *J. Algebra* **1989**, *123*, 397–413. [CrossRef]
7. Rafael, M.; Oinert, J. Separable functors revisited. *Comm. Algebra* **1990**, *18*, 144–1459. [CrossRef]
8. Beggs, E.J. Making non-trivially associated tensor categories from left coset representatives. *J. Pure Appl. Algebra* **2003**, *177*, 5–41. [CrossRef]
9. Al-Shomrani, M.M.; Beggs, E.J. Making nontrivially associated modular categories from finite groups. *Int. J. Math. Math. Sci.* **2004**, *2004*, 2231–2264. [CrossRef]
10. Al-Shomrani, M.M. A construction of graded rings using a set of left coset representatives. *JP J. Algebra Number T.* **2012**, *25*, 105–112.
11. Al-Shomrani, M.M.; Al-Subaie, N. A Generalization of Group-Graded Modules. *Symmetry* **2022**, *14*, 835. [CrossRef]

12. Cohen, M.; Westreich, S. Solvability for semisimple Hopf algebras via integrals. *J. Algebra* **2017**, *472*, 67–94. [CrossRef]
13. Alyoubi, B.M.; Al-Shomrani, M.M. The groups of prime power order and the structure of hopf algebras. *JP J. Algebra Number T.* **2021**, *51*, 145–182. [CrossRef]

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