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Mathematical Analysis of the Poiseuille Flow of a Fluid with Temperature-Dependent Properties

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Abstract: This article is devoted to the mathematical analysis of a heat and mass transfer model for the pressure-induced flow of a viscous fluid through a plane channel subject to Navier's slip conditions on the channel walls. The important feature of our work is that the used model takes into account the effects of variable viscosity, thermal conductivity, and slip length, under the assumption that these quantities depend on temperature. Therefore, we arrive at a boundary value problem for strongly nonlinear ordinary differential equations. The existence and uniqueness of a solution to this problem is analyzed. Namely, using the Galerkin scheme, the generalized Borsuk theorem, and the compactness method, we proved the existence theorem for both weak and strong solutions in Sobolev spaces and derive some of their properties. Under extra conditions on the model data, the uniqueness of a solution is established. Moreover, we considered our model subject to some explicit formulae for temperature dependence of viscosity, which are applied in practice, and constructed corresponding exact solutions. Using these solutions, we successfully performed an extra verification of the algorithm for finding solutions that was applied by us to prove the existence theorem.

Keywords: heat and mass transfer model; viscous fluid; Poiseuille flow; Navier slip condition; nonlinear boundary value problem; variable coefficients; weak and strong solutions; exact solutions; Galerkin method; existence and uniqueness theorem.

MSC: 76D03; 34A34; 34A05



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1. Introduction

Flow and heat transfer in a channel, a tube, or a pipeline, containing a non-uniformly heated moving fluid or a gas, is one of the most common situations encountered in engineering practice. This paper deals with the nonlinear heat and mass transfer model describing the steady unidirectional pressure-driven flow of an incompressible viscous fluid in the channel between the two fixed plates $y = \pm h$ under Navier-type slip (in other terminology, “free slip”) boundary conditions and Robin's conditions for the heat flux on the channel walls:

$$\left\{ \begin{array}{l} -(\mu(\theta)u')' = \xi \quad \text{in } (-h, h), \\ -(k(\theta)\theta')' = \omega \quad \text{in } (-h, h), \\ (\mu(\theta)u' + \lambda(\theta)u)|_{y=h} = 0, \\ (\mu(\theta)u' - \lambda(\theta)u)|_{y=-h} = 0, \\ (k(\theta)\theta' + \beta\theta)|_{y=h} = 0, \\ (k(\theta)\theta' - \beta\theta)|_{y=-h} = 0, \end{array} \right. \quad (1)$$

where

- $u = u(y)$ is the velocity component in the x -direction;
- $\theta = \theta(y)$ is the deviation from the reference temperature;
- $-\zeta$ is the pressure gradient along the x -direction ($\zeta > 0$), that is, $\partial p / \partial x = -\zeta$;
- $\mu = \mu(\theta)$ is the viscosity coefficient, $\mu(\theta) > 0$;
- $k = k(\theta)$ is the thermal conductivity coefficient, $k(\theta) > 0$;
- $\omega = \omega(y)$ is the heat source intensity;
- β is the heat transfer coefficient, $\beta > 0$;
- $\lambda = \lambda(\theta)$ is the friction coefficient, $\lambda(\theta) > 0$;
- the symbol $'$ denotes the differentiation with respect to y .

The flow configuration is shown in Figure 1. In the literature, this kind of fluid flow is commonly referred to as the “plane Poiseuille flow”.

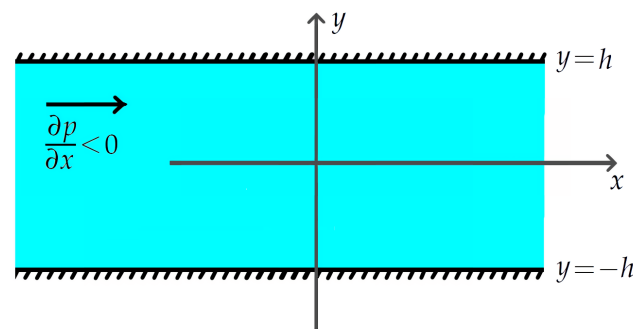


Figure 1. Fluid flow between the fixed plates $y = \pm h$ driven by a given pressure gradient $\partial p / \partial x$.

When studying model (1), we suppose that $u: [-h, h] \rightarrow \mathbb{R}$ and $\theta: [-h, h] \rightarrow \mathbb{R}$ are unknown functions, while all the other functions and constants are given.

For the sake of simplicity, in the following it will be assumed that the function $\omega: [-h, h] \rightarrow \mathbb{R}$, describing the heat source intensity, is even. Thus, the model under consideration has the symmetry property with respect to the plane $y = 0$.

The main feature of the present paper is that we consider the heat and mass transfer model with variable coefficients, assuming that the viscosity μ , the thermal conductivity k , and the slip length $1/\lambda$ depend on the temperature θ , and these dependencies are formulated in the most general form. In spite of the simplicity of formulation, problem (1) is challenging from a point of view of applying both analytical and numerical methods to it. The presence of variable coefficients in the model significantly complicates its theoretical study and requires the development of a special mathematical apparatus for constructing solutions of the corresponding boundary value problem (BVP). To overcome related difficulties, we will use some of the following authors' findings and approaches [1–3].

Note that taking into account the dependence of physical properties of a fluid on temperature is critical for many applications. For example, it is important for engine lubricants, which should perform well under varying temperature conditions, cooling and heating systems, food processing, medical and biological sectors, pollution control, as well as wastewater treatment. Also, in studying many natural phenomena, it is needed to consider temperature-dependent properties of fluids. For example, for geysers and volcanic eruptions, the radius of impact on the environment of their plume is significantly influenced by temperature-dependent variable viscosity. For more detail, we refer to the papers [4–7].

Another feature of our work is that we use a nonlinear temperature-dependent Navier's slip condition on the channel walls (see the pioneering work [8]) instead of the standard no-slip condition. Mathematically speaking, the zero Dirichlet condition for the velocity field is replaced by a nonlinear Robin condition. The motivation of this lies in covering situations in which fluids exhibit phenomena inconsistent with the assumption of no-slip [9]. In this regard, we can mention the cases when boundaries are rough [10], for

instance, the skin of sharks or golf balls [11,12]. One more application of interest can also be found in drag control of aircraft wings; in order to reduce the drag, small injection jets are introduced over wings of planes [13]. In such cases, it is really important to introduce other interfacial conditions to adequately predict the behavior of a fluid on boundaries of a flow domain. A short survey of popular slip boundary conditions, among which the Navier condition occupies one of the central places, is given in [14].

The main aim of this paper is to give a comprehensive mathematical study of problem (1), primarily focusing on the existence and uniqueness theorems (for both the weak and strong formulations of this BVP) and algorithms for constructing solutions. We also establish some qualitative and quantitative properties of solutions. Moreover, considering our model under some (physically meaningful) analytical formulae for temperature dependence of viscosity, we construct corresponding solutions in an explicit form. These solutions favor a better understanding of the qualitative features of fluid flows with variable physical properties and can be used for verification of relevant approximate analytical and numerical methods.

It should be mentioned here that non-isothermal viscous flows through conduits having simple cross-section geometry with temperature-dependent viscosity has been studied by many authors in various aspects for both Newtonian and non-Newtonian fluids; see, for example, the papers [15–29] and the references therein. In most of these works, it is assumed that the fluid viscosity μ is given by Reynold’s model [30] (also known as the Nahme-type law [31]):

$$\mu(\theta) = \mu_1 \exp(-B\theta) + \mu_2, \quad \mu_1 > 0, \mu_2 \geq 0, B > 0,$$

or by the power law equation [32]:

$$\mu(\theta) = \frac{\mu_0}{(1 + A\theta)^\gamma}, \quad \mu_0 > 0, \gamma > 0, A > 0.$$

In the above relations, $A, B, \mu_0, \mu_1, \mu_2$, and γ are substance-dependent constants. Sometimes the temperature-dependent viscosity $\mu(\theta)$ and thermal conductivity $k(\theta)$ are organized as a power series [33].

In the framework of the Stokes approximation, an unsteady flow problem for a non-uniformly heated fluid with constant viscosity under specified inflow–outflow regimes on parts of the boundary of the flow region was considered in [34]. The articles [35–38] and [39–41] are devoted to the study of BVPs for the generalized Boussinesq system with variable coefficients under the zero and non-zero Dirichlet boundary conditions for the velocity field, respectively. The initial-boundary value problem modeling an unsteady unidirectional convective flow of incompressible fluids in a vertical heat exchanger having an arbitrary cross section was studied in [42]. The paper [43] deals with a boundary control problem for the model describing the motion of a viscous heat conduction fluid under slip boundary conditions. Some models for complex heat transfer were investigated in the series of works [44–47]. The “complete” Boussinesq system involving the viscous dissipation, which is represented in the heat equation by the quadratic Rayleigh function, has been recently studied in [48–52]. We also mention the works [53–58], which are within the framework of the analytical approach to solving non-standard problems of hydrodynamics.

After this introduction, our article is organized as follows. In Section 2, we introduce the notations and needed function spaces, along with some important preliminary results such as the generalized Borsuk theorem (see Proposition 1), which will be used in proving the solvability of problem (1). Moving on to Section 3, we give both the weak and strong formulations of our BVP in suitable subspaces of Sobolev spaces. Section 4 contains the main results of this work—Theorems 1–3 about the existence, uniqueness, and properties of solutions to problem (1). These theorems are proved in Section 5. Finally, in Section 6, we construct some solutions in the closed form and, using them, perform an extra verification of the algorithm of finding solutions, developed by us to prove Theorem 1.

2. Preliminaries

2.1. Notations and Function Spaces

Let E and F be real Banach spaces. By $\mathcal{L}(E, F)$ denote the space of continuous linear mappings from E into F . This space is equipped with the operator norm

$$\|A\|_{\mathcal{L}(E, F)} \stackrel{\text{def}}{=} \sup_{\|w\|_E \neq 0} \frac{\|Aw\|_F}{\|w\|_E}.$$

The symbol $\hookrightarrow$ denotes a continuous embedding, while $\hookrightarrow\hookrightarrow$ stands for a compact embedding.

The symbol $\rightharpoonup$ ($\rightarrow$, $\rightrightarrows$, resp.) denotes weak (strong, uniform, resp.) convergence.

Let $h \in (0, \infty)$. For a function $v: [-h, h] \rightarrow \mathbb{R}$, by $\bar{v}_h$ denote the mean of v , that is,

$$\bar{v}_h \stackrel{\text{def}}{=} \frac{1}{2h} \int_{-h}^h v(y) \, dy.$$

By $\mathcal{D}(-h, h)$ denote the space of C^∞ -smooth real-valued functions with compact support contained in the open interval $(-h, h)$.

For any $\alpha \in (0, 1]$, $p, q \in [1, \infty]$ and $m, n \in \mathbb{N} \cup \{0\}$, $L^p[-h, h]$ ($W^{m, q}[-h, h]$, $C^{n, \alpha}[-h, h]$, resp.) denotes the Lebesgue (Sobolev, Hölder, resp.) space of real-valued functions defined on $[-h, h]$. Moreover, we will use the standard notation

$$H^m[-h, h] \stackrel{\text{def}}{=} W^{m, 2}[-h, h].$$

By $C[-h, h]$ ($AC[-h, h]$, resp.) denote the space of continuous (absolutely continuous, resp.) functions on $[-h, h]$. Detailed descriptions of properties of these spaces can be found in the books [59–61].

We define a special scalar product in the Sobolev space $H^1[-h, h]$ by the formula:

$$(v, u)_{H^1[-h, h]} \stackrel{\text{def}}{=} \int_{-h}^h v'(y)u'(y) \, dy + v(h)u(h) + v(-h)u(-h).$$

The expediency of using this scalar product in solving our BVP will become clear in the following.

Note that any function belonging to the space $W^{1, 1}[-h, h]$ is absolutely continuous and, moreover, the equality

$$W^{1, 1}[-h, h] = AC[-h, h]$$

holds. Therefore, we can naturally determine values of functions from $W^{1, 1}[-h, h]$ at the endpoints $y = -h$ and $y = h$.

The following statement is useful in studying strong solutions.

Lemma 1. *If $v \in W^{2, 1}[-h, h]$, then $v \in AC[-h, h]$ and $v' \in AC[-h, h]$.*

Further, let us formulate some results on the embedding of the space $W^{1, p}[-h, h]$ into Hölder's spaces.

Lemma 2. *Suppose $p \in (1, \infty)$, then*

- $W^{1, p}[-h, h] \hookrightarrow C^{0, \alpha}[-h, h]$, for $\alpha \in (0, 1 - 1/p]$;
- $W^{1, p}[-h, h] \hookrightarrow\hookrightarrow C^{0, \alpha}[-h, h]$, for $\alpha \in (0, 1 - 1/p)$.

This lemma is a particular case of the Sobolev embedding theorem (see, for instance, [62], Chapter III, § 2.8).

Corollary 1. *Let $v_0 \in H^1[-h, h]$ and $\{v_n\}_{n=1}^\infty \subset H^1[-h, h]$. If $v_n \rightharpoonup v_0$ in $H^1[-h, h]$ as $n \rightarrow \infty$, then $v_n \rightrightarrows v_0$ as $n \rightarrow \infty$.*

For studying fluid flows that exhibit the symmetry property with respect to the plane $y = 0$, we introduce the following spaces:

$$W_{even}^{m,q}[-h, h] \stackrel{\text{def}}{=} \{v \in W^{m,q}[-h, h] : v(-y) = v(y), \forall y \in [-h, h]\},$$

$$H_{even}^m[-h, h] \stackrel{\text{def}}{=} \{v \in H^m[-h, h] : v(-y) = v(y), \forall y \in [-h, h]\}.$$

Moreover, we also will use three subspaces of the Sobolev space $H^1[-h, h]$:

$$K[-h, h] \stackrel{\text{def}}{=} \{v \in H^1[-h, h] : v = \text{const}\},$$

$$H_0^1[-h, h] \stackrel{\text{def}}{=} \{v \in H^1[-h, h] : v(-h) = v(h) = 0\},$$

$$H_{0,even}^1[-h, h] \stackrel{\text{def}}{=} H_0^1[-h, h] \cap H_{even}^1[-h, h],$$

which are Hilbert spaces with respect to the scalar product that is induced from $H^1[-h, h]$.

It is easily shown that the following decomposition takes place:

$$H_{even}^1[-h, h] = H_{0,even}^1[-h, h] \oplus K[-h, h], \quad (2)$$

where the symbol $\oplus$ denotes the orthogonal sum.

Consider the sequence $\{\psi_j\}_{j=1}^\infty$, where

$$\psi_j : [-h, h] \rightarrow \mathbb{R}, \quad \psi_j(y) \stackrel{\text{def}}{=} \frac{2\sqrt{h}}{\pi(2j-1)} \sin\left(\frac{(2j-1)\pi}{2h}(y+h)\right), \quad \forall y \in [-h, h]. \quad (3)$$

This sequence is an orthonormal basis of the space $H_{0,even}^1[-h, h]$.

Lastly, let $\psi_0 \equiv 1/\sqrt{2}$. Taking into account the decomposition (2), one can easily see that the sequence $\{\psi_j\}_{j=0}^\infty$ is an orthonormal basis of the space $H_{even}^1[-h, h]$.

2.2. Embedding Operator

Lemma 3. Let $I : H_{even}^1[-h, h] \rightarrow C[-h, h]$ be the embedding operator. Then

$$\|I\|_{\mathcal{L}(H_{even}^1[-h, h], C[-h, h])} \leq \max\{1, 2\sqrt{h}\}. \quad (4)$$

Proof. Suppose v is an arbitrary function from the space $H_{even}^1[-h, h]$. Using the following representation

$$v(\tau) = v(-h) + \int_{-h}^{\tau} v'(s) \, ds, \quad \forall \tau \in [-h, h],$$

which holds due to the Newton–Leibniz formula for absolutely continuous functions, we derive

$$\begin{aligned} \|v\|_{C[-h, h]}^2 &= \max_{\tau \in [-h, h]} |v(\tau)|^2 \\ &= \max_{\tau \in [-h, h]} \left| \underbrace{v(-h)}_{=:a} + \underbrace{\int_{-h}^{\tau} v'(s) \, ds}_{=:b} \right|^2. \end{aligned} \quad (5)$$

Further, using the well-known equality

$$(a+b)^2 \leq 2a^2 + 2b^2, \quad \forall a, b \in \mathbb{R},$$

we derive from equality (5) that

$$\|v\|_{C[-h, h]}^2 \leq 2v^2(-h) + 2 \max_{\tau \in [-h, h]} \left(\int_{-h}^{\tau} v'(s) \, ds \right)^2. \quad (6)$$

Moreover, applying Hölder's inequality, we obtain

$$\begin{aligned}
 \left| \int_{-h}^{\tau} v'(s) \, ds \right| &\leq \int_{-h}^{\tau} |v'(s)| \, ds \\
 &\leq \left(\int_{-h}^{\tau} 1 \, ds \right)^{1/2} \left(\int_{-h}^{\tau} |v'(s)|^2 \, ds \right)^{1/2} \\
 &= (\tau + h)^{1/2} \left(\int_{-h}^{\tau} |v'(s)|^2 \, ds \right)^{1/2} \\
 &\leq \sqrt{2h} \left(\int_{-h}^h |v'(s)|^2 \, ds \right)^{1/2}, \quad \forall \tau \in [-h, h].
 \end{aligned} \tag{7}$$

Combining inequalities (6) and (7), one can obtain

$$\begin{aligned}
 \|v\|_{C[-h,h]}^2 &\leq 2v^2(-h) + 2 \max_{\tau \in [-h,h]} \left(\int_{-h}^{\tau} v'(s) \, ds \right)^2 \\
 &\leq 2v^2(-h) + 4h \int_{-h}^h |v'(s)|^2 \, ds.
 \end{aligned}$$

Since

$$\begin{aligned}
 &2v^2(-h) + 4h \int_{-h}^h |v'(s)|^2 \, ds \\
 &\leq \max\{1, 4h\} \left(2v^2(-h) + \int_{-h}^h |v'(s)|^2 \, ds \right) \\
 &= \max\{1, 4h\} \|v\|_{H_{even}^1[-h,h]}^2,
 \end{aligned}$$

we see that

$$\|v\|_{C[-h,h]}^2 \leq \max\{1, 4h\} \|v\|_{H_{even}^1[-h,h]}^2,$$

or equivalently,

$$\|v\|_{C[-h,h]} \leq \max\{1, 2\sqrt{h}\} \|v\|_{H_{even}^1[-h,h]}.$$

This yields the required estimate (4) for the norm of the embedding operator I . Thus, Lemma 3 is proved. $\square$

2.3. Abstract Result on Solvability of One-Parameter Family of Equations

When proving the well-posedness of problem (1) by Galerkin's scheme, we will need the following proposition.

Proposition 1 (Generalized Borsuk's theorem). *Let ρ be a positive number and let*

$$\mathfrak{B}_\rho \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n : |x| < \rho\}.$$

Suppose $\{\mathbf{f}_t : \mathbb{R}^n \rightarrow \mathbb{R}^n\}_{t \in [0,1]}$ is a family of mappings such that

- *the mapping $\mathbf{f} : \mathbb{R}^n \times [0, 1] \rightarrow \mathbb{R}^n$, $\mathbf{f}(x, t) \stackrel{\text{def}}{=} \mathbf{f}_t(x)$ is continuous;*
- *$\mathbf{f}_t(x) \neq \mathbf{0}$ for any $(x, t) \in \partial \mathfrak{B}_\rho \times [0, 1]$;*
- *the mapping $\mathbf{f}_0 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is linear.*

Then, for any $t \in [0, 1]$, the equation $\mathbf{f}_t(x) = \mathbf{0}$ has at least one solution in the ball $\mathfrak{B}_\rho$. In particular, the equation $\mathbf{f}_1(x) = \mathbf{0}$ is solvable in this ball.

This abstract result can be proved by applying methods of the theory of topological degree [63,64].

3. Weak and Strong Solutions

In the sequel, we assume that the functions μ , λ , k , ω satisfy the following four conditions:

- (H.1) the functions $\mu: \mathbb{R} \rightarrow \mathbb{R}$, $\lambda: \mathbb{R} \rightarrow \mathbb{R}$, and $k: \mathbb{R} \rightarrow \mathbb{R}$ are continuous;
- (H.2) the strict inequalities $0 < \mu(s)$ and $0 < \lambda(s)$ hold for any $s \in \mathbb{R}$;
- (H.3) there exists a constant $k_{\min}$ such that $0 < k_{\min} \leq k(s)$ for any $s \in \mathbb{R}$;
- (H.4) the function $\omega: [-h, h] \rightarrow \mathbb{R}$ is even and belongs to the space $L^1[-h, h]$.

Definition 1 (Weak solution). We shall say that a pair (u, θ) is a weak solution to problem (1) if

- $(u, \theta) \in H_{\text{even}}^1[-h, h] \times H_{\text{even}}^1[-h, h]$;
- for any test functions $v \in H_{\text{even}}^1[-h, h]$ and $\eta \in H_{\text{even}}^1[-h, h]$, the following equalities hold:

$$\int_{-h}^h \mu(\theta(y)) u'(y) v'(y) dy + 2\lambda(\theta(h)) u(h) v(h) = \xi \int_{-h}^h v(y) dy, \quad (8)$$

$$\int_{-h}^h k(\theta(y)) \theta'(y) \eta'(y) dy + 2\beta\theta(h) \eta(h) = \int_{-h}^h \omega(y) \eta(y) dy. \quad (9)$$

It can easily be checked that any classical solution to problem (1) is a weak solution. On the other hand, if (u, θ) is a weak solution and the functions u , θ , μ , k are sufficiently smooth, then (u, θ) is a classical solution to this problem.

Definition 2 (Strong solution). We shall say that a pair (u, θ) is a strong solution to problem (1) if the following conditions hold:

- $(u, \theta) \in W_{\text{even}}^{2,1}[-h, h] \times W_{\text{even}}^{2,1}[-h, h]$;
- the equality $-(\mu(\theta)u')' = \xi$ holds almost everywhere in $(-h, h)$;
- the equality $-(k(\theta)\theta')' = \omega$ holds almost everywhere in $(-h, h)$;
- the four boundary conditions are true:

$$\mu(\theta(h))u'(h) + \lambda(\theta(h))u(h) = 0,$$

$$\mu(\theta(-h))u'(-h) - \lambda(\theta(-h))u(-h) = 0,$$

$$k(\theta(h))\theta'(h) + \beta\theta(h) = 0,$$

$$k(\theta(-h))\theta'(-h) - \beta\theta(-h) = 0.$$

Taking into account Lemma 1, we see that $u, u', \theta, \theta' \in AC[-h, h]$ in the last definition. In particular, this means that values of these functions in the endpoints of $[-h, h]$ are well defined. Therefore, the above boundary conditions, which contain these functions, are meaningful.

If (u, θ) is a strong solution of problem (1), then this solution is weak, but the converse statement is not true. However, under extra smoothness assumptions on the functions μ and k , a weak solution becomes a strong solution.

4. Main Results

4.1. Well-Posedness of BVP

In the following two theorems, we collect our results about the existence, uniqueness, and regularity of a solution to the BVP under consideration.

Theorem 1 (Existence and regularity). Suppose that conditions (H.1)–(H.4) hold. Then:

- (a) problem (1) has at least one weak solution;
- (b) if (u, θ) is a weak solution to problem (1) and the functions μ and k are of class C^1 , then (u, θ) is a strong solution of this problem.

Theorem 2 (Uniqueness). *If, in addition to conditions (H.1)–(H.4), we require that the function k satisfies the Lipschitz condition*

$$|k(s_1) - k(s_2)| \leq \ell_k |s_1 - s_2|, \quad \forall s_1, s_2 \in \mathbb{R}, \quad (10)$$

with a positive constant ℓ_k such that

$$\ell_k < \frac{\min\{k_{\min}^2, \beta^2\}}{\|\omega\|_{L^1[-h,h]} \max\{1, 4h\}}, \quad (11)$$

then problem (1) admits a unique weak solution.

4.2. Boundary Values of Solutions, Energy Equalities, and Extrema

The next theorem summarizes our results related to studying properties of weak and strong solutions to system (1).

Theorem 3 (Properties of solutions). *Under the conditions of Theorem 1, the following statements hold.*

(i) *If (u, θ) is a weak solution to problem (1), then*

$$u(h) = u(-h) = \frac{\xi h}{\lambda(\frac{\bar{\omega}_h h}{\beta})}, \quad (12)$$

$$\theta(h) = \theta(-h) = \frac{\bar{\omega}_h h}{\beta}. \quad (13)$$

(ii) *If (u, θ) is a weak solution to problem (1), then the two energy equalities hold:*

$$\int_{-h}^h \mu(\theta(y)) |u'(y)|^2 dy + 2\lambda(\theta(h)) u^2(h) = 2h \xi \bar{u}_h, \quad (14)$$

$$\int_{-h}^h k(\theta(y)) |\theta'(y)|^2 dy + 2\beta \theta^2(h) = \int_{-h}^h \omega(y) \theta(y) dy. \quad (15)$$

(iii) *If (u, θ) is a strong solution to problem (1), then*

$$\begin{aligned} \arg \max_{y \in [-h, h]} u(y) &= \{0\}, \\ \max_{y \in [-h, h]} u(y) &= \frac{\xi h}{\lambda(\frac{\bar{\omega}_h h}{\beta})} + \xi \int_0^h \frac{y}{\mu(\theta(y))} dy. \end{aligned} \quad (16)$$

(iv) *If (u, θ) is a strong solution to problem (1) and there exists a function $\varsigma \in AC[-h, h]$ such that $\varsigma' = \omega$ a.e. in $(-h, h)$ and*

$$\begin{aligned} \varsigma(y) &> 0 \quad \text{if } y \in (h_0 - \epsilon, h_0), \\ \varsigma(y) &= 0 \quad \text{if } y = h_0, \\ \varsigma(y) &< 0 \quad \text{if } y \in (h_0, h_0 + \epsilon), \end{aligned}$$

or

$$\begin{aligned} \varsigma(y) &< 0 \quad \text{if } y \in (h_0 - \epsilon, h_0), \\ \varsigma(y) &= 0 \quad \text{if } y = h_0, \\ \varsigma(y) &> 0 \quad \text{if } y \in (h_0, h_0 + \epsilon), \end{aligned}$$

for some $h_0 \in (-h, h)$ and $\epsilon > 0$, then h_0 is a strict local extrema point of the function θ .

5. Proof of Main Results

5.1. Proof of Theorem 1

We will prove the existence result (a) by the Galerkin method, using Proposition 1 and compactness arguments for an appropriate limit passage. For convenience, our proof is divided into three steps.

Step 1: Galerkin's approximation. Let m be an arbitrary integer. Galerkin's approximate solutions will be sought in the form

$$(u_m, \theta_m) \stackrel{\text{def}}{=} \left(\sum_{j=0}^m a_{mj} \psi_j(y), \sum_{j=0}^m b_{mj} \psi_j(y) \right), \quad \forall y \in [-h, h],$$

where ψ_j is defined in (3), while a_{mj} and b_{mj} are unknown coefficients. To determine these coefficients, we consider a finite-dimensional problem with the parameter $t \in [0, 1]$:

Find a vector $\gamma_m = (a_{m0}, a_{m1}, \dots, a_{mm}, b_{m0}, b_{m1}, \dots, b_{mm})^\top \in \mathbb{R}^{2m+2}$ such that the functions u_m and θ_m satisfy the following equalities:

$$\begin{aligned} \int_{-h}^h \mu(t\theta_m(y)) u'_m(y) \psi'_j(y) dy + 2\lambda(t\theta_m(h)) u_m(h) \psi_j(h) \\ = \xi \int_{-h}^h \psi_j(y) dy, \quad j = 0, 1, \dots, m, \end{aligned} \quad (17)$$

and

$$\begin{aligned} \int_{-h}^h k(t\theta_m(y)) \theta'_m(y) \psi'_j(y) dy + 2\beta \theta_m(h) \psi_j(h) \\ = \int_{-h}^h \omega(y) \psi_j(y) dy, \quad j = 0, 1, \dots, m. \end{aligned} \quad (18)$$

Step 2: A priori estimates of Galerkin's solutions. Let us assume that γ_m satisfies relations (17) and (18) for some $t \in [0, 1]$. Taking into account that $\{\psi_j\}_{j=0}^\infty$ is an orthonormal basis of the space $H_{\text{even}}^1[-h, h]$, we obtain

$$\|\gamma_m\|_{\mathbb{R}^{2m+2}}^2 = \sum_{j=0}^m a_{mj}^2 + \sum_{j=0}^m b_{mj}^2 = \|u_m\|_{H_{\text{even}}^1[-h, h]}^2 + \|\theta_m\|_{H_{\text{even}}^1[-h, h]}^2. \quad (19)$$

Further, we multiply j th equality of system (18) by b_{mj} and sum up the results over j from 0 to m . This gives

$$\int_{-h}^h k(t\theta_m(y)) |\theta'_m(y)|^2 dy + 2\beta |\theta_m(h)|^2 = \int_{-h}^h \omega(y) \theta_m(y) dy.$$

Using the last relation, we derive

$$\begin{aligned} \min\{k_{\min}, \beta\} \|\theta_m\|_{H_{\text{even}}^1[-h, h]}^2 &= \min\{k_{\min}, \beta\} \left(\int_{-h}^h |\theta'_m(y)|^2 dy + 2|\theta_m(h)|^2 \right) \\ &\leq \int_{-h}^h k(t\theta_m(y)) |\theta'_m(y)|^2 dy + 2\beta |\theta_m(h)|^2 \\ &= \int_{-h}^h \omega(y) \theta_m(y) dy \\ &\leq \int_{-h}^h |\omega(y)| dy \max_{s \in [-h, h]} |\theta_m(s)| \\ &= \|\omega\|_{L^1[-h, h]} \|\theta_m\|_{C[-h, h]} \\ &\leq \|\omega\|_{L^1[-h, h]} \|I\|_{\mathcal{L}(H_{\text{even}}^1[-h, h], C[-h, h])} \|\theta_m\|_{H_{\text{even}}^1[-h, h]}, \end{aligned}$$

whence

$$\|\theta_m\|_{H_{\text{even}}^1[-h,h]} \leq \frac{\|\omega\|_{L^1[-h,h]} \|I\|_{\mathcal{L}(H_{\text{even}}^1[-h,h], C[-h,h])}}{\min\{k_{\min}, \beta\}}. \quad (20)$$

Moreover, taking into account the estimate

$$\|\vartheta\|_{C[-h,h]} \leq \|I\|_{\mathcal{L}(H_{\text{even}}^1[-h,h], C[-h,h])} \|\vartheta\|_{H_{\text{even}}^1[-h,h]}, \quad \forall \vartheta \in H_{\text{even}}^1[-h,h],$$

and inequality (20), we obtain

$$\|\theta_m\|_{C[-h,h]} \leq \frac{\|\omega\|_{L^1[-h,h]} \|I\|_{\mathcal{L}(H_{\text{even}}^1[-h,h], C[-h,h])}^2}{\min\{k_{\min}, \beta\}}. \quad (21)$$

Next, we multiply j th equality of system (17) by a_{mj} and sum up the results over j from 0 to m . As a result, we arrive at the following equality

$$\int_{-h}^h \mu(t\theta_m(y)) |u'_m(y)|^2 dy + 2\lambda(t\theta_m(h)) u_m^2(h) = \xi \int_{-h}^h u_m(y) dy. \quad (22)$$

In view of condition (H.2), for any $r > 0$, we have

$$\mu_r \stackrel{\text{def}}{=} \min_{s \in [-r,r]} \mu(s) > 0, \quad \lambda_r \stackrel{\text{def}}{=} \min_{s \in [-r,r]} \lambda(s) > 0. \quad (23)$$

Using relations (21)–(23), we derive

$$\begin{aligned} \min\{\mu_r, \lambda_r\} \|u_m\|_{H_{\text{even}}^1[-h,h]}^2 &\leq \int_{-h}^h \mu(t\theta_m(y)) |u'_m(y)|^2 dy + 2\lambda(t\theta_m(h)) u_m^2(h) \\ &= \xi \int_{-h}^h u_m(y) dy \\ &\leq \xi \int_{-h}^h \max_{s \in [-h,h]} |u_m(s)| dy \\ &= 2h\xi \max_{s \in [-h,h]} |u_m(s)| \\ &= 2h\xi \|u_m\|_{C[-h,h]} \\ &\leq 2h\xi \|I\|_{\mathcal{L}(H_{\text{even}}^1[-h,h], C[-h,h])} \|u_m\|_{H_{\text{even}}^1[-h,h]} \end{aligned} \quad (24)$$

with

$$r = \frac{\|\omega\|_{L^1[-h,h]} \|I\|_{\mathcal{L}(H_{\text{even}}^1[-h,h], C[-h,h])}^2}{\min\{k_{\min}, \beta\}}.$$

From (24) it follows that

$$\|u_m\|_{H_{\text{even}}^1[-h,h]} \leq \frac{2h\xi \|I\|_{\mathcal{L}(H_{\text{even}}^1[-h,h], C[-h,h])}}{\min\{\mu_r, \lambda_r\}}. \quad (25)$$

Finally, combining relations (19), (20), and (25), we derive the estimate for the Euclidean norm of the vector γ_m :

$$\|\gamma_m\|_{\mathbb{R}^{2m+2}}^2 \leq \frac{4h^2\xi^2 \|I\|_{\mathcal{L}(H_{\text{even}}^1[-h,h], C[-h,h])}^2}{\min\{\mu_r^2, \lambda_r^2\}} + \frac{\|\omega\|_{L^1[-h,h]}^2 \|I\|_{\mathcal{L}(H_{\text{even}}^1[-h,h], C[-h,h])}^2}{\min\{k_{\min}^2, \beta^2\}}. \quad (26)$$

It should be noticed at this point that the right-hand side of estimate (26) does not depend on the parameters m and t . Therefore, applying Proposition 1, we deduce that system (17) and (18) is solvable for any $m \in \mathbb{N}$ and $t \in [0, 1]$.

Step 3: Passage to the limit as $m \rightarrow \infty$. By $\hat{\gamma}_m = (\hat{a}_{m0}, \hat{a}_{m1}, \dots, \hat{a}_{mm}, \hat{b}_{m0}, \hat{b}_{m1}, \dots, \hat{b}_{mm})^\top$ we denote a solution of problem (17) and (18) with $t = 1$. Moreover, let us introduce the functions $\hat{u}_m: [-h, h] \rightarrow \mathbb{R}$ and $\hat{\theta}_m: [-h, h] \rightarrow \mathbb{R}$ as follows:

$$\hat{u}_m(y) \stackrel{\text{def}}{=} \sum_{j=0}^m \hat{a}_{mj} \psi_j(y), \quad \hat{\theta}_m(y) \stackrel{\text{def}}{=} \sum_{j=0}^m \hat{b}_{mj} \psi_j(y), \quad \forall y \in [-h, h].$$

Then, we obviously have

$$\begin{aligned} \int_{-h}^h \mu(\hat{\theta}_m(y)) \hat{u}'_m(y) \psi'_j(y) dy + 2\lambda(\hat{\theta}_m(h)) \hat{u}_m(h) \psi_j(h) \\ = \xi \int_{-h}^h \psi_j(y) dy, \quad j = 0, 1, \dots, m, \end{aligned} \quad (27)$$

and

$$\begin{aligned} \int_{-h}^h k(\hat{\theta}_m(y)) \hat{\theta}'_m(y) \psi'_j(y) dy + 2\beta \hat{\theta}_m(h) \psi_j(h) \\ = \int_{-h}^h \omega(y) \psi_j(y) dy, \quad j = 0, 1, \dots, m. \end{aligned} \quad (28)$$

Let us consider the Galerkin solutions set $\{(\hat{u}_m, \hat{\theta}_m)\}_{m=1}^\infty$. From inequalities (20) and (25) it follows that this set is bounded in the space $H_{\text{even}}^1[-h, h] \times H_{\text{even}}^1[-h, h]$. Therefore, there exist real-valued functions u_* and θ_* belonging to the space $H_{\text{even}}^1[-h, h]$ and a subsequence $\{m_i\}_{i=1}^\infty$ such that $\hat{u}_{m_i} \rightharpoonup u_*$ and $\hat{\theta}_{m_i} \rightharpoonup \theta_*$ weakly in $H_{\text{even}}^1[-h, h]$ as $i \rightarrow \infty$. Without loss of generality, we can assume that

$$\hat{u}_m \rightharpoonup u_* \text{ weakly in } H_{\text{even}}^1[-h, h] \text{ as } m \rightarrow \infty, \quad (29)$$

$$\hat{\theta}_m \rightharpoonup \theta_* \text{ weakly in } H_{\text{even}}^1[-h, h] \text{ as } m \rightarrow \infty. \quad (30)$$

Moreover, since the embedding

$$H_{\text{even}}^1[-h, h] \hookrightarrow C[-h, h]$$

holds, we have

$$\hat{u}_m \rightarrow u_* \text{ strongly in } C[-h, h] \text{ as } m \rightarrow \infty, \quad (31)$$

$$\hat{\theta}_m \rightarrow \theta_* \text{ strongly in } C[-h, h] \text{ as } m \rightarrow \infty. \quad (32)$$

Using the convergences (29)–(32), we pass to the limit in equalities (27) and (28) as $m \rightarrow \infty$; this gives

$$\int_{-h}^h \mu(\theta_*(y)) u'_*(y) \psi'_j(y) dy + 2\lambda(\theta_*(h)) u_*(h) \psi_j(h) = \xi \int_{-h}^h \psi_j(y) dy, \quad (33)$$

$$\int_{-h}^h k(\theta_*(y)) \theta'_*(y) \psi'_j(y) dy + 2\beta \theta_*(h) \psi_j(h) = \int_{-h}^h \omega(y) \psi_j(y) dy, \quad (34)$$

for any $j \in \mathbb{N} \cup \{0\}$.

Because $\{\psi_j\}_{j=0}^\infty$ is an orthonormal basis of the space $H_{\text{even}}^1[-h, h]$, equality (33) will be true if we replace the basis function ψ_j in it with an arbitrary function $v \in H_{\text{even}}^1[-h, h]$. Moreover, we can replace ψ_j by an arbitrary function $\eta \in H_{\text{even}}^1[-h, h]$ in equality (34). Thus, it is proved that the pair (u_*, θ_*) is a weak solution of problem (1).

Now we show that the statement (b) of Theorem 1 holds. Suppose (u, θ) is a weak solution to problem (1) and the functions μ and k are of class C^1 .

From equality (8) it follows that

$$\int_{-h}^h \mu(\theta(y)) u'(y) \sigma'(y) dy = \xi \int_{-h}^h \sigma(y) dy, \quad \forall \sigma \in \mathfrak{D}(-h, h). \quad (35)$$

This yields the equality

$$(\mu(\theta) u')' = -\xi \quad \text{in the distribution sense in } (-h, h),$$

and hence

$$(\mu(\theta) u' + \xi y)' = 0 \quad \text{in the distribution sense in } (-h, h).$$

Therefore, we have

$$\mu(\theta) u' = -\xi y + c \quad \text{almost everywhere in } (-h, h)$$

with some constant c .

Dividing both sides of the last equality by the positive quantity $\mu(\theta)$, we obtain

$$u' = -\frac{\xi y - c}{\mu(\theta)} \quad \text{almost everywhere in } (-h, h). \quad (36)$$

Since the inclusion $\theta \in AC[-h, h]$ holds and the function μ is of class C^1 , we see that the composition $\mu \circ \theta$ is an absolutely continuous function. Therefore, from equality (36) it follows that $u' \in AC[-h, h]$. Recalling that

$$AC[-h, h] = W^{1,1}[-h, h],$$

we see that $u' \in W^{1,1}[-h, h]$, and hence $u \in W^{2,1}[-h, h]$. In a manner analogous, one can show that the inclusion $\theta \in W^{2,1}[-h, h]$ holds.

Using integration by parts, we derive from equality (35) the following relation

$$-\int_{-h}^h (\mu(\theta(y)) u'(y))' \sigma(y) dy = \xi \int_{-h}^h \sigma(y) dy, \quad \forall \sigma \in \mathfrak{D}(-h, h),$$

whence

$$-(\mu(\theta) u')' = \xi \quad \text{almost everywhere in } (-h, h). \quad (37)$$

Further, applying integration by parts to the first term in equality (8), we arrive at

$$\begin{aligned} -\int_{-h}^h (\mu(\theta(y)) u'(y))' v(y) dy + 2\mu(\theta(h)) u'(h) v(h) + 2\lambda(\theta(h)) u(h) v(h) \\ = \xi \int_{-h}^h v(y) dy, \quad \forall v \in H_{\text{even}}^1[-h, h]. \end{aligned} \quad (38)$$

Combining relations (37) and (38), we obtain

$$\mu(\theta(h)) u'(h) v(h) + \lambda(\theta(h)) u(h) v(h) = 0, \quad \forall v \in H_{\text{even}}^1[-h, h].$$

Choosing $v \equiv 1$ yields

$$\mu(\theta(h)) u'(h) + \lambda(\theta(h)) u(h) = 0.$$

Moreover, since u and θ are even functions, we see that

$$-\mu(\theta(-h)) u'(-h) + \lambda(\theta(-h)) u(-h) = 0,$$

or equivalently,

$$\mu(\theta(-h)) u'(-h) - \lambda(\theta(-h)) u(-h) = 0.$$

Using the same arguments as above, one can obtain

$$-(k(\theta)\theta')' = \omega \quad \text{almost everywhere in } (-h, h)$$

and

$$k(\theta(h))\theta'(h) + \beta\theta(h) = 0, \quad k(\theta(-h))\theta'(-h) - \beta\theta(-h) = 0.$$

Thus, we have established that the pair (u, θ) is a strong solution to problem (1). This completes the proof of Theorem 1.

5.2. Proof of Theorem 2

Suppose that the conditions of Theorem 2 hold and pairs (u_1, θ_1) and (u_2, θ_2) are weak solutions of problem (1). Let us prove the equality $(u_1, \theta_1) = (u_2, \theta_2)$.

First, we observe that if pairs (u, θ) and (w, θ) are weak solutions of problem (1), then $u = w$. Therefore, it suffices to only prove $\theta_1 = \theta_2$.

Since (u_1, θ_1) and (u_2, θ_2) are weak solutions to problem (1), we have

$$\int_{-h}^h k(\theta_1(y))\theta_1'(y)\psi'(y) \, dy + 2\beta\theta_1(h)\psi(h) = \int_{-h}^h \omega(y)\psi(y) \, dy, \quad (39)$$

$$\int_{-h}^h k(\theta_2(y))\theta_2'(y)\psi'(y) \, dy + 2\beta\theta_2(h)\psi(h) = \int_{-h}^h \omega(y)\psi(y) \, dy, \quad (40)$$

for any $\psi \in H_{\text{even}}^1[-h, h]$.

Subtracting (40) from (39) gives

$$\int_{-h}^h \{k(\theta_1(y))\theta_1'(y) - k(\theta_2(y))\theta_2'(y)\}\psi'(y) \, dy + 2\beta\{\theta_1(h) - \theta_2(h)\}\psi(h) = 0.$$

Setting $\psi = \theta_1 - \theta_2$ into the last equality, we obtain

$$\int_{-h}^h \{k(\theta_1(y))\theta_1'(y) - k(\theta_2(y))\theta_2'(y)\}(\theta_1'(y) - \theta_2'(y)) \, dy + 2\beta(\theta_1(h) - \theta_2(h))^2 = 0,$$

whence, by condition $(\mathcal{H}.3)$ and inequalities (4), (10), and (20), we derive

$$\|\theta_1 - \theta_2\|_{H_{\text{even}}^1[-h, h]}^2 \leq \frac{\ell_k \|\omega\|_{L^1[-h, h]} \max\{1, 4h\}}{\min\{k_{\min}^2, \beta^2\}} \|\theta_1 - \theta_2\|_{H_{\text{even}}^1[-h, h]}^2,$$

and hence

$$\left(1 - \frac{\ell_k \|\omega\|_{L^1[-h, h]} \max\{1, 4h\}}{\min\{k_{\min}^2, \beta^2\}}\right) \|\theta_1 - \theta_2\|_{H_{\text{even}}^1[-h, h]}^2 \leq 0. \quad (41)$$

On the other hand, from inequality (11) it follows that

$$\frac{\ell_k \|\omega\|_{L^1[-h, h]} \max\{1, 4h\}}{\min\{k_{\min}^2, \beta^2\}} < 1,$$

whence

$$1 - \frac{\ell_k \|\omega\|_{L^1[-h, h]} \max\{1, 4h\}}{\min\{k_{\min}^2, \beta^2\}} > 0.$$

The last inequality together with relation (41) imply

$$\|\theta_1 - \theta_2\|_{H_{\text{even}}^1[-h, h]} = 0$$

and consequently, we have $\theta_1 = \theta_2$. This gives the claimed result.

5.3. Proof of Theorem 3

Let (u, θ) be a weak solution to problem (1). Setting $v \equiv 1$ and $\eta \equiv 1$ into equalities (8) and (9), respectively, we obtain

$$2\lambda(\theta(h))u(h) = \xi \int_{-h}^h 1 \, dy, \quad 2\beta\theta(h) = \int_{-h}^h \omega(y) \, dy,$$

whence one can easily derive relations (12) and (13). Moreover, by letting $v = u$ and $\eta = \theta$ in equalities (8) and (9), respectively, it is not hard to obtain energy equalities (14) and (15).

Now suppose that (u, θ) is a strong solution to problem (1). First, we observe that

$$W^{2,1}[-h, h] \subset C^1[-h, h],$$

and hence the function u is of class C^1 .

Second, since the function u is even, we see that u' is an odd function. It follows that $u'(0) = 0$. On the other hand, setting $y = 0$ into equality (36), we obtain $u'(0) = c/\mu(\theta(0))$, whence $c = 0$. Therefore, relation (36) can be rewritten as follows

$$u'(y) = -\frac{\xi y}{\mu(\theta(y))} \quad \text{in } (-h, h). \quad (42)$$

Consequently,

$$\begin{aligned} u'(y) &> 0 \quad \text{if } y \in (-h, 0), \\ u'(y) &< 0 \quad \text{if } y \in (0, h), \end{aligned}$$

which implies that the function u has the global maximum point at 0.

Further, let us integrate both sides of equality (42) with respect to y from 0 to h . Taking into account relation (12), we arrive at equality (16). This means that property (iii) is proved.

Finally, arguing as above, one can establish property (iv). Thus, the proof of Theorem 3 is complete.

6. Some Exact Solutions

Let us suppose that $k \equiv k_0$ and $\omega \equiv \omega_0$, where k_0 and ω_0 are constants. Then, the unique solution of problem (1) can be written in an explicit form as follows:

$$u(y) = \frac{\xi h}{\lambda(\frac{\omega_0 h}{\beta})} - \int_{-h}^y \frac{\xi s}{\mu(\theta(s))} \, ds, \quad (43)$$

$$\theta(y) = -\frac{\omega_0}{2k_0}(y^2 - h^2) + \frac{\omega_0 h}{\beta}, \quad (44)$$

for any $y \in [-h, h]$. The derivation of the above formulae is based on direct calculations, symmetry properties, and simple algebraic transformations. Therefore we omit it here.

Further, taking into account relations (43) and (44), we consider five important partial cases. Let μ_0, μ_1, A , and B be positive constants.

Case 1. If $\mu(\theta) \equiv \mu_0$, then

$$u(y) = -\frac{\xi}{2\mu_0}(y^2 - h^2) + \frac{\xi h}{\lambda(\frac{\omega_0 h}{\beta})}.$$

Case 2. If $\mu(\theta) \stackrel{\text{def}}{=} \mu_0 / \sqrt{1 + A\theta}$, then

$$u(y) = \frac{\xi}{6\mu_0 A \beta \omega_0} \left(\sqrt{\frac{2}{k_0 \beta}} q^{3/2}(y) - 4k_0(Ah\omega_0 + \beta) \sqrt{\frac{Ah\omega_0}{\beta} + 1} \right) + \frac{\xi h}{\lambda(\frac{\omega_0 h}{\beta})},$$

where

$$q(y) \stackrel{\text{def}}{=} -A\beta\omega_0(y^2 - h^2) + 2k_0(Ah\omega_0 + \beta).$$

Case 3. If $\mu(\theta) \stackrel{\text{def}}{=} \mu_0/(1 + A\theta)$, then

$$u(y) = \frac{\xi A \omega_0}{8\mu_0 k_0} (y^4 - h^4) - \frac{\xi}{2\mu_0} \left(1 + A \left(\frac{\omega_0 h^2}{2k_0} + \frac{\omega_0 h}{\beta} \right) \right) (y^2 - h^2) + \frac{\xi h}{\lambda \left(\frac{\omega_0 h}{\beta} \right)}.$$

Case 4. If $\mu(\theta) \stackrel{\text{def}}{=} \mu_0/(1 + A\theta)^2$, then

$$u(y) = -\frac{\xi A^2 \omega_0^2}{24\mu_0 k_0^2} (y^6 - h^6) + \frac{\xi A \omega_0}{4\mu_0 k_0} \left(1 + A \left(\frac{\omega_0 h^2}{2k_0} + \frac{\omega_0 h}{\beta} \right) \right) (y^4 - h^4) - \frac{\xi}{2\mu_0} \left(1 + A \left(\frac{\omega_0 h^2}{2k_0} + \frac{\omega_0 h}{\beta} \right) \right)^2 (y^2 - h^2) + \frac{\xi h}{\lambda \left(\frac{\omega_0 h}{\beta} \right)}.$$

Case 5. If $\mu(\theta) \stackrel{\text{def}}{=} \mu_1 \exp(-B\theta)$, then

$$u(y) = \frac{\xi k_0}{\omega_0 \mu_1 B} \left(1 - \exp \left(\frac{B \omega_0 (y^2 - h^2)}{2k_0} \right) \right) \exp \left(\frac{B \omega_0 (2hk_0 - \beta(y^2 - h^2))}{2k_0 \beta} \right) + \frac{\xi h}{\lambda \left(\frac{\omega_0 h}{\beta} \right)}.$$

As can be seen from Figure 2, the constructed solutions differ significantly from each other. Notably, in Cases 2–4, the velocity profiles are not parabolic, unlike the standard Poiseuille flow with a constant viscosity (Case 1). This is a consequence of taking into account the dependence of the fluid viscosity on the temperature in the heat and mass transfer model for a non-uniformly heated fluid.

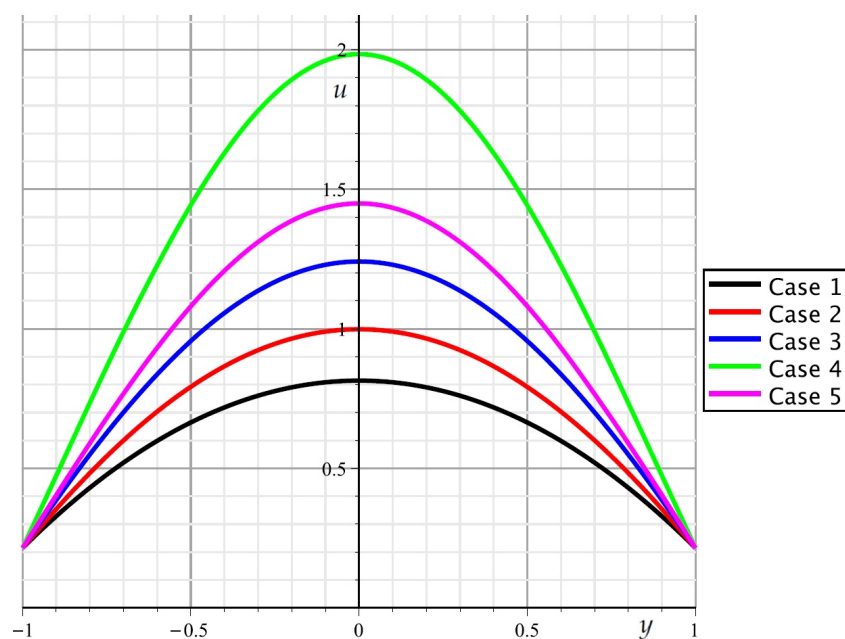


Figure 2. Velocity profiles for Cases 1–5 with $h = 1$, $\xi = 1.2$, $k_0 = 1$, $\omega_0 = 0.6$, $\beta = 1.3$, $\lambda \equiv 5.6$, $\mu_0 = 1$, $\mu_1 = 1$, $A = 1$, and $B = 1$.

Using the above exact solutions, we have performed an extra verification of the algorithm of constructing weak solutions that was applied by us in Section 5.1 for proving Theorem 1(a). As expected, we have obtained a good agreement between the exact solutions and the suitable Galerkin approximate solutions. This is well illustrated by Figure 3. In the corresponding numerical experiment, the relative percentage error of calculation of the velocity field, constructed by Galerkin's method with the six basis functions $\psi_0, \psi_1, \dots, \psi_5$, is equal to 1.4% with respect to the $L^2[-1, 1]$ -norm.

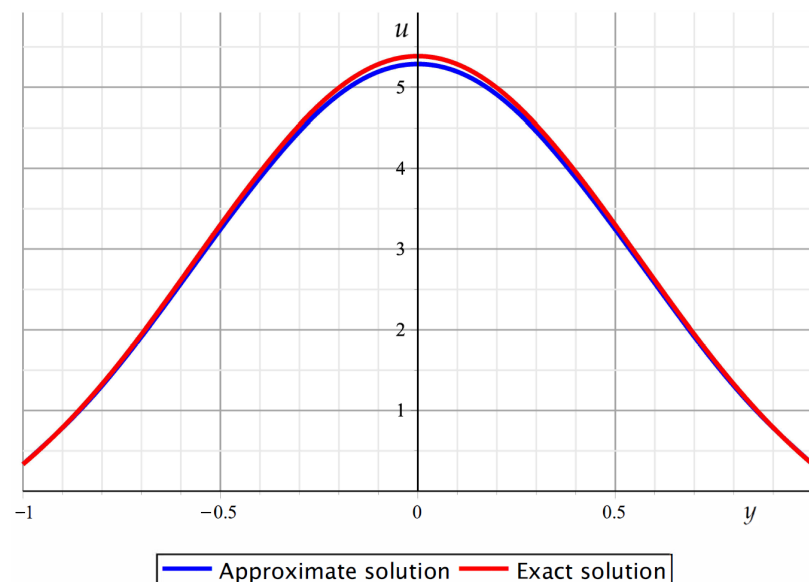


Figure 3. Velocity profiles for the plane Poiseuille flow with $h = 1$, $\xi = 0.2$, $k_0 = 1$, $\omega_0 = 0.8$, $\beta = 1$, $\lambda(\theta) \equiv 0.6$, and $\mu(\theta) \equiv 1.2 \exp(-4\theta)$.

7. Conclusions

In this paper, we have investigated the nonlinear heat and mass transfer model for the unidirectional flow of a viscous fluid through a plane channel under Navier's slip conditions on the channel walls. The effects of variable viscosity, thermal conductivity, and slip length are taken into account. Using the Galerkin scheme, the generalized Borsuk theorem, and the compactness arguments, we have proved the existence and uniqueness theorems for both weak and strong solutions of the corresponding BVP and derived some properties of solutions. In addition, considering our model subject to some analytical relations for temperature dependence of viscosity, we have obtained the explicit expressions for flow velocity and temperature. The approach proposed in the present work advances the insight of complex heat and mass transfer and is quite universal. This can be applied to studying many other models for temperature-dependent flows of both Newtonian and non-Newtonian fluids in channels and tubes. The authors suggest the following directions for further research:

- well-posedness of initial-boundary value problems for unsteady flows;
- modeling of flows under other kinds of realistic boundary conditions, in particular, under threshold-type slip conditions;
- continuous dependence of solutions on model data;
- stability/instability and limiting behavior of solutions;
- optimization and flow control problems.

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