mathematics

MDPI

Article

A Sixth-Order Cubic B-Spline Approach for Solving Linear Boundary Value Problems: An In-Depth Analysis and Comparative Study

Ram Kishun Lodhi [1], Moustafa S. Darweesh [2], Abdelkarim Aydi [3], Lioua Kolsi [4], Anil Sharma [5] and Katta Ramesh [6,7,8,*]

1 Symbiosis Institute of Technology, Pune Campus, Symbiosis International (Deemed University), Pune 412115, Maharashtra, India; ramkishun.lodhi@sitpune.edu.in
2 Civil Engineering Department, College of Engineering, Northern Border University, Arar 73213, Saudi Arabia; mustafa.samir@nbu.edu.sa
3 French International School Victor Hugo, Gontardstraße 11, 60488 Frankfurt am Main, Germany; abdelkarim.aydi@lfvh.net
4 Department of Mechanical Engineering, College of Engineering, University of Ha'il, Ha'il City 81451, Saudi Arabia; lioua_enim@yahoo.fr
5 Department of Mathematics, University Institute of Sciences, Apex Institute of Technology—Computer Science and Engineering, Chandigarh University, Mohali 140413, Punjab, India; anilsharma3091991@gmail.com
6 Department of Pure and Applied Mathematics, School of Mathematical Sciences, Sunway University, Bandar Sunway, Petaling Jaya 47500, Selangor, Malaysia
7 Department of Mathematics, Graphic Era (Deemed to be University), Dehradun 248002, Uttarakhand, India
8 Department of Mathematics, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara 144411, Punjab, India
* Correspondence: rameshka@sunway.edu.my

Abstract: This research presents an efficient and highly accurate cubic B-spline method (CBSM) for solving second-order linear boundary value problems (BVPs). The method achieves sixth-order convergence, supported by rigorous error analysis, ensuring rapid error reduction with mesh refinement. The effectiveness of the CBSM is validated through four numerical examples, showcasing its accuracy, reliability, and computational efficiency, making it well-suited for large-scale problems. A comparative analysis with existing methods confirms the superior performance of the CBSM, positioning it as a practical and powerful tool for solving second-order BVPs.

Keywords: sixth-order convergence; cubic B-spline method; boundary value problems; error analysis

MSC: 65L10

check for updates

Citation: Lodhi, R.K.; Darweesh, M.S.; Aydi, A.; Kolsi, L.; Sharma, A.; Ramesh, K. A Sixth-Order Cubic B-Spline Approach for Solving Linear Boundary Value Problems: An In-Depth Analysis and Comparative Study. *Mathematics* **2024**, *12*, 3274. https://doi.org/10.3390/math12203274

Academic Editor: Zacharias Anastassi

Received: 24 August 2024
Revised: 11 October 2024
Accepted: 16 October 2024
Published: 18 October 2024

Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

In various scientific and engineering fields, such as electro-hydrodynamics, control theory, biomathematics, and chemical reactor theory, we frequently encounter problems involving boundary value problems (BVPs). Finding analytical solutions for these problems can often be extremely challenging or even impossible, which makes numerical methods essential. Numerous techniques have been developed to tackle these challenges, including the variational approach, finite difference method, finite element method, finite volume method, upwind finite difference scheme, nonstandard finite difference scheme, compact finite difference method, shooting method, cubic spline method, quartic spline method, and quantic spline method.

In the current century, numerical techniques have become crucial for solving BVPs. The pioneering work of Bickley [1] introduced the cubic spline technique for numerical solutions of two-point BVPs. Subsequent developments include Munguia and Bhatta's [2]

extension to second-order BVPs, and Latif et al.'s [3] exploration of cubic B-splines for various BVPs. Akram and Rehman [4] introduced the reproducing kernel space method for eighth-order BVPs, while Akram and Siddiqi [5] applied a non-polynomial spline technique to sixth-order BVPs. Other notable contributions include Hussin and Kilicman's [6] use of the differential transformation method and Adomian decomposition method for higher-order BVPs, Khan et al.'s [7] application of intelligent computing techniques for delay differential equations, and Sabir et al.'s [8] innovative approach using Hermite polynomial neural networks for singular periodic systems. Kumar [9] developed a spline function for singular two-point BVPs, and Caglar et al. [10] proposed a non-polynomial spline technique for nonlinear second-order BVPs, which was further applied by Caglar and Ucar [11] to the time-dependent heat-like Lane–Emden equation. Additional work includes Elahi et al.'s [12] Galerkin method based on Legendre polynomials for eighth-order BVPs, Iqbal et al.'s [13] polynomial and non-polynomial cubic spline techniques for tenth-order BVPs, and Khalid et al.'s [14–17] significant contributions to fourth-order, sixth-order, and eighth-order BVPs using cubic B-splines. Loghmani and Ahmadinia [18] developed a sixth-degree spline function for sixth-order BVPs, and Khan et al. [19] applied the residual power series method to twelfth-order BVPs. Ramadan et al. [20] explored polynomial and non-polynomial spline functions for second-order BVPs, and Tassddiq et al. [21] introduced a new cubic B-spline technique for tenth-order nonlinear differential equations related to hydrodynamics and hydro-magnetic stability. Roul et al. [22] introduced a sixth-order numerical method based on B-spline functions to address the complexities of nonlinear BVPs arising in electro-hydrodynamic flow within circular cylindrical conduits. Similarly, Malale et al. [23] developed a sixth-order compact finite difference technique, effectively tackling both linear and nonlinear ordinary differential equations under Robin boundary conditions. Expanding upon finite difference methodologies, Zhanlav et al. [24] applied this approach to obtain numerical solutions for one-dimensional BVPs, including the Brantu, Troesch, and Lane–Emden equations. Setia and Mohanty [25] offered third-order numerical schemes designed to solve second-order nonlinear two-point BVPs, incorporating mixed boundary conditions into their analysis. In a novel approach, Liu and Change [26] employed boundary shape function techniques to transform BVPs into initial value problems, subsequently utilizing the half-interval method and derivative-free Newton method for solutions. Dallashi and Syam [27] further explored numerical solutions by discussing the reproducing kernel technique applied to second-order fuzzy initial value problems. Moreover, Latif et al. [28] developed a symmetric cubic B-spline technique to address linear systems of second-order BVPs, while Liu and Ku [29] presented an artificial neural network-based radial basis function collocation technique for solving elliptic BVPs. Lastly, Ali et al. [30] utilized cubic B-spline techniques to effectively solve Burgers' equation, illustrating the versatility and robustness of B-spline methods in addressing complex differential equations. Spline techniques have thus become a prevalent tool for solving various BVPs, with additional details available in references [31–37].

In this work, we present a sixth-order cubic B-spline method (CBSM) for addressing the following second-order linear boundary value models:

$$w''(t) + p(t)w'(t) + q(t)w(t) = f(t),\ t \in [a,\ b], \tag{1}$$

with the following boundary conditions:

$$w(a) = \lambda_0, \quad w(b) = \lambda_1. \tag{2}$$

The paper is organized as follows: Section 2 introduces the cubic B-spline method, Section 3 describes the procedure of the sixth-order CBSM to second-order BVPs, Section 4 presents the error analysis of the proposed method, Section 5 discusses the numerical results obtained with the proposed method, and Section 6 concludes with a summary of findings and their implications.

2. Cubic B-Spline Method

In this section, we present the basic properties of CBSM which are used to find the numerical solution of second-order BVPs. Let P $= \{a = t_0 < t_1 < \ldots < t_{N-1} < t_N = b\}$ be a partition of the interval $[a, b]$ with $t_r = t_0 + rh$, $r = 0, 1, \ldots, N$ and interval space $h = \frac{b-a}{N}$, where N is the number of subintervals of the $N + 1$ mesh points. To define the basis function of Cubic B-spline (CBS), we consider six more mesh points outside of the interval $[a, b]$ as $t_{-3} < t_{-2} < t_{-1} < t_0$ and $t_N < t_{N+1} < t_{N+2} < t_{N+3}$. Let us define $S_{3,\,P} = \{y(t) \in C^2[a, b]\}$, where $y(t)$ is cubic polynomial on each sub interval. Now the CBS basis functions are defined [22,28] as follows:

$$\beta_r(t) = \frac{1}{6h^3}\begin{cases} \phi_1(t - t_{r-2}), & t \in [t_{r-2}, t_{r-1}] \\ \phi_2(t - t_{r-1}), & t \in [t_{r-1}, t_r] \\ \phi_2(t_{r+1} - t), & t \in [t_r, t_{r+1}] \\ \phi_1(t_{r+2} - t), & t \in [t_{r+1}, t_{r+2}] \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

with $\phi_1(t) = t^3$ and $\phi_2(t) = h^3 + 3h^2t + 3ht^2 - 3t^3$. Let $\beta = \{\beta_{-1}(t), \beta_{-2}(t), \ldots, \beta_N(t), \beta_{N+1}(t)\}$ be the set of cubic spline functions, and let $\Omega(P) = \text{span}\ \beta$. Since all β_r are linearly independent on $[a, b]$, $\Omega(P)$ is $N + 3$ dimensional. Thus, $\Omega(P) = S_{3,\ \ P}$ [1]. The values of $\beta_r(t)$, $\beta_r'(t)$, and $\beta_r''(t)$ at the mesh point t_r are displayed in Table 1.

Table 1. The values of $\beta_r(t)$, $\beta_r'(t)$, and $\beta_r''(t)$ at the mesh point t_r.

	t_{r-2}	t_{r-1}	t_r	t_{r+1}	t_{r+2}
$\beta_r(t)$	0	1/6	2/3	1/6	0
$\beta_r'(t)$	0	$1/2h$	0	$-1/2h$	0
$\beta_r''(t)$	0	$1/h^2$	$-2/h^2$	$1/h^2$	0

Let us define the cubic spline interpolation (CSI) function of the form as follows:

$$s(t) = \sum_{r=-1}^{N+1} \delta_r \beta_r(t) \tag{4}$$

where $\delta_r's$ are unknown coefficients and $s(t)$ satisfies the following interpolating conditions:

$$s_m = w_m,\ m = 0, 1, 2, \ldots, N. \tag{5}$$

$$s_m'' = w_m'' - \frac{w_m^{(iv)}}{12}h^2 + \frac{w_m^{(vi)}}{12}h^4, \quad m = 0, N. \tag{6}$$

The discrete operate Λ is defined for the function ζ in the following domain $[0,\ 1]$:

$$\Lambda\zeta \equiv \zeta_{m-1} + 4\zeta_m + \zeta_{m+1}, \quad m = 1, 2, \ldots, N - 1. \tag{7}$$

We emplace the consistency relation [4] of CSI s as follows:

$$\Lambda s_m' = \frac{3}{h}(-s_{m-1} + s_{m+1}),\ m = 1, 2, \ldots, N - 1. \tag{8}$$

$$\Lambda s_m'' = \frac{6}{h^2}(s_{m-1} - 2s_m + s_{m+1}),\ m = 1, 2, \ldots, N - 1. \tag{9}$$

Lemma 1. *Let $w(t) \in C^8[0,\ 1]$ and $s(t)$ be the CSI of $w(t)$, then, we obtain the following:*

$$\Lambda s_m' = 6w_m' + h^2 w_m^{(iii)} + \frac{w_m^{(v)}}{20}h^5 + O\left(h^6\right). \tag{10}$$

$$\Lambda s''_m = 6w'_m + \frac{w_m^{(iv)}}{2}h^2 + \frac{w_m^{(vi)}}{60}h^4 + O\left(h^6\right). \tag{11}$$

Proof. Expanding the right-hand side of Equation (9) using Taylor's series, we obtain the following:

$$\begin{aligned}\Lambda s''_m = \frac{6}{h^2}\Big\{ & s_m - \tfrac{s'_m}{1!}h + \tfrac{s''_m}{2!}h^2 - \tfrac{s'''_m}{3!}h^3 + \tfrac{s_m^{(iv)}}{4!}h^4 - \tfrac{s_m^{(v)}}{5!}h^5 + \tfrac{s_m^{(vi)}}{6!}h^6 - \tfrac{s_m^{(vii)}}{7!}h^7 + O\left(h^8\right) - 2s_m \\ & + s_m + \tfrac{s'_m}{1!}h + \tfrac{s''_m}{2!}h^2 + \tfrac{s'''_m}{3!}h^3 + \tfrac{s_m^{(iv)}}{4!}h^4 + \tfrac{s_m^{(v)}}{5!}h^5 + \tfrac{s_m^{(vi)}}{6!}h^6 + \tfrac{s_m^{(vii)}}{7!}h^7 + O\left(h^8\right)\Big\}.\end{aligned} \tag{12}$$

From Equations (5) and (12), we obtain the following:

$$\Lambda s''_m = \frac{6}{h^2}\left\{ w''_m h^2 + \frac{w_m^{(iv)}}{12}h^4 + \frac{w_m^{(vi)}}{360}h^6 + O\left(h^8\right)\right\} \tag{13}$$

$$\Lambda s''_m = 6w''_m + \frac{w_m^{(iv)}}{2}h^2 + \frac{w_m^{(vi)}}{60}h^4 + O\left(h^6\right). \tag{14}$$

Therefore, Equation (11) has been proven, and, in a similar manner, we can establish the validity of Equation (10). □

Lemma 2. *Let B be the coefficient matrix of the system $\Lambda\zeta_j = 0$, for $j = 1, 2, \ldots, N-1$ with end conditions $\zeta_0 = \zeta_N = 0$, then $\|B\|_\infty \leq \frac{1}{2}$.*

Proof. The coefficient matrix B is as follows:

$$B = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 4 \end{bmatrix}.$$

Since B is a non-singular matrix. Therefore, we obtain the following:

$$\left\|B^{-1}\right\|_\infty \leq \max_{1\leq m\leq N-1} 1 \leq m \leq N-1 \frac{1}{\rho_m(B)}, \tag{15}$$

where $\rho_m(B) = |B_{mm}| - \sum_{m\neq k} |B_{mk}|, \quad m = 1, 2, \ldots, N-1.$

Hence, the following is true:

$$\left\|B^{-1}\right\|_\infty \leq \max_{1\leq m\leq N-1} 1 \leq m \leq N-1 \frac{1}{\rho_m(B)} = \frac{1}{\min\limits_{1\leq m\leq N-1}\rho_m(B)} = \frac{1}{2}. \tag{16}$$

We provide an estimation of the error bounds for the derivatives of the cubic spline, up to the second order, in the following theorem, evaluated at the mesh points t_m, $m = 0, 1, 2, \ldots, N$ of the partition P. □

Theorem 1. *Let s be the CSI of $w \in C^6[0,\ 1]$ which satisfies the interpolation conditions (5) and (6); then, the following relationships hold at the mesh points $t_m,\ m = 0, 1, 2, \ldots, N$:*

$$s'_m = w'_m - \frac{w_m^{(v)}}{180}h^4 + O\left(h^6\right), \tag{17}$$

$$s''_m = w''_m - \frac{w_m^{(iv)}}{12}h^2 + \frac{w_m^{(vi)}}{360}h^4 + O\left(h^6\right). \tag{18}$$

Proof. From the definition of Λ, for any function $z \in C^6[0,\ 1]$, we have the following:

$$\Lambda z_m \equiv z_{m-1} + 4z_m + z_{m+1},\ m = 1, 2, \ldots, N-1. \tag{19}$$

Expanding Equation (19) in terms of z_m using the Taylor series, we obtain the following:

$$\Lambda z_m = 6z_m + z''_m h^2 + \frac{z_m^{(iv)}}{12}h^4 + O\left(h^6\right). \tag{20}$$

By substituting $z_m = w''_m - \frac{w_m^{(iv)}}{12}h^2 + \frac{w_m^{(vi)}}{360}h^4$ into Equation (20) and simplifying it, we obtain the following:

$$\Lambda\left(w''_m - \frac{w_m^{(iv)}}{12}h^2 + \frac{w_m^{(vi)}}{360}h^4\right) = 6w''_m + \frac{w_m^{(iv)}}{2}h^2 + \frac{w_m^{(vi)}}{60}h^4 + O\left(h^6\right). \tag{21}$$

Subtracting Equation (21) from Equation (20), we obtain the following:

$$\Lambda\left(s''_m - w''_m + \frac{w_m^{(iv)}}{12}h^2 - \frac{w_m^{(vi)}}{360}h^4\right) = O\left(h^6\right). \tag{22}$$

Let us define $d_m \equiv s''_m - w''_m + \frac{w_m^{(iv)}}{12}h^2 - \frac{w_m^{(vi)}}{360}h^4,\ m = 1, 2, \ldots N-1$. From Equations (6) and (22), the following system of equations is obtained as detailed below:

$$\begin{aligned} \Lambda d_m &= O\left(h^6\right),\ m = 1, 2, \ldots, N-1, \\ d_0 &= d_N = 0. \end{aligned} \tag{23}$$

It is clear from Lemma 2 that the inverse of the coefficient matrix in Equation (23) possesses a bounded norm. Consequently, we have the following:

$$d_m = \left\|B^{-1}\right\| = O\left(h^6\right),\ m = 0, 1, \ldots, N. \tag{24}$$

Hence, this is a complete proof of relation (18).

To prove Equation (17), we begin by considering the following cubic spline consistency relations, as follows:

$$s'_m = \frac{1}{h}(s_{m+1} - s_m) - \frac{h}{6}\left(s''_{m+1} + 2s''_m\right),\ m = 0, 1, \ldots, N-1, \tag{25}$$

$$s'_m = \frac{1}{h}(s_m - s_{m-1}) - \frac{h}{6}\left(s''_m + 2s''_{m-1}\right),\ m = 1, 2, \ldots, N. \tag{26}$$

From Equations (5), (18) and (25), we have the following:

$$\begin{aligned} s'_m = \tfrac{1}{h}(w_{m+1} - w_m) - \tfrac{h}{6}\Bigg\{ &\left(w''_{m+1} - \tfrac{w_{m+1}^{(iv)}}{12}h^2 + \tfrac{w_{m+1}^{(vi)}}{360}h^4 + o\left(h^6\right)\right) + 2\Bigg(w''_m - \tfrac{w_m^{(iv)}}{12}h^2 \\ &+ \tfrac{w_m^{(vi)}}{360}h^4 + O\left(h^6\right)\Bigg)\Bigg\}. \end{aligned} \tag{27}$$

Expanding the right-hand side (RHS) of Equation (27) using Taylor's series, we obtain the following:

$$s'_m = w'_m - \frac{w_m^{(v)}}{180}h^4 + O\left(h^6\right),\ m = 0, 1, \ldots, N-1. \tag{28}$$

Equation (28) is the proof of Equation (17) for $m = 0, 1, \ldots, N-1$. We need to prove Equation (17) for $m = N$.

Using Equations (5) and (18) in (26) for $m = N$, we obtain the following:

$$\begin{aligned} s'_m = \tfrac{1}{h}(w_m - w_{m-1}) + \tfrac{h}{6}\Bigg\{ \left(w''_{m-1} - \tfrac{w_{m-1}^{(iv)}}{12}h^2 + \tfrac{w_{m-1}^{(vi)}}{360}h^4 + O\left(h^6\right) \right) + 2\Bigg(w''_m - \tfrac{w_m^{(iv)}}{12}h^2 \\ + \tfrac{w_m^{(vi)}}{360}h^4 + O\left(h^6\right) \Bigg) \Bigg\}. \end{aligned} \tag{29}$$

Expanding the RHS of Equation (29) using Taylor's series, we obtain the following:

$$s'_m = w'_m - \frac{w_m^{(v)}}{180}h^4 + O\left(h^6\right),\ m = N. \tag{30}$$

Thus, we have obtained Equation (17) from Equations (28) and (29) for $m = 0, 1, 2, \ldots, N$. Hence, Theorem 1 is proven. □

Theorem 2. *Let s be the CSI of $w \in C^6[0,\ 1]$, then $\left\| s^{(k)} - w^{(k)} \right\|_\infty = o\left(h^{4-k}\right),\ k = 0, 1, 2.$*

Proof. Using Theorem 1, one may prove the theorem. □

Now, we define some important relation that connects $w_m^{(iv)}$, $w_m^{(v)}$, $w_m^{(vi)}$ with $s(t)$, $s'(t)$, $s''(t)$ at the mesh point of the partition P. To find the relation, we first define the difference operators η and η^2, as follows:

$$\eta g_m = g_{m-1} - 2g_m + g_{m+1}, \quad m = 1, 2, \ldots, N-1, \tag{31}$$

$$\eta^2 g_m = g_{m-2} - 4g_{m-1} + 6g_m - 4g_{m+1} + g_{m+2}, \quad m = 2, 3, \ldots, N-2. \tag{32}$$

Lemma 3. *Let s be the CSI of $w \in C^8[0,\ 1]$. Then, we have the following relation at the interior point of the partition P :*

$$w_m^{(iv)} = \frac{1}{h^2}\eta s''_m + O\left(h^4\right),\ m = 1, 2, \ldots, N-1, \tag{33}$$

$$w_m^{(v)} = \frac{1}{h^4}\eta^2 s'_m + O\left(h^2\right),\ m = 2, 3, \ldots, N-2, \tag{34}$$

$$w_m^{(vi)} = \frac{1}{h^4}\eta^2 s''_m + O\left(h^2\right),\ m = 2, 3, \ldots, N-2. \tag{35}$$

Proof. Since the proof of these relations is straightforward, we have omitted it. □

In the corollary, we derive the approximations for w'_m and w''_m at the interior mesh point of the partition P for $m = 2, 3, \ldots, N-2$.

Corollary 1. *Let $w \in C^8[0,\ 1]$, then for $m = 2, 3, \ldots, N-2$, we have the following approximations for w'_m and w''_m :*

$$w'_m = \left(1 + \frac{\eta^2}{180}\right)s'_m + O\left(h^6\right) = s'_m + \frac{1}{180}\left(s'_{m-2} - 4s'_{m-1} + 6s'_m - 4s'_m + s'_{m+1}\right) + O\left(h^6\right), \tag{36}$$

$$w''_m = \left(1 + \frac{\eta}{12} - \frac{\eta^2}{360}\right)s''_m + O(h^6) = s''_m + \frac{1}{12}\left(s''_{m-1} - s''_m + s''_{m+1}\right) - \frac{1}{360}\left(s''_{m-2} - 4s''_{m-1} + 6s''_m - 4s''_{m+1} + s''_{m+2}\right) + O(h^6). \tag{37}$$

Proof. Using Theorem 1 and Lemma 3, this corollary can be proven. □

We observe that the above relations are valid at the interior mesh points of the partition P. However, to complete our analysis, we need analogous relations for the boundary and near-boundary points of the partition P for $m = 0, 1, N-1, N$. The following lemma addresses this need by providing approximations for the derivatives $w_m^{(iv)}$, $w_m^{(v)}$, $w_m^{(vi)}$ at the boundary and near boundary points.

Lemma 4. *Let* $w \in C^8[0,\ 1]$. *Then, the following relations hold:*

$$w_{x_0}^{(iv)} = \frac{1}{h^2}\left(s''_{x_0} - 14s''_{x_1} + 20s''_{x_2} - 15s''_{x_3} + 6s''_{x_4} - s''_{x_5}\right) + O\left(h^4\right),\ m = 0, N, \tag{38}$$

$$w_{x_0}^{(n)} = \frac{1}{h^4}\left(3s_{x_0}^{(n-iv)} - 14s_{x_1}^{(n-iv)} + 26s_{x_2}^{(n-iv)} - 24s_{x_3}^{(n-iv)} + 11s_{x_4}^{(n-iv)} - 2s_{x_5}^{(n-iv)}\right) + O\left(h^2\right),\ m = 0, N,\ n = v,\ vi, \tag{39}$$

$$w_{x_1}^{(n)} = \frac{1}{h^4}\left(2s_{x_0}^{(n-iv)} - 9s_{x_1}^{(n-iv)} + 16s_{x_2}^{(n-iv)} - 14s_{x_3}^{(n-iv)} + 6s_{x_4}^{(n-iv)} - s_{x_5}^{(n-iv)}\right) + O\left(h^2\right),\ m = 1, N-1,\ n = v,\ vi, \tag{40}$$

where $x_m \equiv m$ *near the left boundary point and* $x_m \equiv N - m$ *near the right boundary point.*

Proof. First of all, we have to prove Equation (38) for $x_0 = 0$. Let us assume the approximation for $w_0^{(iv)}$, as follows:

$$w_0^{(iv)} = 4w_1^{(iv)} - 6w_2^{(iv)} + 4w_3^{(iv)} - w_4^{(iv)} + O\left(h^4\right). \tag{41}$$

From Equations (33) and (41), we obtain the following:

$$w_0^{(iv)} = \frac{1}{h^2}\left(4\eta s''_1 - 6\eta s''_2 + 4\eta s''_3 - \eta s''_4\right) + O\left(h^4\right)$$

$$= \frac{1}{h^2}\left[4\left(s''_0 - 2s''_1 + s''_2\right) - 6\left(s''_1 - 2s''_2 + s''_3\right) + 4\left(s''_2 - 2s''_3 + s''_4\right) - \left(s''_3 - 2s''_4 + s''_5\right)\right] + O\left(h^4\right)$$

$$w_0^{(iv)} = \frac{1}{h^2}\left[4s''_0 - 14s''_1 + 20s''_2 - 15s''_3 + 6s''_4 - s''_5\right] + O\left(h^4\right). \tag{42}$$

Hence, Equation (38) is proven for $x_0 = 0$. Similarly, we can prove Equation (38) for $x_0 = N$.

Now, we have to prove Equation (40) for $x_1 = 1$ and $n = v$. We define the approximate $w_1^{(v)}$ as follows:

$$w_1^{(v)} = 2w_2^{(v)} - w_3^{(v)} + O\left(h^2\right). \tag{43}$$

By substituting the values in Equation (43) from Equation (34) for $m = 2,\ 3$, we obtain the following:

$$w_1^{(v)} = \frac{1}{h^4}\left(2\eta^2 s'_2 - \eta^2 s'_3\right) + O\left(h^2\right)$$

$$w_1^{(v)} = \frac{1}{h^4}\left[2\left(s'_0 - 4s'_1 + 6s'_2 - 4s'_3 + s'_4\right) - \left(s'_1 - 4s'_2 + 6s'_3 - 4s'_4 + s'_5\right)\right] + O\left(h^2\right)$$

$$w_1^{(v)} = \frac{1}{h^4}\left(2s'_0 - 9s'_1 + 16s'_2 - 14s'_3 + 6s'_4 - s'_5\right) + O\left(h^2\right). \tag{44}$$

Hence, Equation (40) is proven for $x_1 = 1$ and $n = v$. Similarly, we can prove Equation (40) for $x_1 = 1,\ n = vi$ and $x_1 = N - 1,\ n = v,\ vi$.

Finally, we have to prove Equation (39) for $x_0 = 1$ and $n = v$. We define the approximate $w_0^{(v)}$ as follows:

$$w_0^{(v)} = 2w_1^{(v)} - w_2^{(v)} + O\left(h^2\right). \tag{45}$$

From Equations (34), (40), and (45) for $x_1 = 1$, $n = v$, and $m = 2$, we obtain the following:

$$w_0^{(v)} = \frac{1}{h^4}\left[2\left(2s_0' - 9s_1' + 16s_2' - 14s_3' + 6s_4' - s'\right) - \left(s_0' - 4s_1' + 6s_2' - 4s_3' + s_4'\right)\right] + O\left(h^2\right).$$

$$w_0^{(v)} = \frac{1}{h^4}\left(3s_0' - 14s_1' + 26s_2' - 24s_3' + 11s_4' - 2s_5'\right) + O\left(h^2\right). \tag{46}$$

Hence, Equation (39) is proved for $x_0 = 0$ and $n = v$. Similarly, we can prove Equation (39) for $x_1 = 1,\ n = vi$ and $x_0 = N,\ n = v,\ vi$. □

Corollary 2. *Let* $w \in C^8[0,\ 1]$; *then for* $m = 0, 1, N-1, N$, *we have the following approximations for* w_m' *and* w_m'' :

$$w_m' = s_m' + \frac{1}{180}\eta^2\left((3-m)s_2' - (2-m)s_3'\right) + O\left(h^6\right), \quad m = 0, 1, \tag{47}$$

$$w_m' = s_m' + \frac{1}{180}\eta^2\left((3-n)s_{N-2}' - (2-n)s_{N-3}'\right) + O\left(h^6\right), \quad (m,\ n) = (N-1,\ 1), (N, 0), \tag{48}$$

$$w_0'' = s_0'' + \frac{1}{12}\eta\left(4s_1'' - 6s_2'' + 4s_3'' - s_4''\right) - \frac{1}{360}\eta^2\left(3s_2'' - 2s_3''\right) + O\left(h^6\right), \tag{49}$$

$$w_1'' = s_1'' + \frac{\eta s_1''}{12} - \frac{1}{360}\eta^2\left(2s_2'' - s_3''\right) + O\left(h^6\right), \tag{50}$$

$$w_{N-1}'' = s_{N-1}'' + \frac{\eta s_{N-1}''}{12} - \frac{1}{360}\eta^2\left(2s_{N-2}'' - s_{N-3}''\right) + O\left(h^6\right), \tag{51}$$

$$w_N'' = s_N'' + \frac{1}{12}\eta\left(4s_{N-1}'' - 6s_{N-2}'' + 4s_{N-3}'' - s_{N-4}''\right) - \frac{1}{360}\eta^2\left(3s_{N-2}'' - 2s_{N-3}''\right) + O\left(h^6\right). \tag{52}$$

Proof. Using Theorem 1 and Lemma 4, we can prove Corollary 2. □

3. Description of Cubic B-Spline Method

In this section, we outline the procedure for the sixth-order cubic B-spline method (CBSM) to obtain numerical solutions for second-order linear boundary value problems (BVPs).

Discretizing Equation (1) at the mesh points t_m, we obtain the following:

$$Lw \equiv w_m'' + p_m w_m' + q_m w_m = f_m, \quad m = 0, 1, \ldots, N, \tag{53}$$

with the following boundary condition:

$$B(w) \equiv [w_0 = \lambda_0,\ \ w_N = \lambda_1], \tag{54}$$

where $w_m'' = w''(t_m),\ w_m' = w'(t_m),\ w_m = w(t_m),\ p_m = p(t_m),\ q_m = q(t_m)$, and $f_m = f(t_m)$.

From Equations (5), (17), (18), and (53), we have

$$s_m'' - \frac{s_m^{(iv)}}{12}h^2 + \frac{s_m^{(vi)}}{360}h^4 + O\left(h^6\right) + p_m\left(s_m' - \frac{s_m^{(v)}}{180}h^4 + O\left(h^6\right)\right) + q_m s_m = f_m,\ \ m = 0, 1, \ldots, N. \tag{55}$$

Substituting Corollary 1 and Lemmas 3 and 4 into the above equation, we obtain the following:

$$\begin{aligned}\tfrac{1}{360}\left(477s_0'' - 406s_1'' + 574s_2'' - 426s_3'' + 169s_4'' - 28s_5''\right) + \tfrac{p_0}{180}(183s_0' \\ -14s_1' + 26s_2' - 24s_3' + 11s_4' - 2s_5') + q_0 s_0 = f_0 + O\left(h^6\right),\end{aligned} \tag{56}$$

$$\begin{aligned}\tfrac{1}{360}\left(28s_0'' + 309s_1'' + 14s_2'' + 14s_3'' - 6s_4'' + s_5''\right) + \tfrac{p_1}{180}(2s_0' + 171s_1' \\ +16s_2' - 14s_3' + 6s_4' - s_5') + q_1 s_1 = f_1 + O\left(h^6\right),\end{aligned} \tag{57}$$

$$\begin{aligned}\tfrac{1}{360}\left(-s_{m-2}'' + 34s_{m-1}'' + 294s_m'' + 34s_{m+1}'' - s_{m-2}''\right) + \tfrac{p_m}{180}\left(s_{m-2}' - 4s_{m-1}' + 186s_m'\right. \\ \left.-4s_{m+1}' + s_{m+2}'\right) + q_m s_m = f_m + O\left(h^6\right), \qquad m = 2, 3, \ldots, N-2,\end{aligned} \tag{58}$$

$$\begin{aligned}\tfrac{1}{360}\left(s_{N-5}'' - 6s_{N-4}'' + 14s_{N-3}'' + 14s_{N-2}'' + 309s_{N-1}'' + 28s_N''\right) + \tfrac{p_{N-1}}{180}\left(-s_{N-5}' + 6s_{N-4}'\right. \\ \left.-14s_{N-3}' + 16s_{N-2}' + 171s_{N-1}' + 2s_N'\right) + q_{N-1}s_{N-1} = f_{N-1} + O\left(h^6\right),\end{aligned} \tag{59}$$

$$\begin{aligned}\tfrac{1}{360}\left(-28s_{N-5}'' + 169s_{N-4}'' - 426s_{N-3}'' + 574s_{N-2}'' - 406s_{N-1}'' + 477s_N''\right) + \tfrac{p_N}{180}\left(-2s_{N-5}'\right. \\ \left.+11s_{N-4}' - 24s_{N-3}' + 26s_{N-2}' - 14s_{N-1}' + 183s_N'\right) + q_N s_N = f_N + O\left(h^6\right).\end{aligned} \tag{60}$$

From Equations (5) and (54), we obtain the following:

$$s_0 = \lambda_0 + O\left(h^6\right), \tag{61}$$

$$s_N = \lambda_1 + O\left(h^6\right). \tag{62}$$

Let L' be the approximate operator of the original operator L defined by the RHS of Equations (56)–(60) and B' be the approximate operator of the original operator B defined in Equations (61) and (62). Then, since the CSI s satisfies these relations, we obtain the following:

$$L's_m = f_m + O\left(h^6\right), \quad m = 0, 1, \ldots N, \tag{63}$$

$$B'(s_0) = \lambda_0 + O\left(h^6\right), \tag{64}$$

$$B'(s_N) = \lambda_1 + O\left(h^6\right), \tag{65}$$

where $L'\phi_m = L\phi_m + \frac{1}{12}\eta\phi_m'' + \frac{1}{360}\left[2p_m\eta^2\phi_m' - \eta^2\phi_m''\right] + O\left(h^6\right), \ m = 2, 3, \ldots, N-2.$

The approximate cubic B-spline is defined as follows:

$$\bar{s}(t) = \sum_{r=-1}^{N+1} \bar{\delta}_r \beta_r(t) \tag{66}$$

Now, we calculate spline approximation $\bar{s}(t)$ to the solution of Equations (53) and (54) by satisfying the following relations:

$$L'\bar{s}_m = f_m + O\left(h^6\right), \quad m = 0, 1, \ldots N, \tag{67}$$

$$B'(\bar{s}_0) = \lambda_0 + O\left(h^6\right), \tag{68}$$

$$B'(\bar{s}_N) = \lambda_1 + O\left(h^6\right). \tag{69}$$

By substituting the values of spline approximation $\bar{s}$ from Equation (66) into Equations (67)–(69) and simplifying, we obtain $(N+3)$ equations with $(N+3)$ unknowns. By solving the obtained linear equations using the Matrix method for the unknowns $\bar{\delta}_{-1}, \bar{\delta}_0, \ldots, \bar{\delta}_N$ and by substituting these unknown values into Equation (66), we obtain the spline solution of Equations (1) and (2).

4. Error Analysis

Using Theorem 1, Lemma 1, and Equations (56)–(60), we obtain the following equations:

$$s'(t_{m-1}) + 4s'(t_m) + s'(t_{m+1}) = \frac{1}{60h}\{-w(t_{m-3}) + 4w(t_{m-2}) - 185w(t_{m-1}) + 185w(t_{m+1}) - 4w(t_{m+2}) + w(t_{m-3})\}, \tag{70}$$

$$s''(t_{m-1}) + 4s''(t_m) + s''(t_{m+1}) = \frac{1}{60h^2}\{-w(t_{m-3}) + 36w(t_{m-2}) + 225w(t_{m-1}) - 520w(t_m) + 225w(t_{m+1}) + 36w(t_{m+2}) - w(t_{m-3})\}. \tag{71}$$

Using the operator notation [38], Equations (70) and (71) can be written as follows:

$$s'(t_m) = \frac{1}{60h}\left(\frac{-E^{-3} + 4E^{-2} - 185E^{-1} + 185E - 4E^2 + E^3}{E^{-1} + 4I + E}\right)w(t_m), \tag{72}$$

$$s''(t_m) = \frac{1}{60h^2}\left(\frac{-E^{-3} + 36E^{-2} + 225E^{-1} - 520I + 225E + 36E^2 - E^3}{E^{-1} + 4I + E}\right)w(t_m), \tag{73}$$

where the operators are defined as $Ew(t_m) = w(t_m + h)$, $Dw(t_m) = w'(t_m)$ and $Iw(t_m) = w(t_m)$. Let $E = e^{hD}$ and expand them in powers of hD; then, we obtain the following:

$$s'(t_m) = w'(t_m) + \frac{1}{630}h^6 w^{(7)}(t_m) + \frac{1}{3780}h^8 w^{(9)}(t_m) + O\left(h^9\right), \tag{74}$$

$$s''(t_m) = w''(t_m) - \frac{1}{15120}h^6 w^{(9)}(t_m) - \frac{19}{45360}h^8 w^{(11)}(t_m) + O\left(h^9\right). \tag{75}$$

We define $e(t) = s(t) - w(t)$ and substitute Equations (74) and (75) into the Taylor series expansion of $e(t_m + \theta h)$; we obtain the following:

$$e(t_m + \theta h) = \frac{\theta}{630}h^7 w^{(7)}(t_m) + \left(\frac{\theta}{3780} - \frac{\theta^2}{30240}\right)h^9 w^{(9)}(t_m) + O\left(h^{10}\right), \tag{76}$$

where $a \le \theta \le b$.

5. Numerical Results

In this section, the proposed method is implemented on four linear BVPs, which demonstrates the efficiency and accuracy of the method. The efficiency and accuracy of the method is assessed using the maximum absolute error (MAE), pointwise absolute error, and order of convergence (OC). The MAE and OC are calculated as follows:

$$\text{MAE}\left(E^N\right) = \max_{a \le m \le b} |w(t_m) - \bar{s}(t_m)|,$$

$$\text{Order of Convergence (OC)} = \frac{\ln\left(E^N\right) - \ln\left(E^{2N}\right)}{\ln(2)}.$$

where $w(t)$ demotes the exact solution and $\bar{s}(t)$ denotes the approximate solution.

Example 1. *Consider the following second-order differential equation:*

$$w''(t) + w'(t) - 6w(t) = t, \quad t \in [0, 1],$$

with the following boundary conditions:

$$w(0) = 0, \qquad w(1) = 1.$$

The exact solution of Example 1 is as follows:

$$w(t) = \frac{(43 - e^2)e^{-3t} - (43 - e^{-3})e^{2t}}{36(e^{-3} - e^2)} - \frac{1}{6}t - \frac{1}{36}.$$

The numerical results for Example 1 are presented in Tables 2–4. Table 2 provides a comparison betweenof the mMean aAbsolute eError (MAE) andwith other existing methods. Table 3 outlines the MAE and the order of convergence for the proposed method. Lastly, Table 4 displays the pointwise absolute error, alongside a comparison with other existing methods.

Table 2. Comparison between MAE and existing methods of Example 1 for different values of N.

N	Munguia and Bhatta [2]	Latif et al. [3]	Proposed Method
20	9.53×10^{-5}	6.23×10^{-8}	2.56×10^{-10}
50	1.52×10^{-5}	1.63×10^{-9}	1.11×10^{-12}
100	3.81×10^{-6}	1.02×10^{-9}	9.64×10^{-15}

Table 3. MAE and OC of Example 1 for different values of N.

N	Maximum Absolute Error	Order of Convergence
8	4.05462×10^{-8}	5.43130
16	9.39649×10^{-10}	5.87746
32	1.59836×10^{-11}	5.95578
64	2.57516×10^{-13}	

Table 4. Pointwise absolute error of example 1 for $N = 20$.

t	Exact Solution	Numerical Solution by Latif et al. [3]	Numerical Solution by Proposed Method	Absolute Error by Latif et al. [3]	Absolute Error by Proposed Method
0.00	0.000000000	0.0000000000	0.000000000	0.000000000	0.0000000000
0.10	0.054257003	0.0542569644	0.054257000	3.587×10^{-8}	$1.31724812 \times 10^{-10}$
0.20	0.107428562	0.1074285058	0.107428562	5.592×10^{-8}	$2.14480128 \times 10^{-10}$
0.30	0.163625544	0.1636254812	0.163625544	6.234×10^{-8}	$2.49995746 \times 10^{-10}$
0.40	0.226741215	0.2267411540	0.226741215	6.063×10^{-8}	$2.55983179 \times 10^{-10}$
0.50	0.300695369	0.3006953149	0.300695369	5.433×10^{-8}	$2.42144915 \times 10^{-10}$
0.60	0.389656735	0.3896566891	0.389656735	4.567×10^{-8}	$2.13493834 \times 10^{-10}$
0.70	0.498258499	0.4982584629	0.498258499	3.594×10^{-8}	$1.71650194 \times 10^{-10}$
0.80	0.631819979	0.6318199531	0.631819979	2.587×10^{-8}	$1.15522147 \times 10^{-10}$
0.90	0.796586570	0.7965865545	0.796586570	1.573×10^{-8}	$4.19659862 \times 10^{-10}$
1.00	1.000000000	1.0000000000	1.000000000	0.000000000	0.0000000000

Example 2. *Consider the following second-order differential equation:*

$$w''(t) + 2w'(t) + 5w(t) = 6\cos(2t) - 7\sin(2t), \quad t \in \left[0, \frac{\pi}{4}\right],$$

with the following boundary conditions:

$$w(0) = 4, \qquad w\left(\frac{\pi}{4}\right) = 1.$$

The exact solution of Example 2 is as follows:

$$w(t) = 2\left(1 + e^{-t}\right)\cos(2t) + \sin(2t).$$

Tables 5–7 present the numerical results for Example 2. Table 5 provides a comparison betweenof the mMean aAbsolute eError (MAE) andwith other existing methods. Table 6 highlights the MAE and the order of convergence for the proposed method. Finally, Table 7 presents the pointwise comparison of absolute error with other existing methods.

Table 5. Comparison between MAE and existing methods of Example 2 for different values of N.

N	Munguia and Bhatta [2]	Latif et al. [3]	Proposed Method
20	6.30×10^{-5}	9.31×10^{-8}	2.24×10^{-10}
50	1.01×10^{-5}	2.40×10^{-9}	8.63×10^{-13}
100	2.52×10^{-6}	1.50×10^{-10}	2.50×10^{-13}

Table 6. MAE and OC of Example 2 for different values of N.

N	Maximum Absolute Error	Order of Convergence
8	3.86150×10^{-8}	5.53894
16	8.30558×10^{-10}	5.90005
32	1.39084×10^{-11}	5.31636
64	3.49054×10^{-13}	

Table 7. Pointwise absolute error of Example 2 for $N = 20$.

t	Exact Solution	Numerical Solution by Latif et al. [3]	Numerical Solution by Proposed Method	Absolute Error by Latif et al. [3]	Absolute Error by Proposed Method
0	4.00000000	4.0000000000	4.00000000	0.000000000	0.000000000
$\pi/40$	3.95797824	3.9579782897	3.95797824	4.523×10^{-8}	$8.65836292 \times 10^{-11}$
$\pi/20$	3.83674430	3.8367443738	3.83674430	7.468×10^{-8}	$1.62325264 \times 10^{-10}$
$3\pi/40$	3.64393870	3.6439387932	3.64393870	8.959×10^{-8}	$2.07015738 \times 10^{-10}$
$\pi/10$	3.38763562	3.3876357138	3.38763562	9.313×10^{-8}	$2.23521646 \times 10^{-10}$
$\pi/8$	3.07624246	3.0762425517	3.07624246	8.804×10^{-8}	$2.16055618 \times 10^{-10}$
$3\pi/20$	2.71841203	2.7184121103	2.71841203	7.666×10^{-8}	$1.89658955 \times 10^{-10}$
$7\pi/40$	2.32296292	2.3229629854	2.32296292	6.091×10^{-8}	$1.49854351 \times 10^{-10}$
$\pi/5$	1.89880428	1.8988043202	1.89880428	4.227×10^{-8}	$1.02212683 \times 10^{-10}$
$9\pi/40$	1.45486147	1.4548614961	1.45486147	2.189×10^{-8}	$5.18372012 \times 10^{-11}$
$\pi/4$	1.00000000	1.0000000000	1.00000000	0.000000000	0.000000000

Example 3. *Consider the following second-order differential equation:*

$$t^2 w''(t) + 3tw'(t) + 3w(t) = 0, \quad t \in [1,\ 2],$$

with the following boundary conditions:

$$w(1) = 5, \qquad w(2) = 0.$$

The exact solution of Example 3 is as follows:

$$w(t) = \frac{5}{t}\left[\cos\left(\sqrt{2}\ln t\right) - \cot\left(\sqrt{2}\ln 2\right)\sin\left(\sqrt{2}\ln t\right)\right].$$

Tables 8 and 9 present the numerical results for Example 3. Table 8 displays the MAE and the order of convergence of the proposed method across different mesh points. Table 9 illustrates the exact solution, approximate solution, and the pointwise comparison between absolute error and other existing methods.

Table 8. MAE and OC of Example 3 for different values of N.

N	Maximum Absolute Error	Order of Convergence
8	6.78226×10^{-6}	5.62071
16	1.37839×10^{-7}	5.85592
32	2.37992×10^{-9}	5.96743
64	3.80354×10^{-11}	

Table 9. Pointwise absolute error of Example 3 for $N = 20$.

t	Exact Solution	Numerical Solution by Proposed Method	Absolute Error by Proposed Method	Absolute Error [2]
1.00	5.00000	5.00000	0.000000000	0.000000000
1.10	4.09477	4.09477	2.97837×10^{-8}	1.60920×10^{-4}
1.20	3.31671	3.31671	3.79969×10^{-8}	3.06556×10^{-4}
1.30	2.64961	2.64961	3.56229×10^{-8}	3.98072×10^{-4}
1.40	2.07798	2.07798	3.00196×10^{-8}	4.32761×10^{-4}
1.50	1.58798	1.58798	2.37521×10^{-8}	4.19774×10^{-4}
1.60	1.16762	1.16762	1.77670×10^{-8}	3.70749×10^{-4}
1.70	0.80667	0.80667	1.23790×10^{-8}	2.96408×10^{-4}
1.80	0.49644	4.96442	7.65325×10^{-9}	2.05565×10^{-4}
1.90	0.22961	2.29614	3.55810×10^{-9}	1.05058×10^{-4}
2.00	0.00000	0.00000	0.000000000	0.000000000

Example 4. *Consider the following second-order differential equation:*

$$tw''(t) + w'(t) = t, \qquad t \in [1,\ 2],$$

with the following boundary conditions:

$$w(1) = 1, \qquad w(2) = 1.$$

The exact solution of Example 4 is as follows:

$$w(t) = \frac{t^2}{4} - \frac{3\ln t}{4\ln 2} + \frac{3}{4}.$$

Table 10 presents the MAE and the order of convergence for Example 4 across different mesh points. Table 11 provides a comparison of the pointwise absolute error with other existing methods.

Table 10. MAE and OC of Example 4 for different values of N.

N	Maximum Absolute Error	Order of Convergence
8	1.22945×10^{-7}	6.40126
16	1.45458×10^{-9}	6.20488
32	1.97189×10^{-11}	5.97484
64	3.13527×10^{-13}	

Table 11. Pointwise absolute error of Example 4 for $N = 20$.

t	Exact Solution	Numerical Solution by Proposed Method	Absolute Error by Proposed Method	Absolute Error [2]
1.00	1.000000	1.000000	0.000000000	0.000000000
1.10	0.949372	0.949372	2.89599×10^{-10}	2.38675×10^{-5}
1.20	0.912724	0.912724	3.582189×10^{-10}	3.66902×10^{-5}
1.30	0.888616	0.888616	3.568669×10^{-10}	4.21471×10^{-5}
1.40	0.875930	0.875930	3.247159×10^{-10}	4.25917×10^{-5}
1.50	0.873778	0.873778	2.778369×10^{-10}	3.95759×10^{-5}
1.60	0.881446	0.881446	2.242379×10^{-10}	3.41494×10^{-5}
1.70	0.898349	0.898349	1.681079×10^{-10}	2.70371×10^{-5}
1.80	0.924002	0.924002	1.116069×10^{-10}	1.87491×10^{-5}
1.90	0.958000	0.958000	5.583129×10^{-11}	9.64910×10^{-6}
2.00	1.000000	1.000000	0.000000000	0.000000000

6. Conclusions

In this manuscript, we introduce a sixth-order cubic B-spline method (CBSM) designed to solve second-order linear boundary value problems (BVPs) on a uniform mesh. The proposed method is applied to four linear BVPs to demonstrate its effectiveness. We provide a theoretical proof of the method's convergence, showing that it achieves sixth-order accuracy. This theoretical result is further validated through numerical examples, which confirm the high accuracy of the method. Additionally, we compare the numerical results obtained with the proposed CBSM against those from existing methods [2,3], using the same number of mesh points. The comparison highlights the superior performance of our method, establishing it as a more effective solution for these types of problems.

Author Contributions: Conceptualization, R.K.L. and A.S.; methodology, M.S.D.; software, A.A. and K.R.; formal analysis, L.K.; writing—original draft preparation, R.K.L.; writing—review and editing, K.R. and L.K.; funding acquisition, M.S.D. All authors have read and agreed to the published version of the manuscript.

Funding: The authors extend their appreciation to the Deanship of Scientific Research at Northern Border University, Arar, KSA, for funding this research work through the project number "NBU-FPEJ-2024-2947-02".

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors confirm that there are no conflicts of interest to declare for this publication.

References

1. Bickley, W.C. Piecewise cubic interpolation and two-point boundary value problems. *Comput. J.* **1968**, *11*, 206–208. [CrossRef]
2. Munguia, M.; Bhatta, D. Use of cubic B-spline in approximating solutions of boundary value problem. *Appl. Appl. Math. Int. J.* **2015**, *10*, 750–771.
3. Latif, B.; Abdul Karim, S.A.; Hashim, I. New cubic B-spline approximation for solving linear two-point boundary-value problems. *Mathematics* **2021**, *9*, 1250. [CrossRef]
4. Akram, G.; Rehman, H.U. Numerical solution of eighth order boundary value problems in reproducing Kernel space. *Numer. Algorithms* **2013**, *62*, 527–540. [CrossRef]
5. Akram, G.; Siddiqi, S.S. Solution of sixth order boundary value problems using non-polynomial spline technique. *Appl. Math. Comput.* **2006**, *181*, 708–720. [CrossRef]
6. Hussin, C.H.C.; Kiliçman, A. On the solutions of nonlinear higher-order boundary value problems by using differential transformation method and adomian decomposition method. *Math. Probl. Eng.* **2011**, *2011*, 724927. [CrossRef]
7. Khan, I.; Raja, M.A.Z.; Khan, M.A.R.; Shoaib, M.; Islam, S.; Shah, Z. Design of backpropagated intelligent networks for nonlinear second-order Lane-Emden pantograph delay differential systems. *Arab. J. Sci. Eng.* **2022**, *47*, 1197–1210. [CrossRef]

8. Sabir, Z.; Baleanu, D.; Ali, M.R.; Sadat, R. A novel computing stochastic algorithm to solve the nonlinear singular periodic boundary value problems. *Int. J. Comput. Math.* **2022**, *99*, 2091–2104. [CrossRef]
9. Kumar, M. A fourth-order spline finite difference method for singular two-point boundary value problems. *Int. J. Comput. Math.* **2003**, *80*, 1499–1504. [CrossRef]
10. Caglar, H.; Caglar, N.; Akkoyunlu, C. Non-polynomial spline method of a non-linear system of second-order boundary value problems. *J. Comput. Anal. Appl.* **2010**, *12*, 544–559.
11. Caglar, S.H.; Ucar, M.F. Non-polynomial spline method for a time dependent heat like Lane Emden equation. *Acta Phys. Pol. A* **2012**, *121*, 262. [CrossRef]
12. Elahi, Z.; Akram, G.; Siddiqi, S.S. Numerical solution for solving special eighth-order linear boundary value problems using Legendre Galerkin method. *Math. Sci.* **2016**, *10*, 201–209. [CrossRef]
13. Iqbal, M.; Rehman, S.; Pervaiz, A.; Hakeem, A. Approximations for linear tenth-order boundary value problems through polynomial and non-polynomial cubic spline techniques. *Proc. Pak. Acad. Sci.* **2015**, *52*, 389–396.
14. Khalid, A.; Naeem, M.; Agarwal, P.; Ghaffar, A.; Ullah, Z.; Jain, S. Numerical approximation for the solution of linear sixth order boundary value problems by cubic B-spline. *Adv. Differ. Equ.* **2019**, *2019*, 492. [CrossRef]
15. Khalid, A.; Naeem, M.N. Cubic B-spline solution of nonlinear sixth order boundary value problems. *J. Math.* **2018**, *50*, 91–103.
16. Khalid, A.; Naeem, M.N. Cubic Spline Solution of linear fourteenth order boundary value problems. *Ceylon J. Sci.* **2018**, *47*, 253–261. [CrossRef]
17. Khalid, A.; Naeem, M.N.; Ullah, Z.; Ghaffar, A.; Baleanu, D.; Nisar, K.S.; Al-Qurashi, M.M. Numerical solution of the boundary value problems arising in magnetic fields and cylindrical shells. *Mathematics* **2019**, *7*, 508. [CrossRef]
18. Loghmani, G.B.; Ahmadinia, M. Numerical solution of sixth order boundary value problems with sixth degree B-spline functions. *Appl. Math. Comput.* **2007**, *186*, 992–999. [CrossRef]
19. Khan, H.; Ullah, I.; Ali, J.; Jamshed, W.; Aziz, A.; Shah, S.I. The solution of twelfth order boundary value problems by the improved residual power series method: New approach. *Int. J. Model. Simul.* **2022**, *43*, 64–74. [CrossRef]
20. Ramadan, M.; Lashien, I.; Zahra, W.K. Polynomial and nonpolynomial spline approaches to the numerical solution of second order boundary value problems. *Appl. Math. Comput.* **2007**, *184*, 476–484. [CrossRef]
21. Tassaddiq, A.; Khalid, A.; Naeem, M.N.; Ghaffar, A.; Khan, F.; Karim, S.A.A.; Nisar, K.S. A New Scheme Using Cubic B-Spline to Solve Non-Linear Differential Equations Arising in Visco-Elastic Flows and Hydrodynamic Stability Problems. *Mathematics* **2019**, *7*, 1078. [CrossRef]
22. Roul, P.; Prasad-Goura, V.M.K.; Kassner, K. A high accuracy numerical approach for electro-hydrodynamic flow of a fluid in an ion-drag configuration in a circular cylindrical conduit. *Appl. Numer. Math.* **2021**, *165*, 303–321. [CrossRef]
23. Malele, J.; Dlamini, P.; Simelane, S. Highly accurate compact finite difference schemes for two-point boundary value problems with robin boundary conditions. *Symmetry* **2022**, *14*, 1720. [CrossRef]
24. Zhanlav, T.; Batgerel, B.; Otgondorj, K.; Buyantogtokh, D.; Ulziibayar, V.; Mijiddorj, R.O. Higher-Order Finite-Difference Schemes for Nonlinear Two-Point Boundary Value Problems. *J. Math. Sci.* **2024**, *279*, 850–865. [CrossRef]
25. Setia, N.; Mohanty, R.K. A high accuracy variable mesh numerical approximation for two-point nonlinear BVPs with mixed boundary conditions. *Soft Comput.* **2022**, *26*, 9805–9821. [CrossRef]
26. Liu, C.-S.; Chang, C.-W. Lie-group shooting/boundary shape function methods for solving nonlinear boundary value problems. *Symmetry* **2022**, *14*, 778. [CrossRef]
27. Dallashi, Q.; Syam, M.I. An efficient method for solving second-order fuzzy order fuzzy initial value problems. *Symmetry* **2022**, *14*, 1218. [CrossRef]
28. Latif, B.; Misro, M.Y.; Abdul Karim, S.A.; Hashim, I. An improved symmetric numerical approach for systems of second-order two-point BVPs. *Symmetry* **2023**, *15*, 1166. [CrossRef]
29. Liu, C.-Y.; Ku, C.-Y. A novel ANN-based radial basis function collocation method for solving elliptic boundary value problems. *Mathematics* **2023**, *11*, 3935. [CrossRef]
30. Ali, I.; Yaseen, M.; Abdullah, M.; Khan, S.; Belgacem, F.B.M. An innovative numerical method utilizing novel cubic B-spline approximations to solve burgers' equation. *Mathematics* **2023**, *11*, 4079. [CrossRef]
31. Prenter, P.M. *Spline and Variational Methods*; Wiley: New York, NY, USA, 1975.
32. Lodhi, R.K.; Mishra, H.K. Quintic B-spline method for solving second order linear and nonlinear singularly perturbed two-point boundary value problems. *J. Comput. Appl. Math.* **2017**, *319*, 170–187. [CrossRef]
33. Roul, P.; Kiran, T. A fourth-order B-spline collocation method and its error analysis for Bratu-type and Lane-Emden problems. *Int. J. Comput. Math.* **2019**, *96*, 85–104. [CrossRef]
34. Kadalbajoo, M.K.; Patidar, K.C. A survey of numerical techniques for solving singularly perturbed ordinary differential equations. *Appl. Math. Comput.* **2002**, *130*, 457–510. [CrossRef]
35. Majeed, A.; Kamran, M.; Asghar, N.; Baleanu, D. Numerical approximation of inhomogeneous time fractional Burgers-Huxley equation with B-spline functions and Caputo derivative. *Eng. Comput.* **2022**, *38*, 885–900. [CrossRef]
36. Shafiq, M.; Abbas, M.; Abdullah, F.A.; Majeed, A.; Abdeljawad, T.; Alqudah, M.A. Numerical solutions of time fractional Burgers' equation involving Atangana-Baleanu derivative via cubic B-spline functions. *Results Phys.* **2022**, *34*, 105244. [CrossRef]

37. Boor, C.D. *A Practical Guide to Splines*; Springer: New York, NY, USA, 1978.
38. Locas, T.R. Error bounds for interpolating cubic splines under various end conditions. *SIAM J. Numer. Anal.* **1974**, *11*, 569–584. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.