

Article

Polynomials Counting Nowhere-Zero Chains Associated with Homomorphisms

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Abstract: A regular matroid M on a finite set E is represented by a totally unimodular matrix. The set of vectors from $\mathbb{Z}^E$ orthogonal to rows of the matrix form a regular chain group N . Assume that ψ is a homomorphism from N into a finite additive Abelian group A and let $A_\psi[N]$ be the set of vectors g from $(A-0)^E$, such that $\sum_{e \in E} g(e) \cdot f(e) = \psi(f)$ for each $f \in N$ (where $\cdot$ is a scalar multiplication). We show that $|A_\psi[N]|$ can be evaluated by a polynomial function of $|A|$. In particular, if $\psi(f) = 0$ for each $f \in N$, then the corresponding assigning polynomial is the classical characteristic polynomial of M .

Keywords: regular matroid; regular chain group; totally unimodular matrix; homomorphism; assigning polynomial

MSC: 05C31; 05B35

1. Introduction

Nowhere-zero flows on graphs present a dual concept to graph coloring problems. Numbers of nowhere-zero group-valued flows on graphs are evaluated by flow polynomials (see cf. [1,2]).

A regular matroid M on a finite set E is represented by a totally unimodular matrix D , an integral matrix, such that the determinant of each minor is either ± 1 or 0. For example, graphic and cographic matroids are regular. The set of integral vectors orthogonal to each row of D is a regular chain group N . Analogously, we can define $A(N)$ considering vectors with coordinates from a finite Abelian group A with additive notation. Nowhere-zero elements of $A(N)$ generalize the notion of nowhere-zero group-valued flows on graphs, and their numbers are evaluated by characteristic polynomials of regular matroids (see [3,4]).

There are many generalizations of flows and flow polynomials on graphs (see cf. [1,5–15]). Recently, we have introduced polynomials counting nowhere-zero chains in graphs—nonhomogeneous analogues of nowhere-zero flows (see [16]). The aim of this paper is to generalize this approach for regular matroids. For this reason, we need to deal with group-valued vectors accompanied with homomorphisms.

Let $\psi : N \rightarrow A$ be a homomorphism. By an (N, ψ) -chain, we mean a vector g indexed by E , with coordinates from A , and satisfying $\sum_{e \in E} g(e) \cdot f(e) = \psi(f)$ for each $f \in N$ (where $\cdot$ denotes a scalar multiplication). An (N, ψ) -chain g is called nowhere-zero if $g(e) \neq 0$ for each $e \in E$. In case $\psi_0(f) = 0$ for each $f \in N$, then (N, ψ_0) -chains coincide with vectors from $A(N^\perp)$, where $N^\perp$ denotes the regular chain group orthogonal to N (the set of integral vectors orthogonal to each element of N).

We introduce a polynomial $p(n)$ that depends on the regular matroid M associated with N and a zero-one mapping α on the set of circuits of M determined by N and ψ . In the main result of this paper, Theorem 1 proves that $p(n)$ equals the number of nowhere-zero (N, ψ) -chains for each regular chain group N associated with M , each homomorphism ψ associated with α , and each Abelian group A of order n (where $\psi : N \rightarrow A$). Polynomial $p(n)$ reduces to the characteristic polynomial of M (see [17,18]) if $\psi_0(f) = 0$ for each $f \in N$.



Citation: Kochol, M. Polynomials Counting Nowhere-Zero Chains Associated with Homomorphisms. *Mathematics* **2024**, *12*, 3218. <https://doi.org/10.3390/math12203218>

Academic Editor: Bo Zhou

Received: 4 September 2024

Revised: 3 October 2024

Accepted: 4 October 2024

Published: 14 October 2024



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Theorem 1 also introduces a recursive formula that generalizes the well-known deletion–contraction rule of the characteristic polynomial. Another formula for the polynomial $p(n)$ (using the concept of compatible sets as presented in [19]) is introduced in Theorem 2.

In the last section, we discuss applications of our results for graphic and cographic matroids and show how the results introduced here correspond with the polynomials studied in [16].

2. Preliminaries

In this section, we recall some basic properties of regular matroids and regular chain groups presented in [3,4,20–24].

Throughout this paper, E denotes a finite nonempty set. The collection of mappings from E to a set S is denoted by S^E . If R is a ring, the elements of R^E are considered as vectors indexed by E , and we will use notations $f + g$, $-f$, and sf for $f, g \in R^E$, and $s \in R$. A *chain* on E (over R , or simply an R -chain) is $f \in R^E$, and the *support* of f is $\sigma(f) = \{e \in E; f(e) \neq 0\}$. We say that f is *proper* if $\sigma(f) = E$. The *zero chain* has null support. Given $X \subseteq E$ and $f \in R^E$, let $f^{\setminus X} \in R^{E \setminus X}$, such that $f^{\setminus X}(e) = f(e)$ for each $e \in E \setminus X$.

A matroid M on E of rank $r(M)$ is *regular* if there exists an $r \times n$ ($r = r(M)$, $n = |E|$) totally unimodular matrix D (called a *representative* matrix of M), such that independent sets of M correspond to independent sets of columns of D . For any basis B of M , D can be transformed to a form $(I_r | U)$, such that I_r corresponds to B and U is totally unimodular. The dual of M is a regular matroid M^* with a representative matrix $(-U^T | I_{n-r})$, where I_{n-r} corresponds to $E \setminus B$.

A *regular chain group* N on E associated with D is the set of chains on E over $\mathbb{Z}$ that are orthogonal to each row of D . Throughout this paper, we always assume that a regular chain group N is associated with a matrix $D = D(N)$ representing a matroid $M = M(N)$. In fact, a regular matroid M can be represented by different matrices D (see cf. [20,21] for more details).

The set of chains orthogonal to every chain of N is a chain group called *orthogonal* to N and is denoted by $N^\perp$ ($N^\perp$ is the set of integral vectors from the linear hull of the rows of D). By *rank* of N , we mean $r(N) = n - r(M) = r^*(M)$. Then, $r(N^\perp) = n - r(N) = r(M)$ (where M is the regular matroid having the representative matrix D).

For any $X \subseteq E$, define by

$$\begin{aligned} N - X &= \{f^{\setminus X}; f \in N, \sigma(f) \cap X = \emptyset\}, \\ N/X &= \{f^{\setminus X}; f \in N\}. \end{aligned} \quad (1)$$

Clearly, $M(N - X) = M - X$ and $D(N - X)$ arises from $D(N)$ after deleting the columns corresponding to X . Furthermore, $(N - X)^\perp = N^\perp / X$, $(N/X)^\perp = N^\perp - X$, and $M(N/X) = M/X$ (where $M - X$ and M/X denote the deletion and contraction of X from M , respectively, see cf. [20]).

For any $X \subseteq E$, denote by $\chi_X \in \mathbb{Z}^E$ such that $\chi_X(e) = 1$ for each $e \in X$ and $\chi_X(e) = 0$ for each $e \in E \setminus X$. We say that $e \in E$ is a *loop* (*isthmus*) of N if $\chi_e \in N$ ($\chi_e \in N^\perp$), i.e., if e is a loop (*isthmus*) of $M(N)$.

A chain f of N is *elementary* if there is no nonzero g of N such that $\sigma(g) \subset \sigma(f)$. An elementary chain f is called a *primitive* chain of N if the coefficients of f are restricted to the values 0, 1, and -1 . We say that a chain g *conforms* to a chain f if $g(e)$ and $f(e)$ are nonzero and have the same sign for each $e \in E$ such that $g(e) \neq 0$. By ([22] Formula (6.2)) and ([23] Formula (5.43)),

$$\text{every chain } f \text{ of } N \text{ can be expressed as a sum of} \quad (2)$$

primitive chains in N that conform to f .

Let A be an Abelian group with additive notation. We shall consider A as a (right) $\mathbb{Z}$ -module such that the scalar multiplication $a \cdot z$ of $a \in A$ by $z \in \mathbb{Z}$ is equal to 0 if $z = 0$, $\sum_1^z a$ if $z > 0$, and $\sum_1^{-z}(-a)$ if $z < 0$. Similarly, if $a \in A$ and $f \in \mathbb{Z}^E$, then define $a \cdot f \in A^E$ so that $(a \cdot f)(e) = a \cdot f(e)$ for each $e \in E$. If N is a regular chain group on E , define by

$$\begin{aligned} A(N) &= \{\sum_{i=1}^m a_i \cdot f_i; a_i \in A, f_i \in N, m \geq 1\}, \\ A[N] &= \{f \in A(N); \sigma(f) = E\}. \end{aligned}$$

Notice that $\mathbb{Z}(N) = N$ and $\mathbb{R}(N)$ coincides with the linear hull of N , which has a dimension of $r(N)$. By ([3] Proposition 1),

$$\begin{aligned} g \in A^E \text{ is from } A(N) \text{ if and only if for each } f \in N^\perp, \\ \sum_{e \in E} g(e) \cdot f(e) = 0. \end{aligned} \quad (3)$$

A *unimodular basis* of N is any basis of $\mathbb{R}(N)$ such that the matrix whose rows are the row vectors of the basis is totally unimodular.

Lemma 1. Let $\{q_1, \dots, q_r\}$ be a unimodular basis of N , $r = r(N)$. Then, for each $f \in N$, there are unique integers $z_1, \dots, z_r$ such that $f = \sum_{i=1}^r z_i q_i$.

Proof. Let B be a base of $M(N)$ and D' denote the matrix with rows $q_1, \dots, q_r$. By assumptions, D' is totally unimodular and the columns corresponding to $E \setminus B$ form a regular submatrix C of D' (notice that $|E \setminus B| = r$). Thus, C is totally unimodular, C^{-1} is integral, and $C^{-1}D'$ has form $(I_r | U)$ where I_r corresponds to $E \setminus B$ and U is totally unimodular. Each $f \in N$ is a linear combination of the rows of D' , and thus also of $C^{-1}D' = (I_r | U)$, where $f = f^{\setminus B}(I_r | U) = f^{\setminus B}C^{-1}D'$. This implies the existence of integers $z_1, \dots, z_r$ (where $(z_1, \dots, z_r) = f^{\setminus B}$) and that the uniqueness follows from properties of bases. $\square$

3. Assigning Polynomials

A *homomorphism* of a regular chain group N into A is a mapping $\psi : N \rightarrow A$ such that $\psi(f_1) + \psi(f_2) = \psi(f_1 + f_2)$ for $f_1, f_2 \in N$ (it is well known that $\psi(0) = 0$, $\psi(-f) = -\psi(f)$, and $\psi(zf) = \psi(f) \cdot z$ for each $f \in N$ and $z \in \mathbb{Z}$). By Lemma 1, ψ is uniquely determined by its values on a unimodular basis $Q = \{q_1, \dots, q_r\}$ of N , i.e., $\psi(f) = \sum_{i=1}^r \psi(q_i) \cdot z_i$ if $f = \sum_{i=1}^r z_i q_i$ and $z_i \in \mathbb{Z}$ for $i \in \{1, \dots, r\}$. Define by

$$\begin{aligned} A_\psi(f) &= \{g \in A^E; \sum_{e \in E} g(e) \cdot f(e) = \psi(f)\}, & f \in N, \\ A_\psi[f] &= \{g \in A_\psi(f); \sigma(g) = E\}, & f \in N, \\ A_\psi(S) &= \bigcap_{f \in S} A_\psi(f), & S \subseteq N, \\ A_\psi[S] &= \bigcap_{f \in S} A_\psi[f], & S \subseteq N. \end{aligned} \quad (4)$$

By Lemma 1, $A_\psi(N) = A_\psi(Q)$ and $A_\psi[N] = A_\psi[Q]$. Elements from $A_\psi(N)$ we call (N, ψ) -chains and elements from $A_\psi[N]$ we call *proper* (N, ψ) -chains. By (3), $A(N^\perp) = A_{\psi_0}(N)$ and $A[N^\perp] = A_{\psi_0}[N]$ where $\psi_0(f) = 0$ for each $f \in N$.

Notice that homomorphisms into integers are studied also in ([22], Section 7).

Denote by $\mathcal{P}(N)$ the set of primitive chains of N and let $\mathcal{C}(M)$ denote the family of circuits of $M = M(N)$. As pointed out in [22,23],

$$\mathcal{C}(M) = \{\sigma(c); c \in \mathcal{P}(N)\}. \quad (5)$$

An *assigning* of M is any mapping α from $\mathcal{C}(M)$ to $\{0, 1\}$. We write $\alpha \equiv 0$ if $\alpha(C) = 0$ for each $C \in \mathcal{C}(M)$.

Let $e \in E$. Then, $\mathcal{C}(M-e) \subseteq \mathcal{C}(M)$ (in fact, $\mathcal{C}(M-e) = \{C \in \mathcal{C}(M); e \notin C\}$), and for any assigning α of M , we can define an assigning $\alpha[-e]$ of $M-e$ so that $\alpha[-e](C) = \alpha(C)$ for each $C \in \mathcal{C}(M-e)$. By ([20] Proposition 3.1.1),

if $C \in \mathcal{C}(M/e)$, then either $C \in \mathcal{C}(M)$, or $C \cup e \in \mathcal{C}(M)$. (6)

Thus, for any $C \in \mathcal{C}(M/e)$, denote by C_e the unique element of $\mathcal{C}(M)$ such that either $C_e = C$ or $C_e = C \cup e$. Define by $\alpha[/math>]/e] an assigning of M/e such that $\alpha[/math>]/e](C) = $\alpha(C_e)$ for each $C \in \mathcal{C}(M/e)$. Notice that $M/e = M - e$ and $\alpha[/math>]/e] = $\alpha[-e]$ if e is an isthmus or a loop of M .$$$

Given a regular chain group N and a homomorphism $\psi : N \rightarrow A$, let $\alpha_{N,\psi}$ denote the assigning of $\mathcal{C}(M)$, $M(N) = M$, such that for each $c \in \mathcal{P}(N)$, $\alpha_{N,\psi}(\sigma(c)) = 0$ if $\psi(c) = 0$ and $\alpha_{N,\psi}(\sigma(c)) = 1$ if $\psi(c) \neq 0$. Notice that $\alpha_{N,\psi}$ is well defined because if $c, c' \in \mathcal{P}(N)$ and $\sigma(c) = \sigma(c')$, then either $c = c'$ or $-c = c'$ (otherwise, at least one of $\sigma(c - c')$ or $\sigma(c + c')$ is a proper nonzero subset of $\sigma(c)$ —a contradiction with the fact that c is a primitive chain) and $\alpha_{N,\psi}(\sigma(-c)) = \alpha_{N,\psi}(\sigma(c))$. We say that an assigning α of M is *homogeneous* if $\alpha = \alpha_{N,\psi}$ for some homomorphism $\psi : N \rightarrow A$, $M(N) = M$. Moreover, if $A_\psi[N] \neq \emptyset$, ψ is called *proper* and $\alpha = \alpha_{N,\psi}$ is called a *proper assigning* of M .

Denote by I_M the set of isthmuses of M . Let $e \in E \setminus I_M$ and N be a regular chain group, $M(N) = M$. Then, there exists a basis B of M^* such that $e \in B$, M^* has a representative matrix $D = (I_r, U)$ where I_r corresponds to B , U is totally unimodular, and N is associated with D . The rows of D correspond to primitive chains of N , forming a unimodular basis $\{q_1, \dots, q_r\}$ of N , $r = r(N)$. Moreover, we can assume that $q_r(e) = 1$, and in this case, $\{q_1, \dots, q_r\}$ is called a (B, e) -basis of N or simply a B -basis of N .

Then, $N - e$ has rank $r - 1$ and $\{q_1^e, \dots, q_{r-1}^e\}$ is a unimodular basis of $N - e$. If e is a loop of M (i.e., $q_r = \chi_e \in N$), then $N/e = N - e$ and $\{q_1^e, \dots, q_{r-1}^e\}$ is also a unimodular basis of N/e . If e is not a loop of M (i.e., $q_r \neq \chi_e \notin N$), then N/e has rank r and $\{q_1^e, \dots, q_r^e\}$ is a unimodular basis of N/e .

Assume that $\psi : N \rightarrow A$ is a homomorphism and $\alpha = \alpha_{N,\psi}$. By Lemma 1 and (4), there is a homomorphism $\psi[/math>]/e] : $N/e \rightarrow A$, such that $\psi[/math>]/e](q_i^e) = $\psi(q_i)$, $i = 1, \dots, r'$ (where $r' = r - 1$ if $q_r = \chi_e \in N$ and $r' = r$ otherwise), and $\alpha[/math>]/e] = $\alpha_{N/e,\psi[/math>]/e]}$. If $q_r \neq \chi_e \notin N$ (resp. $q_r = \chi_e \in N$), then $\psi[/math>]/e](f^e) = $\psi(f)$ for each $f \in N$ (resp. $f \in N/e = N - e$) and $\psi[/math>]/e] does not depend on the choice of the (B, e) -basis.$$$$$

Similarly, $\psi[-e] : N - e \rightarrow A$ is a homomorphism such that $\psi[-e](q_i^e) = \psi(q_i)$, $i = 1, \dots, r''$, and $\alpha[-e] = \alpha_{N-e,\psi[-e]}$. Also, now $\psi[-e](f^e) = \psi(f)$ for each $f \in N - e$.

Theorem 1. Suppose that M is a regular matroid on E and α is a homogeneous assigning of M . Then, there exists a polynomial $p(M, \alpha; k)$, such that $p(M, \alpha; k) = |A_\psi[N]|$ for every regular chain group N on E and $M(N) = M$, every Abelian group A of order k , and every homomorphism $\psi : N \rightarrow A$ satisfies $\alpha_{N,\psi} = \alpha$. If α is proper, then $p(M, \alpha; k)$ has degree $r(M)$, and $p(M, \alpha; k) = 0$ otherwise. Furthermore, for any $e \in E$, $\alpha[-e]$ and $\alpha[/math>]/e] are homogeneous assignments of $M - e$ and M/e , respectively, and$

$$\begin{aligned} p(M, \alpha; k) &= 1, & \text{if } E = \emptyset, \\ p(M, \alpha; k) &= (k - 1)p(M - e, \alpha[-e]; k), & \text{if } e \in I_M, \\ p(M, \alpha; k) &= \alpha(e)p(M - e, \alpha[-e]; k), & \text{if } e \in I_M^*, \\ p(M, \alpha; k) &= p(M - e, \alpha[-e]; k) - p(M/e, \alpha[/math>]/e]; k), & \text{otherwise.} \end{aligned} \quad (7)$$

Proof. Since α is homogeneous, there exist a regular chain group N on E and a homomorphism $\psi : N \rightarrow A$, such that $M(N) = M$ and $\alpha = \alpha_{N,\psi}$.

We use induction by $|E|$. If $|E| = 0$, then N consists of zero chains, $\mathcal{C}(M) = \emptyset$, and $p(M, \alpha; k) = 1$.

Let $E \neq \emptyset$ and $e \in E \setminus I_M$. We can choose a basis B of M^* , $e \in B$, and a (B, e) -basis $\{q_1, \dots, q_r\}$ of N , i.e., $q_r(e) = 1$, $r = r(N)$. Then, $\alpha[/math>]/e] = $\alpha_{N/e,\psi[/math>]/e]}$ and $\alpha[-e] = \alpha_{N-e,\psi[-e]}$, where $\alpha[/math>]/e] and $\alpha[-e]$ are homogeneous assignments of M/e and $M - e$, respectively.$$

By Lemma 1, $A_\psi[N] = \bigcap_{i=1}^r A_\psi[q_i]$ and $A_{\psi[/e]}[N/e] = \bigcap_{i=1}^{r'} A_{\psi[/e]}[q_i^{\setminus e}]$ where $r' = r(N/e)$ (as mentioned before, $r' = r$ if e is not a loop of M and $r' = r - 1$ if e is a loop of M). Similarly, $A_{\psi[-e]}[N-e] = \bigcap_{i=1}^{r-1} A_{\psi[-e]}[q_i^{\setminus e}]$ (because $r(N-e) = r-1$). Hence, $A_{\psi[/e]}[N/e] \subseteq A_{\psi[-e]}[N-e]$ and $\{f^{\setminus e}; f \in A_\psi[N]\} \subseteq A_{\psi[-e]}[N-e]$. For each $g \in A_{\psi[-e]}[N-e]$, there exists a unique $\tilde{g} \in A^E$, such that $\tilde{g}^{\setminus e} = g$ and

$$\sum_{e' \in E} \tilde{g}(e') \cdot q_r(e') = \psi(q_r), \quad (8)$$

i.e., $\tilde{g}(e) = \psi(q_r) - \sum_{e' \in E \setminus e} \tilde{g}(e') \cdot q_r(e')$ and $\tilde{g}(e') = g(e')$ for $e' \in E \setminus e$.

Assume that $e \notin I_{M^*}$, i.e., $q_r \neq \chi_r \notin N$, $r = r(N) = r(N/e)$, and $r(N-e) = r-1$. Let $g \in A_{\psi[-e]}[N-e]$. If $\tilde{g}(e) \neq 0$, then $\tilde{g} \in A_\psi[N]$ and $g \notin A_{\psi[/e]}[N/e]$, because $\sum_{e' \in E \setminus e} \tilde{g}(e') \cdot q_r(e') \neq \psi(q_r)$ by (8). If $\tilde{g}(e) = 0$, then $g \in A_{\psi[/e]}[N/e]$ and $\tilde{g} \notin A_\psi[N]$, because $\sigma(\tilde{g}) = E \setminus e$. Thus,

$$|A_{\psi[-e]}[N-e]| = |A_\psi[N]| + |A_{\psi[/e]}[N/e]|. \quad (9)$$

Let α be proper. Then, by (9), $\alpha[-e]$ is also proper. By the induction hypothesis and (9), $|A_\psi[N]| = p(M-e, \alpha[-e]; k) - p(M/e, \alpha[/e]; k)$ (no matter whether $\alpha[/e] is proper or not). Thus, $p(M, \alpha; k) = p(M-e, \alpha[-e]; k) - p(M/e, \alpha[/e]; k)$ has degree $r(M)$ (because $r(M) = r(M-e) = r(M/e) + 1$ in this case). If α is not proper, then $A_\psi[N] = \emptyset$, where by (9) and the induction hypothesis, $p(M/e, \alpha[/e]; k) = p(M-e, \alpha[-e]; k)$. Therefore, $p(M, \alpha; k) = p(M-e, \alpha[-e]; k) - p(M/e, \alpha[/e]; k) = 0$.$

If $e \in I_{M^*}$, then $r(M) = r(M/e) = r(M-e)$, $q_r = \chi_e$, $A_{\psi[-e]}[N-e] = A_{\psi[/e]}[N/e]$, and by (8), $\tilde{g}(e) = \psi(\chi_e)$ for each $g \in A_{\psi[-e]}[N-e]$. If $\psi(\chi_e) = 0$, then $A_\psi[N] = \emptyset$. If $\psi(\chi_e) \neq 0$, then $|A_\psi[N]| = |A_{\psi[-e]}[N-e]| = |A_{\psi[/e]}[N/e]|$. Hence, $p(M, \alpha; k) = \alpha(e)p(M-e, \alpha[-e]; k)$ and $p(M, \alpha; k)$ has degree $r(M)$ if α is proper.

Let $e \in I_M$, i.e., $\chi_e \in N^\perp$, $f(e) = 0$ for each $f \in N$, and $r(M) = r(M-e) + 1$. Hence, by (4), $p(M, \alpha; k) = (k-1)p(M-e, \alpha[-e]; k)$ and α is proper if and only if $\alpha[-e]$ is proper. Thus, the statement holds true in this case. $\square$

We call $p(M, \alpha; k)$ an α -assigning polynomial of M .

Corollary 1. For any regular matroid M on E , there exists a polynomial $p_M(k)$ such that $p_M(k) = |A[N^\perp]|$ for any Abelian group A of order k and regular chain group N satisfying $M(N) = M$. Furthermore, $p_M(k)$ has degree $r(M)$ if M has no loop, $p_M(k) = 0$ if M has a loop, $p_M(k) = 1$ if $E = \emptyset$, and for any $e \in E$,

$$\begin{aligned} p_M(k) &= (k-1)p_{M-e}, & \text{if } e \text{ is an isthmus of } M, \\ p_M(k) &= p_{M-e}(k) - p_{M/e}(k), & \text{otherwise.} \end{aligned}$$

Proof. This follows from Theorem 1 after setting $p_M(k) = p(M, \alpha_0; k)$ for $\alpha_0 \equiv 0$. $\square$

Notice that $p_M(k)$ is the characteristic polynomial of M (see cf. [4,17–19]).

4. Another Formula for Assigning Polynomials

Let $<$ be a linear ordering of E . For any $X \subseteq E$, denote by $\min(X)$ and $\max(X)$ the minimal and maximal element of X with respect to $<$, respectively.

We say that $X \subseteq E$ is $(M, <)$ -compatible if $C \cap X \neq \{\min(C)\}$ for each $C \in \mathcal{C}(M)$. Clearly, no $(M, <)$ -compatible set can contain a loop of M . Denote by $\mathcal{E}(M, <)$ the family of all $(M^*, <)$ -compatible subsets of E .

Given $X \subseteq E$ and an assigning α of M , let $\delta(M, \alpha; X)$ be defined so that $\delta(M, \alpha; X) = 0$ if there exists $C \subseteq X$ and $C \in \mathcal{C}(M)$, such that $\alpha(C) = 1$, and $\delta(M, \alpha; X) = 1$ otherwise.

Theorem 2. Let α be a homogeneous assigning of a regular matroid M on E and $<$ be a linear ordering of E . Then,

$$p(M, \alpha; k) = \sum_{X \in \mathcal{E}(M, <)} \delta(M, \alpha; X) (-1)^{|X|} (k-1)^{r(M/X)}. \quad (10)$$

Proof. Since α is homogeneous, there exist a regular chain group N on E and a homomorphism $\psi : N \rightarrow A$ such that $M(N) = M$ and $\alpha = \alpha_{N, \psi}$.

We use the induction on $|E \setminus I_M|$. If $E = I_M$, then $\mathcal{E}(M, <) = \{\emptyset\}$, $\delta(M, \alpha; \emptyset) = 1$, and $p(M, \alpha; k) = (k-1)^{r(M)}$ as claimed.

If $E \neq I_M$, choose $e = \max(E \setminus I_M)$ and denote by $\mathcal{E}^+ = \{X \in \mathcal{E}(M, <); e \notin X\}$, $\mathcal{E}^- = \{X \in \mathcal{E}(M, <); e \in X\}$. We prove that

$$\begin{aligned} \mathcal{E}(M-e, <) &= \mathcal{E}^+, \\ \mathcal{E}(M/e, <) &= \{X' \setminus e; X' \in \mathcal{E}^-\}. \end{aligned} \quad (11)$$

Let $X \in \mathcal{E}(M-e, <)$. If $C \in \mathcal{C}(M^*)$, then by (5), $C = \sigma(c)$ where $c \in \mathcal{P}(N^\perp)$, and by (1) and (2), $\sigma(c^{\setminus e})$ is a disjoint union of $C_1, \dots, C_n \in \mathcal{C}(M^*/e)$ such that $c^{\setminus e} = \sum_{i=1}^n c_i$ and $\sigma(c_i) = C_i$, $i = 1, \dots, n$. Then, $e \neq \min(C) = \min(C_j)$ for some $j \in \{1, \dots, n\}$ (since $e = \max(E \setminus I_M)$), and if $X \cap C = \{\min(C)\}$, then $X \cap C_j = \{\min(C_j)\}$ —a contradiction because $X \in \mathcal{E}(M-e, <)$ and $C_j \in \mathcal{C}(M^*/e)$. Thus, $X \cap C \neq \{\min(C)\}$ for each $C \in \mathcal{C}(M^*)$, i.e., X is $(M^*, <)$ -compatible and $X \in \mathcal{E}^+$.

Let $X \in \mathcal{E}^+$. If $C \in \mathcal{C}(M^*/e)$, then by (6), either $C \cup e \in \mathcal{C}(M^*)$ or $C \in \mathcal{C}(M^*)$. If $C \cup e \in \mathcal{C}(M^*)$, then $X \cap C = X \cap (C \cup e) \neq \{\min(C \cup e)\} = \min(C)$ (because $e \notin X$ and $e = \max(E \setminus I_M)$). If $C \in \mathcal{C}(M^*)$, then $X \cap C \neq \{\min(C)\}$. Thus, X is $(M^*/e, <)$ -compatible and $X \in \mathcal{E}(M-e, <)$. This proves the first row of (11).

Let $X \in \mathcal{E}(M/e, <)$ and $C \in \mathcal{C}(M^*)$. If $e \in C$, then $e \in (X \cup e) \cap C \neq \{\min(C)\}$, because $e \neq \min(C)$. If $e \notin C$, we have $(X \cup e) \cap C = X \cap C \neq \{\min(C)\}$. Thus, $X \cup e$ is $(M^*, <)$ -compatible.

If $X' \in \mathcal{E}^-$, then for each $C \in \mathcal{C}(M^*-e)$, $(X' \setminus e) \cap C = X' \cap C \neq \{\min(C)\}$, where $X' \setminus e$ is $(M^*-e, <)$ -compatible. Thus, $\mathcal{E}(M/e, <) = \{X' \setminus e; X' \in \mathcal{E}^-\}$, concluding the proof of (11).

Let $e \notin I_{M^*}$. We claim that for each $X' \in \mathcal{E}^-$,

$$\delta(M/e, \alpha[e]; X' \setminus e) = \delta(M, \alpha; X'). \quad (12)$$

To prove the claim, consider $X' \in \mathcal{E}^-$. If $\delta(M, \alpha; X') = 0$, there exists $C \subseteq X$ such that $C \in \mathcal{C}(M)$ and $\alpha(C) = 1$. By (5), $C = \sigma(c)$ where $c \in \mathcal{P}(N)$, and by (1) and (2), $\sigma(c^{\setminus e})$ is a disjoint union of $C_1, \dots, C_n \in \mathcal{C}(M/e)$ such that $c^{\setminus e} = \sum_{i=1}^n c_i$ and $\sigma(c_i) = C_i$, $i = 1, \dots, n$. If $n = 1$, then by (6), either $C_1 \in \mathcal{C}(M)$ or $C_1 \cup e \in \mathcal{C}(M)$. Since $\psi(c) = \psi[e](c^{\setminus e})$, we obtain $\alpha[e](C_1) = \alpha(C) = 1$ and $\delta(M/e, \alpha[e]; X' \setminus e) = 0$.

If $n \geq 2$, then $C_i \notin \mathcal{C}(M)$ for each $i = 1, \dots, n$, where by (6), $C_i \cup e \in \mathcal{C}(M)$. We can assume that $c_1(e) = c_2(e)$, where $\sigma(c_1 - c_2) = C_1 \cup C_2$, and by applying (2) and (6) for $c_1 - c_2$, we obtain $C' \in \mathcal{C}(M)$ such that $C' \subseteq C_1 \cup C_2$. Thus, $C' = C$, $n = 2$, $e \notin C$, and $\psi(c) = \psi[e](c^{\setminus e}) = \psi[e](c_1) + \psi[e](c_2)$. Since $\alpha(C) = 1$, $\psi[e](c_1) + \psi[e](c_2) = \psi(c) \neq 0$, where either $\alpha[e](C_1) = 1$ or $\alpha[e](C_2) = 1$, i.e., $\delta(M/e, \alpha[e]; X' \setminus e) = 0$.

If $\delta(M/e, \alpha[e]; X' \setminus e) = 0$, there exists $C \subseteq X' \setminus e$ such that $C \in \mathcal{C}(M/e)$ and $\alpha[e](C) = 1$. By (6), either $C \in \mathcal{C}(M)$ or $C \cup e \in \mathcal{C}(M)$, i.e., either $\alpha(C) = \alpha[e](C) = 1$ or $\alpha(C \cup e) = \alpha[e](C) = 1$, i.e., $\delta(M, \alpha; X') = 0$. This proves (12).

For each $X \in \mathcal{E}(M-e, <) = \mathcal{E}^+$, $\delta(M-e, \alpha[-e]; X) = \delta(M, \alpha; X)$, and by ([19] Equation (9)), $r((M-e)/X) = r(M/X)$ (because $e \notin I_M, X$). Thus, by (7) and the induction hypothesis,

$$\begin{aligned} p(M, \alpha; k) &= p(M-e, \alpha[-e]; k) - p(M/e, \alpha[/e]; k) \\ &= \sum_{X \in \mathcal{E}(M-e, <)} \delta(M-e, \alpha[-e]; X) (-1)^{|X|} (k-1)^{r((M-e)/X)} \\ &\quad - \sum_{X \in \mathcal{E}(M/e, <)} \delta(M/e, \alpha[/e]; X) (-1)^{|X|} (k-1)^{r((M/e)/X)} \\ &= \sum_{X \in \mathcal{E}^+} \delta(M, \alpha; X) (-1)^{|X|} (k-1)^{r(M/X)} \\ &\quad - \sum_{X' \in \mathcal{E}^-} \delta(M, \alpha; X') (-1)^{|X' \setminus e|} (k-1)^{r(M/X')} \\ &= \sum_{X \in \mathcal{E}(M, <)} \delta(M, \alpha; X) (-1)^{|X|} (k-1)^{r(M/X)}. \end{aligned}$$

Let $e \in I_{M^*}$. Then, $M-e = M/e$, $\alpha[-e] = \alpha[/math] and $C \cap e = \emptyset$ for each $C \in \mathcal{C}(M^*)$, where $\mathcal{E}^+ = \{X' \setminus e; X' \in \mathcal{E}^-\}$. Denote by $g_{M, \alpha, <}(k)$ the right hand side of (10).$

If $\alpha(e) = 0$, then $\delta(M, \alpha; X') = \delta(M, \alpha; X' \setminus e)$ for each $X' \in \mathcal{E}^-$, where $g_{M, \alpha, <}(k) = 0$ (because $\mathcal{E}^+ = \{X' \setminus e; X' \in \mathcal{E}^-\}$). By the third row of (7), $p(M, \alpha; k) = 0$. Thus, $p(M, \alpha; k) = g_{M, \alpha, <}(k)$ and (10) holds true.

If $\alpha(e) = 1$, then $\delta(M, \alpha; X') = 0$ for each $X' \in \mathcal{E}^-$, where $g_{M-e, \alpha[-e], <}(k) = g_{M, \alpha, <}(k)$. By the induction hypothesis, $p(M-e, \alpha[-e]; k) = g_{M-e, \alpha[-e], <}(k)$, and by the third row of (7), $p(M, \alpha; k) = p(M-e, \alpha[-e]; k)$, i.e. (10) holds true. $\square$

Denote by $\mathcal{E}_1(M, \alpha, <) = \{X \in \mathcal{E}(M, <); \delta(M, \alpha; X) = 1\}$. In the proof of Theorem 2, it was in fact proved that

$$p(M, \alpha; k) = \sum_{X \in \mathcal{E}_1(M, \alpha, <)} (-1)^{|X|} (k-1)^{r(M/X)} \quad (13)$$

and that $|\mathcal{E}_1(M, <, \alpha)|$ satisfies the following recursive rules:

$$\begin{aligned} |\mathcal{E}_1(M, <, \alpha)| &= 1 && \text{if } E = \emptyset, \\ |\mathcal{E}_1(M, <, \alpha)| &= |\mathcal{E}_1(M-e, <, \alpha[-e])| && \text{if } e \in I_M, \\ |\mathcal{E}_1(M, <, \alpha)| &= 2|\mathcal{E}_1(M-e, <, \alpha[-e])| && \text{if } e \in I_{M^*}, \alpha(e) = 0, \\ |\mathcal{E}_1(M, <, \alpha)| &= |\mathcal{E}_1(M-e, <, \alpha[-e])| && \text{if } e \in I_{M^*}, \alpha(e) = 1, \\ |\mathcal{E}_1(M, <, \alpha)| &= |\mathcal{E}_1(M-e, <, \alpha[-e])| && \\ &\quad + |\mathcal{E}_1(M/e, <, \alpha[/e])| && \text{if } e \notin I_M \cup I_{M^*}. \end{aligned} \quad (14)$$

Theorem 3. A homogeneous assigning α of a regular matroid M is proper if and only if $\alpha(e) = 1$ for each loop e of M .

Proof. Necessity follows directly from (7). To prove sufficiency, assume that $\alpha(e) = 1$ for each $e \in I_{M^*}$. Then, $\delta(M, \alpha; X) = 0$ if $X \subseteq E$ and $X \cap I_{M^*} \neq \emptyset$. On the other hand, $r(M) > r(M/X)$ if $X \setminus I_{M^*} \neq \emptyset$. Thus, for each ordering $<$ of E and each nonempty $X \in \mathcal{E}(M, <)$, we have either $\delta(M, \alpha; X) = 0$ or $r(M/X) < r(M)$. Clearly, $\emptyset \in \mathcal{E}(M, <)$ and $\delta(\emptyset, \alpha) = 1$. Thus, the right hand side of (10) is the sum of powers of $(k-1)$ such that exactly one of them (corresponding to $X = \emptyset$) has the maximal possible degree $r(M)$. Hence, $p(M, \alpha; k) \neq 0$ and α is proper. $\square$

5. Assigning Polynomials for Graphic and Cographic Matroids

Let $G = (V, E)$ be a graph with the vertex set $V = V(G)$ and edge set $E = E(G)$. Consider an arbitrary (but fixed) orientation of G , i.e., for $e \in E$, one of the end vertices of e becomes an initial end vertex and the second one becomes a terminal end vertex of e .

For all $S \subseteq V$, we shall denote by ω_S the chain on E , defined by the following: for $e \in E$, $\omega_S(e) = 1$ if e has its initial end in S and its terminal end in $V \setminus S$, $\omega_S(e) = -1$ if e has its initial end in $V \setminus S$ and its terminal end in S , and $\omega_S(e) = 0$ otherwise. Denote by $\Delta(G)$ the regular chain group generated by $\{\omega_S; S \subseteq V\}$ (see [3,22]). The elements of $\Delta(G)$ are called *coboundaries* (but also *tensions* or *potential differences*) of G . The orthogonal chain group $(\Delta(G))^\perp$ is denoted by $\Gamma(G)$ and its elements are called *cycles* (and also *flows* or *circulations*) of G . The rank of $\Delta(G)$ is $r(G) = |V| - c(G)$, where $c(G)$ denotes the number of components of G and $\Gamma(G)$ has rank $m(G) = |E| - |V| + c(G)$ (see [3,20,24]).

Assume that T is a spanning (maximal) forest of G . Notice that in every connected component of G , the spanning forest is a spanning tree. Then, $|E(T)| = r(G)$.

Let $t \in E(T)$. Deleting t from T divides the component of T containing t into the two subtrees T_t and T'_t . Assume that t has its initial end in $V(T_t)$. Then, $\{\omega_{V(T_t)}; t \in E(T)\}$ is the $E(T)$ -basis of $\Delta(G)$.

Let $e \in E \setminus E(T)$. Then, $T \cup e$ contains exactly one circuit $C_{T,e}$. Denote by $c_{T,e}$ the chain on E such that for any edge e' of $C_{T,e}$, we have $c_{T,e}(e') = 1$ if e' is oriented in $C_{T,e}$ in the same way as e , $c_{T,e}(e') = -1$ if e' is oriented in $C_{T,e}$ in the opposite way as e , and as $c_{T,e}(e) = 0$ if e' is not covered by $C_{T,e}$. Then, $\{c_{T,e}; e \in E \setminus E(T)\}$ is the $(E \setminus E(T))$ -basis of $\Gamma(G)$.

Denote by $D_{M(G),T}$ the $r(G) \times |E|$ matrix whose row vectors are $\omega_{V(T_t)}$ and $t \in E(T)$. Then, regular chain group $\Gamma(G)$ is associated with matrix $D_{M(G),T}$, representing the cycle matroid $M(G)$ of G . Analogously, let $D_{M^*(G),T}$ be the $m(G) \times |E|$ matrix whose row vectors are $c_{T,e}$ and $e \in E \setminus E(T)$. Then, regular chain group $\Delta(G)$ is associated with matrix $D_{M^*(G),T}$, representing the bond matroid $M^*(G)$ of G .

Let A be an Abelian group and $\iota : E(T) \rightarrow A$. By Lemma 1, mapping ι defines a unique homomorphism $\psi : \Delta(G) \rightarrow A$ such that $\psi(\omega_{V(T_t)}) = \iota(t)$ for each $t \in E(T)$. By Theorem 1, $\alpha_{\Delta(G),\psi}$ induces polynomial $p(M^*(G), \alpha_{\Delta(G),\psi}; k)$ such that $|A_\psi[\Delta(G)]| = p(M^*(G), \alpha_{\Delta(G),\psi}; |A|)$.

Denote by $b : V \rightarrow A$ such that $b(v) = \psi(\omega_v)$ for every $v \in V(G)$. We claim that

$$\psi(\omega_S) = \sum_{v \in S} b(v) = \sum_{t \in E(T)} \omega_S(t) \iota(t) \quad (15)$$

for each $S \subseteq V$. Clearly, $\sum_{t \in E(T)} \omega_S(t) \iota(t)$ equals the sum of $\iota(e)$ where $e \in E(T)$ has its initial end in S minus the sum of $\iota(e)$ where $e \in E(T)$ has its terminal end in S . Using this fact and definition $b(v) = \psi(\omega_v)$, $v \in V(G)$, we conclude that (15) holds true if $S = \{v\}$. We can prove (15) by induction on $|S|$ applying the induction hypothesis for $S \setminus v$ and $\{v\}$. In particular, for each component H of G ,

$$\sum_{v \in V(H)} b(v) = 0. \quad (16)$$

Denote by $\Lambda(G)$ the family of subsets S of V such that ω_S is a primitive chain in $\Delta(G)$, i.e., by (5), $\mathcal{P}(\Delta(G)) = \{\omega_S; S \in \Lambda(G)\}$ and $\mathcal{C}(M^*(G)) = \{\sigma(\omega_S); S \in \Lambda(G)\}$. Moreover, we also assume that $\emptyset \in \Lambda(G)$ (though $\emptyset \notin \mathcal{C}(M^*(G))$). Let $\alpha_{G,b}$ denote the mapping $\Lambda(G) \rightarrow \{0, 1\}$ such that for any $S \in \Lambda(G)$, $\alpha_{G,b}(S) = 0$ if $\psi(\omega_S) = 0$ and $\alpha_{G,b}(S) = 1$ if $\psi(\omega_S) \neq 0$. For any $v \in V$, denote by D_v^+ (resp. D_v^-) the set of edges of G with its initial (resp. terminal) end v . Let $A_b[G]$ be the set of proper chains φ from A^E (i.e., $\sigma(\varphi) = E$) such that for each $v \in V$,

$$\sum_{e \in D_v^+} \varphi(e) - \sum_{e \in D_v^-} \varphi(e) = b(v),$$

or, in other words,

$$\sum_{e \in E} \varphi(e) \omega_v(e) = \psi(\omega_v).$$

Clearly, this is equivalent to the condition that for each $t \in E(T)$,

$$\sum_{e \in E} \varphi(e) \omega_{V(T_t)}(e) = \psi(\omega_{V(T_t)}).$$

Hence, by (4), $A_b[G] = A_\psi[\Delta(G)]$ and $\alpha_{G,b} = \alpha_{\Delta(G),\psi}$. In ([16] Theorem 1), we proved that $|A_b[G]| = F(G, \alpha_{G,b}; |A|)$, where $F(G, \alpha; k)$ is an α -assigning polynomial of G . Thus, $F(G, \alpha_{G,b}; k) = p(M^*(G), \alpha_{\Delta(G),\psi}; k)$. An analogous version of Theorem 2 is presented in ([16] Theorem 3).

In [16], we did not use homomorphisms and $\alpha = \alpha_{G,b}$ was determined directly from mappings $b : V \rightarrow A$ (not necessary satisfying (16)) such that for each $X \in \Lambda(G)$, $\alpha_{G,b} = 0$ if $\sum_{v \in X} b(v) = 0$, and $\alpha_{G,b} = 1$ otherwise. Similarly, we defined $\alpha_{/e}$ and α_{-e} in graphs G/e and $G-e$, respectively. We can check that if $b' : V \rightarrow A$ satisfies (16), then there exists a homomorphism $\psi' : \Delta(G) \rightarrow A$ such that $b'(v) = \psi'(\omega_v)$ for $v \in V$ and $\alpha_{G,b'} = \alpha_{\Delta(G),\psi'}$, but also $\alpha_{/e} = \alpha_{\Delta(G),\psi'}[-e] = \alpha_{(\Delta(G))/e,\psi'_{/e}}$. Similarly, $\alpha_{-e} = \alpha_{\Delta(G),\psi'}[e] = \alpha_{(\Delta(G))-e,\psi'_{-e}}$ if e is not a bridge of G . But the situation differs if e is a bridge of G , because then (16) is satisfied in the graph $G-e$ if and only if $\psi'(e) = 0$, or, in other words, $F(G-e, \alpha_{-e}; k) \neq 0$ implies that $F(G, \alpha; k) = 0$ and $F(G, \alpha; k) \neq 0$ implies that $F(G-e, \alpha_{-e}; k) = 0$ (see also ([16] Theorem 4)). Thus, in the cases of $E(G) = \emptyset$ and $\alpha = \alpha_{G,b}$, we have $F(G, \alpha; k) = 1$ if (16) holds true for b and $F(G, \alpha; k) = 0$ if (16) does not hold true for b . By ([16] Theorem 1), for any edge e of G ,

$$\begin{aligned} F(G, \alpha; k) &= (k-1)F(G-e, \alpha_{-e}; k) \text{ if } e \text{ is a loop of } G, \\ F(G, \alpha; k) &= F(G/e, \alpha_{/e}; k) - F(G-e, \alpha_{-e}; k) \text{ otherwise,} \end{aligned}$$

i.e., the contraction–deletion rule differs from (7) if e is a bridge of G . Anyway, if we define α from homomorphisms, we can apply Theorem 1.

Finally, by mapping $b : V \rightarrow A$ to satisfy (16) for each component of G and by using a spanning forest T of G , there exists a unique $\iota : E(T) \rightarrow A$ such that (15) holds true. Specifically, for $t \in E(T)$, define by

$$\iota(t) = \sum_{v \in V(T_t)} b(v) \quad (17)$$

(as mentioned previously, T_t denotes the component of $T-t$ with the initial end of t in $V(T_t)$). Clearly, (15) coincides with (17) if $S = V(T_t)$, $t \in V(T)$. Thus, $b(v) = \psi(\omega_v)$ for every $v \in V(G)$ and (15) holds true for each $S \subseteq V(G)$.

Example 1. Let C be a circuit of order 6 with vertices $v_1, \dots, v_6$ and edges $e_1, \dots, e_6$ such that e_i has ends v_i, v_{i+1} , $i = 1, \dots, 6$ (considering the sum mod 6). Then, $T = C - e_6$ is a spanning forest of C , and let ι and ι' be mappings from $E(T)$ to $\mathbb{Z}_5$ such that $\iota(e_1), \dots, \iota(e_5)$ equal 1, 2, 3, 2, 1, respectively, and $\iota'(e_1), \dots, \iota'(e_5)$ equal 1, 3, 4, 3, 1, respectively. Consider an orientation of C such that e_i has its initial end in v_i , $i = 1, \dots, 6$, and denote by ψ and ψ' the homomorphisms from $\Delta(C)$ to $\mathbb{Z}_5$ such that $\psi(\omega_{V(T_t)}) = \iota(t)$ and $\psi'(\omega_{V(T_t)}) = \iota'(t)$ for $t \in E(T)$. By (15), ψ and ψ' correspond to mappings b and b' from $V(C)$ to $\mathbb{Z}_5$, respectively, such that $b(v_1) = b(v_2) = b(v_3) = 1$, $b(v_4) = b(v_5) = b(v_6) = 4$, $b'(v_2) = 2$, $b'(v_5) = 3$, and $b'(v_i) = b(v_i)$ for $i \in \{1, 3, 4, 6\}$. By ([16] Example 4), $\alpha_{C,b} = \alpha_{C,b'}$, where $\alpha_{\Delta(C),\psi} = \alpha_{\Delta(C),\psi'}$. Thus, by Theorem 1, $(\mathbb{Z}_5)_\psi[\Delta(C)] = (\mathbb{Z}_5)_{\psi'}[\Delta(C)]$.

There are similar applications for regular chain group $\Gamma(G)$. For an Abelian group A and a mapping $\varepsilon : E \setminus E(T) \rightarrow A$, there is a homomorphism $\psi : \Gamma(G) \rightarrow A$ such that $\psi(c_{T,e}) = \varepsilon(e)$ for $e \in E \setminus E(T)$. By Theorem 1, $\alpha_{\Gamma(G),\psi}$ induces polynomial $p(M(G), \alpha_{\Gamma(G),\psi}; k)$ such that $|A_\psi[\Gamma(G)]| = p(M(G), \alpha_{\Gamma(G),\psi}; |A|)$. We are not aware of whether the general form of this polynomial has been studied somewhere in the literature. But, if $\varepsilon_0(e) = 0$ for each $e \in E \setminus E(T)$, then ε_0 indicates an homomorphism ψ_0 such that $\psi_0(f) = 0$ for each

$f \in \Gamma(G)$ and $p(M(G), \alpha_{\Gamma(G), \psi_0}; k)$ is the tension polynomial of G , the nontrivial divisor of the chromatic polynomial of G (see cf. [17,18,25]).

Example 2. Let H be a graph consisting from parallel edges $e_1, \dots, e_6$ joining two vertices u_1, u_2 . Clearly, H is the dual graph of planar graph C from Example 1 and $\Gamma(H)$ coincides with $\Delta(C)$. Let T' be the spanning forest of H such that $E(T') = \{e_6\}$. Consider an orientation of H such that e_i has its initial end in u_i , $i = 1, \dots, 6$, and mappings $\varepsilon, \varepsilon'$ from $E(H) \setminus E(T')$ to $\mathbb{Z}_5$ that coincide with ι, ι' , respectively. By applying duality, we can conclude that ψ and ψ' from Example 1 can be considered as homomorphisms from $\Gamma(H)$ to $\mathbb{Z}_5$ such that $\psi(c_{T',e}) = \varepsilon(e)$ and $\psi'(c_{T',e}) = \varepsilon'(e)$ for each $e \in E(H) \setminus E(T')$. Thus, by Example 1, $\alpha_{\Gamma(H), \psi} = \alpha_{\Gamma(H), \psi'}$, where by Theorem 1, $(\mathbb{Z}_5)_\psi[\Gamma(H)] = (\mathbb{Z}_5)_{\psi'}[\Gamma(H)]$.

There are possibilities for future research regarding this topic. We can study relations of the assigning polynomials with the Möbius functions, as were studied for characteristic polynomials in [18]. It would be of some interest to study integral variants of (N, ψ) -chains and assigning polynomials in a similar way as presented in [3,4] for regular chain groups and characteristic polynomials. Another direction can be the study of dichromatic variants of the chain polynomials that are analogous to the Tutte polynomials (see cf. [10,17]).

Funding: This research was partially supported by grant VEGA 2/0042/22.

Data Availability Statement: The original contributions presented in the study are included in the article, further inquiries can be directed to the corresponding author.

Conflicts of Interest: The author declares no conflict of interest.

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