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Quantum Leap: A Price Leap Mechanism in Financial Markets

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Abstract: This study explores the quantum leapfrog mechanism within the context of quantum finance and presents a new interpretation of established financial models through a quantum perspective. In quantum physics, the well-documented phenomenon of particles tunneling through energy barriers has a parallel in finance. We propose a quantum financial leapfrog model in which asset prices make quantum leaps, penetrating market "energy barriers" in non-sequential advances. By leveraging the Hamiltonian operator and the Schrödinger equation, our approach simulates the dynamics of asset prices in a manner akin to the trajectories of particles in quantum mechanics. We draw an analogy between financial markets and gravitational fields, and from this we derive energy equations for pricing orbits. Using path integration techniques, we map out potential price transitions between these orbits, which are guided by the calculation of minimal energy barriers. Furthermore, we introduce a market "propagator" that aligns with the uncertainty principle, identifying the optimal price pathways. Our findings provide new insights and methodologies for navigating the complexities of financial markets, underscoring the significant potential of quantum approaches in the field of finance. These findings have theoretical implications for a variety of market stakeholders, offering strategic guidance and a reference point. We expect that the advancement of the quantum financial leapfrog theory will refine analytical methods and enhance investment strategies in practical financial applications.

Keywords: quantum transition; price movement mechanism; quantum finance; path integral

MSC: 81S40

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1. Introduction

The main research work of this paper includes the following: (1) comparing the market to a gravitational field, comparing the motion of prices in the financial market to the motion of particles in the physical world, and establishing an energy equation to solve for the orbits of price jumps; (2) describing the theoretical mechanism of the path of price jumps between orbits by the method of path integration, which can provide a new perspective for understanding complex phenomena in the financial market; and (3) comparing the motion of prices in the market to the motion of microscopic particles and calculating the probability of the path of price movement by solving the propagator in quantum mechanics. In practice, the results of this study are helpful for individual, institutional, and corporate investors.

The germination of quantum finance originated in the 1950s. French financier Bernard Schmidt first analyzed the quantum characteristics of currency and debt in his doctoral thesis in 1960 [1]. Then, in the 1980s and 1990s, the concept of quantum computing began to emerge, and scientists proposed the first quantum algorithm and decomposed large prime numbers. After the end of the twentieth century, results were scarce, and concept formation was fragmented [2]. Some basic assumptions belonged to individual pioneering work, and the discovery was made that Copenhagen quantum formalism was more suitable for describing economics [3].

Quantum finance gained momentum as we neared the new millennium, with foundational research and the integration of quantum mechanics into financial algorithms. Kirill Ilinsky, in 1997 and 2001 [4], applied quantum field concepts to financial markets, while M. Shubik, in 1999 [5], found parallels to the uncertainty principle in economics. In 1998, Stauffer and Penna [6] built a stock price model from quantum theory, and Schaden, in 2002 [7], used wave functions for asset modeling, diverging from conventional economics. In 2002, Haven [8] infused quantum physics into Black–Scholes models, treating option prices as state functions for arbitrage analysis, with Planck's constant playing a key role. In 2003, Belal E. Baaquied et al. [9] investigated quantum finance in derivative pricing, pushing the field's boundaries further.

In 2007, Emmanuel E [10] and Berard Baquet began exploring quantum finance, publishing works on futures and interest rates using path integrals. Choustova, in 2007 [11], introduced a quantum-like model for market dynamics, incorporating a quantum potential for behavioral economics. S. J. Blundell, in 2018 [12], applied quantum field theory's mathematical tools to finance, influencing financial instrument design with quantum techniques [13]. In 2009 and 2015 [14,15], the Russian scholar Kondratenko examined economic systems through both classical and quantum finance lenses, advancing a probabilistic economic theory and refining quantum finance.

In 2007, Emmanuel E [10] and Berard Baquet's research laid early foundations in quantum finance, with Choustova [11] introducing a quantum-like model for market dynamics in the same year. By 2018, S. J. Blundell [12] had applied quantum field theory to financial instruments.

In 2009, Vladimir and Tishen Alexander [16] explored quantum mechanics for business predictions alongside Bagarello [17,18], who proposed a securities market model. Ataullah [19] compared stocks to charged particles in 2009, empirically testing his model, while Baaquie [20] offered a quantum perspective on LIBOR. In 2009, Jochen Rau [21] highlighted the role of quantum probability theory.

In 2012, Aerts [22] questioned the random walk model, supporting quantum finance models. L.-A. Cotfas [23] (2012) and Pedram [24] (2012) advanced quantum stock models. Baaquie [25] (2013) analyzed stochastic financial evolution quantumly, and Romero et al. [26] (2014) studied quantum financial symmetries. Nastasiuk [27] (2015) and Su [28] (2015) expanded upon quantum probability models, with Tahmasebi et al. [29] (2015) investigating financial market anomalies.

Sarkissian [30] (2016) created a quantum securities market model, and Christof [31] (2016) modeled price jumps. Singber [32] (2017) applied numerical path integration techniques to various systems, while Nasiri et al. [33] (2018) examined the quantum impact on trading volume. Zurek [34] (2018). discussed information processing in complex systems from the perspectives of information theory and statistical physics, with a particular focus on the potential of quantum algorithms to enhance the efficiency of information processing.Patrick Rebentrost [35,36] (2018) proposed quantum algorithms for financial derivatives and portfolio optimization. Irina Basieva et al. [37] (2018) built a model on supervisor expectations. In 2019, Ivan Arraut et al. [38] introduced a tool for option pricing prediction, and Iordanis Kerenidis [39] developed a quantum algorithm for portfolio optimization. Sebastian et al. [40] (2019) used quantum random number generators for option pricing. Baaquie [41] (2019) developed a quantum harmonic oscillator-based model for corporate bond pricing, effectively aligning with market data and offering a new quantum approach to financial engineering.

Wenyan Hao et al. [42] (2019) proposed quantum financial models, and the UK government launched a quantum technology plan [43].

Lee's monograph [44] (2020) detailed quantum finance applications, while Belal [45] (2020) applied quantum mathematics to bond pricing, and Jack Sarkissian [46] (2020) utilized quantum solutions for financial analysis. Atanassov [47] (2020) proposed a quantum quasi-Monte Carlo simulation, and Samuel Mugel et al. [48] (2020) reviewed quantum

optimization in finance. David Orrell [49] (2020) introduced a quantum economic supply and demand model.

Vasileiou [50] (2021) questioned market efficiency through quantum concepts. Hao Tang et al. [51] (2021) applied quantum theory to Monte Carlo financial derivative pricing.

1.1. The Main Methods of Quantum Finance

Quantum finance uses the tools of quantum mechanics to model financial markets, covering probability, state evolution, uncertainty, and asset correlations with techniques like wave functions, Schrödinger's equation, and entanglement, offering insights into market behavior.

1.1.1. Market State Wavefunction

In quantum finance, the state of the market is portrayed as a wavefunction ψ, which belongs to a Hilbert space H, or more generally, a functional analysis space F. The wavefunction ψ encodes the probability distribution information of various variables in the market (e.g., asset prices, trading volumes, etc.).

1.1.2. Schrödinger Equation

The Schrödinger equation dictates the dynamical evolution of the wavefunction over time. Its universal form is as follows: $i\hbar\partial\psi/\partial t = \hat{H}\psi$, where $\hat{H}$ is the Hamiltonian operator, reflecting the intrinsic energy structure of the market. The Schrödinger equation delineates how an initial market state ψ evolves over time, unveiling the dynamical mechanism of market variations.

1.1.3. Path Integral

The path integral calculates transition probabilities between different market states. Its expression is as follows: $K(q',t';q,t) = \int D[q]\exp(iS[q]/\hbar)$. The quantity S[q] inside the integral is the action over all feasible trajectories. The path integral aggregates contributions from all viable paths to compute the complete transition amplitude.

1.1.4. Uncertainty Principle

The uncertainty principle reflects the trade-off between complementary variables in the quantum world. In financial markets, for instance, price p and trading volume q satisfy the following:

$\Delta p \Delta q \geq \hbar/2$. This relation reveals the intrinsic uncertainty of markets.

1.1.5. Random Walk

A random walk depicts the evolution of a variable under random influences: $qn + 1 = qn + \xi n$, where ξn is a random perturbation term. This can simulate the random fluctuations of prices, etc., under market noise.

1.1.6. Quantum Entanglement

Quantum entanglement portrays correlated quantum states of multiple particles in a quantum system. For example, the entanglement state of two particles, A, B, can be written as: $\psi AB = \psi A \otimes \psi B$, where $\otimes$ denotes a tensor product. This reflects correlations between assets in complex markets.

The above elaborates the principal mathematical formalism of quantum finance methodology and their physical interpretations in financial markets. These quantum techniques provide new perspectives for comprehending the statistical behaviors of markets.

1.2. Differences between Quantum Finance and Traditional Financial Hypothesis

1.2.1. Stochastic Processes in Traditional Finance

In traditional financial theory, the dynamics of financial asset prices are often described by geometric Brownian motion, which is a canonical stochastic process model in financial

mathematics. It is widely applied to depict the random evolution of stocks or other financial asset prices in continuous time. This model takes the form of a stochastic differential equation (SDE) as follows:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{1}$$

where μ denotes the expected rate of return, σ represents the volatility of the asset price, and W_t is a Wiener process representing standard Brownian motion. This model captures not only the continuous part of asset price changes, but also characterizes the discontinuous jumps in prices, thereby more comprehensively capturing the random nature of financial markets.

1.2.2. Quantum Finance

Quantum finance, on the other hand, adopts a different methodology from traditional finance by drawing on the fundamental principles of quantum mechanics and using wavefunctions to describe the state of financial markets and their inherent uncertainty. In quantum financial models, the state of the market can be represented in a form similar to Equation (2):

$$\frac{\partial p(x,t)}{\partial t} = -\mu \frac{\partial}{\partial x}(xp(x,t)) + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2}\left(x^2 p(x,t)\right) \tag{2}$$

Note the second-order derivative term, which is formally analogous to the term describing particle kinetic energy in the Schrödinger equation. To smoothly transition from the stochastic differential equation to the quantum finance framework, several considerations need to be made: first, explicitly identify the corresponding "potential" concept in a financial context, which may be closely related to the drift term in the SDE; second, explain the "mass" concept in the quantum framework, associating it with some notion of market "inertia" or a financial "mass" metric; third, consider transforming the real diffusion term into a complex form to accommodate the complex nature involved in the Schrödinger equation; and fourth, recognize the correspondence between the probability density function in stochastic processes and the square of the modulus of the wavefunction in quantum mechanics. In practice, mapping financial quantities to quantum mechanical analogies needs to be done with care, as the fundamental dynamics of random processes and quantum systems are inherently different. Although mathematical similarities exist between the two, quantum financial models provide unique advantages in capturing market uncertainty. By incorporating noise and jump factors from random processes, the expressiveness of quantum financial models can be enriched. Further, combining theoretical derivations with empirical data analysis can help to optimize model details for greater universality and accuracy, thereby unlocking greater value in the realm of finance. We have placed the transition relationship between Equations (1) and (2) in the Appendix A.

In order to ensure the mathematical process and rationality, this article extends the quantum method to the Schwartz distribution space. The specific process is detailed in Appendix B.

1.3. Quantum Transition Theory

The foundational premise of this theory posits that the financial market is an entity driven by both energy and information, exhibiting dynamism in its response to the influx of data. Within the framework of the price energy absorption hypothesis, market prices are theorized to rise following the assimilation of energy from positive market sentiment and information. Conversely, the price energy release hypothesis suggests that prices decline, emitting energy in reaction to market panic or adverse news. Moreover, it is postulated that price fluctuations can be theoretically mapped by a wave function that evolves in accordance with the principles of quantum mechanics, an idea encapsulated in the price wave function evolution hypothesis. Lastly, the price superposition state hypothesis contends that market prices may coexist in several coherent states simultaneously, each endowed with a distinct probability amplitude. These amplitudes confer the likelihood of

each potential outcome until an observation precipitates the collapse of the wave function into a definitive state.

Quantum leap, as the name suggests, is a description of the properties of the motion of electrons outside the nucleus of an atom. In the microscopic world, the electrons outside the nucleus of an atom rotate around the nucleus jumping into different orbits, and different orbits correspond to different energy states. The energy states of each orbital are intermittent and discontinuous. Its electrons can only jump one energy up or down from one energy level. If irradiated by a very strong external photon, the electron absorbs the powerful energy of the photon and will cross several levels and reach a very high energy level. The interval between energy levels can be calculated according to the equations of quantum theory. In a similar manner, the money within the market absorbs or releases energy, resulting in the rise or fall of prices. This process can produce movements akin to the quantum jumps observed in the microscopic world. The movement of electrons in the atom is discontinuous, and its energy is also intermittent. In this paper, the authors found that the movement of prices is also discontinuous and discrete.

As shown in Figure 1, the up and down of the diagram is the candle line in the financial market, and the horizontal line in the diagram is the price jump "track". The price in the process of movement also made a jump from one track to another track movement. In this model, the essential elements of the financial market are the "market atoms"; the securities are the core of the value, analogous to the nucleus of an atom. As shown in Figure 2, in the atomic model of the market, energy is the driver of price, analogous to the energy absorbed or released by electron leaps in the microscopic world. In natural science, visible matter and invisible energy are the two most fundamental elements; in economics and finance, visible data and invisible information are also two fundamental elements. This leads to the primary hypothesis of this paper: the money in the financial market has energy, and the market price will rise when energy is absorbed, and fall when energy is released.

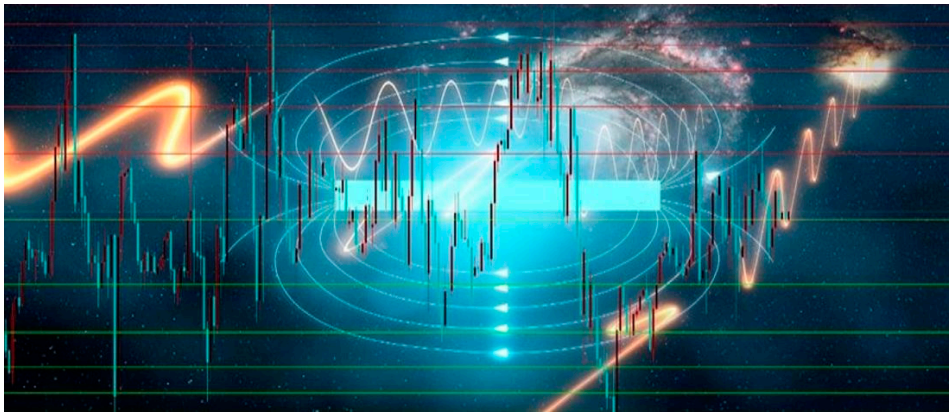


Figure 1. Price transitions between orbits.

The Hamiltonian operator of the system is defined as $\hat{\mathrm{H}} = -\left(\hbar^2/2m\right)\left(\partial^2/\partial \mathrm{p}^2\right) + \mathrm{V(p)}$, where p represents the price, m is the virtual mass, and $V(p)$ is the potential energy function. To express the transition probability using the integral kernel in quantum mechanics, we consider the probability $\mathrm{P[p(t+\delta t), t+\delta t; p(t), t]}$ that is equal to the modulus squared of the inner product $\left\langle p(t+\delta t)\middle|\widehat{U}(t+\delta t, t)\middle|p(t)\right\rangle$. The quantum evolution operator from time t to $t+\delta t$ is denoted by $\widehat{U}(t+\delta t, t)$ and is given by the exponential $\mathrm{e}^{\wedge}\left(-i\widehat{H}(t+\delta t-t)/\hbar\right)$. Expanding this operator $\widehat{U}$ in terms of the propagator expression, we obtain $\mathrm{P[p(t+\delta t), t+\delta t; p(t), t]} = \left|\left\langle p(t+\delta t)\middle|\int D[p]e^{\wedge}(iS[p]/\hbar)\middle|p(t)\right\rangle\right|^2$, where $S[p]$ is the action along the path from $p(t)$ to $p(t+\delta t)$. By computing the transition probability via path integrals, we can quantify the leap behaviors between different price states in the financial market. This integral kernel approach provides a tool to predict and understand complex market behaviors.

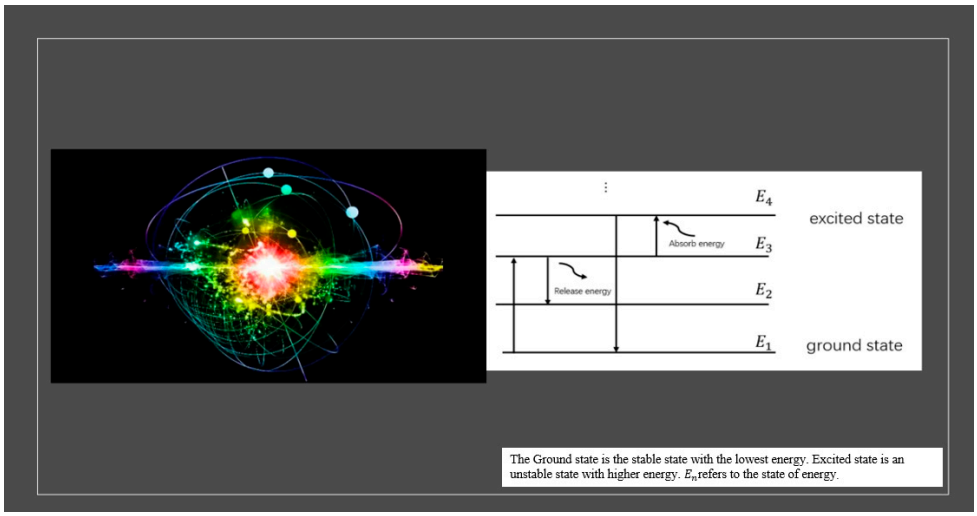


Figure 2. Energy level transitions of microscopic particles.

1.4. Adaptability and Advantages of Quantum Finance

Quantum finance has a richer state space, including superposition states describing complex markets. This enhances adaptability to changing environments. It expresses asset interactions and entanglements, better capturing complexity. Superposition and entanglement depict nonlinear dynamics, improving models and forecasts.

Advanced forecasting models like quantum random walks utilize quantum computing benefits like parallelism and entanglement for higher precision. Employing such models, quantum finance is expected to outperform traditional ones.

It also models micro phenomena, leading to accurate macro models. Simulating individual behavior and interactions enhances the understanding of overall market and risk behavior. This enables better forecasts and risk management strategies, potentially exposing new opportunities.

A fundamental property of quantum mechanics allows for the instantaneous exchange of information between different parts of a system that transcends classical laws of physics. In financial markets, this can be used to describe and analyze instantaneous fluctuations and correlations between asset prices. The probabilistic process description in Brownian motion faces some difficulties, such as the potential conflict between determinism and locality. It is also suggested that these difficulties somehow anticipate the corresponding situation in quantum mechanics. Therefore, it is necessary to use probability densities in spatial coordinates ρ(q) instead of probability densities defined on phase space ρ(q, p). The q space is often referred to as the configuration space, and in phase space mechanics it is always possible to obtain the configuration space density from the phase space:

$$\rho(t,q) = \int \rho(t,q,p)dp \tag{3}$$

The Liouville dynamics on the phase space gives rise to the dynamics of the probability density on the configuration space. Liouville's theorem is an important theorem in classical mechanics and statistical physics. It basically states that in a Hamiltonian system, the volume element (that is, a small region in the phase space) along the flow is constant over time. In other words, the probability density in the phase space obeys the Liouville equation:

$$\frac{\partial \rho}{\partial t} + \{H, \rho\} = 0 \tag{4}$$

where {H, ρ} is the Poisson bracket of the Hamiltonian H and the probability density ρ.

In contrast, quantum finance uses the framework of quantum mechanics to describe the dynamics of financial markets, allowing for a more sophisticated description of the

market that captures instantaneous interactions and correlations between asset prices. The methodology of quantum finance, utilizing Schrödinger's equation and Hilbert space, effectively describes the nonlocality and entanglement inherent in quantum systems. This allows for a more accurate and comprehensive representation of market dynamics, leading to more effective risk management and investment decision-making.

Additionally, the quantum random walk model, which is a quantum analog of the classical random walk model used in Brownian motion, can provide more accurate predictions of market dynamics. By taking advantage of quantum superposition and entanglement, the quantum random walk model can explore multiple possibilities in parallel, leading to more accurate and efficient market forecasts.

The quantum finance methodology offers several advantages over classical Brownian motion, including the ability to describe instantaneous interactions and correlations between asset prices, as well as the potential for more accurate market forecasts using quantum random walk models. These advantages make quantum finance a promising approach for improving risk management and investment decision-making in financial markets.

1.5. The Framework Paradigm of Quantum Finance

In quantum finance, we typically use the mathematical framework of Hilbert space to describe the dynamics of financial markets. This framework can be expressed using many different formulations, but the following is a common form that uses the Schrödinger equation from quantum mechanics:

$$\hbar \frac{\partial \psi}{\partial t} = H\psi \tag{5}$$

The variables in this equation are defined as follows:

i is an imaginary unit (satisfying i2 = -1).

ħ is the reduced Planck constant, which is the fundamental constant in quantum mechanics.

ψ is a wave function in Hilbert space that represents the probability distribution of the financial market states.

H is the Hamiltonian operator, which represents the dynamics of the financial market.

∂/∂t is the partial derivative with respect to time t.

This equation describes the evolution of the financial market state ψ with time t, where the dynamics are determined by the Hamiltonian operator H. The resolution of ψ allows us to obtain the probability distribution of asset prices, etc., in the market for risk management and investment decisions.

1.6. Basic Market Energy Hypothesis

In physics, the kinetic energy theorem is generally used to represent the change in the kinetic energy of an object. Kinetic energy is instantaneous and generally used in physics to mean that the work performed on an object during a process equals the change in kinetic energy during that process. Since the motion of an object brings about a change in mass, kinetic energy is a state quantity, and its formula is as follows:

$$E = \frac{1}{2}(m + m_k)v^2 = \frac{1}{2}mv^2 + o\left(\frac{v^2}{c^2}\right) \tag{6}$$

where m is the object's mass, m_k is the extra mass due to the stretching of the mass brought about by the motion of the object based on special relativity, v is the velocity of the object's motion, and $o\left(\frac{v^2}{c^2}\right)$ is the abbreviation for the stretched mass derived by substituting the Lorentz transformation formula. Particles in the physical world have kinetic and potential energy; analogously, the kinetic energy of the financial market can be expressed as follows:

$$E = \frac{1}{2}(\mu + \mu_k)v^2 = \frac{1}{2}\mu\omega^2 + o\left(\frac{\omega^2}{c^2}\right) \tag{7}$$

In our study, we propose that price movements possess analogous characteristics to the kinetic and potential energies in physics. The current price μ corresponds to the trading volume and can be viewed as the "particle" of the price fluctuation. μ_k denotes the additional kinetic energy obtained by the price fluctuation "particle" due to inherent market mechanisms. Specifically, when the market trading frequency increases, as in high frequency algorithmic trading, it induces greater volatility. This implies heightened uncertainty and risk in price changes. Such markets attract more speculative traders, dramatically increasing trading volumes. Hence, μ_k can be seen as the extra kinetic energy attained by the price fluctuation "particle" from internal market dynamics, exhibiting a positive correlation with market trading frequency and volume. In summary, price fluctuations can be regarded as a complex physical process driven by intrinsic market mechanisms, with prices converting between potential and kinetic energy as per quantum theory tenets during trading activities.

w is the current fluctuation frequency of a single financial marker for the system trading speed for a very small value; so, $o\left(\frac{\omega^2}{c^2}\right)$ tends to 0. For the financial market, it has been believed that there is an energy relationship. Finance has the energy, including static energy and kinetic energy. The sum is equal to the total energy of finance. The formula can be expressed as follows:

$$E = E_o + E_k \tag{8}$$

where E_o is the potential energy of the underlying financial price itself, and E_k is the energy transformed into the financial price stretch by the speed of the transaction. Specifically, the static energy is expressed as follows:

$$E_0 = \mu\omega r \tag{9}$$

The kinetic energy is expressed as follows:

$$E_\mathrm{k} = \frac{1}{2}\mu\omega^2 \tag{10}$$

The dimension of E is given in terms of monetary units times (capital flow rate)^2 where the static energy is represented by $E_0 = \mu\omega r$. Here, r denotes the asset price corresponding to the current trading volume. Kinetic energy is represented as $E_\mathrm{k} = \frac{1}{2}\mu\omega^2$. This formulation quantitatively establishes the physical relationship between financial prices and trading volumes in financial markets.

2. Model

2.1. Define the Price Hub

Just as electrons in the microscopic world revolve around the nucleus of an atom, the market can be seen as a central force field formed by the fluctuation of prices around a price pivot. When things move in a regular cycle, there is often a center, and the range of motion does not deviate from this center. The average price of a candlestick line defines the "nucleus" of the price P. Since the conditions assume that money has mass, just as the mass of a molecule is concentrated in the nucleus, the "mass" of the price field is concentrated in the dense "mass" of the price field. This is concentrated in the dense trading area, which defines the pivot of the price.

$$P = \sum_{n=1}^{N} \frac{x + w + y + z}{4} / N \tag{11}$$

Let 'x' represent the highest price within the last 'n' candles; let 'y' represent the lowest price within the same range; let 'w' represent the opening price of the first candle in this

range; and let 'z' represent the closing price of the last candle. Let 'P' denote the pivot price and let 'N' be the total number of candles considered.

2.2. Transition Orbit

The formula below represents the wave function φ(r), which mirrors the condition of the financial market by drawing parallels with the oscillatory nature of the cosine function and the market's own fluctuation patterns.

$$\varphi(r) = \cos\left(\frac{wr}{d}\right) \tag{12}$$

The wave function φ(r) is a mathematical expression that encapsulates the state of the financial market. The variable ω signifies the fundamental oscillation frequency that is characteristic of the market and varies across different data types. Meanwhile, d is indicative of the maximal differential in price that can occur during the shift between nodes and the movement along the candlestick chart. The total energy of the current financial market can be expressed as follows:

$$E = \frac{1}{2}u\omega^2 d^2 \tag{13}$$

$$E(r) = \frac{1}{2}u\omega^2 r \tag{14}$$

In this context, $\hbar$ symbolizes the aggregate fund volume within the financial market, quantifying the amount of capital exchanged each second, while μ denotes the transaction volume. The financial energy operator can thus be articulated as follows.

$$H = -\frac{\hbar^2}{2\mu}\nabla^2 + V(r) \tag{15}$$

$$V(r) = \mu\omega r \tag{16}$$

In this framework, 'H' represents the function of the price's current spatial state. The term 'V(r)' corresponds to the potential energy affecting the funds due to the gravitational influence of the price pivot at this moment, with 'r' indicating the distance from the current price to the pivot point. This distance, reflecting the deviation between the current price and the orbit's central price, serves as the potential energy, suggesting that the propensity for the price to revert increases as it moves further from the pivot. The operator '∇^2' is a differential operator, which describes the evolutionary state of the price. Rigorous proof has been provided in the document, where '$\hbar$' denotes the market's monetary scale, specifically the amount of capital traded in the market every second.

Moving forward, the quantum mechanical spherical harmonic function is utilized for precise computations, as denoted by spherical coordinate Equation (15). This operation is selected due to the quantum model's analogy of financial movements to those within a spatial field. Given the tri-dimensional nature of physical fields, spherical coordinates are thus employed. In this analogy, monetary flows are likened to movements within a three-dimensional price field. Financial market data are inherently multi-dimensional, and their representation should be a three-dimensional composition of these multiple dimensions. Quantum mechanics introduces three spherical coordinate operators—$\hat{L}^2$, L^2 and ∇^2—along with an angular momentum L. Within spherical coordinates (r,θ,ψ), these operators are defined as follows.

$$\hat{L}^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{2\theta^2}\right) \tag{17}$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) \tag{18}$$

$$L^2 = l(l+1)\hbar^2, \quad l = 0, 1, 2, \cdots, \tag{19}$$

Equations (17) and (18) are operators in quantum mechanics, where θ denotes the angle in Figure 3 and r denotes the distance measure to the next node. $\hbar$ is defined as the measure representing the amount of funds. In this case, $\hbar$ represents the fund amount measure. Equation (19) delineates the angular momentum corresponding to the particle's microstate. Within this equation, the integer 1 is always positive, reflecting the diversity in the fund's application categories. This integer serves as an analogy for the various utilization characteristics of the funds. Consequently, the comprehensive energy operator in the final Equation (15) can be abbreviated as follows:

$$\hat{H} = -\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2} + \frac{1}{2\mu r^2}\hat{L}^2 + V(r) \tag{20}$$

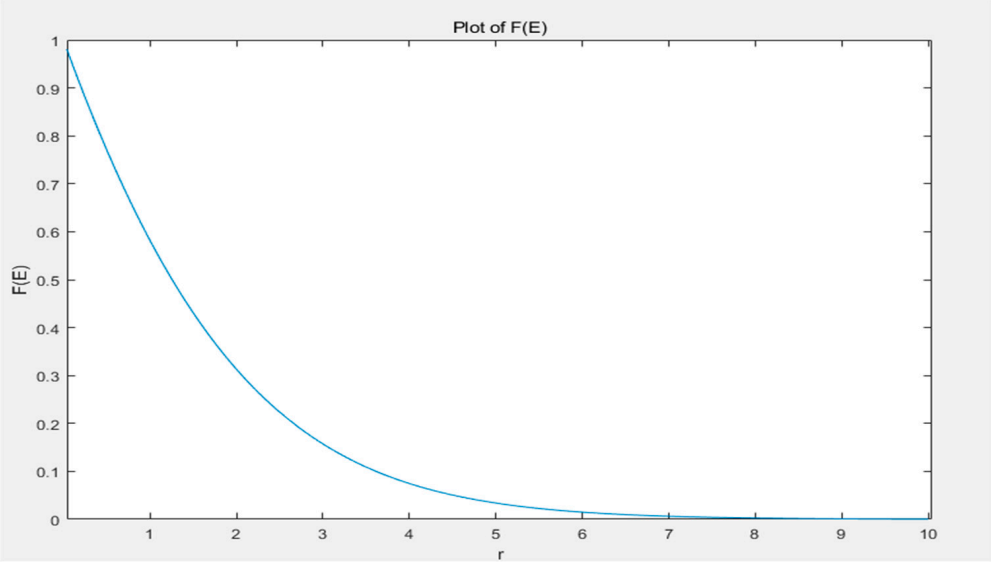


Figure 3. Energy level transitions of microscopic particles.

Typically, the initial term is referred to as the radial kinetic energy operator, and the subsequent term is known as the centrifugal potential energy operator. This implies that the dynamics of capital within the financial market are influenced by both the potential energy of reversion and the inertial kinetic energy of the flow of funds. Regarding Equation (20), the operator from quantum mechanics and the energy equation Hψ = Eψ are further condensed.

$$H\psi = E\psi, \tag{21}$$

Substitute the above expression (17) into (21) to obtain the following equation:

$$\left[\frac{\hbar^2}{\mu}\frac{1}{r}\frac{\partial}{\partial r} + \frac{l(l+1)\hbar^2}{2\mu r^2} + V(r)\right]\psi(r) = E\psi(r) \tag{22}$$

because

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\psi(r) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}[r\psi(r)] \tag{23}$$

So, Equation (22) simplifies to the following:

$$\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2} + \frac{l(l+1)\hbar^2}{2\mu r^2} + V(r) = E. \tag{24}$$

The latter term indicates that the centrifugal potential energy's radial wave function, ψ(r), must comply with the radial equation as well as adhere to the normalization condition.

This requirement is to streamline the computational process and to harmonize the varying dimensions present within the data.

$$\int_0^\infty |\psi(r)|^2 dr = 1 \tag{25}$$

Within the framework of quantum mechanics, the probability of locating a particle within a specific region of space is depicted by the modulus squared of the wave function. Correspondingly, this paper utilizes $|\psi(r)|^2$ to represent the likelihood of funds transitioning to a particular node within the payment network. As for the financial market, as r approaches infinity, F converges to zero, and it is customary to consider ψ(r→∞) = 0 as a benchmark for the market's energy state. Under such conditions, Equation (21) is reduced to a parameter describing the transition radius. Ultimately, expression (24) is articulated as follows.

$$\hat{H} = -\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{2\mu r^2}\hat{L}^2 + V(r) \tag{26}$$

To simplify the calculation, let Equation (22) be equal to 1 as a whole. Then, take the different l values corresponding to different transition distance coefficients as $w = 1\text{hbar} = 1$, and take the value of l as an integer between 1–10 to solve for ten different transition distance parameters (see Table 1). Since the capital transition is in two directions on the straight line, the $|r|/2$ value is equivalent to the single-direction transition dimension.

Table 1. The distance and probability of price jumps.

l	1	2	3	4	5	6	7	8	9	10
r	−0.93	2.62	3.18	3.49	1.62	3.86	3.99	−1.49	2	−0.90
$\|^r$ I	0.93	2.62	3.18	3.49	1.62	3.86	3.99	1.49	2	0.90
Ir| /2	0.47	1.31	1.59	1.745	0.81	1.93	2.00	0.745	1	0.45

As shown in Table 1, the values of the leap distance *r* are somewhat very close to the golden mean values of $1.5, 1.618, 2, 2.618$, etc. Regarding the distribution of energy levels between orbitals, this paper draws on previous scholarly research [52] with slight modifications in order to accommodate the distribution of prices between leap orbitals in the following states.

$$\mathrm{F(E)} = \frac{\mu \mathrm{r}}{\mathrm{e}^{\mu \mathrm{r}/C} - 1} \tag{27}$$

The image of the function is shown in Figure 3, where the horizontal coordinate is the distance, and the vertical coordinate is the probability of the price to reach the trajectory. It can be seen that the probability of the price jumping to the near distance is the largest, and the probability becomes smaller as the distance gets farther.

Figure 4 displays transitions among the ten orbits listed in Table 1, with the horizontal axis indicating time and the vertical axis showing the asset's price.

In Figure 5, data from the US Dollar Index spanning the period from 1 January 2017 to 31 December 2022 are selected for analysis. The vertical axis of the graph quantifies the frequency of price transitions to the orbits, while the horizontal axis represents the respective orbits under consideration. The dots in different colors indicate outlier values above or below 1.5 times the interquartile range above the upper quartile and below the lower quartile, respectively, for different transition orbits. These outliers represent extreme values or anomalous observations in the data distribution. The box plots depict the quartiles to illustrate the data distribution, while the dots help identify outlier values outside the quartile ranges to provide a more comprehensive view of the full range of the data distribution.

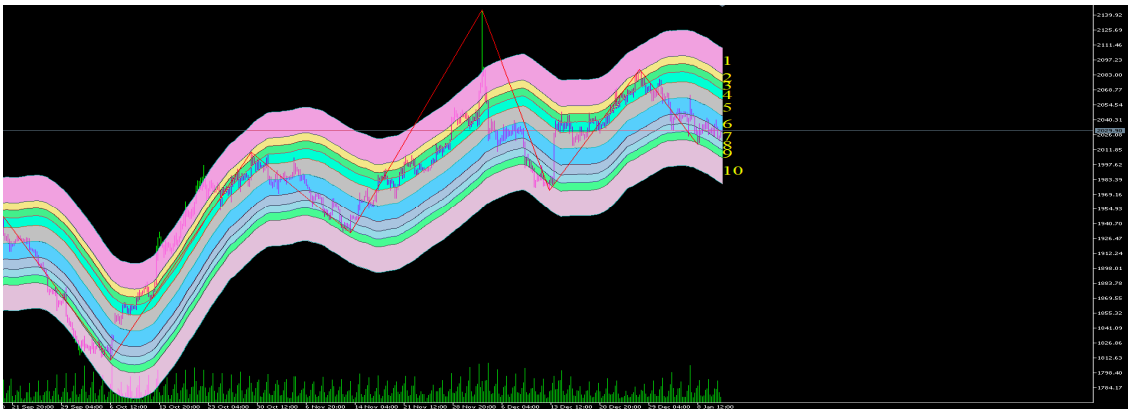


Figure 4. Energy level transitions of microscopic particles.

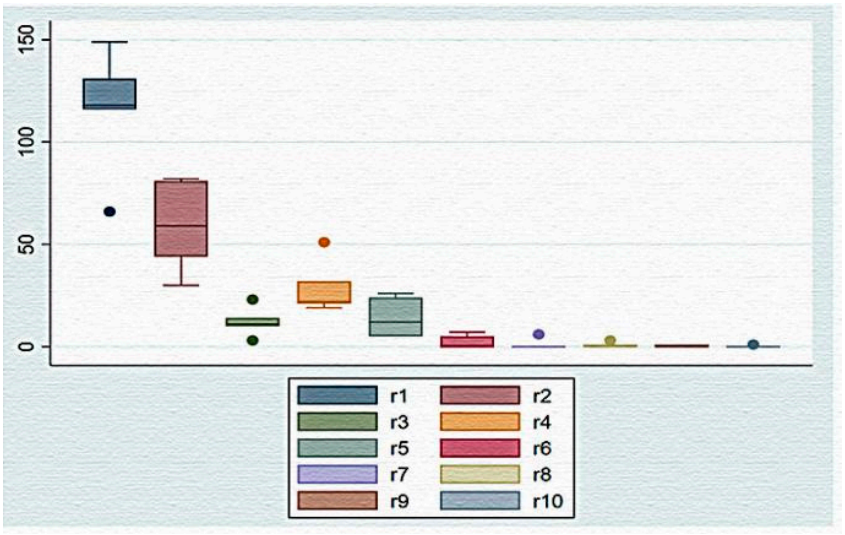


Figure 5. Frequency of price transitions between orbits.

2.3. Transition Path Integral

Path integrals are ideal tools for describing quantum mechanical, thermodynamic, or statistical path integrals. This section uses path integrals to describe the trajectory of price transitions between orbits.

As shown in Figure 6, prices jump from the abutment at the moment t_n to the t_{n+1} excited state for all possible paths, where compliance has the following to show: in the physical world, all possible paths between two points would be the actual paths possible for the particle. Of course, the particle goes with different probabilities and association probabilities for different possible paths. There is no one path, like the classical particle identified by classical physics, in the following order of precedence from left to right:

$$\begin{array}{lll} r_1 & r_2 & r_3 \\ r_1 & r_3 & r_2 \\ r_2 & r_1 & r_3 \\ r_2 & r_3 & r_1 \\ r_3 & r_1 & r_2 \\ r_3 & r_2 & r_1 \end{array}$$

In calculating the path integral of price transitions between different orbits, since the quantum state of the price is uncertain, it can exist in multiple superposition states. That is, the price may take different potential paths before transitioning from the initial orbit to the target orbit. In order to consider all possible paths, this paper enumerates six possible

order combinations according to the mathematical permutation principles, representing all the potential paths that the price can take to go through these three orbits during the transition. Although the final result is the superposition of the integrals of all paths, the integral calculations corresponding to different orders in the computational process are not the same, which will affect the final value of the integral. This reflects the quantum effects in complex systems, where the quantum mechanics results corresponding to different path integrals will also differ. Therefore, this paper calculates the quantum leap behavior of prices by enumerating all possible orders, which is consistent with the ideas of path integrals and also reveals the quantum nature of price movements.

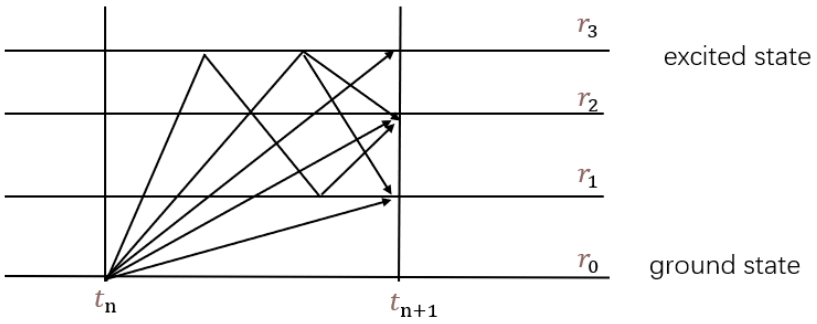


Figure 6. Price transition path diagram.

In this paper, taking three orbits between orbits as an example, the equivalence relation of the path integral is, in the middle, all possible. Its mathematical expression relation is obtained by first constructing a matrix of the path integral $(H(r_1)H(r_2)H(r_3))$. The results are as follows:

$$\begin{aligned}\hat{T}(H(r_1)H(r_2)H(r_3)) &= \theta(r_3-r_2)\theta(r_2-r_1)H(r_3)H(r_2)H(r_1)+\theta(r_2-r_3)\theta(r_3-r_1)H(r_2)H(r_3)H(r_1)\\ &+\theta(r_3-r_1)\theta(r_1-r_2)H(r_3)H(r_1)H(r_2)+\theta(r_1-r_3)\theta(r_3-r_2)H(r_1)H(r_3)H(r_2)\\ &+\theta(r_2-r_1)\theta(r_1-r_3)H(r_2)H(r_1)H(r_3)+\theta(r_1-r_2)\theta(r_2-r_3)H(r_1)H(r_2)H(r_3)\end{aligned} \tag{28}$$

where $\Theta(r-r')$ is the Heaviside step function, its presence indicates the law of causality. Assuming time from t_n to t_{n+1}, the expression corresponding to the integration of all paths of the price between the three energy levels is as follows:

$$\begin{aligned}&\hat{T}\Big(\int_{r_0}^{r}dr_1\int_{r_0}^{r}dr_2\int_{r_0}^{r}dr_3H(r_1)H(r_2)H(r_3)\Big)\\ &=\int_{r_0}^{r}dt_3\int_{r_0}^{r}dt_2\int_{r_0}^{r}dr_1\theta(r_3-r_2)\theta(r_2-r_1)H(r_3)H(r_2)H(r_1)\\ &\quad+\int_{r_0}^{r}dr_2\int_{r_0}^{r}dt_3\int_{r_0}^{r}dr_1\theta(r_2-r_3)\theta(r_3-r_1)H(r_2)H(r_3)H(r_1)\\ &+\int_{r_0}^{r}dr_3\int_{r_0}^{r}dr_1\int_{r_0}^{r}dr_2\theta(r_3-r_1)\theta(r_1-r_2)H(r_3)H(r_1)H(r_2)+\int_{r_0}^{r}dr_1\int_{r_0}^{r}dr_3\int_{r_0}^{r}dr_2\theta(r_1-r_3)\theta(r_3-r_2)H(r_1)H(r_3)H(r_2)\\ &\quad+\int_{r_0}^{r}dr_2\int_{r_0}^{r}dr_1\int_{r_0}^{r}dr_3\theta(r_2-r_1)\theta(r_1-r_3)H(r_2)H(r_1)H(r_3)\\ &\quad+\int_{r_0}^{r}dr_1\int_{r_0}^{r}dr_2\int_{r_0}^{r}dr_3\theta(r_1-r_2)\theta(r_2-r_3)H(r_1)H(r_2)H(r_3)\\ &\int_{r_0}^{r}dr_3\Big(\int_{r_0}^{r_3}+\int_{r_3}^{r}\Big)dr_2\Big(\int_{r_0}^{r_2}+\int_{r_2}^{r}\Big)dr_1\theta(r_3-r_2)\theta(r_2-r_1)H(r_3)H(r_2)H(r_1)\\ &+\int_{r_0}^{r}dr_2\Big(\int_{r_0}^{r_2}+\int_{r_2}^{r}\Big)dr_3\Big(\int_{r_0}^{r_3}+\int_{r_3}^{r}\Big)dr_1\theta(r_2-r_3)\theta(r_3-r_1)H(r_2)H(r_3)H(r_1)\\ &+\int_{r_0}^{r}dr_3\Big(\int_{r_0}^{r_3}+\int_{r_3}^{r}\Big)dr_1\Big(\int_{r_0}^{r_1}+\int_{r_1}^{r}\Big)dr_2\theta(r_3-r_1)\theta(r_1-r_2)H(r_3)H(r_1)H(r_2)\\ &+\int_{r_0}^{r}dr_1\Big(\int_{r_0}^{r_1}+\int_{r_1}^{r}\Big)dr_3\Big(\int_{r_0}^{r_3}+\int_{r_3}^{r}\Big)dr_2\theta(r_1-r_3)\theta(r_3-r_2)H(r_1)H(r_3)H(r_2)\\ &+\int_{r_0}^{r}dr_2\Big(\int_{r_0}^{r_2}+\int_{r_2}^{r}\Big)dr_1\Big(\int_{r_0}^{r_1}+\int_{r_1}^{r}\Big)dr_3\theta(r_2-r_1)\theta(r_1-r_3)H(r_2)H(r_1)H(r_3)\\ &+\int_{r_0}^{r}dr_1\Big(\int_{r_0}^{r_1}+\int_{r_1}^{r}\Big)dt_2\Big(\int_{r_0}^{r_2}+\int_{r_2}^{r}\Big)dr_3\theta(r_1-r_2)\theta(r_2-r_3)H(r_1)H(r_2)H(r_3)\\ &\quad=\int_{r_0}^{r}dr_3\int_{r_0}^{r_3}dr_2\int_{r_0}^{r_2}dr_1H(r_3)H(r_2)H(r_1)+\int_{r_0}^{r}dr_2\int_{r_0}^{r_2}dr_3\int_{r_0}^{r_3}dr_1H(r_2)H(r_3)H(r_1)\\ &+\int_{r_0}^{r}dr_3\int_{r_0}^{r_3}dr_1\int_{r_0}^{r_1}dr_2H(r_3)H(r_1)H(r_2)+\int_{r_0}^{r}dr_1\int_{r_0}^{r_1}dr_3\int_{r_0}^{r_3}dr_2H(r_1)H(r_3)H(r_2)\\ &+\int_{r_0}^{r}dr_2\int_{r_0}^{r_2}dr_1\int_{r_0}^{r_1}dr_3H(r_2)H(r_1)H(r_3)\int_{r_0}^{r}dr_1\int_{r_0}^{r_1}dr_2\int_{r_0}^{r_2}dr_3H(r_1)H(r_2)H(r_3)\\ &\quad=3!\int_{r_0}^{r}dr_3\int_{r_0}^{r_3}dr_2\int_{r_0}^{r_2}dr_1H(r_3)H(r_2)H(r_1)\end{aligned} \tag{29}$$

In our proposed quantum financial model, we conceptualize each financial market as a circular propagator system with a minimum radius of σ. This minimum radius σ represents the boundary within which the price fluctuations occur in each market. Specifically, the propagator $K(r,t;r_0)$ encapsulates the dynamics of price transitions, describing how an initial price r0 evolves to a final price r over a time period t. For computational simplicity, we make the assumption that the radius of the propagator circle for each market is a standardized σ. This constructs a well-defined circular geometry for each financial market to analyze its internal price fluctuation mechanisms.

Based on this circular quantum propagator representation, we can leverage the mathematical apparatus of path integrals from quantum mechanics to characterize the action along all feasible price fluctuation trajectories between the initial and final prices across multiple orbital energy levels. The path integral aggregates the transition amplitudes along every potential price fluctuation path into an overall probability amplitude for the entire price movement. By integrating the action function, defined by the Lagrangian L, over the entire spectrum of permissible price paths, the path integral formulation allows us to compute the propagator $K(r,t;r_0)$. This further enables the deduction of the probabilities associated with the price dynamics.

3. A method for Solving Transition Paths

3.1. Price Fluctuation Particles

The path integral approach, a cornerstone of quantum mechanics, can be innovatively applied to financial markets, as shown in Figure 7. The main tenet of this approach is that the propagator, which characterizes the transition probabilities between different quantum states, is calculated as the accumulation of all possible paths that satisfy certain constraints. These paths can be visualized as particle fluctuations akin to price movements.

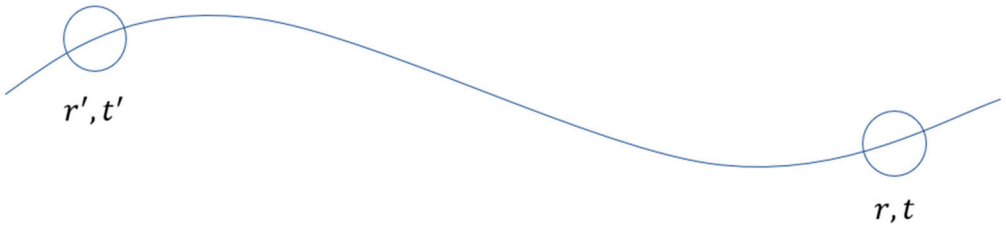


Figure 7. Price changes are seen as random particle movement.

By modeling each financial market as a quantum circular propagator system with a fixed radius, we can apply the principles of quantum mechanical path integration to calculate the total action. This approach allows us to determine the transition probabilities for all conceivable price fluctuation trajectories within each market, providing crucial insights into integral price behaviors and orbital mechanisms.

The adoption of a standardized propagator circle, as illustrated in Figure 7, provides a robust mathematical framework for analyzing the complex quantum finance system. This unified perspective bridges the world of quantum mechanics with finance, facilitating a coherent understanding of market dynamics and opening new avenues for market prediction and risk management.

The propagator in quantum mechanics can be expressed in mathematical form as follows:

$$U(r',t';r,t) = \left\langle r' \middle| e^{-\frac{i}{\hbar}\hbar(t'-t)} \middle| r \right\rangle \Theta(t'-t) \tag{30}$$

The following integral can express the physical meaning of the above propagator:

$$\Psi(r',t') = \int dq U(r',t';r,t)\Psi(r,t) \tag{31}$$

This means that the above propagator is precisely the integral kernel of the wave function evolution. The core task of the path integral is to compute the above propagator. First, we consider dividing the time uniformly into N » 1 parts so that we can obtain the following:

$$e^{-i\hat{H}t/\hbar} = \left[e^{-i\hat{H}\Delta t/\hbar}\right]^N \tag{32}$$

where $\Delta t = t/N$ is a minimal time interval. The evolution operator can be further decomposed as follows:

$$e^{-i\hat{H}\Delta t/\hbar} = e^{-i\hat{T}\Delta t/\hbar}e^{-i\hat{V}\Delta t/\hbar} + O\left(\Delta t^2\right) \tag{33}$$

3.2. Integral Measure of Action

This part introduces the path fractional approach, which involves splitting up the time-evolution operator U(t) into small increments and then applying integration to these increments to obtain an integral path representation of the propagator K. This technique is analogous to the stochastic calculus used in the equation $|\psi_t\rangle = U(t)|\psi_0\rangle$, where ψ_0 denotes the wave function defined earlier to depict the market's energy state at that moment. Since the original equation did not specify a time term t, an additional formula is introduced here to represent how prices evolve over time by incorporating a time factor. In essence, the path fractional approach splits the time-evolution operator into small segments, integrates over these segments to obtain the propagator path integral, and introduces a time component to model price changes, similar to techniques used in stochastic calculus. This allows for the representation of the time evolution of prices in the quantum finance framework.

$$U(t) = e^{-itH/\hbar} = e^{-\frac{it(T+V)}{\hbar}} \tag{34}$$

This formula represents the definition of the time evolution operator (also known as the propagator) in quantum mechanics. Each symbol has the following meaning:

$U(t)$: This is the time evolution operator. In quantum mechanics, it describes how a system evolves over time. Given an initial state $|\psi(0)\rangle$, after a time t, the state of the system will be $|\psi(t)\rangle = U(t)|\psi(0)\rangle$.

e : This is the base of the natural logarithm. It is a constant approximately equal to 2.71828.

$-i$: Here, i is the unit imaginary number satisfying $i^2 = -1$. The symbol—represents a negative sign.

t: This is time, indicating how long the system has evolved.

H: This is the Hamiltonian operator, a key concept in quantum mechanics. It corresponds to the total energy (kinetic plus potential) in classical mechanics and describes the energy and time evolution of a system.

ħ: This is the reduced Planck constant, a physical constant, often represented as an h-bar. It plays a crucial role in quantum mechanics, bridging wave–particle duality.

T and V: These represent the kinetic and potential energy of the system, respectively. In quantum mechanics, the Hamiltonian operator is often decomposed into kinetic and potential parts, i.e., H = T + V.

The meaning of this formula is that the way a quantum system's state evolves over time is determined by its Hamiltonian operator. Specifically, the time evolution operator is an exponential function controlled by the Hamiltonian operator. This is a fundamental principle of quantum mechanics, known as the Schrödinger equation. We bring the above equation into the Schrödinger equation for the state vector to also obtain the equation for the propagator [46]:

$$i\hbar\frac{\partial}{\partial t}K(r,t;r_0) = \left(-\frac{\hbar^2}{2\mu}\nabla^2 + V(r)\right)K(r,t;r_0) \tag{35}$$

$$U(t) = e^{-i/\hbar \sum_{j=0}^{N-1; t_N = t} H(t_{j+1} - t_j)} = \Pi_j e^{-iH\Delta t_j/\hbar} \tag{36}$$

The propagator can therefore be expressed as follows:

$$\begin{aligned} K(r,t,x_0) &= \int dr_1 dr_2 \cdots dr_{N-1} \langle r|U(t_N - t_{N-1})|r_{N-1}\rangle \langle r_{N-1}|U(t_{N-1} - t_{N-2})|r_{N-2}\rangle \\ &\quad \cdots \langle r_1|U(t_1)|r_0\rangle \\ &= \int dr_1 dr_2 \cdots dr_{N-1} K(r, \Delta t; r_{N-1}) K(r_{N-1}, \Delta t; r_{N-2}) \cdots K(r_1, \Delta t; r_0) \end{aligned} \tag{37}$$

Suppose the kinetic energy part is only related to momentum. In that case, we can use the Fourier transform to study it under the momentum table, while the potential energy part is only related to the position.

$$\begin{aligned} K(r_2, \Delta t; r_1) &= \left\langle r_2 \middle| e^{-\frac{ip^2}{\hbar 2\mu}\Delta t} e^{-\frac{i}{\hbar}V(r_1)\Delta t} \middle| r_1 \right\rangle + O(\Delta t^2) \\ &= e^{-\frac{i}{\hbar}V(r_1)\Delta t} \left\langle r_2 \middle| e^{-\frac{ip^2}{\hbar 2\mu}\Delta t} \middle| r_1 \right\rangle + O(\Delta t^2) \end{aligned} \tag{38}$$

For the momentum part, we switch to the momentum table and further solve for the following:

$$\left\langle r_2 \middle| e^{-\frac{ip^2}{\hbar 2\mu}\Delta t} \middle| r_1 \right\rangle = \int dp e^{-\frac{ip^2}{\hbar 2\mu}\Delta t} \langle r_2 \mid p\rangle \langle p \mid r_1\rangle = \int \frac{dp}{(2\pi\hbar)^n} e^{\frac{i}{\hbar}(-\frac{p^2\Delta t}{2\mu} + p(r_2 - r_1))} \tag{39}$$

The Fourier transform was partially used to switch to the momentum manifold. The above integral can be directly rounded to a Gaussian integral to obtain the following:

$$\left\langle r_2 \middle| e^{-\frac{ip^2}{\hbar 2\mu}\Delta t} \middle| r_1 \right\rangle = \left(\frac{\mu}{2\pi i\hbar\Delta t}\right)^{n/2} e^{\frac{im\mu}{2\hbar\Delta t}} \tag{40}$$

Therefore, the propagator of the leap from orbit r_2 tor r_1 is as follows:

$$K(r_2, \Delta t; r_1) = \left(\frac{\mu}{2\pi i\hbar\Delta t}\right)^{n/2} e^{\frac{i}{\hbar}(\frac{\mu(r_2 - r_1)^2}{2\Delta t} - V(r_1)\Delta t)} + O\left(\Delta t^2\right) \tag{41}$$

The above integrals can be directly rounded to Gaussian integrals to obtain the following:

$$\begin{aligned} \left\langle r_2 \middle| e^{-\frac{ip^2}{\hbar 2\mu}\Delta t} \middle| r_1 \right\rangle &= \left(\frac{\mu}{2\pi i\hbar\Delta t}\right)^{\frac{n}{2}} e^{\frac{i\mu(r_2 - r_1)^2}{2\hbar\Delta t}} \\ K(r_2, \Delta t; r_1) &= \left(\frac{\mu}{2\pi i\hbar\Delta t}\right)^{\frac{n}{2}} e^{\frac{i}{\hbar}(\frac{\mu(r_2 - r_1)^2}{2\Delta t} - V(r_1)\Delta t)} + O(\Delta t^2) \end{aligned} \tag{42}$$

In order to eliminate the residual term, we take the limit of infinitesimal time increments, thereby expressing the propagator as a path integral over continuous price trajectories between the initial and final market states. This path integral sums the probability amplitudes for all possible price fluctuation paths connecting the initial and final prices. Calculating the overall integration enables canceling out the residual and derives the integral path representation for the propagator $K(r, t; r_0)$. This path integral formulation aggregates amplitude contributions across the spectrum of price paths within the bounded market system. Introducing infinitesimal time slices allows for the representation of the propagator as a path integral, which yields the cancellation of the residual term and derives the mathematical formulation of the propagator.

$$K(r,t;r_0) = \lim_{N\to+\infty}\left(\frac{\mu}{2\pi i\hbar\Delta t}\right)^{\frac{nN}{2}} \int dr_{N-1}dr_{N-2}$$
$$dr_1 e^{\frac{i\Delta t}{\hbar}\sum_{j=0}^{N-1}\left(\frac{\mu(r_{j+1}-z_j)^2}{2(\Delta t)^2}-V(r_j)\right)} \tag{43}$$

The upper part of exp can be understood as the integral of a Lagrangian quantity, that is, the action quantity:

$$\lim_{\Delta t\to 0}\sum_{j=0}^{t/\Delta t}\left(\frac{\mu}{2}\left(\frac{r_{j+1}-r_j}{\Delta t}\right)^2 - V\left(r_j\right)\right) = \int_0^t ds\left(\frac{\mu}{2}\dot{x}(s)^2 - V(r(s))\right) =$$
$$\int_0^t dsL\left(r,\dot{r},s\right) = S \tag{44}$$

Therefore, the path integral representation can be written as follows:

$$K(r,t;r_0) = C\int D[r]e^{\frac{i}{\hbar}\int_0^t dsL} \tag{45}$$

The propagator $K(r,t;r_0)$ describes the transition amplitude from an initial position r_0 to a final position r over time t. It is derived using the time evolution operator U(t), which is approximated by dividing time into infinitesimal intervals Δt. The position at each interval is denoted by r_i. Terms involving mass μ, Planck's constant ħ, Hamiltonian H, potential V(r), and Lagrangian L characterize the kinetics and dynamics between intervals. A Fourier transform aids solving the propagator in momentum space p. Finally, the path integral formulation expresses the propagator as a functional integration over all possible paths r(s) from 0 to t, represented by the action S, which sums the transition amplitudes between intervals to obtain the full propagation dynamics.

Where C is a normalized constant function that can define metrics, such as continuous functions, and taking upper bounds can form the complete metric space, and so lead to various other mathematical structures, so here, D[r] is the Lebesgue measure defined on the path space, which is the exponentially weighted sum of the actions of the paths.

3.3. Discrete Selection Model to Solve the Orbit Probability

In this section, we will use a discrete selection model to calculate the probability of price transitions between different trajectories. The discrete selection model is a powerful tool that can handle discrete selection behavior.

We first define the utility function for each orbit. Its utility function usually includes a system and random utility for a specific price trajectory. System utility is the part we can predict and explain from observable data, while random utility is the random variation caused by influencing factors that we cannot observe.

Subsequently, we assume that the principle of maximizing utility drives the price transition between different orbits. This means that prices will jump to the trajectory where their utility is maximized. Therefore, we can calculate the probability of price transition to each orbit by comparing the utility functions of different rotations.

In the next section, we introduce using a discrete selection model to calculate the transition probability between price trajectories and apply this to actual financial market data. The utility function includes the systematic utility that a decision maker gets from choosing a price trajectory, and the random utility that accounts for unpredictable factors.

$$U_{ij} = V_{ij} + \varepsilon_{ij}$$
$$P_{ij} = \Pr\left(V_{ij} + \varepsilon_{ij} > V_{ik} + \varepsilon_{ik}, \forall k \in K - \{j\}\right) \tag{46}$$

U_{ij}: this is the total utility of the trajectory i to j. V_{ij}: this represents the systematic utility of the trajectory i to j. ε_{ij}: this denotes the random utility of the trajectory i to j. This part of the utility is random and cannot be predicted from our observable data. It is generally assumed to follow a certain probability distribution. P_{ij}: this represents the probability of the trajectory i selecting option j. In the random utility model, it is assumed that the trajectory will select the option with the maximum total utility. Hence, P_{ij} is the probability that U_{ij} is the maximum among all options. The meaning of this formula is that the total utility U_{ij} of the trajectory i to j is the sum of the systematic utility V_{ij} and the random utility ε_{ij}; the probability P_{ij} of the trajectory i choosing option j is the probability that U_{ij} is the maximum among all options.

$$F(\varepsilon) = \mathrm{e}^{-\mathrm{e}^{-\varepsilon}} \tag{47}$$

Then the probability of choosing the leap orbit j is as follows:

$$\begin{aligned} P_{ij} &= \Pr\left(\varepsilon_{ik} < V_{ij} - V_{ik} + \varepsilon_{ij}, \forall k \in K - \{j\}\right) \\ &= \Pr\left(\varepsilon_{ij} = r\right) \cdot \prod_{k \neq j} \Pr\left(\varepsilon_{ik} < V_{ij} - V_{ik} + r\right) \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{d}F(r)}{\mathrm{d}x} F\left(V_{ij} - V_{ik} + r\right) \mathrm{d}r \\ &= \int_{-\infty}^{\infty} \mathrm{e}^{-r} \mathrm{e}^{-\mathrm{e}^{-r}} \prod_{k \neq j} \mathrm{e}^{-\mathrm{e}^{-(v_{ij} - V_{ik} + r)}} \mathrm{d}r \end{aligned} \tag{48}$$

$$y = F(r)F(V_1 - V_2 + r) = \mathrm{e}^{-\mathrm{e}^{-\pi}(1 + \mathrm{e}^{V_1 - V_2})} \tag{49}$$

$$\frac{\mathrm{d}y}{|\mathrm{d}r} = \mathrm{e}^{-\mathrm{e}^{-r}(1 + \mathrm{e}^{v_1 - V_2})} \mathrm{e}^{-r} \left(1 + \mathrm{e}^{V_1 - V_2}\right) = y\mathrm{e}^{-r} \left(1 + \mathrm{e}^{V_2 - V_1}\right) \tag{50}$$

Since, the above equation can be written as follows:

$$\begin{aligned} &\int_{-\infty}^{\infty} \mathrm{e}^{-\sum_k \mathrm{e}^{-(v_{ij} - v_{ik} + \infty)}} \mathrm{e}^{-r} |\mathrm{d}r \\ &= \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{e}^{-*\sum_k \mathrm{e}^{-(v_{ij} - v_{ik})}} \mathrm{e}^{-r} |\mathrm{d}r \end{aligned} \tag{51}$$

In the mathematical derivation process, we define the variables and integral operations.

$$\begin{aligned} P_{ij} &= \int_{\infty}^{0} e^{-t\sum_k e^{-(v_{ij} - v_{ikt})}} \cdot (-\mathrm{d}t) \\ &= \int_{0}^{\infty} e^{-t\sum_k e^{-(v_{ij} - v_{it})}} \mathrm{d}t \\ &= \frac{e^{-t\sum\limits_k e^{-(v_{ij} - v_{ik})}}}{-\sum_k e^{-(v_{ij} - V_{it})}\Big|_0^{\infty}} \\ &= \frac{e^{-\infty} - e^{0}}{-\sum_k e^{-(v_{ij} - V_{ikt})}} \\ &= \frac{e^{v_{ij}}}{\sum_k e^{V_{ik}}} \end{aligned} \tag{52}$$

Formula (52) gives the expression for calculating the probability P_{ij} of the price transitioning from orbit i to orbit j. Here, v_{ij} and v_{ik} represent the systematic utility of orbit i to j and orbit i to k, respectively, while $\varepsilon^{\exists j}$ and $\varepsilon^{\exists k}$ denote the random utility. Through a series of mathematical derivations, it is obtained that the transition probability P_{ij} from i to j can be expressed as the ratio between the exponential of the systematic utility of orbit i to j and the sum of exponentials of the systematic utility of all possible orbits.

This formula reflects a basic idea: when price particles transition between different orbits, they will choose the orbit that maximizes their utility. By comparing the systematic utility of different orbits, the probability of transitioning to each orbit can be solved. This provides a theoretical approach to calculate the transition probability of prices between different orbits.

This chapter employs a discrete choice model from random utility theory, combining the concepts of systematic utility and random utility, and derives the calculation formula for transition probability from the principle of maximum utility. This method offers a new perspective for simulating and analyzing the quantum leap behavior of prices and lays the theoretical foundation for subsequent model calculations.

4. Demonstration

To empirically observe the nature of price transitions in financial markets, the closing price time series data of the US Dollar Index are utilized, spanning the period from 1 January 2017 to 31 December 2022. The US Dollar Index is selected as it effectively reflects the volatility and price leap characteristics in the foreign exchange market. The US Dollar Index closing price dataset used in this study covers 1827 trading days of data from 1 January 2017 to 31 December 2022, with daily frequency. Using the US Dollar Index closing prices, we can effectively study the behavior of price transitions between different orbits and validate the established quantum leap theoretical model. The analysis in this section references the algorithm outlined previously in Algorithm 1.

Algorithm 1. Quantitative description of price transitions

Ensure: data 1 January 2017–31 December 2022
Volatile particles that define prices;
Generate define different orbital parameters;
{$r1$, $r2$, $r3$, $r4$, $r5$, $r6$, $r7$, $r8$, $r9$, $r10$}
if the price fluctuation stabilizes temporarily within an orbit
then
Conduct statistics on the current orbit's price fluctuation status, including fluctuation amplitude, frequency distribution;
Analyze the distribution of price transition locations within the current orbit;
Display the range of price fluctuation amplitude in the current orbit;
end

We expanded the data in the third section of the theory to 0–100 and observed the actual transition data, as shown in Table 2 below.

Table 2. Different parameter selection for the model.

l	r	l	r	l	r	l	r	l	r
0	0.001	20	2.0008	40	4.0006	60	6.0004	80	8.0002
1	0.10099	21	2.10079	41	4.10059	61	6.10039	81	8.10019
2	0.20098	22	2.20078	42	4.20058	62	6.20038	82	8.20018
3	0.30097	23	2.30077	43	4.30057	63	6.30037	83	8.30017
4	0.40096	24	2.40076	44	4.40056	64	6.40036	84	8.40016
5	0.50095	25	2.50075	45	4.50055	65	6.50035	85	8.50015
6	0.60094	26	2.60074	46	4.60054	66	6.60034	86	8.60014
7	0.70093	27	2.70073	47	4.70053	67	6.70033	87	8.70013
8	0.80092	28	2.80072	48	4.80052	68	6.80032	88	8.80012
9	0.90091	29	2.90071	49	4.90051	69	6.90031	89	8.90011
10	1.0009	30	3.0007	50	5.0005	70	7.0003	90	9.0001

Table 2. *Cont.*

l	r	l	r	l	r	l	r	l	r
11	1.10089	31	3.10069	51	5.10049	71	7.10029	91	9.10009
12	1.20088	32	3.20068	52	5.20048	72	7.20028	92	9.20008
13	1.30087	33	3.30067	53	5.30047	73	7.30027	93	9.30007
14	1.40086	34	3.40066	54	5.40046	74	7.40026	94	9.40006
15	1.50085	35	3.50065	55	5.50045	75	7.50025	95	9.50005
16	1.60084	36	3.60064	56	5.60044	76	7.60024	96	9.60004
17	1.70083	37	3.70063	57	5.70043	77	7.70023	97	9.70003
18	1.80082	38	3.80062	58	5.80042	78	7.80022	98	9.80002
19	1.90081	39	3.90061	59	5.90041	79	7.90021	99	9.90001
100	10.0								

The strategy can be optimized by increasing expected returns, reducing volatility/drawdowns, optimizing parameters to lower trading frequency, and determining effective stoploss methods. This reflects continuously improving strategies using quantum transition theory. It models the financial market as a complex quantum system using theory to analyze uncertainty and nonlinearity. The growing complexity of strategies suggests participants apply the theory to build advanced strategies and risk management, as implied by a complex linear regression of the data.

$$\begin{aligned} \text{Profit} \; = \quad & \beta_0 + \beta_1 r + \beta_2 d + \beta_3 \text{ stoploss } + \beta_4 r^2 \\ & + \beta_5 d^2 + \beta_6 (\text{stoploss })^2 + \beta_7 rd + \beta_8 r \text{ stoploss} \\ & + \beta_9 d \text{ stoploss} \end{aligned} \tag{53}$$

where the variables are defined as follows:

gain is the dependent variable we want to predict;
r, d and stoploss are independent variables (characteristics);
β_0 is a constant term;
$\beta_1, \beta_2, \beta_3 \ldots \beta_9$ are the coefficients estimated from the data;
r^2, d^2, stoploss^2 are the squared terms of the independent variables;
rd, r stoploss, d stoploss are the interaction terms of the independent variables.

As more terms and interaction terms are brought into the model, the complexity of the model increases. This attempts to model nonlinear relationships and interactions between predictor variables.

The complete model is as follows:

$$\begin{aligned} \text{Profit} \; = \quad & -2432.54 + 158.13r + 82.438d + 0.0034 \text{ stoploss } - 840.45r^2 \\ & - 19.567d^2 + 0.005(\text{ stoploss })^2 + 5.574rd + 0.09r \text{ stoploss} \\ & - 0.0003d \text{ stoploss} \end{aligned} \tag{54}$$

Due to the limited sample size of the data, it is not possible to estimate these 13 coefficients precisely. We can only make a rough estimate based on the sample data.

$$\begin{aligned}
\beta_0(\text{constant term}) &= -2432.54\\
\beta_1(\text{linear coefficient of } r) &= 158.13\\
\beta_2(\text{linear coefficient of } d) &= 82.438\\
\beta_3(\text{linear coefficient of stoploss}) &= 0.0034\\
\beta_4(\text{squared coefficient of } r) &= -840.45\\
\beta_5(\text{squared coefficient of } d) &= -19.567\\
\beta_6(\text{squared coefficient of stoploss}) &= 0.005\\
\beta_7(rd \text{ interaction coefficient}) &= 5.574\\
\beta_8(\text{rstoploss interaction coefficient}) &= 0.09\\
\beta_9(\text{interaction term coefficient of } d \text{ stoploss}) &= -0.0003
\end{aligned}$$

The US Dollar Index closing prices during the sample period are predominantly concentrated around a mean of 0.072, exhibiting a bell-curve distribution resembling a normal distribution. Values are mainly distributed from 0.072 to 0.075, with sparser distributions below and above. A long right tail indicates positive skewness. The peak is 0.072 with an estimated standard deviation of 0.0025. The bell-shape pattern with denser central values and sparser sides is indicative of a normal distribution, albeit with narrower intervals and a skewed higher tail. Overall, the data are clustered around 0.072, but they are progressively sparser on both sides, which is consistent with the characteristics of a normal but skewed distribution.

This study found that the statistical distribution of the US Dollar Index closing prices does not follow a typical normal distribution. Further tests revealed a large number of zero and repeated values, which are less common in the expected normal distributions of classical statistical models. In addition, the distribution shape exhibits significant right skewness, differing from the symmetrical shape of common normal distributions. The dataset also contains many discrete and extreme values, again deviating from normal distribution models. These suggest the complex market behaviors reflected in the dollar index closing price sequences may not be fully explained by classical statistical models. From a quantum mechanics perspective, price fluctuations exhibit quantum superposition states, and transitions between different orbits lead to statistical distributions no longer following classical ones, presenting more complex "non-normal" characteristics instead. This verifies the necessity of analyzing financial time series from a quantum leap lens and lays the foundation for building quantum financial models.

Figure 8 reveals a non-linear relationship between profit and r. As profit increases from low to high values, r initially rises rapidly, then increases slowly, with most data at profits <200 and r from 0.07 to −0.072. Higher profit outliers above 200 occur with r up to 0.073. Negative profits exhibit scattered r values mainly from 0.07 to −0.072, limiting loss impacts on r. While a non-linear relationship exists, r remains in the 0.07–0.072 range.

The analysis discovered that although there is some discrete nonlinearity, the price fluctuation range and leap radius r generally demonstrate a positive correlation. As the price fluctuation increases, the leap radius r also shows an increasing trend. By plotting a scatter diagram of price fluctuations against the leap radius, as depicted in Figure 9, it can be observed that most data points ascend along the vertical axis as the price fluctuation on the horizontal axis increases. This indicates a characteristic of positive correlation. Figure 9

shows the actual leap radius ortho-terrestrial distribution, with each point representing the price fluctuation range and corresponding orbital leap radius of a trade. Meanwhile, the low correlation coefficient also suggests this correlation is heavily influenced by errors and other factors. Further complexity analysis is needed to describe the exact relationship between the two. But this preliminarily verifies a certain positive correlation trend between the price fluctuation range and leap radius r.

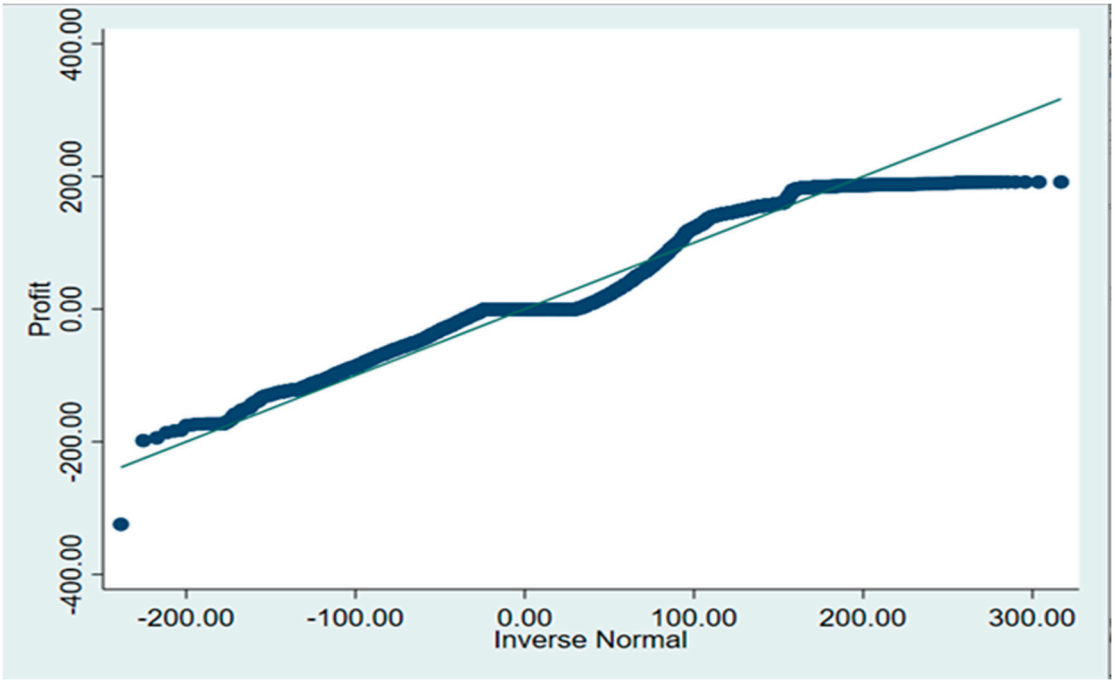


Figure 8. distribution chart.

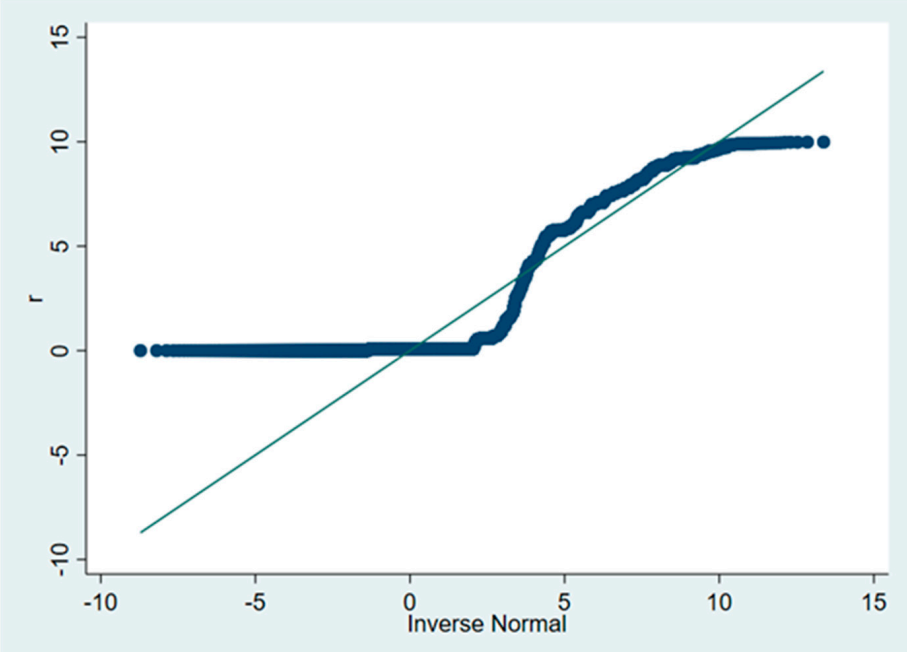


Figure 9. Price Fluctuation and Orbit Leap Correlation Chart.

Figure 10 shows the relationship between time and price fluctuation amplitude between energy levels. The x-axis represents time and y-axis represents the fluctuation mag-

nitude. Price variations exhibit non-uniform behavior, with localized periods of heightened fluctuations contrasting intervals of relative stability. This local clustering phenomenon occurs within the price transition process.

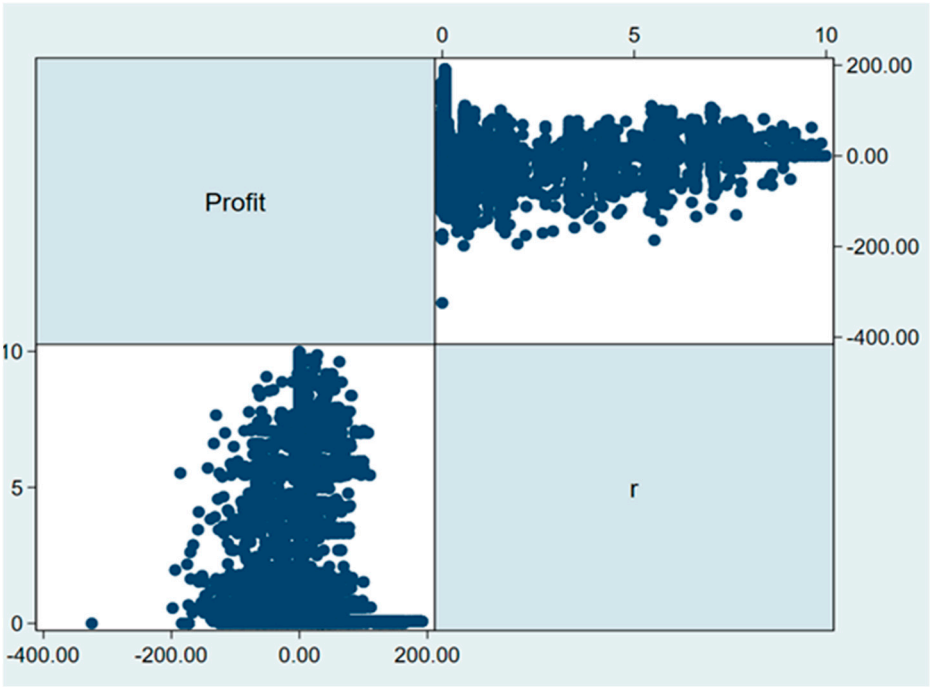


Figure 10. Leap radius and scatter matrix correlation diagram.

Figure 11 depicts a two-dimensional simulation of a two-level quantum system with the initial state preset at the first energy level. The probabilities of the system locating at each level over time are computed and charted. Periodic transitions are discerned between the two levels. Such regularity was presumed for this idealized system devoid of external interference.

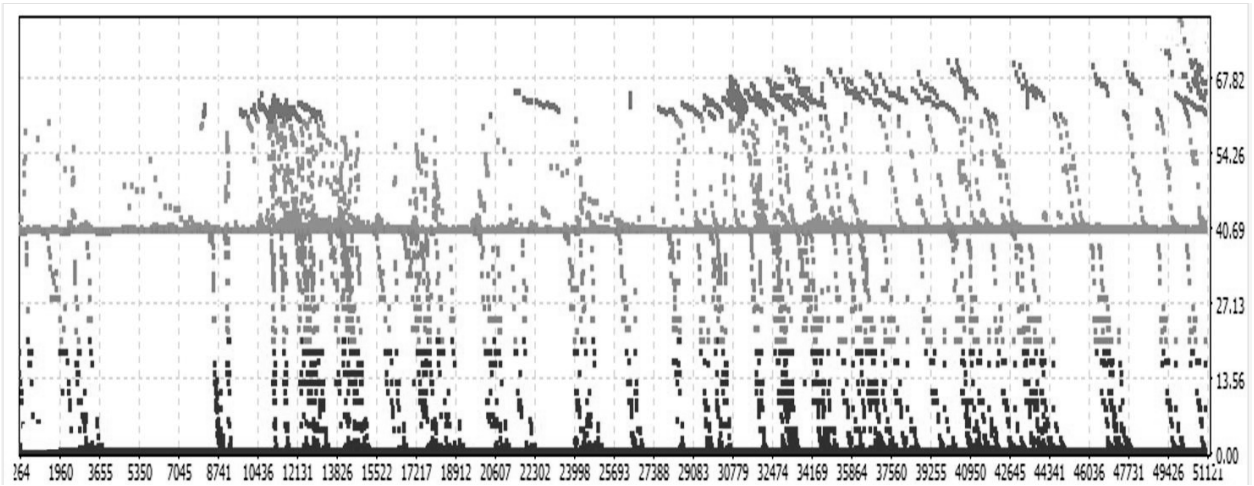


Figure 11. Transition state.

Specifically, an x,y coordinate system is established where the x-axis denotes time and the y-axis represents the probabilities of the two energy levels. Subsequently, at different time points, the probabilities of the system occupying each energy level are calculated and two probability curves are plotted. As time elapses, the two probability curves exhibit alternating peaks, thereby constituting a two-dimensional image of the periodic transitions.

For this idealized quantum system, the transitions between the two levels are very regular due to the assumption of no external interference. However, in real financial markets, more intricate influencing factors lead to more complex evolutionary dynamics of the system. Overall, such two-dimensional simulations assist in examining the probability variations between different levels, providing valuable references for constructing quantum financial models.

5. Conclusions

Our study demonstrates the potential of applying quantum transition theory to model the price dynamics in financial markets. The quantitative models developed can characterize the transition behaviors and optimal paths underlying complex price fluctuations.

We recognize certain limitations exist due to market noise and the discretization of financial variables, which may cause deviations from the theoretical mechanisms. More rigorous empirical validations on broader market data would further verify the robustness of the quantum models proposed.

Nonetheless, this research lays a solid foundation to utilize quantum finance theory for practical pricing and risk management applications. There remain ample promising directions, such as incorporating market frictions, expanding the models for high-dimensional scenarios, and exploring agent-based simulations.

Financial systems exhibit entanglement, emergence, and nonlinearity similar to complex phenomena in nature. Quantum finance provides a fresh perspective reflecting the new era of financial development. With innovations in concepts, interpretations, and models, quantum finance will likely grow as an active interdisciplinary field tackling real-world challenges in risk, portfolio optimization, and beyond.

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Appendix A. Mapping Stochastic Differential Equations to Quantum Finance Domain

Appendix A.1. Stochastic Differential Equations and the Fokker–Planck Equation

Initially, consider the stochastic differential equation for asset pricing:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

The objective is to derive the evolution equation for the probability density function $p(x,t)$. This is achieved by employing the Itô lemma and the theory of continuous-time stochastic processes. The Itô lemma facilitates the transformation of stochastic processes and yields the differential form of the transformed process.

Let $f(x,t)$ be a differentiable function of S_t. According to the Itô lemma, the differential $df(x,t)$ is given by the following:

$$df(S_t,t) = \left(\frac{\partial f}{\partial t} + \mu S_t \frac{\partial f}{\partial x} + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma S_t \frac{\partial f}{\partial x} dW_t$$

Subsequently, consider the probability distribution of f relative to S_t. Its evolution is described by the Fokker–Planck equation, which is derived by taking the expected value of the expression obtained from the Itô lemma and applying the law of total expectation from probability theory:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}(\mu x p) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2 x^2 p\right)$$

This represents the evolution equation for the probability density function $p(x,t)$.

Appendix A.2. Analogy with the Schrödinger Equation

The one-dimensional time-dependent Schrödinger equation is as follows: $i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi$.

An attempt is made to recast the Fokker–Planck equation into a form analogous to the Schrödinger equation by considering the following transformations:

1. Introducing the complex domain: consider $\Psi(x,t) = \sqrt{p(x,t)}e^{iS(x,t)/\hbar}$, where $S(x,t)$ is a real function related to x and t.
2. Associating volatility with Planck's constant: set $\sigma^2 = \hbar/m$.
3. Introducing potential energy: for a complete mathematical analogy, one may posit that $V(x)$ corresponds to a form of potential energy within the financial model.

Through these transformations, the Fokker–Planck equation can be rewritten in terms of the wave function Ψ, thereby mathematically simulating the Schrödinger equation. In this process, the relationship between $p(x,t)$ and $S(x,t)$ must be handled correctly to ensure they satisfy the appropriate boundary and normalization conditions. This process involves complex arithmetic and differential equations.

By mathematical transformation and analogy, the stochastic differential equation from the realm of finance is formally mapped onto the quantum mechanics equation of the physical domain. This analogy highlights the similarities in mathematical form between different disciplines, but it must be emphasized that such similarity does not imply equivalence in physical or financial interpretation. In fact, they describe entirely different phenomena: one characterizes the stochastic behavior of macro-scale markets, while the other describes the quantum behavior of micro-scale particles. Thus, while the mathematical analogy is insightful and appealing, when applying it to real-world problems, one must carefully delineate the specific contexts and foundational assumptions to which they pertain.

Appendix B. Extending Quantum Square on Schwartz Distribution Space

Quantum finance employs tools from quantum mechanics to model financial markets, in which the concept of Schwartz space plays a significant role. The Schwartz space $S(R\wedge n)$ includes all rapidly decreasing smooth functions $f : R\wedge n \to R$, such that for every non-negative integer α and β, the functions and all their derivatives satisfy the following:

$$p_(\alpha,\beta)(f) = \sup_{_}\left(x \in R\wedge n\right)\left|x\wedge\alpha D\wedge\beta f(x)\right| < \infty$$

Financial models often use Brownian motion W_t to simulate changes in asset prices, which can be represented by the following stochastic differential equation (SDE):

$$\mathrm{dSt} = \mu\mathrm{S_tdt} + \sigma\mathrm{S_t}d\mathrm{W_t}$$

Quantum finance characterizes quantum stochastic processes using noncommutative operators, with the Wigner function $\mathrm{W(x,p)}$ representing its probability distribution. The connection between Schwartz space and quantum finance is established through characteristic functions, where the characteristic function $\phi(t)$ is defined as follows:

$$\phi(\mathrm{t}) = \mathrm{E}\left[\mathrm{e}\wedge(\mathrm{itX})\right] = \int \mathrm{e}\wedge(\mathrm{itx})\rho(\mathrm{x})\mathrm{dx}$$

A focal point in quantum finance is the Black–Scholes–Merton partial differential equation (PDE) for pricing:

$$\partial\mathrm{V}/\partial\mathrm{t} + 1/2\sigma\wedge 2\mathrm{S}\wedge 2\left(\partial\wedge 2\mathrm{V}\right)/\left(\partial\mathrm{S}\wedge 2\right) + \mathrm{rS}(\partial\mathrm{V}/\partial\mathrm{S}) - \mathrm{rV} = 0$$

The path integral method is used to solve the above PDE, analogous to quantum particle paths, computing the evolution of the quantum state of the system:

$$\Psi(x,t) = \int D[x(t')]e^{\wedge}(iS[x(t')]/\hbar)\Psi(x_0,0)$$

The Fourier transform of a function in the Schwartz space is given by the following:

$$\psi(p) = 1/\left((2\pi\hbar)^{\wedge}(n/2)\right)\int e^{\wedge}(-i(p\cdot x)/\hbar)\psi(x)dx$$

The state of an asset is described by the wave function $\psi(S)$, which evolves according to a quantum stochastic differential equation. The characteristic function $\Phi(k,t)$ is obtained through the Fourier transform of the log-price X:

$$\Phi(k,t) = \int e^{\wedge}(ikX)\rho(X,t)dX$$

Option pricing can be expressed as a sum over probability amplitudes using path integrals:

$$C(S,t) = e^{\wedge}(-r(T-t))\int F(S_T)P(S_T,T \mid S_1 t)dS_T$$

Quantum financial models may include quantum noise, with option values calculated using quantum path integrals:

$$C(S,t) = e^{\wedge}(-r(T-t))\int D[S(t')]e^{\wedge}(iS_\text{"quantum"}\ [S(t')]/\hbar)F(S_T)$$

In summary, quantum finance integrates principles of quantum mechanics and financial theory, utilizing functions within the Schwartz space and characteristic functions to address the pricing of financial derivatives.

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