

Article

Common Best Proximity Point Theorems for Generalized Dominating with Graphs and Applications in Differential Equations

Watchareepan Atiponrat ^{1,2}, Anchalee Khemphet ^{1,2}, Wipawinee Chaiwino ^{1,2}, Teeranush Suebcharoen ^{1,2} and Phakdi Charoensawan ^{1,2,*}

¹ Advanced Research Center for Computational Simulation, Chiang Mai University, Chiang Mai 50200, Thailand; watchareepan.a@cmu.ac.th (W.A.); anchalee.k@cmu.ac.th (A.K.); wipawinee_chai@cmu.ac.th (W.C.); teeranush.s@cmu.ac.th (T.S.)

² Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand

* Correspondence: phakdi@hotmail.com

Abstract: In this paper, we initiate a concept of graph-proximal functions. Furthermore, we give a notion of being generalized Geraghty dominating for a pair of mappings. This permits us to establish the existence of and unique results for a common best proximity point of complete metric space. Additionally, we give a concrete example and corollaries related to the main theorem. In particular, we apply our main results to the case of metric spaces equipped with a reflexive binary relation. Finally, we demonstrate the existence of a solution to boundary value problems of particular second-order differential equations.

Keywords: dominating proximal; generalized Geraghty; best proximity point; common proximity point; graph; climate change

MSC: 47H10; 47H09; 54H25



Citation: Atiponrat, W.; Khemphet, A.; Chaiwino, W.; Suebcharoen, T.; Charoensawan, P. Common Best Proximity Point Theorems for Generalized Dominating with Graphs and Applications in Differential Equations. *Mathematics* **2024**, *12*, 306. <https://doi.org/10.3390/math12020306>

Academic Editors: Mihai Postolache, Jen-Chih Yao and Yonghong Yao

Received: 19 December 2023

Revised: 10 January 2024

Accepted: 16 January 2024

Published: 17 January 2024



Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

Fixed point theory is one of the most powerful research fields that proves very useful in both pure and applied mathematics aspects. In recent decades, an abundance of real-world problems have been treated from the perspective of fixed point theory. This is one of the reasons why fixed point theory keeps expanding in both popularity among researchers and the breadth of research topics. Recently, applications of fixed point theory have emerged in almost every branch of science by transforming the original questions into fixed point problems. One of its prominent applications is that mathematicians usually employ fixed point theory to establish the existence and uniqueness of solutions to differential and integral equations. Fixed point theory is employed to ascertain the identity and existence of fractional order models in the context of the climate change model under fractional derivatives, which includes investigating the effects of accelerating global warming on aquatic ecosystems by considering variables that change over time [1] and exploring how atmospheric carbon dioxide can be controlled through planting genetically modified trees [2].

Other notable advantages of fixed point theory lie among signal recovery problems involving several blurred filters. In addition, the theory also shows great involvement in the attempt to restore original images, solving image restoration problems. It is worth pointing out that what we have listed here is only a small part of the advantages of fixed point theory. For other amazing applications of the field, we encourage the reader to explore [3–9]. Furthermore, other celebrated works in the field can also be found in the references mentioned in these papers.

As mentioned above, fixed point theory expands its research topics in various dimensions. Specifically, one of the most active fields among researchers is solving best proximity point problems; see for instance [10–18]. This is because best proximity points generalize the

idea of fixed points by permitting mathematicians to consider the closest points to being fixed points in the case when the existence of a fixed point fails to be achieved. In this work, we study the case that extends the previous idea by considering a point that is the best proximity point of two functions simultaneously, namely, a common best proximity point. To be more precise, we recall that a common best proximity point is the point that, together with its image, realizes the distance between the domain and codomain of a pair of functions. This allows us to obtain the most achievable solution to a common fixed point problem in the case when there are no fixed points. It is worth noting that one of the significant benefits of studying common best proximity problems also emerges in guaranteeing the existence of solutions to differential equations, which we will explicitly illustrate in this paper.

Alternatively, researchers can impose various conditions to attain proof of the existence and uniqueness of a common best proximity point. One of the techniques that we will employ here is to adjust a factor that dominates the key inequality in the Banach contraction principle. In other words, we construct a particular class of functions that will play an important role as a generalized idea of contractions. Our idea is inspired by the work of M. A. Geraghty. First mentioned in [19], M. A. Geraghty initiated the concept of a function class S consisting of mappings $\alpha : [0, \infty) \rightarrow [0, 1)$ satisfying

$$\lim_{n \rightarrow \infty} \alpha(t_n) = 1 \implies \lim_{n \rightarrow \infty} t_n = 0.$$

This provided the existence of fixed points for self-mappings in metric spaces and generalized the Banach contraction principle. Later, in [20], M. I. Ayari extended previous works in the literature by introducing the concept of a function class F consisting of mappings $\beta : [0, \infty) \rightarrow [0, 1]$ satisfying

$$\lim_{n \rightarrow \infty} \beta(t_n) = 1 \implies \lim_{n \rightarrow \infty} t_n = 0.$$

As a consequence, this definition established the existence and uniqueness outcomes for the best proximity points in the case of closed subsets of complete metric spaces. Recently, in [21], A. Khemphet et al. generated the idea of dominating proximal generalized Geraghty for pairs of functions by employing the class F above and proved the existence and uniqueness theorems for common best proximity points in complete metric spaces. This work extended previous results in the literature and, in particular, extended recent results by L. Chen; see [22].

In this work, we also adopt the famous notion of metric spaces endowed with directed graphs. This powerful idea was first brought up in the construction of J. Jachymski; see [23]. It is very influential that several research articles with this theme keep emerging repeatedly in the literature; see for instance [24–30].

Our main goal of this paper is to initiate a more general concept of dominating proximal generalized Geraghty pairs, namely Geraghty dominating of type $\Gamma_{\alpha, \beta}$ pairs. With this definition, we aim to extend preceding works in the literature to the case of metric spaces endowed with directed graphs. Indeed, we will prove the existence and uniqueness results for common best proximity points in our settings. To be more specific, we organize this paper into six consecutive sections. Our first section is the introduction that provides the motivation and objectives of this work. In Section 2, we offer our main results including important definitions and the main theorem, which asserts the existence and uniqueness results for a common best proximity point of a pair of functions (α, β) in complete metric space. In Section 3, we provide a concrete example supporting our main results along with the consequent corollaries. This establishes coincidence point and fixed point theorems as being direct outcomes of the main part. In Section 4, we offer the necessary definitions and prove a common best proximity point theorem for complete metric spaces endowed with reflexive binary relations. Furthermore, in Section 5, we illustrate the application of our main theorem in ordinary differential equations. Finally, we allocate Section 6 to our conclusions.

2. Main Results

In this section, we initiate concepts of being \tilde{G} -proximal for mappings, and Geraghty dominating of type $\Gamma_{\alpha,\beta}$ for a pair of mappings (α, β) . Indeed, we also provide a common best proximity theorem for such mappings.

Hereafter, for a metric space (X, d) with $A, B \subseteq X$, we employ the notations defined as follows:

$$\begin{aligned} d(A, B) &:= \inf\{d(a, b) : a \in A \text{ and } b \in B\}; \\ A_0 &:= \{a \in A : \text{there is } b \in B \text{ with } d(a, b) = d(A, B)\}; \\ B_0 &:= \{b \in B : \text{there is } a \in A \text{ with } d(a, b) = d(A, B)\}. \end{aligned}$$

Throughout this work, let $(X, d, \alpha, \beta, \tilde{G})$ denote a structure having the following five properties:

- (1) X is a nonempty set;
- (2) (X, d) is a metric space;
- (3) $\alpha, \beta : A \rightarrow B$ are functions with (A, B) being a pair of nonempty subsets of X ;
- (4) A_0, B_0 are nonempty and $\alpha(A_0) \subseteq B_0$;
- (5) X is endowed with a directed graph $\tilde{G} = (V_{\tilde{G}}, E_{\tilde{G}})$. Here, the set of vertices, denoted by $V_{\tilde{G}}$, is X . In addition, the set of edges, denoted by $E_{\tilde{G}}$, contains the diagonal of $X \times X$ but excludes parallel edges.

It is worth providing some remarks at this moment that our results will certainly work for the case of undirected graphs. This is because every undirected graph can be treated as a directed graph with its set of edges being symmetric. In addition, it can be seen that every metric space is naturally equipped with a directed graph $\tilde{G} = (V_{\tilde{G}}, E_{\tilde{G}})$ such that $V_{\tilde{G}} = X$ and $E_{\tilde{G}} = X \times X$. For further details, we encourage the readers to investigate the prominent reference [23].

Next, we give a notion of Ω as follows:

$$\Omega := \left\{ \omega : [0, \infty) \rightarrow [0, 1] : \lim_{n \rightarrow \infty} \omega(t_n) = 1 \implies \lim_{n \rightarrow \infty} t_n = 0 \right\}.$$

Definition 1. On $(X, d, \alpha, \beta, \tilde{G})$, the pair (α, β) is said to be Geraghty dominating of type $\Gamma_{\alpha,\beta}$ if there exists $\omega \in \Omega$ such that for any $x_1, x_2, p_1, p_2, q_1, q_2 \in A$ with

$$d(p_1, \alpha x_1) = d(p_2, \alpha x_2) = d(A, B) = d(q_1, \beta x_1) = d(q_2, \beta x_2),$$

the fact that $(\beta x_1, \beta x_2) \in E_{\tilde{G}}$ implies

$$d(p_1, p_2) \leq \omega(\Gamma(p_1, p_2, q_1, q_2))\Gamma(p_1, p_2, q_1, q_2), \tag{1}$$

where

$$\Gamma(p_1, p_2, q_1, q_2) = \max \left\{ \begin{aligned} &d(q_1, q_2) + |d(q_1, p_1) - d(q_2, p_2)|, \\ &d(q_1, p_1) + |d(q_1, q_2) - d(q_2, p_2)|, \\ &d(q_2, p_2) + |d(q_1, q_2) - d(q_1, p_1)| \end{aligned} \right\}.$$

Example 1. Let $X = \mathbb{R}^2$ equipped with the metric d given by

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Let $A = \{(x, -1) : 0 \leq x \leq 9\}$ and $B = \{(y, 1) : 0 \leq y \leq 9\}$. It is easy to see that $d(A, B) = 2$.

Define mappings $\alpha, \beta : A \rightarrow B$ by

$$\alpha(x, -1) = (\ln(x + 1), 1) \text{ and } \beta(x, -1) = (x, 1)$$

for all $(x, -1) \in A$.

It suffices to show that our setting satisfies all the requirements of Definition 1.

(i) By the definitions of A_0 and B_0 , we obtain that $A_0 = A$ is closed and $B_0 = B$. Additionally,

$$\alpha(A_0) = \{(x, 1) : 0 \leq x \leq \ln(10)\} \subseteq \{(x, 1) : 0 \leq x \leq 9\} = B_0 = \beta(A_0).$$

Now, define

$$E_{\tilde{G}} = \left\{ ((x, y), (u, v)) \in \mathbb{R}^2 \times \mathbb{R}^2 : x \geq u, y \leq v \text{ and } x, u \in [0, 1] \right\}.$$

Define a mapping $\omega : [0, \infty) \rightarrow [0, 1]$ by

$$\omega(t) = \begin{cases} 1 & \text{if } t = 0, \\ \frac{\arctan(t)}{t} & \text{if } t > 0. \end{cases}$$

Then it can be checked that $\omega \in \Omega$.

Let $x_1, x_2, p_1, p_2, q_1, q_2 \in A$ such that

$$\begin{aligned} x_1 &= (\hat{x}_1, -1), & x_2 &= (\hat{x}_2, -1), \\ p_1 &= (\hat{p}_1, -1), & p_2 &= (\hat{p}_2, -1), \\ q_1 &= (\hat{q}_1, -1), & q_2 &= (\hat{q}_2, -1) \end{aligned}$$

and

$$d(p_1, \alpha x_1) = d(p_2, \alpha x_2) = d(A, B) = d(q_1, \beta x_1) = d(q_2, \beta x_2).$$

It follows that $\hat{p}_1 = \ln(\hat{x}_1 + 1)$, $\hat{p}_2 = \ln(\hat{x}_2 + 1)$, $\hat{q}_1 = \hat{x}_1$, $\hat{q}_2 = \hat{x}_2$ and $\hat{x}_1, \hat{x}_2 \in [0, 9]$.

Assume that $(\beta x_1, \beta x_2) \in E_{\tilde{G}}$. Then we have $\hat{x}_1 \geq \hat{x}_2$ and $\hat{x}_1, \hat{x}_2 \in [0, 1]$. This implies that $\hat{q}_1 \geq \hat{q}_2$ and $\hat{q}_1, \hat{q}_2 \in [0, 1]$. To obtain the inequality (1), notice that if $p_1 = p_2$ or $q_1 = q_2$, then we are finished. So, assume that $p_1 \neq p_2$ and $q_1 \neq q_2$. Thus, we have

$$\begin{aligned} d(p_1, p_2) &= |\hat{p}_1 - \hat{p}_2| \\ &= |\ln(\hat{x}_1 + 1) - \ln(\hat{x}_2 + 1)| \\ &= |\ln(\hat{q}_1 + 1) - \ln(\hat{q}_2 + 1)| \\ &\leq \ln(1 + |\hat{q}_1 - \hat{q}_2|) \\ &\leq \ln(1 + \Gamma(p_1, p_2, q_1, q_2)) \\ &\leq \arctan(\Gamma(p_1, p_2, q_1, q_2)) \\ &= \left[\frac{\arctan(\Gamma(p_1, p_2, q_1, q_2))}{\Gamma(p_1, p_2, q_1, q_2)} \right] \Gamma(p_1, p_2, q_1, q_2) \\ &= \omega(\Gamma(p_1, p_2, q_1, q_2)) \Gamma(p_1, p_2, q_1, q_2). \end{aligned}$$

Therefore, the pair (α, β) is Geraghty dominating of type $\Gamma_{\alpha, \beta}$.

The following lemma will prove useful in showing the main theorem.

Lemma 1. On $(X, d, \alpha, \beta, \tilde{G})$, for any $x, y \in A$ and any $\omega \in \Omega$,

$$d(x, y) \leq \omega(\Gamma(x, y, x, y)) \Gamma(x, y, x, y) \text{ implies } \Gamma(x, y, x, y) = 0.$$

Proof. Let $x, y \in A$ and $\omega \in \Omega$ such that

$$d(x, y) \leq \omega(\Gamma(x, y, x, y))\Gamma(x, y, x, y).$$

Consider

$$\begin{aligned} \Gamma(x, y, x, y) &= \max \left\{ d(x, y) + |d(x, x) - d(y, y)|, \right. \\ &\quad d(x, x) + |d(x, y) - d(y, y)|, \\ &\quad \left. d(y, y) + |d(x, y) - d(x, x)| \right\} \\ &= d(x, y). \end{aligned}$$

Suppose, on the contrary, that $\Gamma(x, y, x, y) \neq 0$, i.e., $d(x, y) > 0$. We have

$$\begin{aligned} d(x, y) &\leq \omega(\Gamma(x, y, x, y))\Gamma(x, y, x, y) \\ &= \omega(d(x, y))d(x, y) \\ &\leq d(x, y). \end{aligned}$$

This gives

$$1 \leq \omega(d(x, y)) \leq 1.$$

By the property of ω , $d(x, y) = 0$, which is a contradiction. Thus, $\Gamma(x, y, x, y) = 0$. \square

Next, we recall relevant definitions in the literature.

Definition 2 ([18]). Suppose we have a structure $(X, d, \alpha, \beta, \tilde{G})$.

(i) For any $x^* \in A$, x^* is said to be a **common best proximity point** of the pair (α, β) if

$$d(x^*, \alpha x^*) = d(A, B) = d(x^*, \beta x^*).$$

We denote the set of common best proximity points of (α, β) by $C_B(\alpha, \beta)$.

(ii) For any $x^* \in A$, x^* is said to be a **coincidence point** of the pair (α, β) if

$$\alpha x^* = \beta x^*.$$

We denote the set of coincidence points of (α, β) by $C(\alpha, \beta)$.

(iii) For any $x^* \in A$, x^* is said to be a **fixed point** of α if

$$\alpha x^* = x^*.$$

We denote the set of fixed points of α by $Fix(\alpha)$.

Definition 3 ([22]). On $(X, d, \alpha, \beta, \tilde{G})$, we say that the pair (α, β) commutes proximally if for any $x, p, q \in A$,

$$d(q, \alpha x) = d(A, B) = d(p, \beta x) \text{ implies } \alpha p = \beta q.$$

Lemma 2. On $(X, d, \alpha, \beta, \tilde{G})$, assume that the following two conditions hold:

- (1) There exists $u \in A_0 \cap C(\alpha, \beta)$;
- (2) (α, β) commutes proximally.

Then, there exists $v \in A_0$ such that $d(v, \alpha u) = d(A, B) = d(v, \beta u)$ and

$$\Gamma(u, v, u, v) = 0 \text{ if and only if } u = v \in C_B(\alpha, \beta).$$

Proof. Let $u \in A_0 \cap \mathcal{C}(\alpha, \beta)$. Then, we have $\alpha u = \beta u$. Since $u \in A_0$ and $\alpha(A_0) \subseteq B_0$, we have $\alpha u \in B_0$. So, there exists $v \in A_0$ such that

$$d(v, \alpha u) = d(A, B) = d(v, \beta u). \tag{2}$$

Consider

$$\begin{aligned} \Gamma(u, v, u, v) &= \max \left\{ d(u, v) + |d(u, u) - d(v, v)|, \right. \\ &\quad d(u, u) + |d(u, v) - d(v, v)|, \\ &\quad \left. d(v, v) + |d(u, v) - d(u, u)| \right\} \\ &= d(u, v). \end{aligned} \tag{3}$$

If $d(u, v) = \Gamma(u, v, u, v) = 0$, then $u = v$. Furthermore, by (2), we obtain

$$d(v, \alpha v) = d(u, \alpha u) = d(A, B) = d(u, \beta u) = d(v, \beta v).$$

Thus, $u = v \in \mathcal{C}_B(\alpha, \beta)$.

For the inverse, suppose that $u = v$; by (3) we have $\Gamma(u, v, u, v) = d(u, v) = 0$. \square

Lemma 3. On $(X, d, \alpha, \beta, \tilde{G})$, suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in A_0 such that all the following four conditions hold:

- (1) $\beta x_{n+1} = \alpha x_n$ for all $n \geq 0$;
- (2) $d(y_n, y_{n+1}) \neq 0$ and $d(y_n, \alpha x_n) = d(A, B)$ for all $n \geq 0$;
- (3) (α, β) is Geraghty dominating of type $\Gamma_{\alpha, \beta}$;
- (4) $(\beta x_n, \beta x_{n+1}) \in E_{\tilde{G}}$ for all $n \geq 0$.

Then,

$$\lim_{n \rightarrow \infty} d(y_{n-1}, y_n) = 0.$$

Moreover, assume that $\{y_{n_k}\}$ and $\{y_{m_k}\}$ are subsequences of $\{y_n\}$ with $m_k > n_k > k$ for all $k \in \mathbb{N}$, the sequences $\{d(y_{m_k}, y_{n_k})\}$ and $\{d(y_{m_{k+1}}, y_{n_{k+1}})\}$ converge to the same value, and $E_{\tilde{G}}$ has transitive property, i.e., for all $a, b, c \in X$, $(a, b), (b, c) \in E_{\tilde{G}}$ implies $(a, c) \in E_{\tilde{G}}$. Then,

$$\lim_{k \rightarrow \infty} d(y_{m_k}, y_{n_k}) = \lim_{k \rightarrow \infty} d(y_{m_{k+1}}, y_{n_{k+1}}) = 0.$$

Proof. Let $\{x_n\}$ and $\{y_n\}$ be sequences in A_0 satisfying

$$\beta x_{n+1} = \alpha x_n \text{ and } d(y_n, \alpha x_n) = d(A, B) \text{ for all } n \geq 0. \tag{4}$$

Due to (4), we obtain that

$$d(A, B) = d(y_n, \alpha x_n) = d(y_n, \beta x_{n+1}) \text{ for all } n \geq 0. \tag{5}$$

From (5), note that for all $n \geq 1$,

$$d(y_n, \alpha x_n) = d(A, B) = d(y_{n-1}, \beta x_n) \tag{6}$$

and

$$d(y_{n+1}, \alpha x_{n+1}) = d(A, B) = d(y_n, \beta x_{n+1}). \tag{7}$$

Since $(\beta x_n, \beta x_{n+1}) \in E_{\tilde{G}}$ for all $n \geq 0$, the Equations (6), (7), and the fact that (α, β) is Geraghty dominating of type $\Gamma_{\alpha, \beta}$ imply that there exists $\omega \in \Omega$ such that for all $n \geq 1$,

$$\begin{aligned} d(y_n, y_{n+1}) &\leq \omega(\Gamma(y_n, y_{n+1}, y_{n-1}, y_n))\Gamma(y_n, y_{n+1}, y_{n-1}, y_n) \\ &\leq \Gamma(y_n, y_{n+1}, y_{n-1}, y_n), \end{aligned} \tag{8}$$

where

$$\Gamma(y_n, y_{n+1}, y_{n-1}, y_n) = \max \left\{ \begin{aligned} &d(y_{n-1}, y_n) + |d(y_{n-1}, y_n) - d(y_n, y_{n+1})|, \\ &d(y_{n-1}, y_n) + |d(y_{n-1}, y_n) - d(y_n, y_{n+1})|, \\ &d(y_n, y_{n+1}) + |d(y_{n-1}, y_n) - d(y_n, y_{n+1})| \end{aligned} \right\}.$$

Now, for all $n \geq 1$, define

$$Y_n = d(y_{n-1}, y_n).$$

We have

$$\Gamma(y_n, y_{n+1}, y_{n-1}, y_n) = \max\{Y_n + |Y_n - Y_{n+1}|, Y_{n+1}\}.$$

Next, we shall prove that the sequence $\{Y_n\}$ is decreasing. Assume that $\{Y_n\}$ is not decreasing. Then there exists $k_0 \in \mathbb{N}$ such that $Y_{k_0} \leq Y_{k_0+1}$ and

$$\Gamma(y_{k_0}, y_{k_0+1}, y_{k_0-1}, y_{k_0}) = Y_{k_0+1}.$$

Put $n = k_0$ in (8); then we have

$$Y_{k_0+1} \leq \omega(Y_{k_0+1})Y_{k_0+1} \leq Y_{k_0+1}.$$

By assumption (2), $d(y_n, y_{n+1}) \neq 0$ for all $n \geq 0$, so we have $Y_{k_0+1} = d(y_{k_0}, y_{k_0+1}) \neq 0$. From the above inequality, we obtain $\omega(Y_{k_0+1}) = 1$. Also, by the fact that $\omega \in \Omega$, we have

$$Y_{k_0+1} = 0,$$

which is a contradiction. Therefore, $\{Y_n\} = \{d(y_{n-1}, y_n)\}$ is a decreasing sequence. Since it is bounded below, we see that the sequence is convergent. To obtain that $\lim_{n \rightarrow \infty} Y_n = 0$, suppose on the contrary that $\lim_{n \rightarrow \infty} Y_n > 0$. For each $n \geq 1$, since $Y_n > Y_{n+1}$, we determine that

$$\Gamma(y_n, y_{n+1}, y_{n-1}, y_n) = \max\{Y_n + |Y_n - Y_{n+1}|, Y_{n+1}\} = \max\{2Y_n - Y_{n+1}, Y_{n+1}\}$$

and

$$\lim_{n \rightarrow \infty} \max\{2Y_n - Y_{n+1}, Y_{n+1}\} = \lim_{n \rightarrow \infty} Y_n.$$

Letting $n \rightarrow \infty$ in (8), we find that

$$1 = \lim_{n \rightarrow \infty} \frac{Y_{n+1}}{\max\{2Y_n - Y_{n+1}, Y_{n+1}\}} \leq \lim_{n \rightarrow \infty} \omega\left(\max\{2Y_n - Y_{n+1}, Y_{n+1}\}\right) \leq 1,$$

which implies $\lim_{n \rightarrow \infty} \omega\left(\max\{2Y_n - Y_{n+1}, Y_{n+1}\}\right) = 1$. By the definition of ω ,

$$\lim_{n \rightarrow \infty} \max\{2Y_n - Y_{n+1}, Y_{n+1}\} = \lim_{n \rightarrow \infty} Y_n = 0,$$

which is a contradiction. Thus, we obtain

$$\lim_{n \rightarrow \infty} d(y_{n-1}, y_n) = 0. \tag{9}$$

Next, assume that $\{y_{n_k}\}$ and $\{y_{m_k}\}$ are subsequences of $\{y_n\}$ such that for all $k \in \mathbb{N}$, $m_k > n_k > k$ and there exists $\sigma \geq 0$ such that

$$\lim_{k \rightarrow \infty} d(y_{m_k}, y_{n_k}) = \lim_{k \rightarrow \infty} d(y_{m_{k+1}}, y_{n_{k+1}}) = \sigma. \tag{10}$$

Now, suppose that $\sigma > 0$. Since $\{y_{m_k}\}$ and $\{y_{n_k}\}$ satisfy the Equations (6) and (7), we obtain that

$$\begin{aligned} d(y_{n_{k+1}}, \alpha x_{n_{k+1}}) &= d(A, B) = d(y_{n_k}, \beta x_{n_{k+1}}) \text{ and} \\ d(y_{m_{k+1}}, \alpha x_{m_{k+1}}) &= d(A, B) = d(y_{m_k}, \beta x_{m_{k+1}}) \end{aligned} \tag{11}$$

for each $k \geq 1$. Since $(\beta x_n, \beta x_{n+1}) \in E_{\tilde{G}}$ for all $n \geq 0$, and $E_{\tilde{G}}$ has transitive property, we have $(\beta x_{n_{k+1}}, \beta x_{m_{k+1}}) \in E_{\tilde{G}}$ for all $k \in \mathbb{N}$. According to (11) and the fact that (α, β) is Geraghty dominating of type $\Gamma_{\alpha, \beta}$, we have

$$\begin{aligned} d(y_{n_{k+1}}, y_{m_{k+1}}) &\leq \omega(\Gamma(y_{n_{k+1}}, y_{m_{k+1}}, y_{n_k}, y_{m_k})) \Gamma(y_{n_{k+1}}, y_{m_{k+1}}, y_{n_k}, y_{m_k}) \\ &\leq \Gamma(y_{n_{k+1}}, y_{m_{k+1}}, y_{n_k}, y_{m_k}), \end{aligned} \tag{12}$$

where

$$\begin{aligned} \Gamma(y_{n_{k+1}}, y_{m_{k+1}}, y_{n_k}, y_{m_k}) &= \max \left\{ d(y_{n_k}, y_{m_k}) + |d(y_{n_k}, y_{n_{k+1}}) - d(y_{m_k}, y_{m_{k+1}})|, \right. \\ &\quad d(y_{n_k}, y_{n_{k+1}}) + |d(y_{n_k}, y_{m_k}) - d(y_{m_k}, y_{m_{k+1}})|, \\ &\quad \left. d(y_{m_k}, y_{m_{k+1}}) + |d(y_{n_k}, y_{m_k}) - d(y_{n_k}, y_{n_{k+1}})| \right\}. \end{aligned}$$

By using (9) and (10), we have

$$\lim_{k \rightarrow \infty} \Gamma(y_{n_{k+1}}, y_{m_{k+1}}, y_{n_k}, y_{m_k}) = \sigma > 0.$$

On the other hand, (10) also implies that taking $k \rightarrow \infty$ in (12) yields

$$\lim_{k \rightarrow \infty} \omega\left(\Gamma(y_{n_{k+1}}, y_{m_{k+1}}, y_{n_k}, y_{m_k})\right) = 1.$$

By the definition of ω , we conclude that

$$\sigma = \lim_{k \rightarrow \infty} \Gamma(y_{n_{k+1}}, y_{m_{k+1}}, y_{n_k}, y_{m_k}) = 0.$$

This is a contradiction. Therefore, $\sigma = 0$ and

$$\lim_{k \rightarrow \infty} d(y_{m_k}, y_{n_k}) = \lim_{k \rightarrow \infty} d(y_{m_{k+1}}, y_{n_{k+1}}) = 0.$$

The proof is complete. \square

Before we prove our main theorem, let us introduce other related definitions as follows:

Definition 4 ([30]). On $(X, d, \alpha, \beta, \tilde{G})$, we state that α is β -edge preserving with regard to \tilde{G} if for any $x, y \in A$, $(\beta x, \beta y) \in E_{\tilde{G}}$ implies $(\alpha x, \alpha y) \in E_{\tilde{G}}$.

Definition 5 ([23]). Let X be endowed with a graph \tilde{G} . For any $x \in X$, a function $T : X \rightarrow X$ is said to be \tilde{G} -continuous at x if $Tx_n \rightarrow Tx$ for each sequence $\{x_n\}$ in X with $x_n \rightarrow x$ and $(x_n, x_{n+1}) \in E_{\tilde{G}}$ for all $n \in \mathbb{N}$. In addition, we say that T is \tilde{G} -continuous if it is \tilde{G} -continuous at every point in X .

Definition 6. On $(X, d, \alpha, \beta, \tilde{G})$, the function β is said to be \tilde{G} -proximal if for any $x, y, p, q \in A$, $(\beta x, \beta y) \in E_{\tilde{G}}$ and $d(p, \beta x) = d(A, B) = d(q, \beta y)$ together imply $(p, q), (\beta p, \beta q) \in E_{\tilde{G}}$.

The next lemma, which constitutes a result in the existing literature, will play a significant role in our main theorem (see for instance [3,4]).

Lemma 4. Suppose that $\{u_n\}$ is a sequence in a metric space (X, d) . Furthermore, assume that there are subsequences $\{u_{m_k}\}$ and $\{u_{n_k}\}$ of $\{u_n\}$ together with $\epsilon > 0$ such that for all $k \in \mathbb{N}$, $m_k > n_k > k$ while n_k is the tiniest number possible with

$$d(u_{m_k}, u_{n_k}) \geq \epsilon \text{ and } d(u_{m_k}, u_{n_{k-1}}) < \epsilon.$$

If $\lim_{n \rightarrow \infty} d(u_{n-1}, u_n) = 0$, then $\{d(u_{m_k}, u_{n_k})\}$ and $\{d(u_{m_{k+1}}, u_{n_{k+1}})\}$ converge to ϵ .

We are now ready to prove our main theorem as follows:

Theorem 1. On $(X, d, \alpha, \beta, \tilde{G})$ such that (X, d) is a complete metric space, suppose that the following six conditions hold:

- (i) A_0 is closed and $\alpha(A_0) \subseteq \beta(A_0)$;
- (ii) There is $x_0 \in A_0$ such that $(\beta x_0, \alpha x_0) \in E_{\tilde{G}}$;
- (iii) α is β -edge preserving with regard to \tilde{G} , and $E_{\tilde{G}}$ satisfies the transitivity property;
- (iv) β is \tilde{G} -proximal;
- (v) (α, β) is Geraghty dominating of type $\Gamma_{\alpha, \beta}$ and commutes proximally;
- (vi) At least one of the following conditions holds:
 - (a) $(\beta u, \beta v) \in E_{\tilde{G}}$ for any $u, v \in C(\alpha, \beta)$, and α and β are \tilde{G} -continuous;
 - (b) For all sequence $\{y_n\}$ in A such that $y_n \rightarrow y \in A$ and $(\beta y_n, \beta y_{n+1}) \in E_{\tilde{G}}$ for all $n \in \mathbb{N}$, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that for each $k \in \mathbb{N}$,

$$d(y, \alpha y_{n_k}) = d(A, B) = d(y, \beta y_{n_k}) \text{ and } (\beta y_{n_k}, \beta y) \in E_{\tilde{G}}.$$

Then, $C_B(\alpha, \beta) \neq \emptyset$. Moreover, if we have $(\beta u, \beta v) \in E_{\tilde{G}}$ for all $u, v \in C_B(\alpha, \beta)$, then (α, β) has a unique common best proximity point.

Proof. Let $x_0 \in A_0$ such that $(\beta x_0, \alpha x_0) \in E_{\tilde{G}}$. Since $\alpha(A_0) \subseteq \beta(A_0)$, we obtain a sequence $\{x_n\}$ in A_0 satisfying

$$\beta x_{n+1} = \alpha x_n \text{ for all } n \geq 0. \tag{13}$$

For each $n \geq 0$, $x_n \in A_0$ implies $\alpha x_n \in \alpha(A_0) \subseteq \beta(A_0)$. So, for each $n \geq 0$, there is an element $u_n \in A_0$ such that

$$d(u_n, \alpha x_n) = d(A, B). \tag{14}$$

Furthermore, by (13) and (14), we obtain that

$$d(A, B) = d(u_n, \alpha x_n) = d(u_{n-1}, \beta x_n) \text{ for all } n \in \mathbb{N}. \tag{15}$$

Since $(\beta x_0, \beta x_1) = (\beta x_0, \alpha x_0) \in E_{\tilde{G}}$ and α is β -edge preserving with regard to \tilde{G} , we have $(\beta x_1, \beta x_2) = (\alpha x_0, \alpha x_1) \in E_{\tilde{G}}$. Continuing this process inductively, we obtain that

$$(\beta x_n, \beta x_{n+1}) \in E_{\tilde{G}} \text{ for all } n \geq 0. \tag{16}$$

In the case that there exists $n_0 \geq 0$ such that $u_{n_0} = u_{n_0+1}$, by (13) and (15), we determine that

$$d(A, B) = d(u_{n_0+1}, \alpha x_{n_0+1}) = d(u_{n_0}, \alpha x_{n_0}) = d(u_{n_0}, \beta x_{n_0+1}).$$

Since (α, β) commutes proximally, we have

$$\alpha u_{n_0} = \beta u_{n_0+1} = \beta u_{n_0},$$

which implies $u_{n_0} \in A_0 \cap \mathcal{C}(\alpha, \beta)$. Next, since $u_{n_0} \in A_0$ and $\alpha(A_0) \subseteq B_0$, there exists $p \in A_0$ such that

$$d(p, \alpha u_{n_0}) = d(A, B) = d(p, \beta u_{n_0}). \tag{17}$$

Again, because (α, β) commutes proximally, we have

$$\alpha p = \beta p.$$

Therefore, $p \in A_0 \cap \mathcal{C}(\alpha, \beta)$. Since $p \in A_0$ and $\alpha(A_0) \subseteq B_0$, there exists $q \in A_0$ such that

$$d(q, \alpha p) = d(A, B) = d(q, \beta p). \tag{18}$$

Next, if assumption (a) holds, we have $(\beta u_{n_0}, \beta p) \in E_{\tilde{G}}$. Because of (17), (18), and the fact that (α, β) is Geraghty dominating of type $\Gamma_{\alpha, \beta}$, we have

$$d(p, q) \leq \omega(\Gamma(p, q, p, q))\Gamma(p, q, p, q).$$

According to Lemma 1, we obtain that $\Gamma(p, q, p, q) = 0$. At this point, all the assumptions in Lemma 2 hold. It follows from the proof of Lemma 2 that

$$p = q \in \mathcal{C}_B(\alpha, \beta).$$

Next, assume that assumption (b) is true. Since $\{y_n\} = \{u_{n_0}\}$ is a sequence in A such that

$$(\beta u_{n_0}, \beta u_{n_0+1}) = (\beta u_{n_0}, \beta u_{n_0}) \in E_{\tilde{G}}$$

and $y_n \rightarrow u_{n_0} \in A$, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that for all $k \in \mathbb{N}$,

$$d(u_{n_0}, \alpha y_{n_k}) = d(A, B) = d(u_{n_0}, \beta y_{n_k}) \text{ and } (\beta y_{n_k}, \beta u_{n_0}) \in E_{\tilde{G}}. \tag{19}$$

Since (α, β) commutes proximally, we have $\alpha u_{n_0} = \beta u_{n_0}$. Because $u_{n_0} \in A_0$ and $\alpha(A_0) \subseteq B_0$, there exists $v' \in A_0$ such that

$$d(v', \alpha u_{n_0}) = d(A, B) = d(v', \beta u_{n_0}). \tag{20}$$

Note that by (19) and (20) we also have

$$d(v', \alpha u_{n_0}) = d(u_{n_0}, \alpha y_{n_k}) = d(A, B) = d(u_{n_0}, \beta y_{n_k}) = d(v', \beta u_{n_0}). \tag{21}$$

Due to (19), (21), and the fact that (α, β) is Geraghty dominating of type $\Gamma_{\alpha, \beta}$, we obtain

$$d(u_{n_0}, v') \leq \omega(\Gamma(u_{n_0}, v', u_{n_0}, v'))\Gamma(u_{n_0}, v', u_{n_0}, v').$$

Again, by Lemma 1, we obtain $\Gamma(u_{n_0}, v', u_{n_0}, v') = 0$. Now, every assumption in Lemma 2 is satisfied so it is a consequence that

$$u_{n_0} = v' \in \mathcal{C}_B(\alpha, \beta).$$

Now, we consider the case that $u_n \neq u_{n+1}$ for all $n \geq 0$. From (13), (14), (16), and our assumptions, the first part of Lemma 3 implies that

$$\lim_{n \rightarrow \infty} d(u_{n-1}, u_n) = 0. \tag{22}$$

Next, we claim that $\{u_n\}$ is a Cauchy sequence. Suppose, on the contrary, that $\{u_n\}$ is not a Cauchy sequence. Then there exist $\epsilon > 0$ together with subsequences $\{u_{m_k}\}$ and $\{u_{n_k}\}$ of $\{u_n\}$ such that for all $k \in \mathbb{N}$, $m_k > n_k > k$ and

$$d(u_{m_k}, u_{n_k}) \geq \epsilon. \tag{23}$$

In addition, for every $k \in \mathbb{N}$, we can choose the tiniest n_k satisfying (23) so that

$$d(u_{m_k}, u_{n_k-1}) < \epsilon.$$

Hence, (22) and Lemma 4 offer that

$$\lim_{k \rightarrow \infty} d(u_{m_k}, u_{n_k}) = \lim_{k \rightarrow \infty} d(u_{m_{k+1}}, u_{n_{k+1}}) = \epsilon.$$

According to our assumptions and the above proof, all the hypotheses of Lemma 3 hold. As a result, we obtain that

$$\epsilon = \lim_{k \rightarrow \infty} d(u_{n_k}, u_{m_k}) = 0,$$

which is a contradiction. Thence, $\{u_n\}$ becomes a Cauchy sequence in the closed subset A_0 of the complete metric space X . Thus, there is $u \in A_0$ such that

$$\lim_{n \rightarrow \infty} u_n = u. \tag{24}$$

Now, by (15), using the fact that β is \tilde{G} -proximal and (α, β) commutes proximally, we determine that for each $n \geq 0$,

$$(u_n, u_{n+1}), (\beta u_n, \beta u_{n+1}) \in E_{\tilde{G}} \text{ and } \beta u_{n+1} = \alpha u_n. \tag{25}$$

Now, assume that assumption (a) is true. Due to (24), (25), and the \tilde{G} -continuity of α and β , we have

$$\alpha u = \lim_{n \rightarrow \infty} \alpha u_n = \lim_{n \rightarrow \infty} \beta u_{n+1} = \beta u,$$

which implies that $u \in \mathcal{C}(\alpha, \beta)$. Since $u \in A_0$ and $\alpha(A_0) \subseteq B_0$, we have $u \in A_0 \cap \mathcal{C}(\alpha, \beta)$ and there exists $v \in A_0$ such that

$$d(v, \alpha u) = d(A, B) = d(v, \beta u). \tag{26}$$

By the assumption that (α, β) commutes proximally, we have

$$\alpha v = \beta v,$$

which means $v \in A_0 \cap \mathcal{C}(\alpha, \beta)$. Since $v \in A_0$ and $\alpha(A_0) \subseteq B_0$, there exists $w \in A_0$ such that

$$d(w, \alpha v) = d(A, B) = d(w, \beta v). \tag{27}$$

By (26) and (27), we have

$$d(v, \alpha u) = d(w, \alpha v) = d(A, B) = d(v, \beta u) = d(w, \beta v). \tag{28}$$

Since $u, v \in \mathcal{C}(\alpha, \beta)$, by assumption (a), we have $(\beta u, \beta v) \in E_{\tilde{G}}$. Because of (28) and the fact that (α, β) is Geraghty dominating of type $\Gamma_{\alpha, \beta}$, we also have

$$d(v, w) \leq \omega(\Gamma(v, w, v, w))\Gamma(v, w, v, w).$$

Lemma 1 yields $\Gamma(v, w, v, w) = 0$. Now, all conditions in Lemma 2 hold. It follows that

$$v = w \in \mathcal{C}_{\mathcal{B}}(\alpha, \beta).$$

Next, assume that assumption (b) is true. Recall that $\{u_n\}$ is a sequence in A such that $u_n \rightarrow u \in A_0$ and $(\beta u_n, \beta u_{n+1}) \in E_{\tilde{G}}$ for all $n \in \mathbb{N}$. Therefore, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that for all $k \in \mathbb{N}$,

$$d(u, \alpha u_{n_k}) = d(A, B) = d(u, \beta u_{n_k}) \text{ and } (\beta u_{n_k}, \beta u) \in E_{\tilde{G}}. \tag{29}$$

Because (α, β) commutes proximally, we have $\alpha u = \beta u$, which yields $u \in A_0 \cap \mathcal{C}(\alpha, \beta)$. Since $u \in A_0$ and $\alpha(A_0) \subseteq B_0$, there exists $v \in A_0$ such that

$$d(v, \alpha u) = d(A, B) = d(v, \beta u). \tag{30}$$

Note that by (29) and (30), we receive for each $k \in \mathbb{N}$,

$$d(v, \alpha u) = d(u, \alpha u_{n_k}) = d(A, B) = d(u, \beta u_{n_k}) = d(v, \beta u). \tag{31}$$

It is clear that (29), (31), and the fact that (α, β) is Geraghty dominating of type $\Gamma_{\alpha, \beta}$ imply

$$d(u, v) \leq \omega(\Gamma(u, v, u, v))\Gamma(u, v, u, v).$$

So, Lemma 1 yields $\Gamma(u, v, u, v) = 0$. Now, all assumptions in Lemma 2 hold. It follows that

$$u = v \in \mathcal{C}_B(\alpha, \beta).$$

Finally, suppose that $(\beta x^*, \beta y^*) \in E_{\tilde{G}}$ for all $x^*, y^* \in \mathcal{C}_B(\alpha, \beta)$. We have to show that the set $\mathcal{C}_B(\alpha, \beta)$ is a singleton. To this end, let $x^*, y^* \in \mathcal{C}_B(\alpha, \beta)$. By assumption, we have $(\beta x^*, \beta y^*) \in E_{\tilde{G}}$ and

$$d(x^*, \alpha x^*) = d(y^*, \alpha y^*) = d(A, B) = d(x^*, \beta x^*) = d(y^*, \beta y^*).$$

Since (α, β) is Geraghty dominating of type $\Gamma_{\alpha, \beta}$, we obtain that

$$\begin{aligned} d(x^*, y^*) &\leq \omega(\Gamma(x^*, y^*, x^*, y^*))\Gamma(x^*, y^*, x^*, y^*) \\ &\leq \Gamma(x^*, y^*, x^*, y^*). \end{aligned}$$

Due to Lemma 1, the above observation suggests that $d(x^*, y^*) = \Gamma(x^*, y^*, x^*, y^*) = 0$. Thus, $x^* = y^*$ and the proof is complete. \square

In the following section, we offer a supportive example as well as straightforward consequences of Theorem 1.

3. Example and Consequences

The succeeding example demonstrates the case in which Theorem 1 can be applied.

Example 2. Let $X = \mathbb{R}^3$ equipped with the metric d given by

$$d((x_1, y_1, z_1), (x_2, y_2, z_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

for any $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3$. It is well known that (X, d) is a complete metric space.

Next, let $A = \{(1, -1, z) : 0 \leq z \leq 1\}$ and $B = \{(-2, 3, z) : 0 \leq z \leq 1\}$. It is easy to see that $d(A, B) = 5$. Define mappings $\alpha, \beta : A \rightarrow B$ by

$$\alpha(1, -1, z) = (-2, 3, \ln(z^2 + 1)) \text{ and } \beta(1, -1, z) = (-2, 3, z)$$

for all $(1, -1, z) \in A$.

It suffices to show that our setting satisfies all the requirements of Theorem 1.

(i) By the definitions of A_0 and B_0 , we obtain that $A_0 = A$ is closed and $B_0 = B$. Additionally,

$$\alpha(A_0) = \{(-2, 3, z) : 0 \leq z \leq \ln 2\} \subseteq \{(-2, 3, z) : 0 \leq z \leq 1\} = B_0 = \beta(A_0).$$

Now, define

$$E_{\tilde{G}} = \left\{ ((x, y, z), (u, v, w)) \in \mathbb{R}^3 \times \mathbb{R}^3 : x \geq u, y \leq v, z \geq w \text{ and } w, z \in [0, 1/2] \right\}.$$

(ii) There exists $(1, -1, 0) \in A_0$ such that

$$(\beta(1, -1, 0), \alpha(1, -1, 0)) = ((-2, 3, 0), (-2, 3, 0)) \in E_{\tilde{G}}.$$

(iii) It is easy to check that α is β -edge preserving with regard to \tilde{G} , and $E_{\tilde{G}}$ satisfies the transitivity property.

(iv) To see that β is \tilde{G} -proximal, let $x_1, x_2, u, v \in A$ such that

$$d(u, \beta x_1) = d(A, B) = d(v, \beta x_2)$$

and $(\beta x_1, \beta x_2) = ((-2, 3, \hat{x}_1), (-2, 3, \hat{x}_2)) \in E_{\tilde{G}}$. We have $\hat{x}_1 \geq \hat{x}_2$ and $\hat{x}_1, \hat{x}_2 \in [0, 1/2]$. Consequently, $x_1 = (1, -1, \hat{x}_1), x_2 = (1, -1, \hat{x}_2), u = (1, -1, \hat{u}), v = (1, -1, \hat{v})$, where $\hat{u} = \hat{x}_1$ and $\hat{v} = \hat{x}_2$.

Thus, $\hat{u} \geq \hat{v}$ and $\hat{u}, \hat{v} \in [0, 1/2]$. This implies $(u, v) = ((1, -1, \hat{u}), (1, -1, \hat{v})) \in E_{\tilde{G}}$ and $(\beta u, \beta v) = ((-2, 3, \hat{u}), (-2, 3, \hat{v})) \in E_{\tilde{G}}$. Hence, β is \tilde{G} -proximal.

(v) We will show that (α, β) is Geraghty dominating of type $\Gamma_{\alpha, \beta}$ and commutes proximally.

Define a mapping $\omega : [0, \infty) \rightarrow [0, 1]$ by

$$\omega(t) = \begin{cases} 1 & \text{if } t = 0, \\ \frac{\ln(1+t)}{t} & \text{if } t > 0. \end{cases}$$

Then, it can be checked that $\omega \in \Omega$.

Let $x_1, x_2, p_1, p_2, q_1, q_2 \in A$ such that

$$\begin{aligned} x_1 &= (1, -1, \hat{x}_1), & x_2 &= (1, -1, \hat{x}_2), \\ p_1 &= (1, -1, \hat{p}_1), & p_2 &= (1, -1, \hat{p}_2), \\ q_1 &= (1, -1, \hat{q}_1), & q_2 &= (1, -1, \hat{q}_2) \end{aligned}$$

and

$$d(p_1, \alpha x_1) = d(p_2, \alpha x_2) = d(A, B) = d(q_1, \beta x_1) = d(q_2, \beta x_2).$$

It follows that $\hat{p}_1 = \ln(1 + \hat{x}_1^2), \hat{p}_2 = \ln(1 + \hat{x}_2^2), \hat{q}_1 = \hat{x}_1, \hat{q}_2 = \hat{x}_2$ and $\hat{x}_1, \hat{x}_2 \in [0, 1]$.

Assume that $(\beta x_1, \beta x_2) \in E_{\tilde{G}}$. Then we have $\hat{x}_1 \geq \hat{x}_2$ and $\hat{x}_1, \hat{x}_2 \in [0, 1/2]$. This implies that $\hat{q}_1 \geq \hat{q}_2$ and $\hat{q}_1, \hat{q}_2 \in [0, 1/2]$. To obtain the inequality (1), notice that if $p_1 = p_2$ or $q_1 = q_2$, then we are finished. So, assume that $p_1 \neq p_2$ and $q_1 \neq q_2$. Thus, we have

$$\begin{aligned}
 d(p_1, p_2) &= |\hat{p}_1 - \hat{p}_2| \\
 &= \left| \ln(1 + \hat{x}_1^2) - \ln(1 + \hat{x}_2^2) \right| \\
 &= \left| \ln(1 + \hat{q}_1^2) - \ln(1 + \hat{q}_2^2) \right| \\
 &= \left| \ln\left(\frac{1 + \hat{q}_2^2 + \hat{q}_1^2 - \hat{q}_2^2}{1 + \hat{q}_2^2}\right) \right| \\
 &= \left| \ln\left(1 + \frac{\hat{q}_1^2 - \hat{q}_2^2}{1 + \hat{q}_2^2}\right) \right| \\
 &\leq \left| \ln\left(1 + \frac{\hat{q}_1 - \hat{q}_2}{1 + \hat{q}_2^2}\right) \right| \\
 &\leq \ln\left(1 + |\hat{q}_1 - \hat{q}_2| + \left| |\hat{q}_1 - \hat{p}_1| - |\hat{q}_2 - \hat{p}_2| \right| \right) \\
 &\leq \ln(1 + \Gamma(p_1, p_2, q_1, q_2)) \\
 &= \left[\frac{\ln(1 + \Gamma(p_1, p_2, q_1, q_2))}{\Gamma(p_1, p_2, q_1, q_2)} \right] \Gamma(p_1, p_2, q_1, q_2) \\
 &= \omega(\Gamma(p_1, p_2, q_1, q_2)) \Gamma(p_1, p_2, q_1, q_2).
 \end{aligned}$$

Therefore, the pair (α, β) is Geraghty dominating of type $\Gamma_{\alpha, \beta}$.
 Now, we will show that (α, β) commutes proximally. Let $x, u, v \in A$ such that

$$d(u, \alpha x) = d(A, B) = d(v, \beta x).$$

Consequently, $x = (1, -1, \hat{x}), u = (1, -1, \hat{u}), v = (1, -1, \hat{v})$, where $\hat{u} = \ln(1 + \hat{x}^2)$ and $\hat{v} = \hat{x}$. Thus,

$$\alpha v = (-2, 3, \ln(1 + \hat{v}^2)) = (-2, 3, \ln(1 + \hat{x}^2)) = (-2, 3, \hat{u}) = \beta u,$$

which means (α, β) commutes proximally.

(vi) We will prove that condition (a) is true in our case. That is, for all $u, v \in \mathcal{C}(\alpha, \beta)$, we have $(\beta u, \beta v) \in E_{\tilde{G}}$, and α and β are \tilde{G} -continuous.

First, it is not hard to see that α and β are \tilde{G} -continuous.

Second, let $u, v \in \mathcal{C}(\alpha, \beta)$ such that $u = (1, -1, \hat{u}), v = (1, -1, \hat{v})$. Thus,

$$\alpha u = \beta u \text{ and } \alpha v = \beta v.$$

Then, $\ln(1 + \hat{u}^2) = \hat{u}$ and $\ln(1 + \hat{v}^2) = \hat{v}$. It can be deduced that $\hat{u} = 0 = \hat{v}$. This means that there exists only one element $u = v = (1, -1, 0) \in \mathcal{C}(\alpha, \beta)$. So, we have $(\beta u, \beta v) \in E_{\tilde{G}}$.

Finally, to see that $\mathcal{C}_{\mathcal{B}}(\alpha, \beta)$ is a singleton, let $w, z \in \mathcal{C}_{\mathcal{B}}(\alpha, \beta)$. We obtain $w = (1, -1, \hat{w})$ and $z = (1, -1, \hat{z})$, where $\hat{w}, \hat{z} \in [0, 1]$ and

$$d(z, \alpha z) = d(w, \alpha w) = d(A, B) = d(w, \beta w) = d(z, \beta z).$$

As a consequence, $\hat{w} = \ln(1 + \hat{w}^2)$ and $\hat{z} = \ln(1 + \hat{z}^2)$. Again, it can be deduced that $\hat{w} = 0 = \hat{z}$. Therefore, $(\beta w, \beta z) \in E_{\tilde{G}}$. By Theorem 1, we determine that $\mathcal{C}_{\mathcal{B}}(\alpha, \beta)$ is a singleton. In fact, it is clear from the above argument that the point $(1, -1, 0)$ is the unique common best proximity point of (α, β) .

We close this section by showing that the succeeding corollaries are consequences of our main results. To be more specific, we first investigate a special case of Theorem 1 when there is $k \in [0, 1)$ such that $\omega(t) = k$ for each $t \geq 0$.

Corollary 1. On $(X, d, \alpha, \beta, \tilde{G})$ such that (X, d) is a complete metric space, suppose that the following six conditions hold:

- (i) A_0 is closed and $\alpha(A_0) \subseteq \beta(A_0)$;
- (ii) There is $x_0 \in A_0$ such that $(\beta x_0, \alpha x_0) \in E_{\tilde{G}}$;
- (iii) α is β -edge preserving with regard to \tilde{G} , and $E_{\tilde{G}}$ satisfies the transitivity property;
- (iv) β is \tilde{G} -proximal;
- (v) (α, β) commutes proximally, and there exists $k \in [0, 1)$ such that for any $x_1, x_2, p_1, p_2, q_1, q_2 \in A$ with

$$d(p_1, \alpha x_1) = d(p_2, \alpha x_2) = d(A, B) = d(q_1, \beta x_1) = d(q_2, \beta x_2),$$

we have $(\beta x_1, \beta x_2) \in E_{\tilde{G}}$, which implies that

$$d(p_1, p_2) \leq k\Gamma(p_1, p_2, q_1, q_2);$$

- (vi) At least one of the following conditions holds:

- (a) $(\beta u, \beta v) \in E_{\tilde{G}}$ for any $u, v \in \mathcal{C}(\alpha, \beta)$, and α and β are \tilde{G} -continuous;
- (b) For any sequence $\{y_n\}$ in A such that $y_n \rightarrow y \in A$ and $(\beta y_n, \beta y_{n+1}) \in E_{\tilde{G}}$ for all $n \in \mathbb{N}$, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that for each $k \in \mathbb{N}$,

$$d(y, \alpha y_{n_k}) = d(A, B) = d(y, \beta y_{n_k}) \text{ and } (\beta y_{n_k}, \beta y) \in E_{\tilde{G}}.$$

Then, $\mathcal{C}_{\mathcal{B}}(\alpha, \beta) \neq \emptyset$. Moreover, if we have $(\beta u, \beta v) \in E_{\tilde{G}}$ for all $u, v \in \mathcal{C}_{\mathcal{B}}(\alpha, \beta)$, then (α, β) has a unique common best proximity point.

Next, we consider another special case of the main theorem to obtain related results for coincidence points and fixed points. To start with, we investigate a particular situation that $A = B = X$, which provides $d(A, B) = 0$. Thus, we obtain the following corollary, which guarantees the existence and uniqueness of a coincidence point.

Corollary 2. On $(X, d, \alpha, \beta, \tilde{G})$ such that (X, d) is a complete metric space, suppose that the following six conditions hold:

- (i) $\alpha(X) \subseteq \beta(X)$;
- (ii) There is $x_0 \in X$ such that $(\beta x_0, \alpha x_0) \in E_{\tilde{G}}$;
- (iii) α is β -edge preserving with regard to \tilde{G} , and $E_{\tilde{G}}$ satisfies the transitivity property;
- (iv) β is \tilde{G} -proximal ;
- (v) (α, β) commutes, i.e., $\alpha\beta x = \beta\alpha x$ for all $x \in X$, and there exists $\omega \in \Omega$ such that for any $x_1, x_2 \in X$, $(\beta x_1, \beta x_2) \in E_{\tilde{G}}$ implies that

$$d(\alpha x_1, \alpha x_2) \leq \omega(\Gamma(\alpha x_1, \alpha x_2, \beta x_1, \beta x_2))\Gamma(\alpha x_1, \alpha x_2, \beta x_1, \beta x_2),$$

where

$$\Gamma(\alpha x_1, \alpha x_2, \beta x_1, \beta x_2) = \max \left\{ \begin{aligned} & d(\beta x_1, \beta x_2) + |d(\beta x_1, \alpha x_1) - d(\beta x_2, \alpha x_2)|, \\ & d(\beta x_1, \alpha x_1) + |d(\beta x_1, \beta x_2) - d(\beta x_2, \alpha x_2)|, \\ & d(\beta x_2, \alpha x_2) + |d(\beta x_1, \beta x_2) - d(\beta x_1, \alpha x_1)| \end{aligned} \right\};$$

- (vi) At least one of the following conditions holds:

- (a) $(\beta u, \beta v) \in E_{\tilde{G}}$ for any $u, v \in \mathcal{C}(\alpha, \beta)$, and α and β are \tilde{G} -continuous;
- (b) For all sequence $\{y_n\}$ in A such that $y_n \rightarrow y \in A$ and $(\beta y_n, \beta y_{n+1}) \in E_{\tilde{G}}$ for all $n \in \mathbb{N}$, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that for each $k \in \mathbb{N}$,

$$\alpha y_{n_k} = y = \beta y_{n_k} \text{ and } (\beta y_{n_k}, \beta y) \in E_{\tilde{G}}.$$

Then, (α, β) has a unique common fixed point.

Furthermore, in the following corollary, we can assert the existence and uniqueness of a fixed point when $A = B = X$ and β is the identity function I .

Corollary 3. On $(X, d, \alpha, I, \tilde{G})$ such that (X, d) is a complete metric space, suppose that the following four conditions hold:

- (i) There is $x_0 \in X$ such that $(x_0, \alpha x_0) \in E_{\tilde{G}}$;
- (ii) α is I -edge preserving with regard to \tilde{G} , and $E_{\tilde{G}}$ satisfies the transitivity property;
- (iii) There exists $\omega \in \Omega$ such that for any $x_1, x_2 \in X$, $(x_1, x_2) \in E_{\tilde{G}}$ implies that

$$d(\alpha x_1, \alpha x_2) \leq \omega(\Gamma(\alpha x_1, \alpha x_2, x_1, x_2))\Gamma(\alpha x_1, \alpha x_2, x_1, x_2),$$

where

$$\Gamma(\alpha x_1, \alpha x_2, x_1, x_2) = \max \left\{ \begin{aligned} &d(x_1, x_2) + |d(x_1, \alpha x_1) - d(x_2, \alpha x_2)|, \\ &d(x_1, \alpha x_1) + |d(x_1, x_2) - d(x_2, \alpha x_2)|, \\ &d(x_2, \alpha x_2) + |d(x_1, x_2) - d(x_1, \alpha x_1)| \end{aligned} \right\};$$

- (iv) At least one of the following conditions holds:
 - (a) $(u, v) \in E_{\tilde{G}}$ for any $u, v \in \text{Fix}(\alpha)$, and α is \tilde{G} -continuous;
 - (b) For any sequence $\{y_n\}$ in A such that $y_n \rightarrow y \in A$ and $(y_n, y_{n+1}) \in E_{\tilde{G}}$ for all $n \in \mathbb{N}$, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that for each $k \in \mathbb{N}$,

$$\alpha y_{n_k} = y = y_{n_k}.$$

Then, α has a unique fixed point.

In the next section, we affirm that our main results can be applied to the case of complete metric spaces equipped with reflexive binary relations.

4. Common Best Proximity Point Theorem for Reflexive Binary Relation \mathfrak{R}

Here and subsequently, let us denote by $(X, d, \alpha, \beta, \mathfrak{R})$ a mathematical structure such that the following five properties hold:

- (1) X is a nonempty set;
- (2) (X, d) is a metric space;
- (3) $\alpha, \beta : A \rightarrow B$ are functions with (A, B) being a pair of nonempty subsets of X ;
- (4) A_0, B_0 are nonempty and $\alpha(A_0) \subseteq B_0$;
- (5) \mathfrak{R} is a reflexive binary relation on X .

Now, let us introduce other relevant definitions as follows:

Definition 7. Suppose we have a structure $(X, d, \alpha, \beta, \mathfrak{R})$.

- (i) For any $x \in X$, α is said to be **\mathfrak{R} -continuous at x** if $Tx_n \rightarrow Tx$ for each sequence $\{x_n\}$ in X with $x_n \rightarrow x$ and $x_n \mathfrak{R} x_{n+1}$ for all $n \in \mathbb{N}$. In addition, we say that α is **\mathfrak{R} -continuous** if it is \mathfrak{R} -continuous at every point in X .
- (ii) β is said to be **\mathfrak{R} -proximal** if for any $x, y, p, q \in A$, $\beta x \mathfrak{R} \beta y$ and $d(p, \beta x) = d(A, B) = d(q, \beta y)$ together imply $p \mathfrak{R} q$ and $\beta p \mathfrak{R} \beta q$.
- (iii) α is said to be **β preserving with regard to \mathfrak{R}** if for any $x, y \in A$, $\beta x \mathfrak{R} \beta y$ implies $\alpha x \mathfrak{R} \alpha y$.
- (iv) \mathfrak{R} is said to have a transitive property if for any $a, b, c \in X$, $a \mathfrak{R} b$ and $b \mathfrak{R} c$ imply $a \mathfrak{R} c$.

At this moment, we are in a position to prove a common best proximity point theorem for complete metric spaces equipped with reflexive binary relations.

Theorem 2. On $(X, d, \alpha, \beta, \mathfrak{R})$ such that (X, d) is a complete metric space, suppose that the following six conditions hold:

- (i) A_0 is closed and $\alpha(A_0) \subseteq \beta(A_0)$;
- (ii) There is $x_0 \in A_0$ such that $\beta x_0 \mathfrak{R} \alpha x_0$;
- (iii) α is β preserving with regard to \mathfrak{R} , and \mathfrak{R} satisfies the transitivity property;
- (iv) β is \mathfrak{R} -proximal;
- (v) (α, β) commutes proximally, and there exists $\omega \in \Omega$ such that for any $x_1, x_2, p_1, p_2, q_1, q_2 \in A$ with

$$d(p_1, \alpha x_1) = d(p_2, \alpha x_2) = d(A, B) = d(q_1, \beta x_1) = d(q_2, \beta x_2),$$

we have $\beta x_1 \mathfrak{R} \beta x_2$ implies that

$$d(p_1, p_2) \leq \omega(\Gamma(p_1, p_2, q_1, q_2))\Gamma(p_1, p_2, q_1, q_2),$$

where

$$\Gamma(p_1, p_2, q_1, q_2) = \max \left\{ \begin{aligned} & d(q_1, q_2) + |d(q_1, p_1) - d(q_2, p_2)|, \\ & d(q_1, p_1) + |d(q_1, q_2) - d(q_2, p_2)|, \\ & d(q_2, p_2) + |d(q_1, q_2) - d(q_1, p_1)| \end{aligned} \right\};$$

- (vi) At least one of the following conditions holds:
 - (a) $\beta u \mathfrak{R} \beta v$ for all $u, v \in C(\alpha, \beta)$, and α and β are \mathfrak{R} -continuous;
 - (b) For any sequence $\{y_n\}$ in A such that $y_n \rightarrow y \in A$ and $\beta y_n \mathfrak{R} \beta y_{n+1}$ for all $n \in \mathbb{N}$, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that for each $k \in \mathbb{N}$,

$$d(y, \alpha y_{n_k}) = d(A, B) = d(y, \beta y_{n_k}) \text{ and } \beta y_{n_k} \mathfrak{R} \beta y.$$

Then $C_B(\alpha, \beta) \neq \emptyset$. Moreover, if we have $\beta u \mathfrak{R} \beta v$ for all $u, v \in C_B(\alpha, \beta)$, then (α, β) has a unique common best proximity point.

Proof. Let us consider a directed graph $E_{\tilde{G}} = (V_{\tilde{G}}, E_{\tilde{G}})$ such that $V_{\tilde{G}} = X$ and

$$E_{\tilde{G}} = \{(x, y) \in X \times X : x \mathfrak{R} y\}.$$

It is not hard to see that every condition in Theorem 1 is satisfied. \square

To finish our work, we devote the last part of this paper to an application of our results in ordinary differential equations.

5. Application in Ordinary Differential Equations

For this present section, we provide an application in ordinary differential equations of Corollary 3. To begin with, suppose that $u \in C[0, 1]$ and consider a second-order differential equation such that

$$u''(t) = -g(t, u(t)) \text{ for all } t \in [0, 1] \tag{32}$$

with two-point boundary conditions

$$u(0) = 0 \text{ and } u(1) = 0, \tag{33}$$

where $g : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

The important point to note here is that the function $u \in C[0, 1]$ becomes an answer for (32) if and only if it is a solution to the integral equation

$$u(t) = \int_0^1 \psi(s, t)g(s, u(s))ds,$$

where $\psi(t, s)$ is such that

$$\psi(t, s) = \begin{cases} t(1 - s) & \text{if } 0 \leq t \leq s \leq 1, \\ s(1 - t) & \text{if } 0 \leq s \leq t \leq 1. \end{cases}$$

It is easy to see that

$$\sup_{t \in [0,1]} \int_0^1 \psi(t, s) ds = \frac{1}{8}.$$

Next, it is worth pointing out that a normed space $(C[0, 1], \|\cdot\|_\infty)$ is complete. Therefore, the metric space $(C[0, 1], d)$ is also complete. Here, the metric d is defined so that for all $u, v \in C[0, 1]$,

$$d(u, v) := \|u - v\|_\infty = \sup\{|u(t) - v(t)| : t \in [0, 1]\}.$$

In addition, we define a function $F : C[0, 1] \rightarrow C[0, 1]$ such that

$$Fu(t) = \int_0^1 \psi(t, s)g(s, u(s))ds.$$

According to our setting above, it can be verified that the existence of a fixed point of the function F is equivalent to the existence of a function $u \in C[0, 1]$ satisfying (32). In particular, we illustrate this observation as an advantage of our preceding result in the following theorem.

Theorem 3. Given $\delta : \mathbb{R}^2 \rightarrow \mathbb{R}$, suppose that the following conditions (H_1) – (H_4) hold:

(H_1) There exists $u_0 \in C[0, 1]$ with $\delta(u_0(t), Fu_0(t)) \geq 0$ for each $t \in [0, 1]$;

(H_2) For all $v, u \in C[0, 1]$ and $t \in [0, 1]$,

$$\delta(u(t), v(t)) \geq 0 \text{ implies } \delta(Fu(t), Fv(t)) \geq 0;$$

(H_3) For all $v, u, w \in C[0, 1]$ and $t \in [0, 1]$,

$$\delta(u(t), v(t)) \geq 0 \text{ and } \delta(v(t), w(t)) \geq 0 \text{ together imply } \delta(u(t), w(t)) \geq 0;$$

(H_4) For any $t \in [0, 1]$ and any $v, u \in C[0, 1]$ with $\delta(u(a), v(a)) \geq 0$ for each $a \in [0, 1]$, it is satisfied that

$$|g(t, v(t)) - g(t, u(t))| \leq 8 \arctan(\Gamma(u, v)),$$

where

$$\Gamma(u, v) = \max \left\{ \|u - v\| + \left| \|u - Fu\| - \|v - Fv\| \right|, \right. \\ \left. \|u - Fu\| + \left| \|u - v\| - \|v - Fv\| \right|, \right. \\ \left. \|v - Fv\| + \left| \|u - v\| - \|u - Fu\| \right| \right\}.$$

Then, the boundary value problem (32) has a solution.

Proof. We define a directed graph $\tilde{G} = (V_{\tilde{G}}, E_{\tilde{G}})$, where $V_{\tilde{G}} = C[0, 1]$ and

$$E_{\tilde{G}} = \{(u, v) \in C[0, 1] \times C[0, 1] : \delta(u(t), v(t)) \geq 0 \text{ for all } t \in [0, 1]\}.$$

Recall that $F : C[0, 1] \rightarrow C[0, 1]$ is defined by the equation

$$Fu(t) = \int_0^1 \psi(t, s)g(s, u(s))ds.$$

Now, we will show that assumption (iii) in Corollary 3 is fulfilled in our case. Notice that the condition (H₄) suggests that for all $t \in [0, 1]$ and $u, v \in C[0, 1]$ such that $(u, v) \in E_{\tilde{G}}$, we obtain

$$\begin{aligned} |Fu(t) - Fv(t)| &= \left| \int_0^1 \psi(t, s)(g(s, u(s)) - g(s, v(s)))ds \right| \\ &\leq \int_0^1 \psi(t, s)|(g(s, u(s)) - g(s, v(s)))|ds \\ &\leq \int_0^1 \psi(t, s)8 \arctan(\Gamma(u, v))ds \\ &\leq 8 \arctan(\Gamma(u, v)) \sup_{t \in [0, 1]} \int_0^1 \psi(t, s)ds \\ &\leq \arctan(\Gamma(u, v)). \end{aligned}$$

Next, we define $\omega : [0, \infty) \rightarrow [0, 1]$ such that

$$\omega(t) = \begin{cases} \frac{\arctan(t)}{t} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

It can be checked that $\omega \in \Omega$. Now, for any $v, u \in C[0, 1]$ such that $(u, v) \in E_{\tilde{G}}$, we have

$$\begin{aligned} d(Fu, Fv) &\leq \arctan(\Gamma(u, v)) \\ &= \omega(\Gamma(u, v))\Gamma(u, v). \end{aligned}$$

Thus, assumption (iii) in Corollary 3 holds. By assuming assumptions (H₁)–(H₃), all of the requirements of Corollary 3 are fulfilled. Consequently, there is a function $u^* \in C[0, 1]$ satisfying $Fu^* = u^*$. In other words, the boundary value problem (32) has u^* as its solution. □

6. Conclusions

We construct a concept of being \tilde{G} -proximal for mappings. In addition, we introduce a definition of being Geraghty dominating of type $\Gamma_{\alpha, \beta}$ for a pair of functions (α, β) . This allows us to establish the existence and uniqueness results for a common best proximity point of the pair (α, β) in complete metric space. Furthermore, we provide a concrete example and corollaries related to the main theorem. Indeed, we apply our main results to the case of complete metric spaces endowed with reflexive binary relations. Finally, we affirm the existence of a solution to boundary value problems of particular second-order differential equations.

Author Contributions: Conceptualization, W.A. and P.C.; writing—original draft preparation, W.A. and P.C.; writing—review and editing, W.A., A.K., W.C., T.S. and P.C.; supervision, P.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research was partially funded by (1) Fundamental Fund 2024 Chiang Mai University; (2) Chiang Mai University, Chiang Mai, Thailand; and (3) Faculty of Science, Chiang Mai University, Chiang Mai, Thailand.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: This research was partially supported by (1) Fundamental Fund 2024 Chiang Mai University; (2) Chiang Mai University, Chiang Mai, Thailand; and (3) Faculty of Science, Chiang Mai University, Chiang Mai, Thailand.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Farman, M.; Shehzad, A.; Nisar, K.S.; Hincal, E.; Akgul, A.; Hassan, A.M. Generalized Ulam-Hyers-Rassias stability and novel sustainable techniques for dynamical analysis of global warming impact on ecosystem. *Sci. Rep.* **2023**, *2023*, 22441. [CrossRef]
2. Farman, M.; Sarwar, R.; Askar, S.; Ahmad, H.; Sultan, M.; Akram, M.M. Fractional order model to study the impact of planting genetically modified trees on the regulation of atmospheric carbon dioxide by analysis and modeling. *Results Phys.* **2023**, *2023*, 106409. [CrossRef]
3. Karapinar, E.; Abdeljawad, T.; Jarad, F. Applying new fixed point theorems on fractional and ordinary differential equations. *Adv. Differ. Equ.* **2019**, *2019*, 421. [CrossRef]
4. Wongsajjai, B.; Charoensawan, P.; Suebcharoen, T.; Atiponrat, W. Common fixed point theorems for auxiliary functions with applications in fractional differential equation. *Adv. Differ. Equ.* **2021**, *2021*, 503. [CrossRef]
5. Cholamjiak, W.; Khan, S.A.; Yambangwai, D.; Kazmi, K.R. Strong convergence analysis of common variational inclusion problems involving an inertial parallel monotone hybrid method for a novel application to image restoration. *Rev. Real Acad. Cienc. Exactas Fis. Nat. A. Mat.* **2020**, *114*, 99. [CrossRef]
6. Suantai, S.; Yambangwai, D.; Cholamjiak, W. Solving common nonmonotone equilibrium problems using an inertial parallel hybrid algorithm with Armijo line search with applications to image recovery. *Adv. Differ. Equ.* **2021**, *2021*, 410. [CrossRef]
7. Jun-on, N.; Suparatulatorn, R.; Gamal, M.; Cholamjiak, W. An inertial parallel algorithm for a finite family of G -nonexpansive mappings applied to signal recovery. *AIMS Math.* **2022**, *7*, 1775–1790. [CrossRef]
8. Suparatulatorn, R.; Cholamjiak, W.; Gibali, A.; Mouktonglang, T. A parallel Tseng's splitting method for solving common variational inclusion applied to signal recovery problems. *Adv. Differ. Equ.* **2021**, *492*, 492. [CrossRef]
9. Mouktonglang, T.; Suparatulatorn, R.; Park, C. Hyers-Ulam stability of hom-derivations in Banach algebras. *Carpathian J. Math.* **2022**, *38*, 839–846. [CrossRef]
10. Fan, K. Extensions of two fixed point theorems of F. E. Browder. *Math. Z.* **1969**, *122*, 234–240. [CrossRef]
11. Reich, S. Approximate selections, best approximations, fixed points, and invariant sets. *J. Math. Anal. Appl.* **1978**, *62*, 104–113. [CrossRef]
12. Basha, S.S.; Veeramani, P. Best proximity pair theorems for multifunctions with open fibres. *J. Approx. Theory* **2000**, *103*, 119–129. [CrossRef]
13. Kirk, W.A.; Reich, S.; Veeramani, P. Proximinal retracts and best proximity pair theorems. *Numer. Funct. Anal. Optim.* **2003**, *24*, 851–862. [CrossRef]
14. Eldred, A.A.; Veeramani, P. Existence and convergence of best proximity points. *J. Math. Anal. Appl.* **2006**, *323*, 1001–1006. [CrossRef]
15. Mongkolkeha, C.; Cho, Y.J.; Kumam, P. Best proximity points for Geraghty's proximal contraction mappings. *Fixed Point Theory Appl.* **2013**, *180*, 1–17. [CrossRef]
16. Suparatulatorn, R.; Suantai, S. A new hybrid algorithm for global minimization of best proximity points in Hilbert spaces. *Carpathian J. Math.* **2019**, *35*, 95–102. [CrossRef]
17. Basha, S.S. Common best proximity points: Global minimization of multi-objective functions. *J. Glob. Optim.* **2012**, *54*, 367–373. [CrossRef]
18. Kumam, P.; Mongkolekeha, C. Common best proximity points for proximity commuting mapping with Geraghty's functions. *Carpathian J. Math.* **2015**, *31*, 359–364. Available online: <http://www.jstor.org/stable/44000075> (accessed on 13 April 2017). [CrossRef]
19. Geraghty, M.A. On contractive mappings. *Am. Math. Soc.* **1973**, *40*, 604–608. [CrossRef]
20. Ayari, M.I. A best proximity point theorem for α -proximal Geraghty non-self mappings. *Fixed Point Theory Appl.* **2019**, *10*, 1–10. [CrossRef]
21. Khemphet, A.; Chanthorn, P.; Phudolsitthiphath, N. Common best proximity coincidence point theorem for dominating proximal generalized geraghty in complete metric spaces. *J. Funct. Spaces* **2020**, *2020*, 9620254. [CrossRef]
22. Chen, L. Common best proximity points theorems. *J. Math. Res. Appl.* **2019**, *39*, 289–294. [CrossRef]
23. Jachymski, J. The contraction principle for mappings on a metric space with a graph. *Proc. Am. Math. Soc.* **2008**, *136*, 1359–1373. [CrossRef]
24. Cholamjiak, P. Fixed point theorems for Banach type contraction on Tvs -cone metric spaces endowed with a graph. *J. Comput. Anal. Appl.* **2014**, *16*, 338–345.
25. Suparatulatorn, R.; Cholamjiak, W.; Suantai, S. A modified S-iteration process for G -nonexpansive mappings in Banach spaces with graphs. *Numer. Algorithms* **2018**, *77*, 479–490. [CrossRef]
26. Hammad, H.A.; Cholamjiak, W.; Yambangwai, D. A modified shrinking projection methods for numerical reckoning fixed points of G -nonexpansive mappings in Hilbert spaces with graphs. *Miskolc Math. Notes* **2019**, *20*, 941–956. [CrossRef]
27. Suparatulatorn, R.; Suantai, S.; Cholamjiak, W. Hybrid methods for a finite family of G -nonexpansive mappings in Hilbert spaces endowed with graphs. *AKCE Int. J. Graphs Comb.* **2017**, *14*, 101–111. [CrossRef]
28. Suantai, S.; Donganont, M.; Cholamjiak, W. Hybrid methods for a countable family of G -nonexpansive mappings in Hilbert spaces endowed with graphs. *Mathematics* **2019**, *7*, 936. [CrossRef]

29. Klanarong, C.; Suantai, S. Best proximity point theorems for G-proximal generalized contraction in complete metric spaces endowed with graphs. *Thai J. Math.* **2017**, *15*, 261–276.
30. Basha, S.S.; Shahzad, N.; Jeyaraj, R. Common best proximity points: Global optimization of multi-objective functions. *Appl. Math. Lett.* **2011**, *24*, 883–886. [[CrossRef](#)]

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.