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Article

A Validation of the Phenomenon of Linearly Many Faults on Burnt Pancake Graphs with Its Applications

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Abstract: "Linearly many faults" is a phenomenon observed by Cheng and Lipták in which a specific structure emerges when a graph is disconnected and often occurs in various interconnection networks. This phenomenon means that if a certain number of vertices or edges are deleted from a graph, the remaining part either stays connected or breaks into one large component along with smaller components with just a few vertices. This phenomenon can be observed in many types of graphs and has important implications for network analysis and optimization. In this paper, we first validate the phenomenon of linearly many faults for surviving graph of a burnt pancake graph BP_n when removing any edge subset with a size of approximately six times $\lambda(BP_n)$. For graph G, the ℓ-component edge connectivity denoted as $\lambda_\ell(G)$ (resp., the ℓ-extra edge connectivity denoted as $\lambda^{(\ell)}(G)$) is the cardinality of a minimum edge subset S such that $G-S$ is disconnected and has at least ℓ components (resp., each component of $G-S$ has at least $\ell+1$ vertices). Both $\lambda_\ell(G)$ and e$\lambda^{(\ell)}(G)$ are essential metrics for network reliability assessment. Specifically, from the property of "linearly many faults", we may further prove that $\lambda_5(BP_n) = \lambda^{(3)}(BP_n) + 3 = 4n-3$ for $n \geqslant 5$; $\lambda_6(BP_n) = \lambda^{(4)}(BP_n) + 4 = 5n-4$ and $\lambda_7(BP_n) = \lambda^{(5)}(BP_n) + 5 = 6n-5$ for $n \geqslant 6$.

Keywords: burnt pancake graph; component edge connectivity; extra edge connectivity; linearly many faults; conditional connectivity

MSC: 05C40; 05C75; 68R10

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1. Introduction

Investigating interconnection networks and their intrinsic properties is crucial for developing efficient parallel and distributed computer systems. A simple undirected graph represents the underlying topology of such a system called an interconnection network, where vertices represent a processor, and edges represent communication links between processors. Therefore, a well-structured network topology can lead to higher benefits for the system operation, including fault-tolerant data transmission and system reliability. For convenience, the terms graphs and networks are used interchangeably.

1.1. Background

It is almost impossible to design a multiprocessor system without defects. *Connectivity* $\kappa(G)$ and *edge connectivity* $\lambda(G)$ are used to adjudicate a network's reliability and fault tolerance. Usually, a fundamental property of an interconnection network is that it must possess *regularity* (i.e., every vertex in the network has the same degree). In particular, it is better if it meets the *maximal connectivity* (i.e., the connectivity equals the regularity of the graph). It is interesting to think about what would happen if we were to remove more than n vertices or edges from an n-regular graph. In such cases, two possible scenarios could arise. Either the resulting graph would remain connected, or it would split into several components, with the smaller component containing just a singleton. Further, one may wonder what exactly would happen if about $2n, 3n, 4n, 5n, \ldots$ or more vertices or edges were further removed. When multiple failures occur at the same time and the graph becomes disconnected, the best-case scenario is when a large component containing most of the remaining vertices is retained, along with some smaller components. This way, the subnetwork represented by the large component can continue to function effectively. In fact, this phenomenon of a disconnected graph caused by failures was first discovered in the pioneering work of Yang et al. [1]. Later, Cheng and Lipt'ak [2] popularized this concept and formally called this phenomenon the "linear many faults" property. Since then, this property has attracted much attention in the research for other networks, e.g., Cayley graphs generated by transposition trees [2], 2-tree [3], transposition triangle free unicyclic graphs [4], (n,k)-star graphs [5], arrangement graphs [6], augmented cubes [7], and dual-cube-like networks [8]. Mainly, this property can export network metrics related to fault tolerance [9–12].

Regarding Cayley graphs generated by transposition trees, we let Γ be a finite group and S a subset of Γ. The *Cayley digraph of* Γ *generated by* S, denoted by $\mathrm{Cay}(\Gamma, S)$, is digraph with vertex set Γ and arc set $\{(\gamma, \gamma s) \mid \gamma \in \Gamma \text{ and } s \in S\}$. If S does not include the identity and $S = S^{-1} = \{s^{-1} \mid s \in S\}$, then $\mathrm{Cay}(\Gamma, S)$ is an undirected simple graph. We let $[n] = \{1, 2, \ldots, n\}$, $Sym(n)$ be the symmetric group on $[n]$, and T be a set of transpositions of $Sym(n)$. Then, $\mathrm{Cay}(Sym(n), T)$ is called the *Cayley graphs generated by transposition tree* T if $G(T)$ is a tree with the vertex set $[n]$ such that edge $uv \in E(G(T))$ if and only if the corresponding transposition $(uv) \in T$.

To better understand the reliability of networks, Harary [13] proposed a concept called *conditional connectivity* which involves attaching certain conditions to connected components. Additionally, Fábrega and Fiol [14] introduced two generalizations of classical connectivity, namely *extra connectivity* and *extra edge connectivity*, which help to ensure the scale of each component. Later on, Sampathkumar [15] and Chartrand et al. [16] independently introduced a generalization of classical (edge) connectivity regarding the number of components for disconnected graphs, the former called *general connectivity* and the latter called *generalized connectivity*. Henceforth, we adopt appropriate terms called *component connectivity* and *component edge connectivity*, suggested by Hsu et al. [17] and Zhao et al. [18], respectively. For the recent results of interconnection networks, please refer to [19–23] for extra (edge) connectivity, [24–29] for component (edge) connectivity, and [12,30–32] for relationship between these two kinds of (edge) connectivity. In addition, for research on connectivity related to diverse graph indices (such as the Wiener index, the Zagreb index, the Randic index, etc.) with fuzzy information and their applications, please refer to [33–36].

This paper investigates the "linear many faults" property on a burnt pancake graph BP_n, which is the Cayley graph of the group of signed permutations generated by prefix reversals and defined by Gates and Papadimitriou in 1979 [37]. BP_n attracts the attention of researchers mainly because of another accompanying interesting definition called the *pancake graph*, which refers to the mathematic puzzle of sorting a pile of unordered pancakes in the size order. In this case, a spatula could be inserted anywhere in the pancake stack to flip all the pancakes above it. The minimum number of flips required to sort the given pancakes is called the pancake number. Hence, the operation of flips is called the *prefix reversal* when we treat the stack of pancakes as a sequence of symbols, and acquiring the pancake number is equal to obtaining the diameter of the pancake graph. Then, BP_n introduces the change in positive and negative signs, making this question more interesting. However, there has yet to be a general solution to the diameter problem of these two classes of graphs so far [38].

For burnt pancake graphs, the earliest research mainly pursued their diameters, while the current research focuses on exploring fault tolerance [39,40] and diagnosis [40,41]. In addition, many diverse connectivities have been investigated in the literature, including spanning connectivity [42], structure connectivity [43], neighbor connectivity [44,45], and component connectivity [25,28]. Following the direction of probing connectivity, this paper first proves that when removing any edge subset with a size of approximately six times $\lambda(BP_n)$, the surviving graph possesses the "linearly many faults" property. According to this characteristic, we obtain component edge connectivity and extra edge connectivity of BP_n for certain dimensions n, extending the results of [30]. Specifically, we prove that $\lambda_5(BP_n) = \lambda^{(3)}(BP_n) + 3 = 4n - 3$ for $n \geqslant 5$; $\lambda_6(BP_n) = \lambda^{(4)}(BP_n) + 4 = 5n - 4$ and $\lambda_7(BP_n) = \lambda^{(5)}(BP_n) + 5 = 6n - 5$ for $n \geqslant 6$.

1.2. Organization

Section 2 introduces definitions and necessary terminologies and notations. Also, burnt pancake graphs and related properties are given. Section 3 shows the existence of the "linearly many faults" property for the surviving graph of BP_n when the removal of an edge subset with a size of approximately six times $\lambda(BP_n)$. Section 4 obtains some relations between component edge connectivity and extra edge connectivity of BP_n through the derived property. Finally, we add concluding remarks in Section 5.

2. Preliminaries

2.1. Definitions and Terminologies

Let $G = (V(G), E(G))$ be a graph. Two vertices u and v are *adjacent* if they are joined by an edge, where u and v are called *neighbors* to each other. For vertex $u \in V(G)$, let $N_G(u)$ be the set of neighbors of u in G. For $U \subseteq V(G)$, the *open neighborhood* of U in G is defined as $N_G(U) = \cup_{u \in U} N_G(u) - U$. The *edge neighborhood* of U in G, denoted as $N_{E(G)}(U)$ (or $N_E(U)$), is the set of edges incident with at least one vertex of U in G. Also, denote $G[U]$ the subgraph of G induced by U. For two disjoint subgraphs (or vertex sets) H_1 and H_2, let $E(H_1, H_2)$ be the set of edges with one end in H_1 and the other in H_2. A cycle (resp., path) of length k is called a *k-cycle* (resp., *k-path*), denoted by C_k (resp., P_k).

Let G be a graph. The *connectivity* (resp., *edge connectivity*) of G, denoted by $\kappa(G)$ (resp., $\lambda(G)$), is the minimum number of vertices (resp., edges) that need to be removed to disconnect G or become a trivial graph. For $S \subseteq V(G)$ (resp., $S \subseteq E(G)$), let $G - S$ be the graph that removes vertices (resp., edges) of S from G. Particularly, S is a *vertex-cut* (resp., *edge-cut*) of G provided $G - S$ is disconnected. In $G - S$, the component with the largest number of vertices is called the *large component*, and a component that is not the largest one is called the *smaller component*.

Graph G is *super h-vertex-connected* (resp., *super h-edge-connected*) *of order q* if, after deleting at most h vertices (resp., h edges), the resulting graph is either connected or has one large component along with smaller components containing totally at most q vertices. In other words, the resulting graph has a component of size at least $|V(G - F)| - q$ with $|F| \leqslant h$. The following result is helpful throughout the paper.

Proposition 1 ([9])**.** *Let $q \geqslant 1$ be an integer. If a connected graph G with at least* $\max\{m + 2q + 4, 3q + 1\}$ *vertices is super-m-vertex-connected of order q, then G is super-m-edge-connected of order q.*

Definition 1 (see [17])**.** *Let G be a connected graph and $F \subset E(G)$. If $G - F$ is disconnected and has at least ℓ components, then F is called an ℓ-*component edge-cut*. The ℓ-*component edge connectivity* of G, denoted by $\lambda_\ell(G)$, is the cardinality of a minimum ℓ-component edge-cut of G. Obviously, $\lambda_{\ell+1}(G) \geqslant \lambda_\ell(G)$ and $\lambda_2(G) = \lambda(G)$ for every positive integer ℓ.*

Definition 2 (see [14])**.** *Let G be a connected graph and $F \subset E(G)$. If $G - F$ is disconnected and every component of $G - F$ has at least $h + 1$ vertices, then F is called an h-*extra edge-cut*. The h-*extra edge connectivity* of G, denoted by $\lambda^{(h)}(G)$, is the cardinality of a minimum h-extra edge-cut, if it exists. Obviously, $\lambda^{(h+1)}(G) \geqslant \lambda^{(h)}(G)$ and $\lambda^{(0)}(G) = \lambda(G)$.*

Lemma 1 (see [30]). *Let H be a connected graph and $k < |V(H)|/2$ be an integer. Let*

$$X^* = \arg\min_{X \subseteq V(H)} \{|E(X, H - X)|\colon\ |X| = k,\ H[X] \text{ and } H - X \text{ are connected subgraphs}\},$$

$h = |E(X^*, H - X^*)|$, and $\ell = |E(H[X^*])|$. If H fulfills the following:

(i) *For $F \subseteq E(H)$ with $|F| \leqslant h - 1$, $H - F$ has a large component along with small components containing totally at most $k - 1$ vertices;*

(ii) *For $F' \subseteq E(H)$ with $|F'| \leqslant h + \ell - 1$, $H - F'$ has at most k components;*

then $\lambda_{k+1}(H) = h + \ell = \lambda^{(k-1)}(H) + \ell$.

2.2. Burnt Pancake Graphs BP_n

We put a negative sign on the top of a symbol for notational convenience, e.g., $\overline{k} = -k$. We let $[n] = \{1, 2, \ldots, n\}$ and $\langle n \rangle = [n] \cup \{\overline{k}\colon\ k \in [n]\}$. A *signed permutation* of $[n]$ is an permutation $x_1 x_2 \cdots x_n$ of $\langle n \rangle$ such that $|x_1||x_2| \cdots |x_n|$ (each element takes the absolute value) forms a permutation of $[n]$. For signed permutation $x = x_1 x_2 \cdots x_i \cdots x_n$ of $\langle n \rangle$ and integer $i \in [n]$, the *ith prefix reversal* of x is defined by $x^i = \overline{x}_i \overline{x}_{i-1} \cdots \overline{x}_1 x_{i+1} \cdots x_n$.

Definition 3 (see [37]). *An n-regular graph BP_n with $n!2^n$ vertices is called the n-*dimensional burnt pancake network *if every vertex of BP_n has a unique label from the signed permutation of $\langle n \rangle$ such that $uv \in E(BP_n)$ if and only if $u^i = v$ for $i \in [n]$. The edge uv is called an i-*dimensional edge *and u is called the i-*neighbor *of v, and vice versa.*

Figure 1 depicts BP_n for all $n \in [3]$, where we use different types of line to draw distinct dimensional edges. Clearly, every vertex of BP_n has a unique k-neighbor for $k \in [n]$. By definition, BP_n is decomposed into $2n$ vertex-disjoint subgraphs BP_n^k for $k \in \langle n \rangle$ such that every vertex in a subgraph fixes the symbol k in the rightmost position. Clearly, BP_n^k is isomorphic to BP_{n-1}. An *external edge* of BP_n is one whose two ends are in distinct BP_n^ks. For $u \in V(BP_n^i)$, the unique neighbor outside BP_n^k is called the *external neighbor* of u. Indeed, an external edge is an n-dimensional edge. Also, $E_{j,k}(BP_n)$ denotes the set of edges between BP_n^j and BP_n^j for $j, k \in \langle n \rangle$ with $j \neq k$.

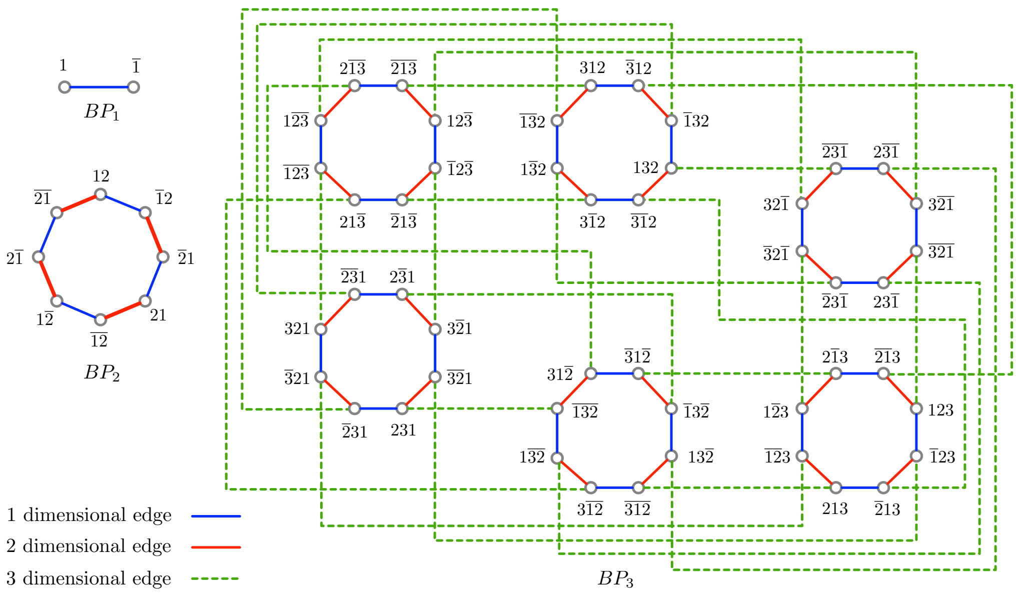

Figure 1. Burnt pancake graphs of small dimensions.

Lemma 2 (see [39,42,46]). *For BP_n, the following properties hold:*

(1) *BP_n is an n-regular graph with $n \times 2^{n-1} \times n!$ edges. $|E_{j,k}(BP_n)| = 2^{n-2} \times (n-2)!$ if $j \neq \overline{k}$, and $|E_{j,\overline{k}}(BP_n)| = 0$.*

(2) *For* $n \geqslant 2$, $\kappa(BP_n) = \lambda(BP_n) = n$.

(3) *For* $n \geqslant 2$, *the girth of* BP_n *is* $g(BP_n) = 8$.

Lemma 3 (see [40,41])**.** *For* $n \geqslant 4$, *we let* F *be a vertex-cut of* BP_n. *The following properties hold:*

(1) *If* $|F| \leqslant 2n-2$, $BP_n - F$ *has two components, one of which is a singleton or an edge. Furthermore, if the small component is an edge, then* F *is the neighborhood of this edge and* $|F| = 2n-2$.

(2) *If* $|F| \leqslant 3n-5$, $BP_n - F$ *has a large component along with smaller components containing totally at most two vertices.*

(3) *If* $|F| \leqslant 4n-7$, $BP_n - F$ *has a large component along with smaller components containing totally at most three vertices.*

Lemma 4 (see [28])**.** *For* $n \geqslant 5$, *we let* F *be a vertex-cut of* BP_n. *If* $|F| \leqslant 5n-9$, $BP_n - F$ *contains a large component along with smaller components containing totally at most four vertices.*

Lemma 5. *For* $n \geqslant 4$, *we let* F *be an edge-cut of* BP_n. *The following properties hold:*

(1) *If* $|F| \leqslant 2n-2$, $BP_n - F$ *has two components, one of which is a singleton or an edge. Furthermore, if the small component is an edge, then* F *is the neighborhood of this edge and* $|F| = 2n-2$.

(2) *If* $|F| \leqslant 3n-5$, $BP_n - F$ *has a large component along with smaller components containing totally at most two vertices.*

(3) *If* $|F| \leqslant 4n-7$, $BP_n - F$ *has a large component along with smaller components containing totally at most three vertices.*

Proof. By Lemma 3, BP_n is super-$(2n-3)$-vertex-connected of order 1, $(3n-5)$-vertex-connected of order 2, and $(4n-7)$-vertex-connected of order 3, respectively. We note that $|V(BP_n)| = n!2^n > \max\{(2n-3)+2\times 1+4, 3\times 1+1\}$ (resp., $n!2^n > \max\{(3n-5)+2\times 2+4, 3\times 2+1\}$ and $n!2^n > \max\{(4n-7)+2\times 3+4, 3\times 3+1\}$) for $n \geq 4$. By Proposition 1, BP_n is super-$(2n-3)$-edge-connected of order 1, $(3n-5)$-edge-connected of order 2, and $(4n-7)$-edge-connected of order 3, respectively. Thus, the lemma follows. □

Lemma 6. *For* $n \geqslant 5$, *we let* F *be an edge-cut of* BP_n. *If* $|F| \leqslant 5n-9$, $BP_n - F$ *has a large component along with smaller components containing totally at most four vertices.*

Proof. By Lemma 4, BP_n is super-$(5n-9)$-vertex-connected of order 4. We note that $|V(BP_n)| = n!2^n > \max\{(5n-9)+2\times 4+4, 3\times 4+1\} = \max\{5n+3, 13\}$ for $n \geq 5$. By Proposition 1, BP_n is super-$(5n-9)$-edge-connected of order 4, and the result holds. □

3. Linearly Many Faults in Burnt Pancake Graphs

In this section, we focus on the linearly many faults in burnt pancake graphs.

Lemma 7. *For* BP_n *with* $n \geqslant 4$ *and* $X \subset V(BP_n)$, *if* $|X| = 4$, *then* $|E(X, BP_n - X)| \geqslant 4n-6$ *and* $|N_{E(BP_n)}(X)| \geqslant 4n-3$.

Proof. Let $X = \{u, v, x, y\}$. Note that BP_n has no k-cycle for $k \leqslant 4$. By Lemma 2, $\kappa(BP_n) = \lambda(BP_n) = n$ and $g(BP_n) = 8$.

If $BP_n[X]$ contains four singletons, then $|E(X, BP_n - X)| = |N_{E(BP_n)}(X)| = 4n$.

If $BP_n[X]$ contains two singletons and an edge, then $|E(X, BP_n - X)| = 2n + 2(n-1) = 4n-2$ and $|N_{E(BP_n)}(X)| = 4n-2+1 = 4n-1$.

If $BP_n[X]$ contains (i) two edges or (ii) a 2-path and a singleton, then $|E(X, BP_n - X)| = 2(n-1) + 2(n-1) = 4n-4$ and $|N_{E(BP_n)}(X)| = 4n-4+2 = 4n-2$.

If $BP_n[X]$ contains (i) a 3-path or (ii) a graph isomorphic to $K_{1,3}$, then $|E(X, BP_n - X)| = 2(n-1) + 2(n-2) = 4n - 6$ and $|N_{E(BP_n)}(X)| = 4n - 6 + 3 = 4n - 3$.

Table 1 lists all cases of $BP_n[X]$. Hence, $|E(X, BP_n - X)| \geqslant 4n - 6$ and $|N_{E(BP_n)}(X)| \geqslant 4n - 3$. $\square$

Table 1. All cases of $BP_n[X]$ for $X = \{x, y, u, v\}$.

	$BP_n[X]$	$E(X, BP_n - X)$	$N_{E(BP_n)}(X)$
1	four singletons	$4n$	$4n$
2	an edge and two singletons	$4n-2$	$4n-1$
3	a 2-path and a singleton	$4n-4$	$4n-2$
4	two edges	$4n-4$	$4n-2$
5	a graph isomorphic to $K_{1,3}$	$4n-6$	$4n-3$
6	a 3-path	$4n-6$	$4n-3$

Lemma 8. *For BP_n with $n \geqslant 4$ and $X \subset V(BP_n)$, if $|X| = 5$, then $|E(X, BP_n - X)| \geqslant 5n - 8$ and $|N_{E(BP_n)}(X)| \geqslant 5n - 4$.*

Proof. Let $X = \{x, y, z, u, v\}$. Note that BP_n has no k-cycle for $k \leqslant 5$. By Lemma 2, $\kappa(BP_n) = \lambda(BP_n) = n$ and $g(BP_n) = 8$.

If $BP_n[X]$ contains five singletons, then $|E(X, BP_n - X)| = |N_{E(BP_n)}(X)| = 5n$.

If $BP_n[X]$ contains three singletons and an edge, then $|E(X, BP_n - X)| = 3n + 2(n-1) = 5n - 2$ and $|N_{E(BP_n)}(X)| = 5n - 2 + 1 = 5n - 1$.

If $BP_n[X]$ contains (i) two edges and a singleton or (ii) a 2-path and two singletons, then $|E(X, BP_n - X)| = 2(n-1) + 2(n-1) + n = 5n - 4$ (resp., $|E(X, BP_n - X)| = 2(n-1) + (n-2) + 2n = 5n - 4$) and $|N_{E(BP_n)}(X)| = 5n - 4 + 2 = 5n - 2$.

If $BP_n[X]$ contains (i) a 2-path and an edge, (ii) a 3-path and a singleton, or (iii) a graph isomorphic to $K_{1,3}$ and a singleton, then $|E(X, BP_n - X)| = 2(n-1) + 2(n-1) + n - 2 = 5n - 6$ and $|N_{E(BP_n)}(X)| = 5n - 6 + 3 = 5n - 3$.

If $BP_n[X]$ contains (i) a 4-path, (ii) a graph isomorphic to $K_{1,4}$, or (iii) a tree with five vertices, then $|E(X, BP_n - X)| = 2(n-1) + 3(n-2) = 5n - 8$ and $|N_{E(BP_n)}(X)| = 5n - 8 + 4 = 5n - 4$.

All cases of the induced subgraph $BP_n[X]$ are listed in Table 2 (see Figure 2a–c). Hence, $|E(X, BP_n - X)| \geqslant 5n - 8$ and $|N_{E(BP_n)}(X)| \geqslant 5n - 4$. $\square$

Table 2. All cases of $BP_n[X]$ for $X = \{x, y, z, u, v\}$.

	$BP_n[X]$	$E(X, BP_n - X)$	$N_{E(BP_n)}(X)$
1	five singletons	$5n$	$5n$
2	an edge and three singletons	$5n-2$	$5n-1$
3	two edges and a singleton	$5n-4$	$5n-2$
4	a 2-path and two singletons	$5n-4$	$5n-2$
5	a 2-path and an edge	$5n-6$	$5n-3$
6	a 3-path and a singleton	$5n-6$	$5n-3$
7	a graph isomorphic to $K_{1,3}$ and a singleton	$5n-6$	$5n-3$
8	a 4-path, Figure 2a	$5n-8$	$5n-4$
9	a graph isomorphic to $K_{1,4}$, Figure 2c	$5n-8$	$5n-4$
10	a tree with 5 vertices, Figure 2b	$5n-8$	$5n-4$

Lemma 9. *For BP_n with $n \geqslant 4$ and $X \subset V(BP_n)$, if $|X| = 6$, then $|E(X, BP_n - X)| \geqslant 6n - 10$ and $|N_{E(BP_n)}(X)| \geqslant 6n - 5$.*

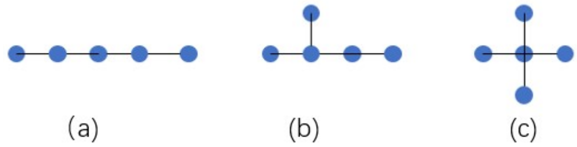

Figure 2. (**a**–**c**) Three trees with five vertices.

Proof. Let $X = \{x, y, z, u, v, w\}$. Note that BP_n has no k-cycle for $k \leqslant 6$. By Lemma 2, $\kappa(BP_n) = \lambda(BP_n) = n$ and $g(BP_n) = 8$.

If $BP_n[X]$ contains six singletons, then $|E(X, BP_n - X)| = |N_{E(BP_n)}(X)| = 6n$.

If $BP_n[X]$ contains four singletons and an edge, then $|E(X, BP_n - X)| = 4n + 2(n - 1) = 6n - 2$ and $|N_{E(BP_n)}(X)| = 6n - 2 + 1 = 6n - 1$.

If $BP_n[X]$ contains (i) two edges and two singleton or (ii) a 2-path and three singletons, then $|E(X, BP_n - X)| = 2(n-1) + 2(n-1) + 2n = 6n - 4$ (resp., $|E(X, BP_n - X)| = 2(n-1) + (n-2) + 3n = 6n - 4$) and $|N_{E(BP_n)}(X)| = 6n - 4 + 2 = 6n - 2$.

If $BP_n[X]$ contains (i) a 3-path and two singletons, (ii) three edges, or (iii) an edge, a 2-path and a singleton, then $|E(X, BP_n - X)| = 2(n-1) + 2(n-2) + 2n = 6n - 6$ and $|N_{E(BP_n)}(X)| = 6n - 6 + 3 = 6n - 3$.

If $BP_n[X]$ contains (i) a singleton and a tree with five vertices Figure 2a–c, (ii) an edge and a 3-path or a graph isomorphic to $K_{1,3}$, or (iii) two 2-paths, then $|E(X, BP_n - X)| = 2(n-1) + 2(n-1) + 2(n-2) = 6n - 8$ and $|N_{E(BP_n)}(X)| = 6n - 8 + 4 = 6n - 4$.

If $BP_n[X]$ contains (i) a 5-path, (ii) a graph isomorphic to $K_{1,5}$, or (iii) a tree with 6 vertices, isomorphic to one of Figure 3b–d, then $|E(X, BP_n - X)| = 2(n-1) + 4(n-2) = 6n - 10$ and $|N_{E(BP_n)}(X)| = 6n - 10 + 5 = 5n - 5$.

All cases of the induced subgraph $BP_n[X]$ are listed in Table 3 (see Figure 3). Hence, $|E(X, BP_n - X)| \geqslant 6n - 10$ and $|N_{E(BP_n)}(X)| \geqslant 6n - 5$. □

Table 3. All cases of $BP_n[X]$ for $X = \{x, y, z, u, v, w\}$.

	$BP_n[X]$	$E(X, BP_n - X)$	$N_{E(BP_n)}(X)$
1	six singletons	$6n$	$6n$
2	an edge and four singletons	$6n - 2$	$6n - 1$
3	a 2-path and three singletons	$6n - 4$	$6n - 2$
4	a 3-path and two singletons	$6n - 6$	$6n - 3$
5	two singletons and two edges	$6n - 4$	$6n - 2$
6	a 4-path and a singleton	$6n - 8$	$6n - 4$
7	a singleton and a graph isomorphic to $K_{1,4}$	$6n - 8$	$6n - 4$
8	a singleton and a tree with 5 vertices, Figure 2b	$6n - 8$	$6n - 4$
9	three edges	$6n - 6$	$6n - 3$
10	an edge and 3-path	$6n - 8$	$6n - 4$
11	an edge and a graph isomorphic to $K_{1,3}$	$6n - 8$	$6n - 4$
12	an edge, a singleton and 2-path	$6n - 6$	$6n - 3$
13	two 2-paths	$6n - 8$	$6n - 4$
14	a 5-path, Figure 3a	$6n - 10$	$6n - 5$
15	a graph isomorphic to $K_{1,5}$, Figure 3f	$6n - 10$	$6n - 5$
16	a tree with 6 vertices, isomorphic to one of Figure 3b–d	$6n - 10$	$6n - 5$

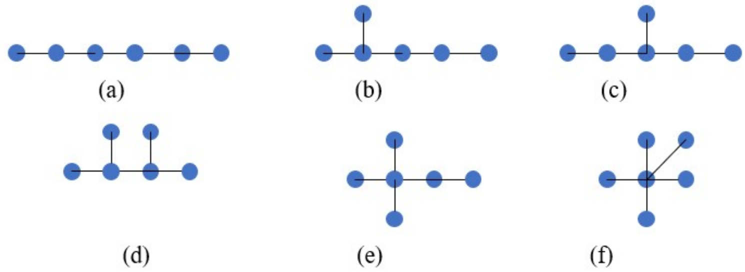

Figure 3. (**a**–**f**) Six trees with six vertices.

Lemma 10. *For BP_n with $n \geqslant 4$ and $W \subset V(BP_n)$, if $|W| = 7$, then $|E(W, BP_n - W)| \geqslant 7n - 12$ and $|N_{E(BP_n)}(W)| \geqslant 7n - 6$.*

Proof. By Lemma 2, we have $g(BP_n) = 8$. We let H be a connected subgraph of BP_n that does not contain a 6-path. Then, any vertex $x \in V(BP_n) \setminus V(H)$ can connect to at most one vertex in H; otherwise, the subgraph induced by $V(H) \cup \{x\}$ produces a cycle of length of less than 8. Particularly, we consider H a component of $BP_n[X]$, where X is a subset of $V(BP_n)$ with $|X| = 6$ shown in Table 3. We let t be the number of components of $BP_n[X]$ and let $W = X \cup \{x\}$, where $x \in V(BP_n) \setminus X$. Clearly, $|E(x, BP_n - \{x\})| = |N_{E(BP_n)}(x)| = n$ and, from the above reasoning, x may connect to at most t vertices of X in BP_n, i.e., $|E(\{x\}, X)| \leqslant t$. By checking all sixteen cases in Table 3, we have $|E(X, BP_n - X)| - 2|E(\{x\}, X)| \geqslant |E(X, BP_n - X)| - 2t \geqslant 6n - 12$ and $|N_{E(BP_n)}(X)| - |E(\{x\}, X)| \geqslant |N_{E(BP_n)}(X)| - t \geqslant 6n - 6$. Thus,

$$\begin{aligned} |E(W, BP_n - W)| &= |E(X \cup \{x\}, BP_n - (X \cup \{x\}))| \\ &= |E(x, BP_n - \{x\})| + |E(U, BP_n - X)| - 2|E(\{x\}, X)| \\ &\geqslant n + (6n - 12) \\ &= 7n - 12 \end{aligned}$$

and

$$\begin{aligned} |N_{E(BP_n)}(W)| &= |N_{E(BP_n)}(X \cup \{x\})| \\ &= |N_{E(BP_n)}(x)| + |N_{E(BP_n)}(X)| - |E(\{x\}, X)| \\ &\geqslant n + (6n - 6) \\ &= 7n - 6, \end{aligned}$$

as desired. □

We recall that BP_n is decomposed into $2n$ vertex-disjoint subgraphs BP_n^i for $i \in \langle n \rangle$ by fixing symbol i in the rightmost position for each vertex where each BP_n^i is isomorphic to BP_{n-1}. Henceforth, we consider F to be an edge-cut of BP_n and let $F_i = F \cap E(BP_n^i)$ and $f_i = |F_i|$ for each $i \in \langle n \rangle$. We let $F_c = F - \sum_{i \in \langle n \rangle} F_i$ and $f_c = |F_c|$. We let $I = \{i \in \langle n \rangle : f_i \geqslant n - 1\}$ and $J = \langle n \rangle \setminus I$. Also, we define

$$F_I = \bigcup_{i \in I} F_i, \;\; F_J = \bigcup_{j \in J} F_j, \;\; f_I = |F_I|, \;\; f_J = |F_J|, \text{ and } BP_n^J = BP_n\Big[\bigcup_{j \in J} V(BP_n^j)\Big].$$

Theorem 1. *For $n \geqslant 5$, we let BP_n be the n-dimensional burnt pancake graph and $F \subset E(BP_n)$ be an arbitrary edge set. If $|F| \leqslant 6n - 11$, then $BP_n - F$ either is connected to or contains a large component along with smaller components containing totally at most five vertices.*

Proof. We suppose that $BP_n - F$ is disconnected and let M be the union of smaller components of $BP_n - F$. By the definition of M, it suffices to show that $|V(M)| \leqslant 5$. Since $|F| \leqslant 6n - 11$ and $f_i \geqslant n-1$ for $i \in I$, we have $|I| \leqslant 5$. Then, $|J| = 2n - |I| \geqslant 2n - 5 \geqslant 5$ when $n \geqslant 5$. For each $j \in J$, as each subgraph BP_n^j is isomorphic to BP_{n-1}, by Lemma 2(2), we have $f_j < n-1 = \lambda(BP_{n-1})$, and thus $BP_n^j - F_j$ is connected. We claim that the following remark holds.

Remark 1. *$BP_n^J - F_J$ is connected.*

For $j,k \in J$ and $j \neq \bar{k}$, by Lemma 2(1), we have $|E_{j,k}(BP_n)| = (n-2)! \times 2^{n-2} > 6n - 11$ when $n \geqslant 5$. Thus, $BP_n^j - F_j$ is connected with $BP_n^k - F_k$ through an external edge. Moreover, since $|J| \geqslant 5$, if $k, \bar{k} \in J$, there exists $j \in J \setminus \{k, \bar{k}\}$ such that $BP_n^j - F_j$ is connected to each of $BP_n^k - F_k$ and $BP_n^{\bar{k}} - F_{\bar{k}}$ through external edges. Therefore, $BP_n^J - F_J$ is connected.

We prove the theorem by induction on n, and the proof is separated into two parts: Part I for base case ($n = 5$) and Part II for induction step ($n \geqslant 6$).

For base case, if $n = 5$, then $|F| \leqslant 6n - 11 = 19$. We note that BP_5 can be decomposed into 10 vertex-disjoint subgraphs, denoted by BP_5^i, by fixing symbol i in the rightmost position of each vertex for $i \in \langle 5 \rangle$. Obviously, BP_5^i is isomorphic to BP_4. As $I = \{i \in \langle 5 \rangle \colon f_i \geqslant n - 1 = 4\}$, we have $|I| \leqslant 4$; otherwise, $|F| \geqslant (n-1)|I| \geqslant 4 \times 5 > 19$. By Remark 1, $BP_5^J - F_J$ is connected. If $|I| = 0$, then $BP_5 - F = BP_5^J - F_J$ is connected; the result holds. We now consider $1 \leqslant |I| \leqslant 4$. For each $i \in I$, we let $S_i \subset V(BP_n^i)$ be the set of vertices that do not belong to the large component of $BP_n^i - F_i$. We consider the following cases.

Case I-1. $|I| = 1$. We let $I = \{i\}$. For $4 \leqslant f_i \leqslant 11 = 5(n-1) - 9$, if $BP_5^i - F_i$ is disconnected, by Lemma 6, it has a large component and is with $|S_i| \leqslant 4$. Since every vertex of BP_5^i has an external edge, there are $2^{n-1}(n-1)! = 4! \times 2^4$ edges between BP_5^i and $BP_5^J - F_J$. Also, since $4! \times 2^4 > 19 \geqslant |F|$, the large component of $BP_5^i - F_i$ is connected to $BP_5^J - F_J$. This implies that $|V(M)| \leqslant |S_i| \leqslant 4$ (see Figure 4a).

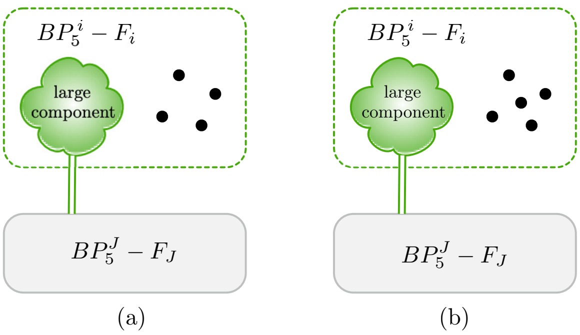

Figure 4. A schematic concept to illustrate the proof of Case I-1: (**a**) $|V(M)| \leqslant 4$ when $4 \leqslant f_i \leqslant 11$; (**b**) $|V(M)| \leqslant 5$ when $12 \leqslant f_i \leqslant 19$.

It remains to consider $12 \leqslant f_i \leqslant 19$. In this case, we have $f_c \leqslant |F| - f_i \leqslant 19 - 12 = 7$. Since every vertex of M has exactly one external neighbor, we have $|V(M)| \leqslant f_c \leqslant 7$. If $|V(M)| = 6$, by Lemma 9, $|F| \geqslant |E(V(M), BP_5 - V(M))| \geqslant 6n - 10 = 20 > 19$, a contradiction. Similarly, if $|V(M)| = 7$, by Lemma 10, $|F| \geqslant |E(V(M), BP_n - V(M))| \geqslant 7n - 12 = 23 > 19$, a contradiction. This implies that $|V(M)| \leqslant 5$ (see Figure 4b).

Case I-2. $|I| = 2$. We let $I = \{i, j\}$ and, without loss of generality, we suppose $f_i \leqslant f_j$. Since $|F| \leqslant 19$, we have $4 \leqslant f_i \leqslant 9$; otherwise, $f_i + f_j \geqslant 2f_i \geqslant 20$. We first consider $4 \leqslant f_j \leqslant 9$. For each $\ell \in I$, as $f_\ell \leqslant 9 = 4(n-1) - 7$, by Lemma 5(3), if $BP_5^\ell - F_\ell$ is disconnected, it has a large component and is with $|S_\ell| \leqslant 3$. Thus, $|V(M)| \leqslant |S_i| + |S_j| \leqslant 6$. A proof similar to Case 1 shows that the large component of $BP_n^\ell - F_\ell$ is connected to

$BP_n^J - F_J$. If $|V(M)| = 6$, by Lemma 9, $|F| \geqslant |E(V(M), BP_n - V(M))| \geqslant 6n - 10 = 20 > 19$, a contradiction. This implies that $|V(M)| \leqslant 5$.

It remains to consider $10 \leqslant f_j \leqslant |F| - f_i \leqslant 19 - 4 = 15$. In this situation, $f_c \leqslant |F| - f_i - f_j \leqslant 19 - 4 - 10 = 5$, which means that at most five faulty external edges. Since every vertex in M has exactly one external neighbor, we have $|V(M)| \leqslant f_c \leqslant 5$.

Case I-3. $|I| = 3$. We let $I = \{i, j, k\}$. Without loss of generality, we suppose $f_i \leqslant f_j \leqslant f_k$. Since $|F| \leqslant 19$, we have $4 \leqslant f_i \leqslant f_j \leqslant f_k \leqslant 19 - 2 \times 4 = 11$. If $f_j \geqslant 8$, then $|F| = f_i + f_j + f_k \geqslant 4 + 2 \times 8 = 20$, a contradiction. Thus, $4 \leqslant f_i \leqslant f_j \leqslant 7$. We first consider $4 \leqslant f_i \leqslant f_j \leqslant f_k \leqslant 7 = 3(n-1) - 5$. For each $\ell \in I$, if $BP_n^\ell - F_\ell$ is disconnected, by Lemma 5(2), it has large components and is with $|S_\ell| \leqslant 2$. Thus, $|V(M)| \leqslant |S_i| + |S_j| + |S_k| \leqslant 6$. A proof similar to Case 1 shows that the large component of $BP_n^\ell - F_\ell$ is connected to $BP_n^J - F_J$. If $|V(M)| = 6$, by Lemma 9, $|F| \geqslant |E(V(M), BP_n - V(M))| \geqslant 6n - 10 = 20 > 19$, a contradiction. This implies that $|V(M)| \leqslant 5$.

It remains to consider $8 \leqslant f_k \leqslant 11$. In this situation, $f_c \leqslant |F| - f_i - f_j - f_k \leqslant 19 - 2 \times 4 - 8 = 3$, which means that at most three faulty external edges. Since every vertex in M has exactly one external neighbor, we have $|V(M)| \leqslant f_c \leqslant 3$.

Case I-4. $|I| = 4$. We let $I = \{i, j, k, m\}$. As $f_\ell \geqslant 4$ for each $\ell \in I$, we have $f_c \leqslant |F| - f_i - f_j - f_k - f_m \leqslant 19 - 4 \times 4 = 3$, which means that at most three faulty external edges. Since every vertex in M has exactly one external neighbor, we have $|V(M)| \leqslant f_c \leqslant 3$.

For induction step, we assume $n \geqslant 6$ and the result holds for BP_{n-1}. That is, for each $i \in I$, if $|F_i| \leqslant 6(n-1) - 11$, then $BP_n^i - F_i$ either is connected or contains a large component and smaller components containing totally at most five vertices. We let $S_i \subset V(BP_n^i)$ be the set of vertices that do not belong to the large component of $BP_n^i - F_i$. Obviously, if $|I| = 0$, then $BP_n - F = BP_n^J - F_J$ is connected and the result holds. We consider the following cases:

Case II-1. $|I| = 1$. We let $I = \{i\}$. There are two subcases depending on the range of f_i.

Case II-1.1. $n - 1 \leqslant f_i \leqslant 6n - 17$.

Since BP_n^i is isomorphic to BP_{n-1} and $f_i \leqslant 6n - 17 = 6(n-1) - 11$, by induction hypothesis, we have $|S_i| \leqslant 5$. Since every vertex of BP_n^i has an external edge, there are $2^{n-1}(n-1)!$ edges between BP_n^i and $BP_n^J - F_J$. Also, since $2^{n-1}(n-1)! - |S_i| \geqslant 2^{n-1}(n-1)! - 5 > 6n - 11 \geqslant |F|$ when $n \geqslant 6$, the large component of $BP_n^i - F_i$ is connected to $BP_n^J - F_J$. As M is the union of smaller components of $BP_n - F$, this implies that $|V(M)| \leqslant |S_i| \leqslant 5$.

Case II-1.2. $6n - 16 \leqslant f_i \leqslant 6n - 11$.

In this case, we have $f_c \leqslant |F| - f_i \leqslant (6n - 11) - (6n - 16) = 5$, which means that F contains at most five faulty external edges. Since every vertex in M has exactly one external neighbor, we have $|V(M)| \leqslant f_c \leqslant 5$.

Case II-2. $|I| = 2$. We let $I = \{i, j\}$ and, without loss of generality, we suppose $f_i \leqslant f_j$. Since $|F| \leqslant 6n - 11$, we have $n - 1 \leqslant f_i \leqslant f_j \leqslant (6n - 11) - (n - 1) = 5n - 10$. If $f_i \geqslant 4n - 11$, then $f_i + f_j \geqslant 2(4n - 11) = 8n - 22 > 6n - 11 \geqslant |F|$ for $n \geqslant 6$. Thus, it requires that $f_i \leqslant 4n - 12$. We consider the following two subcases.

Case II-2.1. $n - 1 \leqslant f_j \leqslant 4n - 11$.

In this case, we have $n - 1 \leqslant f_i \leqslant f_j \leqslant 4n - 11 = 4(n-1) - 7$. For each $\ell \in I$, if $BP_n^\ell - F_\ell$ is disconnected; by Lemma 5(3), it contains a large component and is with $|S_\ell| \leqslant 3$. Then, via a proof similar to Case 1.1, we can show that the large component of $BP_n^\ell - F_\ell$ is connected to $BP_n^J - F_J$. Thus, $|V(M)| \leqslant |S_i| + |S_j| \leqslant 6$. If $|V(M)| = 6$, by Lemma 9, $|F| \geqslant |E(V(M), BP_n - V(M))| \geqslant 6n - 10$, a contradiction. This implies that $|V(M)| \leqslant 5$.

Case II-2.2. $4n - 10 \leqslant f_j \leqslant 5n - 10$.

In this case, we have $n - 1 \leqslant f_i \leqslant \min\{(6n - 11) - (4n - 10), 4n - 12\} = 2n - 1 < 4(n-1) - 7$ for $n \geqslant 6$. By Lemma 5(3), if $BP_n^i - F_i$ is disconnected, it contains a large component and is with $|S_i| \leqslant 3$. For $4n - 10 \leqslant f_j \leqslant 5n - 14 = 5(n-1) - 9$, if $BP_n^j - F_j$ is disconnected, by Lemma 6, it contains a large component and is with $|S_j| \leqslant 4$. Since $2^{n-1}(n-1)! - |S_i| - |S_j| \geqslant 2^{n-1}(n-1)! - 7 > 6n - 11 \geqslant |F|$ when $n \geqslant 6$, the large component of $BP_n^\ell - F_\ell$ for $\ell \in \{i,j\}$ is connected to $BP_n^J - F_J$. Thus, $|V(M)| \leqslant |S_i| + |S_j| \leqslant 7$. If $|V(M)| = 6$, by Lemma 9, $|F| \geqslant |E(V(M), BP_n - V(M))| \geqslant 6n - 10$, a contradiction. Similarly, if $|V(M)| = 7$, by Lemma 10, $|F| \geqslant |E(V(M), BP_n - V(M))| \geqslant 7n - 12 > 6n - 10$ when $n \geqslant 6$, a contradiction. This implies that $|V(M)| \leqslant 5$.

It remains to consider $5n - 13 \leqslant f_j \leqslant 5n - 10$. In this situation, since $|F| \leqslant 6n - 11$ and $f_i \geqslant n - 1$, it follows that $f_c \leqslant |F| - f_i - f_j = (6n - 11) - (n - 1) - (5n - 13) = 3$. Thus, at most three vertices in $BP_n^i \cup BP_n^j - (F_i \cup F_j)$ cannot connect to $BP_n^J - F_J$ in $BP_n - F$, i.e., $|V(M)| \leqslant f_c \leqslant 3$.

Case II-3. $|I| = 3$. We let $I = \{i,j,k\}$. Without loss of generality, we suppose $f_i \leqslant f_j \leqslant f_k$. Since $|F| \leqslant 6n - 11$, we have $n - 1 \leqslant f_i \leqslant f_j \leqslant f_k \leqslant (6n - 11) - 2(n - 1) = 4n - 9$. If $f_j \geqslant 3n - 7$, then $f_i + f_j + f_k \geqslant (n - 1) + 2(3n - 7) = 7n - 15 > 6n - 11 \geqslant |F|$ for $n \geqslant 6$. Thus, it requires that $f_i \leqslant f_j \leqslant 3n - 8$. We consider the following two subcases.

Case II-3.1. $n - 1 \leqslant f_i \leqslant f_j \leqslant f_k \leqslant 3n - 8 = 3(n - 1) - 5$.

For each $\ell \in I$, if $BP_n^\ell - F_\ell$ is disconnected, by Lemma 5(2), it contains a large component and is with $|S_\ell| \leqslant 2$. Then, via a proof similar to Case 1.1, we can show that the large component of $BP_n^\ell - F_\ell$ is connected to $BP_n^J - F_J$. Thus, $|V(M)| \leqslant |S_i| + |S_j| + |S_k| \leqslant 6$. If $|V(M)| = 6$, by Lemma 9, $|F| \geqslant |E(V(M), BP_n - V(M))| \geqslant 6n - 10$, a contradiction. This implies that $|V(M)| \leqslant 5$.

Case II-3.2. $n - 1 \leqslant f_i \leqslant f_j \leqslant 3n - 8 < 3n - 7 \leqslant f_k \leqslant 4n - 9$

For each $\ell \in \{i,j\}$, since $n - 1 \leqslant f_i \leqslant f_j \leqslant 3n - 8 = 3(n - 1) - 5$, if $BP_n^\ell - F_\ell$ is disconnected, by Lemma 5(2), it contains a large component and is with $|S_\ell| \leqslant 2$. For $3n - 7 \leqslant f_k \leqslant 4n - 11 = 4(n - 1) - 7$, if $BP_n^k - F_k$ is disconnected, by Lemma 5(3), it contains a large component and is with $|S_k| \leqslant 3$. Then, via a proof similar to Case 1.1, we can show that the large component of $BP_n^\ell - F_\ell$ for $\ell \in \{i,j,k\}$ is connected to $BP_n^J - F_J$. Thus, $|V(M)| \leqslant |S_i| + |S_j| + |S_k| \leqslant 7$. If $|V(M)| = 6$, by Lemma 9, $|F| \geqslant |E(V(M), BP_n - V(M))| \geqslant 6n - 10$, a contradiction. Similarly, if $|V(M)| = 7$, by Lemma 10, $|F| \geqslant |E(V(M), BP_n - V(M))| \geqslant 7n - 12 > 6n - 10$ when $n \geqslant 6$, a contradiction. This implies that $|V(M)| \leqslant 5$.

It remains to consider $4n - 10 \leqslant f_k \leqslant 4n - 9$. In this situation, since $|F| \leqslant 6n - 11$ and $f_j \geqslant f_i \geqslant n - 1$, it follows that $f_c \leqslant |F| - f_i - f_j - f_k \leqslant (6n - 11) - 2(n - 1) - (4n - 10) = 1$. Thus, at most one vertex in $BP_n^i \cup BP_n^j \cup BP_n^k - (F_i \cup F_j \cup F_k)$ cannot connect with $BP_n^J - F_J$ in $BP_n - F$, i.e., $|V(M)| \leqslant f_c \leqslant 1$.

Case II-4. $|I| = 4$. We let $I = \{i,j,k,m\}$. Without loss of generality, we suppose $f_i \leqslant f_j \leqslant f_k \leqslant f_m$. Since $|F| \leqslant 6n - 11$, we have $n - 1 \leqslant f_i \leqslant f_j \leqslant f_k \leqslant f_m \leqslant (6n - 11) - 3(n - 1) = 3n - 8$. If $f_k \geqslant 2n - 4$, then $f_i + f_j + f_k + f_m \geqslant 2(n - 1) + 2(2n - 4) = 6n - 10 > 6n - 11 \geqslant |F|$ for $n \geqslant 6$. Thus, it requires that $n - 1 \leqslant f_i \leqslant f_j \leqslant f_k \leqslant 2n - 5$. We consider the following two subcases.

Case II-4.1. $n - 1 \leqslant f_i \leqslant f_j \leqslant f_k \leqslant f_m \leqslant 2n - 5 < 2(n - 1) - 2$.

For each $\ell \in I$, if $BP_n^\ell - F_\ell$ is disconnected, by Lemma 5(1), it has two components, one of which is a singleton, i.e., $|S_\ell| = 1$. A proof similar to Case 1.1 shows that the large component of $BP_n^\ell - F_\ell$ is connected to $BP_n^J - F_J$. Clearly, $|V(M)| \leqslant |S_i| + |S_j| + |S_k| + |S_m| \leqslant 4$.

Case II-4.2. $n - 1 \leqslant f_i \leqslant f_j \leqslant f_k \leqslant 2n - 5 < 2n - 4 \leqslant f_m \leqslant 3n - 8$.

For each $\ell \in \{i,j,k\}$, since $n - 1 \leqslant f_\ell \leqslant 2n - 5 < 2(n - 1) - 2$, if $BP_n^\ell - F_\ell$ is disconnected, by Lemma 5(1), it contains a large component and is with $|S_\ell| = 1$. Since $2n - 4 \leqslant f_m \leqslant 3n - 8 = 3(n - 1) - 5$, if $BP_n^m - F_m$ is disconnected, by Lemma 5(2), it contains a large component and is with $|S_m| \leqslant 2$. A proof similar to Case 1.1 shows that the large component of $BP_n^\ell - F_\ell$ for $\ell \in I$ is connected to $BP_n^J - F_J$. Thus, $|V(M)| \leqslant |S_i| + |S_j| + |S_k| + |S_m| \leqslant 5$.

Case II-5. $|I| = 5$. We let $I = \{i,j,k,m,p\}$. For each $\ell \in I$, we let $S_\ell \subset V(BP_n^\ell)$ be the set of vertices that do not belong to the large component of $BP_n^\ell - F_\ell$. Since $|F| \leqslant 6n - 11$, we have $n - 1 \leqslant f_\ell \leqslant (6n - 11) - 4(n - 1) = 2n - 7 < 2(n - 1) - 2$ for $n \geqslant 6$. Since $f_\ell \neq 2(n - 1) - 2$, if $BP_n^\ell - F_\ell$ is disconnected, by Lemma 5(1), it has two components, one of which is a singleton, i.e., $|S_\ell| = 1$. A proof similar to Case 1.1 shows that the large component of $BP_n^\ell - F_\ell$ is connected to $BP_n^J - F_J$. Clearly, $|V(M)| \leqslant |S_i| + |S_j| + |S_k| + |S_m| + |S_p| \leqslant 5$. $\square$

4. Applications to Extra Edge Connectivity and Component Edge Connectivity

As applications of Theorem 1, we determine the relation between $\lambda^{(\ell)}(BP_n)$ and $\lambda_{\ell+2}(BP_n)$ for $3 \leqslant \ell \leqslant 5$.

4.1. Relation between $\lambda^{(3)}(BP_n)$ and $\lambda_5(BP_n)$

Lemma 11. *For $n \geqslant 5$, let BP_n be the n-dimensional burnt pancake graph and $F \subset E(BP_n)$ be an arbitrary edge set. If $|F| \leqslant 4n-4$, then $BP_n - F$ has at most four components.*

Proof. Note that $|F| \leqslant 4n-4 \leqslant 5n-9$ for $n \geqslant 5$. By Lemma 6, if $BP_n - F$ is disconnected, it has a large component along with smaller components containing totally at most four vertices. Suppose that $BP_n - S$ has five components, four of which are singletons. By Lemma 7, isolating these four singletons requires the removal of at least $4n-3$ edges, which contradicts that $|F| \leqslant 4n-4$. $\square$

Theorem 2. $\lambda_5(BP_n) = \lambda^{(3)}(BP_n) + 3 = 4n-3$ *for $n \geqslant 5$.*

Proof. Let $s = 4$ and $S^* = \arg\min_{S \subseteq V(BP_n)} \{|E(S, G-S)|\colon\ |S| = s,\ BP_n[S]$ and $BP_n - S$ are connected subgraphs$\}$. As $|S^*| = 4$ and $BP_n[S^*]$ is connected, observe from Table 1 that $BP_n[S^*]$ is a 3-path or a $K_{1,3}$. By Lemma 7, let $t = |E(S^*, G-S^*)| = 4n-6$ and $m = |E(BP_n[S^*])| = 3$. Let F be an edge-cut of BP_n. By Lemma 5(3), if $|F| \leqslant 4n-7 = (4n-6)-1 = t-1$, then $BP_n - F$ has a large component along with smaller components containing totally at most $s-1 = 3$ vertices. This fulfills the condition of Lemma 1(i). Also, by Lemma 11, if $|F| \leqslant 4n-4 = (4n-6)+3-1 = t+m-1$, then $BP_n - F$ has at most $s = 4$ components. This fulfills the condition of Lemma 1(ii). Therefore, by Lemma 1, have $\lambda_{4+1}(BP_n) = \lambda^{(4-1)}(BP_n) + m = t+m = (4n-6)+3 = 4n-3$ for $n \geqslant 5$. $\square$

4.2. Relation between $\lambda^{(4)}(BP_n)$ and $\lambda_6(BP_n)$

Lemma 12. *For $n \geqslant 6$, let BP_n be the n-dimensional burnt pancake graph and $F \subset E(BP_n)$ be an arbitrary edge set. If $|F| \leqslant 5n-5$, then $BP_n - F$ has at most five components.*

Proof. Note that $|F| \leqslant 5n-5 \leqslant 6n-11$ for $n \geqslant 6$. By Theorem 1, if $BP_n - F$ is disconnected, it has a large component and smaller components containing totally at most five vertices. Suppose that $BP_n - S$ has six components, five of which are singletons. By Lemma 8, isolating these five singletons requires the removal of at least $5n-4$ edges, which contradicts that $|F| \leqslant 5n-5$. $\square$

Theorem 3. $\lambda_6(BP_n) = \lambda^{(4)}(BP_n) + 4 = 5n-4$ *for $n \geqslant 6$.*

Proof. Let $s = 5$ and $S^* = \arg\min_{S \subseteq V(BP_n)} \{|E(S, G-S)|\colon\ |S| = s,\ BP_n[S]$ and $BP_n - S$ are connected subgraphs$\}$. As $|S^*| = 5$ and $BP_n[S^*]$ is connected, observe from Table 2 that $BP_n[S^*]$ is a 4-path or a tree with 5 vertices (including $K_{1,4}$). By Lemma 8, let $t = |E(S^*, G-S^*)| = 5n-8$ and $m = |E(BP_n[S^*])| = 4$. Let F be an edge-cut of BP_n. By Lemma 6, if $|F| \leqslant 5n-9 = (5n-8)-1 = t-1$, then $BP_n - F$ has a large component and smaller components containing totally at most $s-1 = 4$ vertices. This fulfills the condition of Lemma 1(i). Also, by Lemma 12, if $|F| \leqslant 5n-5 = (5n-8)+4-1 = t+m-1$, then $BP_n - F$ has at most $s = 5$ components. This fulfills the condition of Lemma 1(ii). Therefore, by Lemma 1, have $\lambda_{5+1}(BP_n) = \lambda^{(5-1)}(BP_n) + m = t+m = (5n-8)+4 = 5n-4$ for $n \geqslant 6$. $\square$

4.3. Relation between $\lambda^{(5)}(BP_n)$ and $\lambda_7(BP_n)$

Lemma 13. *For $n \geqslant 6$, let BP_n be the n-dimensional burnt pancake graph and $F \subset E(BP_n)$ be an arbitrary edge set. If $|F| \leqslant 6n-6$, then $BP_n - F$ has at most six components.*

Proof. Let M be the union of smaller components of $BP_n - F$ and let $c(M)$ be the such number of components in M. By the definition of M, it suffices to show that $c(M) \leqslant 5$. Since $|F| \leqslant 6n-6$ and $f_i \geqslant n-1$ for $i \in I$, have $|I| \leqslant 6$. Then, $|J| = 2n - |I| \geqslant 2n-6 \geqslant 5$ when $n \geqslant 6$. With reasoning similar to Remark 1 in the proof of Theorem 1, it is shown that $BP_n^j - Fj$ is connected for each $j \in J$ and the following remark is further obtained.

Remark 2. *$BP_n^J - F_J$ is connected.*

Obviously, if $|I| = 0$, then $BP_n - F = BP_n^J - F_J$ is connected and the result holds. Now consider $1 \leqslant |I| \leqslant 6$. For each $i \in I$, let $S_i \subset V(BP_n^i)$ be the set of vertices that do not belong to the large component of $BP_n^i - F_i$.

Case 1. $|I| = 1$. Let $I = \{i\}$. There are two subcases depending on the range of f_i.

Case 1.1. $n-1 \leqslant f_i \leqslant 6n-17 = 6(n-1)-11$.

Since BP_n^i is isomorphic to BP_{n-1}, by Theorem 1, $BP_n^i - F_i$ has a large component and is with $|S_i| \leqslant 5$. As every vertex of BP_n^i has an external edge, there are $2^{n-1}(n-1)!$ edges between BP_n^i and $BP_n^J - F_J$. Also, since $2^{n-1}(n-1)! - 5 > 6n-6 \geqslant |F|$ when $n \geqslant 6$, the large component of $BP_n^i - F_i$ is connected to $BP_n^J - F_J$. This implies that $c(M) \leqslant |V(M)| \leqslant |S_i| \leqslant 5$.

Case 1.2. $6n-16 \leqslant f_i \leqslant 6n-6$.

In this case, there is $f_c \leqslant |F| - f_i \leqslant (6n-6) - (6n-16) = 10$, which means that F contains at most ten faulty external edges. Since every vertex in M has exactly one external neighbor, there is $|V(M)| \leqslant f_c \leqslant 10$. If $|V(M)| \leqslant 5$, it is clear that $c(M) \leqslant |V(M)| \leqslant 5$. If $|V(M)| = 6$, by Lemma 9, $|F| \geqslant |N_{E(BP_n)}(V(M))| \geqslant 6n-5 > 6n-6$, a contradiction. Similarly, if $|V(M)| = 7$, by Lemma 10, $|F| \geqslant |N_{E(BP_n)}(V(M))| \geqslant 7n-6 > 6n-6$, a contradiction. Now deal with the situations for $8 \leqslant |V(M)| \leqslant 10$ as follows.

Case 1.2.1. $|V(M)| = 8$. Let $V(M) = V(M_1) \cup \{x\}$, where $|V(M_1)| = 7$. By Lemma 10, $|N_{E(BP_n)}(V(M_1))| \geqslant 7n-6$. Clearly, $|N_{E(BP_n)}(x)| = n$ and x may connect to at most 7 vertices in M_1, i.e., $|E(\{x\}, M_1)| \leqslant 7$. Thus,

$$\begin{aligned} |N_{E(BP_n)}(V(M))| &= |N_{E(BP_n)}(V(M_1))| + |N_{E(BP_n)}(x)| - |E(\{x\}, M_1)| \\ &\geqslant (7n-6) + n - 7 = 8n - 13 > 6n - 6 \geqslant |F| \end{aligned}$$

when $n \geqslant 6$, a contradiction.

Case 1.2.2. $|V(M)| = 9$. Let $V(M) = V(M_1) \cup \{x\}$, where $|V(M_1)| = 8$. By Case 1.2.1, $|N_{E(BP_n)}(V(M_1))| \geqslant 8n-13$. Clearly, $|N_{E(BP_n)}(x)| = n$ and x may connect to at most 8 vertices in M_1, i.e., $|E(\{x\}, M_1)| \leqslant 8$. Thus,

$$\begin{aligned} |N_{E(BP_n)}(V(M))| &= |N_{E(BP_n)}(V(M_1))| + |N_{E(BP_n)}(x)| - |E(\{x\}, M_1)| \\ &\geqslant (8n-13) + n - 8 = 9n - 21 > 6n - 6 \geqslant |F| \end{aligned}$$

when $n \geqslant 6$, a contradiction.

Case 1.2.3. $|V(M)| = 10$. Let $V(M) = V(M_1) \cup \{x, y\}$, where $|V(M_1)| = 8$. By Case 1.2.1, $|N_{E(BP_n)}(V(M_1))| \geqslant 8n-13$. First, consider xy forms an edge in M. Then, $|N_{E(BP_n)}(\{x,y\})| = 2n-1$ and x (resp., y) may connect to at most eight vertices or $n-1$ vertices (if $6 \leqslant n \leqslant 8$) in M_1. That is, $|E(\{x\}, M_1)| \leqslant \min\{8, n-1\}$ and $|E(\{y\}, M_1)| \leqslant \min\{8, n-1\}$. Since $xy \in E(BP_n)$ and the girth of BP_n is 8, x and y cannot be adjacent to a vertex in M_1 simultaneously. Thus, $|E(\{x,y\}, M_1)| \leqslant \min\{8, n-1\} \leqslant 8$ and

$$\begin{aligned} |N_{E(BP_n)}(V(M))| &= |N_{E(BP_n)}(V(M_1))| + |N_{E(BP_n)}(\{x,y\})| - |E(\{x,y\}, M_1)| \\ &\geqslant (8n-13) + (2n-1) - 8 = 10n - 22 > 6n - 6 \geqslant |F| \end{aligned}$$

when $n \geqslant 6$, a contradiction. Next, suppose x and y are singletons in M. Then, $|N_{E(BP_n)}(\{x,y\})| = 2n$ and $|E(\{x,y\}, M_1)| \leqslant \min\{16, 2(n-1)\} \leqslant 16$. Thus,

$$\begin{aligned} |N_{E(BP_n)}(V(M))| &= |N_{E(BP_n)}(V(M_1))| + |N_{E(BP_n)}(\{x,y\})| - |E(\{x,y\}, M_1)| \\ &\geqslant (8n-13) + 2n - 16 = 10n - 29 > 6n - 6 \geqslant |F| \end{aligned}$$

when $n \geqslant 6$, a contradiction.

Based on the discussion of the above situations, conclude $c(M) \leqslant |V(M)| \leqslant 5$.

Case 2. $|I| = 2$. Let $I = \{i, j\}$ and, without loss of generality, suppose $f_i \leqslant f_j$. Since $|F| \leqslant 6n - 6$, there is $n - 1 \leqslant f_i \leqslant f_j \leqslant 6n - 6 - (n-1) = 5n - 5$. Consider the following three subcases.

Case 2.1. $n - 1 \leqslant f_i \leqslant f_j \leqslant 4n - 11$.

In this case, there is $n - 1 \leqslant f_i \leqslant f_j \leqslant 4n - 11 = 4(n-1) - 7$. For each $\ell \in I$, if $BP_n^\ell - F_\ell$ is disconnected, by Lemma 5(3), it contains a large component and is with $|S_\ell| \leqslant 3$. Then, via a proof similar to Case 1.1, it can be shown that the large component of $BP_n^\ell - F_\ell$ is connected to $BP_n^J - F_J$. Thus, $|V(M)| \leqslant |S_i| + |S_j| \leqslant 6$. If $|V(M)| = 6$, by Lemma 9, $|F| \geqslant |N_{E(BP_n)}(V(M))| = 6n - 5$, a contradiction. This implies that $c(M) \leqslant |V(M)| \leqslant 5$.

Case 2.2. $n - 1 \leqslant f_i \leqslant 4n - 11 < 4n - 10 \leqslant f_j \leqslant 5n - 5$.

In this case, there is $n - 1 \leqslant f_i \leqslant 4n - 11 = 4(n-1) - 7$. By Lemma 5(3), if $BP_n^i - F_i$ is disconnected, it contains a large component and is with $|S_i| \leqslant 3$. For $4n - 10 \leqslant f_j \leqslant 5n - 14 = 5(n-1) - 9$, if $BP_n^j - F_j$ is disconnected, by Lemma 6, it contains a large component and is with $|S_i| \leqslant 4$. Then, via a proof similar to Case 1.1, it can be shown that the large component of $BP_n^\ell - F_\ell$ for $\ell \in I$ is connected to $BP_n^J - F_J$. Thus, $|V(M)| \leqslant |S_i| + |S_j| \leqslant 7$. If $6 \leqslant |V(M)| \leqslant 7$, a contradiction can be acquired through an argument similar to Case 1.2. This implies that $c(M) \leqslant |V(M)| \leqslant 5$.

It remains to consider $5n - 13 \leqslant f_j \leqslant 5n - 5$. In this situation, since $|F| \leqslant 6n - 6$, there is $f_c \leqslant |F| - f_i - f_j \leqslant (6n-6) - (n-1) - (5n-13) = 8$. Thus, at most eight vertices in $BP_n^I - F_I$ cannot connect to $BP_n^J - F_J$ in $BP_n - F$, i.e., $|V(M)| \leqslant f_c \leqslant 8$. If $6 \leqslant |V(M)| \leqslant 8$,a contradiction can be acquired through an argument similar to Case 1.2. Thus, $c(M) \leqslant |V(M)| \leqslant 5$.

Case 2.3. $4n - 10 \leqslant f_i \leqslant f_j \leqslant 5n - 5$.

In this case, $6n - 6 \geqslant f_i + f_j \geqslant 2(4n - 10) = 8n - 20$, which leads to $6 \leqslant n \leqslant 7$. Note that $f_c \leqslant |F| - f_i - f_j \leqslant (6n - 6) - 2(4n - 10) = 14 - 2n$. Thus, $0 \leqslant f_c \leqslant 2$ and at most two vertices in $BP_n^I - F_I$ cannot connect with $BP_n^J - F_J$ in $BP_n - F$, i.e., $|V(M)| \leqslant f_c \leqslant 2$. It is clear that $c(M) \leqslant |V(M)| \leqslant 2$.

Case 3. $|I| = 3$. Let $I = \{i, j, k\}$. Without loss of generality, suppose $f_i \leqslant f_j \leqslant f_k$. Since $|F| \leqslant 6n - 6$, there is $n - 1 \leqslant f_i \leqslant f_j \leqslant f_k \leqslant (6n-6) - 2(n-1) = 4n - 4$. If $f_i \geqslant 3n - 7$, then $f_i + f_j + f_k \geqslant 3(3n - 7) = 9n - 21 > 6n - 6 \geqslant |F|$ for $n \geqslant 6$. Thus, it requires that $n - 1 \leqslant f_i \leqslant 3n - 8$. Consider the following three subcases.

Case 3.1. $n - 1 \leqslant f_i \leqslant f_j \leqslant f_k \leqslant 3n - 8 = 3(n-1) - 5$.

For each $\ell \in I$, if $BP_n^\ell - F_\ell$ is disconnected, by Lemma 5(2), it contains a large component and is with $|S_\ell| \leqslant 2$. Then, via a proof similar to Case 1.1, it can be shown that the large component of $BP_n^\ell - F_\ell$ is connected to $BP_n^J - F_J$. Thus, $|V(M)| \leqslant |S_i| + |S_j| + |S_k| \leqslant 6$. If $|V(M)| = 6$, by Lemma 9, $|F| \geqslant |N_{E(BP_n)}(V(M))| \geqslant 6n - 5 > 6n - 6$, a contradiction. This implies that $c(M) \leqslant |V(M)| \leqslant 5$.

Case 3.2. $n - 1 \leqslant f_i \leqslant f_j \leqslant 3n - 8 < 3n - 7 \leqslant f_k \leqslant 4n - 4$.

For each $\ell \in \{i, j\}$, since $n - 1 \leqslant f_i \leqslant f_j \leqslant 3n - 8 = 3(n-1) - 5$, if $BP_n^\ell - F_\ell$ is disconnected, by Lemma 5(2), it contains a large component and is with $|S_\ell| \leqslant 2$. For $3n - 7 \leqslant f_k \leqslant 4n - 11 = 4(n-1) - 7$, if $BP_n^k - F_k$ is disconnected, by Lemma 5(3), it contains a large component and is with $|S_k| \leqslant 3$. Then, via a proof similar to Case 1.1, it can b shown that the large component of $BP_n^\ell - F_\ell$ for $\ell \in I$ is connected to $BP_n^J - F_J$. Thus, $|V(M)| \leqslant |S_i| + |S_j| + |S_k| \leqslant 7$. If $6 \leqslant |V(M)| \leqslant 7$, a contradiction can be acquired through an argument similar to Case 1.2. Thus, $c(M) \leqslant |V(M)| \leqslant 5$.

It remains to consider $4n-10 \leqslant f_k \leqslant 4n-4$. In this situation, since $|F| \leqslant 6n-6$ and $f_j \geqslant f_i \geqslant n-1$, there is $f_c \leqslant |F| - f_i - f_j - f_k \leqslant (6n-6) - 2(n-1) - (4n-10) = 6$. Thus, at most six vertices in $BP_n^I - F_I$ cannot connect with $BP_n^J - F_J$ in $BP_n - F$, i.e., $|V(M)| \leqslant f_c \leqslant 6$. If $|V(M)| = 6$, by Lemma 9, $|F| \geqslant |N_{E(BP_n)}(V(M))| \geqslant 6n-5 > 6n-6$, a contradiction. This implies that $c(M) \leqslant |V(M)| \leqslant 5$.

Case 3.3. $n-1 \leqslant f_i \leqslant 3n-8 < 3n-7 \leqslant f_j \leqslant f_k \leqslant 4n-4$.

In this case, $6n-6 \geqslant f_i + f_j + f_k \geqslant (n-1) + 2(3n-7) = 7n-15$, which leads to $6 \leqslant n \leqslant 9$. Note that $f_c \leqslant |F| - f_i - f_j - f_k \leqslant (6n-6) - (n-1) - 2(3n-7) = 9-n$. Thus, $0 \leqslant f_c \leqslant 3$ and at most three vertices in $BP_n^I - F_I$ cannot connect with $BP_n^J - F_J$ in $BP_n - F$, i.e., $|V(M)| \leqslant f_c \leqslant 3$. It is clear that $c(M) \leqslant |V(M)| \leqslant 3$.

Case 4. $|I| = 4$. Let $I = \{i, j, k, m\}$. Without loss of generality, suppose $f_i \leqslant f_j \leqslant f_k \leqslant f_m$. Since $|F| \leqslant 6n-6$, there is $n-1 \leqslant f_i \leqslant f_j \leqslant f_k \leqslant f_m \leqslant (6n-6) - 3(n-1) = 3n-3$. If $f_i \geqslant 2n-4$, then $f_i + f_j + f_k + f_m \geqslant 4(2n-4) = 8n-16 > 6n-6 \geqslant |F|$ for $n \geqslant 6$. Thus, it requires that $n-1 \leqslant f_i \leqslant 2n-5$. Also, if $f_k \geqslant 3n-7$, then $f_i + f_j + f_k + f_m \geqslant 2(n-1) + 2(3n-7) = 8n-16 > 6n-6 \geqslant |F|$ for $n \geqslant 6$. Thus, it requires that $n-1 \leqslant f_j \leqslant f_k \leqslant 3n-8$. Consider the following two subcases.

Case 4.1. $n-1 \leqslant f_i \leqslant 2n-5$, $n-1 \leqslant f_j \leqslant f_k \leqslant f_m \leqslant 3n-8$.

In this case, there is $n-1 \leqslant f_i \leqslant 2n-5 < 2(n-1)-2$. If $BP_n^i - F_i$ is disconnected, by Lemma 5(1), it has two components, one of which is a singleton, i.e., $|S_i| = 1$. For $\ell \in \{j, k, m\}$, since $n-1 \leqslant f_\ell \leqslant 3n-8 < 3(n-1)-5$, if $BP_n^\ell - F_\ell$ is disconnected, by Lemma 5(2), it contains a large component and is with $|S_k| \leqslant 2$. A proof similar to Case 1.1 shows that the large component of $BP_n^\ell - F_\ell$ for $\ell \in I$ is connected to $BP_n^J - F_J$. Thus, $|V(M)| \leqslant |S_i| + |S_j| + |S_k| + |S_m| \leqslant 7$. If $6 \leqslant |V(M)| \leqslant 7$, a contradiction can be acquired through an argument similar to Case 1.2. Thus, $c(M) \leqslant |V(M)| \leqslant 5$.

Case 4.2. $n-1 \leqslant f_i \leqslant f_j \leqslant f_k \leqslant 3n-8 < 3n-7 \leqslant f_m \leqslant 3n-3$.

In this case, $f_c \leqslant |F| - f_i - f_j - f_k - f_m \leqslant (6n-6) - 3(n-1) - (3n-7) = 2$. Thus, at most two vertices in $BP_n^I - F_I$ cannot connect with $BP_n^J - F_J$ in $BP_n - F$, i.e., $|V(M)| \leqslant f_c \leqslant 2$. It is clear that $c(M) \leqslant |V(M)| \leqslant 2$.

Case 5. $|I| = 5$. Let $I = \{i, j, k, m, p\}$. Without loss of generality, suppose $f_i \leqslant f_j \leqslant f_k \leqslant f_m \leqslant f_p$. Since $|F| \leqslant 6n-6$, there is $n-1 \leqslant f_i \leqslant f_j \leqslant f_k \leqslant f_m \leqslant f_p \leqslant (6n-6) - 4(n-1) = 2n-2$. If $f_m \geqslant 2n-4$, then $f_i + f_j + f_k + f_m + f_p \geqslant 3(n-1) + 2(2n-4) = 7n-11 > 6n-6 \geqslant |F|$ for $n \geqslant 6$. Thus, it requires that $n-1 \leqslant f_i \leqslant f_j \leqslant f_k \leqslant f_m \leqslant 2n-5$. Consider the following two subcases.

Case 5.1. $n-1 \leqslant f_i \leqslant f_j \leqslant f_k \leqslant f_m \leqslant f_p \leqslant 2n-5$.

For each $\ell \in I$, if $BP_n^\ell - F_\ell$ is disconnected, by Lemma 5(1), $BP_n^\ell - F_\ell$ has two components, one of which is a singleton, i.e., $|S_i| = 1$. A proof similar to Case 1.1 shows that the large component of $BP_n^\ell - F_\ell$ is connected to $BP_n^J - F_J$. Thus, $|V(M)| \leqslant 5$. This leads to $c(M) \leqslant |V(M)| \leqslant 5$.

Case 5.2. $n-1 \leqslant f_i \leqslant f_j \leqslant f_k \leqslant f_m \leqslant 2n-5 < 2n-4 \leqslant f_p \leqslant 2n-2$.

In this case, $f_c \leqslant |F| - f_i - f_j - f_k - f_m \leqslant (6n-6) - 4(n-1) - (2n-4) = 2$. Thus, at most two vertices in $BP_n^I - F_I$ cannot connect with $BP_n^J - F_J$ in $BP_n - F$, i.e., $|V(M)| \leqslant f_c \leqslant 2$. It is clear that $c(M) \leqslant |V(M)| \leqslant 2$.

Case 6. $|I| = 6$. Let $I = \{i, j, k, m, p, q\}$. Since $|F| \leqslant 6n-6$, there is $f_i = f_j = f_k = f_m = f_p = f_q = n-1$ and $f_c = 0$. Since $f_\ell = n-1 < 2(n-1)-2$ for $n \geqslant 6$, by Lemma 5(1), if $BP_n^\ell - F_\ell$ is disconnected, then $BP_n^\ell - F_\ell$ has two components, one of which is a singleton, i.e., $|S_\ell| = 1$. As $n \geqslant 6 = |I|$, by Lemma 2(1), there exists $\ell' \in \langle n \rangle \setminus (I \cup \{\bar{\ell}\})$ such that $|E_{\ell,\ell'}(BP_n)| - (f_\ell + 1) = (n-2)! \times 2^{n-2} - (n-1) - 1 > 0$. This implies that the large component of $BP_n^\ell - F_\ell$ is connected to $BP_n^J - F_J$. Thus, $|V(M)| \leqslant 6$. If $|V(M)| = 6$, by Lemma 9, $|F| \geqslant |N_{E(BP_n)}(V(M))| \geqslant 6n-5 > 6n-6$, a contradiction. This implies that $c(M) \leqslant |V(M)| \leqslant 5$. $\square$

Theorem 4. $\lambda_7(BP_n) = \lambda^{(5)}(BP_n) + 5 = 6n-5$ *for* $n \geqslant 6$.

Proof. Let $s = 6$ and $S^* = \arg \min_{S \subseteq V(BP_n)} \{|E(S, G-S)|: |S| = s, BP_n[S]$ and $BP_n - S$ are connected subgraphs$\}$. As $|S^*| = 6$ and $BP_n[S^*]$ is connected, it can be observed from Table 3 that $BP_n[S^*]$ is a 5-path or a tree with 6 vertices (including $K_{1,5}$). By Lemma 9, let $t = |E(S^*, G - S^*)| = 6n - 10$ and $m = |E(BP_n[S^*])| = 5$. Let F be an edge-cut of BP_n. By Theorem 1, if $|F| \leqslant 6n - 11 = (6n-10) - 1 = t - 1$, then $BP_n - F$ has a large component along with smaller components containing totally at most $s - 1 = 5$ vertices. This fulfills the condition of Lemma 1(i). Also, by Lemma 13, if $|F| \leqslant 6n - 6 = (6n-10) + 5 - 1 = t + m - 1$, then $BP_n - F$ has at most $s = 6$ components. This fulfills the condition of Lemma 1(ii). Therefore, by Lemma 1, there is $\lambda_{6+1}(BP_n) = \lambda^{(6-1)}(BP_n) + m = t + m = (6n - 10) + 5 = 6n - 5$ for $n \geqslant 6$. □

5. Concluding Remarks

For burnt pancake graph BP_n, this paper shows that when removing any edge subset with a size of approximately six times $\lambda(BP_n)$, the surviving graph possesses the "linearly many faults" property. Applying this property, we attain $\lambda^{(h)}(BP_n)$ and $\lambda_r(BP_n)$. Specifically, we prove that $\lambda_5(BP_n) = \lambda^{(3)}(BP_n) + 3 = 4n - 3$ for $n \geqslant 5$; $\lambda_6(BP_n) = \lambda^{(4)}(BP_n) + 4 = 5n - 4$ and $\lambda_7(BP_n) = \lambda^{(5)}(BP_n) + 5 = 6n - 5$ for $n \geqslant 6$, as summarized in Table 4.

Table 4. The comparison of $\lambda^{(h)}(BP_n)$ and $\lambda_r(BP_n)$.

$\boldsymbol{\lambda^{(h)}(BP_n)}$	**Ref.**	$\boldsymbol{\lambda_r(BP_n)}$	**Ref.**
$\lambda^{(1)}(BP_n) = 2n - 2$	[30]	$\lambda_3(BP_n) = 2n - 1$	[30]
$\lambda^{(2)}(BP_n) = 3n - 4$		$\lambda_4(BP_n) = 3n - 2$	
$\lambda^{(3)}(BP_n) = 4n - 6$	Theorem 2	$\lambda_5(BP_n) = 4n - 3$	Theorem 2
$\lambda^{(4)}(BP_n) = 5n - 8$	Theorem 3	$\lambda_6(BP_n) = 5n - 4$	Theorem 3
$\lambda^{(5)}(BP_n) = 6n - 10$	Theorem 4	$\lambda_7(BP_n) = 6n - 5$	Theorem 4

For ℓ-componen edge connectivity and h-extra edge connectivity with higher ℓ and h, e.g., $h = 6$ and $\ell = 8$, since we showed in Lemma 10 that $E(W, BP_n - W)| \geqslant 7n - 12$ and $|N_{E(BP_n)}(W)| \geqslant 7n - 6$ for $W \subset V(BP_n)$ with $|W| = 7$, this prompts us to have the following conjecture:

Conjecture 1. *$\lambda_8(BP_n) = \lambda^{(6)}(BP_n) + 6 = 7n - 6$ for $n \geqslant 6$.*

Obviously, to affirm the above conjecture is equivalent to showing that the following two implications hold for $n \geqslant 7$ and any edge set $F \subset E(BP_n)$: (i) If $|F| \leqslant 7n - 13$, then $BP_n - F$ either is connected or contains a large component along with smaller components containing totally at most six vertices. (ii) if $|F| \leqslant 7n - 7$, then $BP_n - F$ has at most seven components.

Similarly, as BP_n is n-regular and its girth is eight, we are easy to check that $|E(C_8, BP_n - C_8)| = 8n - 16$ and $|N_{E(BP_n)}(C_8)| = 8n - 8$. Based on the relationship of $\lambda^{(h)}(G)$ and $\lambda_r(G)$ for a regular graph G [30], we also have the following conjecture:

Conjecture 2. *$\lambda_9(BP_n) = \lambda^{(7)}(BP_n) + 8 = 8n - 8$ for $n \geqslant 6$.*

To prove this conjecture, we need to show that when removing any edge subset with a size approximately of eight times $\lambda(BP_n)$, the surviving graph still retains the "linearly many faults" property. With the increase in the removal of edges, the situation becomes more complex, and it is an interesting and challenging research topic.

We conclude this paper by discussing some of its limitations against real-world instances. Even though various interconnection networks have specific structural phenomena when a linear number of vertices or edges fail, do these phenomena occur frequently? Since most research considers vertex or edge failures in a network to be random and uncorrelated, it ignores possible events that cause components close to each other to fail simultaneously with a higher probability. In this case, is there a more reasonable evaluation measure combining h-extra edge connectivity or ℓ-component edge connectivity that can genuinely reflect this phenomenon?

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