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Global Regular Axially Symmetric Solutions to the Navier–Stokes Equations: Part 2

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Abstract: The axially symmetric solutions to the Navier–Stokes equations are considered in a bounded cylinder $\Omega \subset \mathbb{R}^3$ with the axis of symmetry. S_1 is the boundary of the cylinder parallel to the axis of symmetry, and S_2 is perpendicular to it. We have two parts of S_2. On S_1 and S_2, we impose vanishing of the normal component of velocity and the angular component of vorticity. Moreover, we assume that the angular component of velocity vanishes on S_1 and the normal derivative of the angular component of velocity vanishes on S_2. We prove the existence of global regular solutions. To prove this, the coordinate of velocity along the axis of symmetry must vanish on it. We have to emphasize that the technique of weighted spaces applied to the stream function plays a crucial role in the proof of global regular axially symmetric solutions. The paper is a generalization of Part 1, where the periodic boundary conditions are prescribed on S_2. The transformation is not trivial because it needs to examine many additional boundary terms and derive new estimates.

Keywords: Navier–Stokes equations; axially symmetric solutions; cylindrical domain; existence of global regular solutions

MSC: 35A01; 35B01; 35B65; 35Q30; 76D03; 76005

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1. Preliminary Remarks

The problem of the existence of global regular axially symmetric solutions to Navier–Stokes equations has a long history. The first such results with vanishing swirl were proved by O.A. Ladyzhenskaya [1] and by M.R. Ukhovskii and V.I. Yudovich [2], independently, in 1968.

The case with nonvanishing swirl is still open. This paper is a step in this direction. We prove the existence of global regular solutions with large swirl. Unfortunately, in the proof the stream function divided by the radius, denoted by ψ_1, must vanish on the axis of symmetry. This means that the coordinate of velocity along the axis of symmetry must vanish on the axis of symmetry too.

The main result of this paper is Theorem 1 and the global esitmate (24).

Since mathematicians were not able to prove the global estimate for regular solutions, they prove it by assuming some different Serrin-type conditions. The conditions are such that some coordinates either of velocity, or of derivatives of velocity, or of vorticity belong to $L_q(0,T;L_p(\Omega))$ spaces for appropriately chosen parameters p and q. A significant literature of this kind is cited in the papers: [3–7].

In papers [8–11], it is shown that a suitable continuity of some coordinates of velocity in a neighborhood of the axis of symmetry implies the regularity of axially symmetric solutions.

In this section, we are going to show the main points of the paper that are crucial for the proof of (24). For this purpose, we only recall the appropriate points of the paper.

Looking at the literature on the regularity problem of axially symmetric solutions, it is clear that to prove it we need additional estimates and appropriate inequalitites. We found them in Sections 4, 6 and 7. The results are original. Let $u = rv_\varphi$, v_φ, the angular component of velocity, be swirl.

In Section 6, we proved the estimate

$$\|u\|_{L_\infty(0,t;H^1(\Omega))} + \|u\|_{L_2(0,t;H^2(\Omega))} \le \phi(\text{data}), \tag{1}$$

where ϕ denotes an increasing positive function.

The most important inequality is proved in Section 7 (see (173)):

$$\begin{aligned} \left\|\frac{\omega_r}{r}\right\|^2_{L_2(\Omega^t)} &\le c\|u\|_{L_\infty(\Omega^t)}(D_1 + \|u\|_{L_2(0,t;H^2(\Omega))})\cdot \\ &\quad \cdot \left\|\frac{\omega_\varphi}{r}\right\|_{L_2(0,t;H^1(\Omega))} + \phi(\text{data}), \end{aligned} \tag{2}$$

where ω_r and ω_φ are radial and angular coordinates of vorticity.

Moreover, $D_1 = \phi(\text{data})$ bounds the weak solution (see Lemma 1), and the estimate (see Lemma 2)

$$\|u\|_{L_\infty(\Omega^t)} \le \phi(\text{data}) \tag{3}$$

is well known.

The most important feature of (2) is such that norm of $\frac{\omega_r}{r}$ appears in the square but also in the norm of $\frac{\omega_\varphi}{r}$ linearly.

Inequality (2) is crucial in the proof of global inequality (24). To prove (2), we need inequalities from Sections 4, 6, and 7.

The main difficulty in the regularity theory of the Navier–Stokes equations is to handle with the nonlinear terms. We need to tranform them in such a way that they can be absorbed by the main linear terms.

In this paper, we consider problem (17)–(20) for functions $\Phi = \frac{\omega_r}{r}$, $\Gamma = \frac{\omega_\varphi}{r}$ defined by (16). The problem was considered in [8].

Applying the energy method, we derive inequality (111) with the strongly nonlinear term denoted by I_3.

The main task of this paper is to estimate I_3 by quantities that can be absorbed by the terms from the l.h.s. of (111).

I_3 is estimated in (122). Using notation (132)

$$X(t) = \|\Phi\|_{V(\Omega^t)} + \|\Gamma\|_{V(\Omega^t)},$$

where

$$\|u\|_{V(\Omega^t)} = \|u\|_{L_\infty(0,t;L_2(\Omega))} + \|u\|_{L_2(0,t;H^1(\Omega))},$$

we derive from (111) and (122) the inequality (see (134))

$$X^2 \le \phi_1 X^{2-\delta} + \phi(\text{data}), \tag{4}$$

where ϕ_1 depends on $|v_\varphi|_{d,\infty,\Omega^t}$, $|v_\varphi|_{\infty,\Omega^t}$, $d > 3$ and $\delta > 0$.

For $\delta > 0$, the Young inequality can be applied in (4) so (133) holds. We write it in the form

$$X^2 \le c|v_\varphi|^{\frac{4\varepsilon}{\theta}}_{d,\infty,\Omega^t}(1 + |v_\varphi|^{\varepsilon_0}_{\infty,\Omega^t}) + (\text{data}), \tag{5}$$

where $\varepsilon = \varepsilon_1 + \varepsilon_2$, $\theta = \left(1 - \frac{3}{d}\right)\varepsilon_1 - \frac{3}{d}\varepsilon_2$ and ε_0 can be chosen as arbitrarily small.

The existence of such positive δ implies (122) is the form (4). For $\delta = 0$, we were not able to apply the Young inequality in (4) so we were not be able to prove Theorem 1.

Applying (2), we were able to prove (4) with positive δ.

Hence, (2) is the most important inequality in this paper. It is a totally new result.

In the next step, we eliminate $|v_\varphi|_{d,\infty,\Omega^t}$, $d=12$ from the r.h.s. of (5). To perform this, we have to enter into the proof of Lemma 13. To derive (141) from (140), we need to satisfy the estimate

$$\int_\Omega \frac{\psi_1^2}{r^\alpha} dx \le c\|\Gamma\|^2_{H^1(\Omega)}, \tag{6}$$

where $\alpha \le 6$. (6) does not hold for $\alpha = 6$ because (202) is not true for $\mu = 0$.

Hence, (202) implies that (6) holds for any number less than 6. It is denoted by $6'$.

In the next step, we recall (147).

$$|v_\varphi|^{6'}_{12,\infty,\Omega^t} \le c|v_\varphi|^{\frac{4\varepsilon}{\theta}}_{12,\infty,\Omega^t} + \phi(\text{data}). \tag{7}$$

To apply the Young inequality in (7), we require that $6' > \frac{4\varepsilon}{\theta}$. In Remark 4, it is shown that the inequality holds for $6'$ sufficiently close to 6. In this case, $|v_\varphi|_{12,\infty,\Omega^t}$ can be eliminated from the r.h.s. of (5).

Eliminating $|v_\varphi|_{\infty,\Omega^t}$ is easy because it appears with the power ε_0, which is assumed to be arbitrarily small.

We have to emphasize that (137) is proved for such solutions to problem (6) that v_φ is not very small. The existence of such local solutions is proved in Appendix A.

Hence, the global estimate (24) holds for these solutions. This means that the local solution can be extended in time.

To describe the transformation from Part 1 to Part 2, we have to examine terms, where the integration by parts with respect to z appears.

To derive the second term in (47), we need the following term to vanish:

$$\int_{S_2} (\bar{n}\cdot\nabla v_r v_r + \bar{n}\cdot\nabla v_\varphi v_\varphi + \bar{n}\cdot\nabla v_z v_z) dS_2 = 0. \tag{8}$$

To satisfy (8), we see that $\bar{n}\cdot\nabla v_r \cdot v_r|_{S_2} = v_{r,z}v_r|_{S_2} = v_{z,r}v_r|_{S_2} = 0$ because we assumed that $v_z|_{S_2} = 0$ so also $v_{z,r}|_{S_2} = 0$.

Moreover, we used that $\omega_\varphi|_{S_2} = v_{r,z} - v_{z,r}|_{S_2} = 0$. Next, we see that

$$\bar{n}\cdot\nabla v_\varphi v_\varphi|_{S_2} = v_{\varphi,z}v_\varphi|_{S_2} = 0$$

because

$$v_{\varphi,z}|_{S_2} = 0. \tag{9}$$

Finally, the last term in (8) vanishes because $v_z|_{S_2} = 0$.

To show (53), we require that

$$\int_{S_2} \bar{n}\cdot\nabla u u|u|^{s-2} dS_2 = 0.$$

It is true because (9) implies that $u_{,z} = rv_{\varphi,z} = 0$ on S_2.

In proofs of (62)–(64), we show that prescribed boundary conditions on S_2 imply the vanishing of all the boundary terms that appeared on S_2.

Comparing Part 2 with Part 1, we see that in (113) an additional boundary term on S_2 appears. Fortunately, it vanishes because $\Phi|_{S_2} = 0$. To prove (138), a boundary term on S_2 also appears. It vanishes because (9) holds.

The same argument works in (154) and in the proofs of (159) and (160).

In (174), two boundary terms on S_2 appear:

$$I_1 = \int\limits_{S_2^t} \bar{n} \cdot \nabla \omega_r \omega_r dS_2 dt', \quad I_2 = \int\limits_{S_2^t} \bar{n} \cdot \nabla \omega_z \omega_z dS_2 dt'.$$

Since $\omega_r = -v_{\varphi,z}$, then (9) implies $I_1 = 0$. To prove $I_2 = 0$, we see that $u_{,zr} = v_{\varphi,z} + v_{\varphi,zr} = 0$ on S_2 by (9).

The boundary term on S_2 in J_1 vanishes because (9) holds. To vanish the boundary term in J_3, we need $\psi_{1,r}|_{S_2} = 0$, $\psi_{1,rr}|_{S_2} = 0$, $\psi_{1,zz}|_{S_2} = \omega_1|_{S_2} = 0$.

We have to emphasize that condition (9) is crucial in the proof of Theorem 1.

Finally, we shortly describe the results of this paper.

In Section 2, the considered problem is formulated in Cartesian coordinates in (6), in cylindrical coordinates of velocity in (7), and in cylindrical coordinates of vorticity in (9). Moreover, we recalled very important relations between cylindrical coordinates of velocity, vorticity, and the stream function (see (13), (15) and (21)). Next, (14) is the problem for the stream function ψ and (22) for the modified stream function $\psi_1 = \psi/r$.

Problem (17)–(20) for functions Φ, Γ defined in (16), is the main problem in this paper. For solutions to this problem, we derive the global estimate (24), which is proved in Theorem 1. Finally, the end of this Section is devoted to the proof of Theorem 1.

In Section 3, there are introduced notations used in this paper. Moreover, we proved the energy estimate (see Lemma 1), the estimate for swirl (see Lemma 2), and the energy estimate for the modified stream function ψ_1 (see Lemma 4). Moreover, in Lemma 5 we recall the interpolation inequality for weighted Sobolev spaces proved in [8]. The inequality is crucial in the proof of (24). At the end of this Section, we recalled the definition and some properties of weighted Sobolev spaces (see [12]).

In Section 4, we derive many new estimates for the modified stream function ψ_1. Since Equation $(22)_1$ has a singular coefficient, the estimates can be derived either by the energy method or by applying the technique of weighted Sobolev spaces developed by Kondratiev [12].

Applying the energy method, some terms on the boundary and on the axis of symmetry appear. The terms on the boundary vanish in view of the boundary conditions. To eliminate the terms on the axis of symmetry, we need expansions of v_r, v_φ, ψ_1 near the axis of symmetry proved by Liu-Wang (see [13]).

Moreover, the expansions hold for the sufficiently regular local solution. The existence of such a local solution is proved in Appendix A.

Section 5 is the most important part in this paper. First, we apply the energy method for solutions to problem (17)–(20). Next, we use the results from Sections 4, 6 and 7. The main points of this proof are described in the proof of Theorem 1 from Section 2 and at the beginning of "Preliminary results".

First, we derive (111), where I_3 is estimated by (122). Using (173) with simplified form (2), we derive (4) with positive δ. Then, by the Young inequality we derive (5). Next, we want to eliminate $|v_\varphi|_{d,\infty,\Omega^t}$ from the r.h.s. of (5). It is possible for $d = 12$ and inequality (7) (see also (137)). In the r.h.s. of (7), $|v_\varphi|_{\infty,\Omega^t}^{\varepsilon_0}$ appears, where ε_0 is small (see (151)). It is estimated by (152). For ε_0 small, we derive (24).

In Section 6, we derive new estimates for swirl (see Lemma 15). The estimates are necessary in the proof of (173) from Section 7.

Inequality (173) is crucial for the proof of (24). It is a new original result.

In Section 8, we proved some estimates for ψ_1 in weighted Sobolev spaces, which are necessary in Section 4.

In Appendix A, we proved the existence of the local regular solutions necessary for deriving expansions near the axis of symmetry shown in [13].

2. Introduction

The Section is divided into the following steps:

1. The formulation of the considered problem in Cartesian coordinates.
2. Formulation in the cylindrical coordinates of velocity and vorticity. Moreover, the important relation between the cylindrical coordinates of velocity, vorticity, and stream function are recalled.
3. Proof of the global estimate.

In this paper, we prove the existence of global regular axially symmetric solutions to the Navier–Stokes equations in a cylindrical domain $\Omega \subset \mathbb{R}^3$:

$$\Omega = \{x \in \mathbb{R}^3 \colon x_1^2 + x_2^2 < R^2, |x_3| < a\},$$

where a, R are given positive numbers. We denote by $x = (x_1, x_2, x_3)$ Cartesian coordinates. It is assumed that the x_3-axis is the axis of symmetry of Ω and $\partial\Omega = S = S_1 \cup S_2$.

Moreover,

$$\begin{aligned} S_1 &= \{x \in \mathbb{R}^3 \colon \sqrt{x_1^2 + x_2^2} = R, x_3 \in (-a, a)\}, \\ S_2(a_0) &= \{x \in \mathbb{R}^3 \colon \sqrt{x_1^2 + x_2^2} < R, x_3 = a_0 \in \{-a, a\}\}, \end{aligned}$$

where S_1 is parallel to the axis of symmetry and $S_2(a_0)$ is perpendicular to it. $S_2(a_0)$ meets the axis of symmetry at a_0.

To describe the considered problem, we introduce cylindrical coordinates r, φ, z by the relations

$$x_1 = r\cos\varphi, \quad x_2 = r\sin\varphi, \quad x_3 = z. \tag{1}$$

The following orthonormal system:

$$\bar{e}_r = (\cos\varphi, \sin\varphi, 0), \;\; \bar{e}_\varphi = (-\sin\varphi, \cos\varphi, 0), \;\; \bar{e}_z = (0, 0, 1) \tag{2}$$

is connected with the cylindrical coordinates.

Any vector u for the axially symmetric motions can be decomposed as follows:

$$u = u_r(r, z, t)\bar{e}_r + u_\varphi(r, z, t)\bar{e}_\varphi + u_z(r, z, t)\bar{e}_z, \tag{3}$$

where u_r, u_φ, u_z are cylindrical coordinates of u.

Therefore, velocity v and vorticity $\omega = \operatorname{rot} v$ are decomposed in the form

$$v = v_r(r, z, t)\bar{e}_r + v_\varphi(r, z, t)\bar{e}_\varphi + v_z(r, z, t)\bar{e}_z \tag{4}$$

and

$$\omega = \omega_r(r, z, t)\bar{e}_r + \omega_\varphi(r, z, t)\bar{e}_\varphi + \omega_z(r, z, t)\bar{e}_z. \tag{5}$$

The paper is devoted to a proof of global regular axially symmetric solutions to the problem

$$\begin{array}{ll} v_{,t} + v \cdot \nabla v - \nu\Delta v + \nabla p = f & \text{in } \Omega^T = \Omega \times (0, T), \\ \operatorname{div} v = 0 & \text{in } \Omega^T, \\ v \cdot \bar{n}|_S = 0, \;\; \omega_\varphi|_S = 0, \;\; v_\varphi|_{S_1} = 0, \;\; v_{\varphi,z}|_{S_2} = 0 & \text{on } S^T = S \times (0, T), \\ v|_{t=0} = v(0) & \text{in } \Omega, \end{array} \tag{6}$$

where $v = (v_1(x, t), v_2(x, t), v_3(x, t)) \in \mathbb{R}^3$ is the velocity of the fluid, $p = p(x, t) \in \mathbb{R}$ is the pressure, $f = (f_1(x, t), f_2(x, t), f_3(x, t)) \in \mathbb{R}^3$ is the external force field, and $\nu > 0$ is the constant viscosity coefficient.

Expressing problem (6) in the cylindrical coordinates of velocity yields

$$
\begin{aligned}
&v_{r,t}+v\cdot\nabla v_r-\frac{v_\varphi^2}{r}-\nu\Delta v_r+\nu\frac{v_r}{r^2}=-p_{,r}+f_r,\\
&v_{\varphi,t}+v\cdot\nabla v_\varphi+\frac{v_r}{r}v_\varphi-\nu\Delta v_\varphi+\nu\frac{v_\varphi}{r^2}=f_\varphi,\\
&v_{z,t}+v\cdot\nabla v_z-\nu\Delta v_z=-p_{,z}+f_z,\\
&(rv_r)_{,r}+(rv_z)_{,z}=0\\
&v_r|_S=0,\ \ v_\varphi|_{S_1}=0,\ \ v_{\varphi,z}|_{S_2}=0,\ \ v_{r,z}-v_{z,r}|_S=0,\\
&v_r|_{t=0}=v_r(0),\ \ v_\varphi|_{t=0}=v_\varphi(0),\ \ v_z|_{t=0}=v_z(0),
\end{aligned}
\tag{7}
$$

and

$$
\begin{aligned}
&v\cdot\nabla=(v_r\bar{e}_r+v_z\bar{e}_z)\cdot\nabla=v_r\partial_r+v_z\partial_z,\\
&\Delta u=\frac{1}{r}(ru_{,r})_{,r}+u_{,zz}.
\end{aligned}
\tag{8}
$$

Formulating problem (6) in terms of the cylindrical coordinates of vorticity implies

$$
\begin{aligned}
&\omega_{r,t}+v\cdot\nabla\omega_r-\nu\Delta\omega_r+\nu\frac{\omega_r}{r^2}=\omega_rv_{r,r}+\omega_zv_{r,z}+F_r,\\
&\omega_{\varphi,t}+v\cdot\nabla\omega_\varphi-\frac{v_r}{r}\omega_\varphi-\nu\Delta\omega_\varphi+\nu\frac{\omega_\varphi}{r^2}=\frac{2}{r}v_\varphi v_{\varphi,z}+F_\varphi,\\
&\omega_{z,t}+v\cdot\nabla\omega_z-\nu\Delta\omega_z=\omega_rv_{z,r}+\omega_zv_{z,z}+F_z,\\
&\omega_r|_{t=0}=\omega_r(0),\ \ \omega_\varphi|_{t=0}=\omega_\varphi(0),\ \ \omega_z|_{t=0}=\omega(0)
\end{aligned}
\tag{9}
$$

and we have boundary conditions $(7)_5$ on S, where $F=\operatorname{rot} f$ and

$$
F=F_r(r,z,t)\bar{e}_r+F_\varphi(r,z,t)\bar{e}_\varphi+F_z(r,z,t)\bar{e}_z. \tag{10}
$$

The function

$$
u=rv_\varphi \tag{11}
$$

is called swirl. It is a solution to the problem

$$
\begin{aligned}
&u_{,t}+v\cdot\nabla u-\nu\Delta u+\frac{2\nu}{r}u_{,r}=rf_\varphi\equiv f_0,\\
&u|_{S_1}=0,\ \ u_{,z}|_{S_2}=0,\\
&u|_{t=0}=u(0).
\end{aligned}
\tag{12}
$$

The cylindrical components of vorticity can be described in terms of the cylindrical components of velocity and swirl in the following form

$$
\begin{aligned}
&\omega_r=-v_{\varphi,z}=-\frac{1}{r}u_{,z},\\
&\omega_\varphi=v_{r,z}-v_{z,r},\\
&\omega_z=\frac{1}{r}(rv_\varphi)_{,r}=v_{\varphi,r}+\frac{v_\varphi}{r}=\frac{1}{r}u_{,r}.
\end{aligned}
\tag{13}
$$

Equation $(7)_4$ implies the existence of the stream function ψ, which is a solution to the problem

$$
\begin{aligned}
&-\Delta\psi+\frac{\psi}{r^2}=\omega_\varphi,\\
&\psi|_S=0.
\end{aligned}
\tag{14}
$$

Moreover, cylindrical components of velocity can be expressed in terms of the stream function in the following way:

$$\begin{aligned} &v_r = -\psi_{,z}, \ \ v_z = \frac{1}{r}(r\psi)_{,r} = \psi_{,r} + \frac{\psi}{r}, \\ &v_{r,r} = -\psi_{,zr}, \ \ v_{r,z} = -\psi_{,zz}, \\ &v_{z,z} = \psi_{,rz} + \frac{\psi_{,z}}{r}, \ \ v_{z,r} = \psi_{,rr} + \frac{1}{r}\psi_{,r} - \frac{\psi}{r^2}. \end{aligned} \tag{15}$$

Introduce the pair

$$(\Phi, \Gamma) = (\omega_r/r, \omega_\varphi/r). \tag{16}$$

Formula (6) from [8] implies that quantities (16) satisfy the following equations:

$$\Phi_{,t} + v\cdot\nabla\Phi - \nu\left(\Delta + \frac{2}{r}\partial_r\right)\Phi - (\omega_r\partial_r + \omega_z\partial_z)\frac{v_r}{r} = F_r/r \equiv \bar{F}_r \tag{17}$$

and

$$\Gamma_{,t} + v\cdot\nabla\Gamma - \nu\left(\Delta + \frac{2}{r}\partial_r\right)\Gamma + 2\frac{v_\varphi}{r}\Phi = F_\varphi/r \equiv \bar{F}_\varphi. \tag{18}$$

We add the following initial and boundary conditions to solutions of (17) and (18)

$$\Phi|_S = 0, \ \ \Gamma|_S = 0, \tag{19}$$

$$\Phi|_{t=0} = \Phi(0), \ \ \Gamma|_{t=0} = \Gamma(0). \tag{20}$$

Next, we express the cylindrical coordinates of velocity in terms of $\psi_1 = \psi/r$

$$\begin{aligned} &v_r = -r\psi_{1,z}, && v_z = (r\psi_1)_{,r} + \psi_1 = r\psi_{1,r} + 2\psi_1, \\ &v_{r,r} = -\psi_{1,z} - r\psi_{1,rz}, && v_{r,z} = -r\psi_{1,zz}, \\ &v_{z,z} = r\psi_{1,rz} + 2\psi_{1,z}, && v_{z,r} = 3\psi_{1,r} + r\psi_{1,rr}. \end{aligned} \tag{21}$$

The aim of this paper is to prove the existence of global regular axially symmetric solutions to problem (6). For this purpose, we have to find a global estimate guaranteeing the existence of global regular solutions.

Function ψ_1 is a solution to the problem

$$\begin{aligned} &-\Delta\psi_1 - \frac{2}{r}\psi_{1,r} = \omega_1 \quad \text{in } \ \Omega = (0,R)\times(-a,a), \\ &\psi_1|_S = 0, \end{aligned} \tag{22}$$

where

$$\omega_1 = \omega_\varphi/r. \tag{23}$$

We have that $\omega_1 = \Gamma$.

This paper is a generalization of paper [14], where the periodic boundary conditions on S_2 are assumed. Since the periodic boundary conditions are mathematical-type conditions, we replaced them in this paper by $v\cdot\bar{n}|_{S_2} = 0$, $\omega_\varphi|_{S_2} = 0$, $v_{\varphi,z}|_{S_2} = 0$. This replacement is not trivial because it needs many additional considerations.

To state the main result, we first introduce necessary assumptions.

Assumption 1. *Assume that the following quantities are finite:*

$$\begin{aligned}
&\psi_1|_{r=0}=0,\\
&D_1=\|f\|_{L_2(\Omega^t)}+\|v(0)\|_{L_2(\Omega)},\\
&D_2=\|f_0\|_{L_{\infty,1}(\Omega^t)}+\|u(0)\|_{L_\infty(\Omega)},\\
&f_0=rf_\varphi,\ \ u=rv_\varphi,\\
&D_3^2=D_1^2D_2^2+\|u_{,z}(0)\|^2_{L_2(\Omega)}+\|f_0\|^2_{L_2(\Omega^t)},\\
&D_4^2=D_1^2(1+D_2)+\|u_{,r}(0)\|^2_{L_2(\Omega)}+\|f_0\|^2_{L_2(\Omega^t)}+\|f_0\|_{L_2(0,t;L_{4/3}(S_1))},
\end{aligned}$$

where D_1, D_2 are introduced in (46) and (52), respectively, and D_3, D_4 in (159) and (160), respectively. Let

$$\begin{aligned}
&D_5=D_2(D_1+D_2+D_3),\\
&D_6=D_2^{1-\varepsilon_0}D_3,
\end{aligned}$$

where ε_0 is arbitrary small positive number. Moreover,

$$\begin{aligned}
D_7=&\|F_r\|^2_{L_2(0,t;L_{6/5}(\Omega))}+\|F_z\|^2_{L_2(0,t;L_{6/5}(\Omega))}\\
&+\|\omega_r(0)\|^2_{L_2(\Omega)}+\|\omega_z(0)\|^2_{L_2(\Omega)}
\end{aligned}$$

is defined in Lemma 16.

Next,

$$\begin{aligned}
D_8=&\phi(D_2)(\|\bar F_r\|^2_{L_2(0,t;L_{6/5}(\Omega))}+\|\bar F_\varphi\|^2_{L_2(0,t;L_{6/5}(\Omega))})\\
&+\|\Phi(0)\|^2_{L_2(\Omega)}+\|\Gamma(0)\|^2_{L_2(\Omega)},
\end{aligned}$$

where $\bar F_r=F_r/r$, $\bar F_\varphi=F_\varphi/r$, $\Phi=\frac{\omega_r}{r}$, $\Gamma=\frac{\omega_\varphi}{r}$ and D_8 appears in (111).

In Lemma 13, the following quantity is defined:

$$D_9(12)=12\|f_\varphi\|_{L_{12}(0,t;L_{36/25}(\Omega))}+\|v_\varphi(0)\|_{L_{12}(\Omega)}.$$

Finally, we have introduced in Lemma 14 the quantity

$$D_{10}=\|f_\varphi/r\|_{L_1(0,t;L_\infty(\Omega))}+\|v_\varphi(0)\|_{L_\infty(\Omega)}.$$

Theorem 1. *Assume that Assumption 1 holds. Then, an increasing positive function ϕ exists such that*

$$\|\Phi\|_{V(\Omega^t)}+\|\Gamma\|_{V(\Omega^t)}\le\phi(D_1,\cdots,D_{10}). \tag{24}$$

Remark 1. *Estimate (24) implies any regularity of solutions to problem (6) assuming sufficient regularity of data.*

To prove (24), we ψ_1 and v_z need to vanish on the axis of symmetry.

Proof of Theorem 1. Inequality (113) in the form

$$\frac{d}{dt}|\Phi|^2_{2,\Omega}+|\nabla\Phi|^2_{2,\Omega}\le I+\int\limits_\Omega\bar F_r\Phi dx \tag{25}$$

is the first step of the proof of (24), where $\Phi=-\frac{v_{\varphi,z}}{r}$, $\bar F_r=\frac{F_r}{r}$ and

$$I\le\int\limits_\Omega\left|v_\varphi\partial_r\frac{v_r}{r}\Phi_{,z}\right|dx+\int\limits_\Omega\left|v_\varphi\partial_z\frac{v_r}{r}\Phi_{,r}\right|dx\equiv I_1+I_2.$$

Our aim is to estimate I_1 and I_2 by a product of norms $\|\Phi\|_{V(\Omega^t)}$, $\|\Gamma\|_{V(\Omega^t)}$.

Since the L_∞-estimate of swirl rv_φ is bounded by D_2 (see Lemma 2) and $\frac{v_r}{r} = -\psi_{1,z}$, we obtain the estimates

$$\begin{aligned} I_1 &\le D_2 \left|\frac{\psi_{1,rz}}{r}\right|_{2,\Omega} |\Phi_{,z}|_{2,\Omega}, \\ I_2 &\le D_2 \left|\frac{\psi_{1,zz}}{r}\right|_{2,\Omega} |\Phi_{,r}|_{2,\Omega}. \end{aligned} \tag{26}$$

To examine estimate (26), we recall that ψ_1 is a solution to problem (22).

We prove the existence of weak solutions to problem (22) in Lemma 4 and derive the estimate (56)

$$\|\psi_1\|_{1,\Omega} \le c|w_1|_{6/5,\Omega}. \tag{27}$$

In Section 4, we increase the regularity of weak solutions by deriving estimates for higher derivatives.

From (82), we have

$$\left|\frac{\psi_{1,rz}}{r}\right|_{2,\Omega} \le c|\Gamma_{,z}|_{2,\Omega}. \tag{28}$$

The estimate holds for the weak solutions to problem (22) because [13] yields the expansion of ψ_1 near the axis of symmetry

$$\psi_1 = a_1(z,t) + a_2(z,t)r^2 + a_3(z,t)r^4 + \cdots \tag{29}$$

Hence, $\psi_{1,r} = 2a_2(z,t)r$ and the norm $|\frac{\psi_{1,rz}}{r}|_{2,\Omega}$ can be finite.

To estimate I_2, we need

$$\left|\frac{\psi_{1,zz}}{r}\right|_{2,\Omega} \le c|\Gamma_{,z}|_{2,\Omega}. \tag{30}$$

The estimate holds for such a class of regularized weak solutions to problem (22) that

$$\psi_1|_{r=0} = 0. \tag{31}$$

It means that in expansion (29), we have that $a_1(z,t) = 0$.

The existence of solutions to problem (22) (see also (61)) satisfying restriction (31) and estimate (30) follows from the theory developed by Kondratiev (see [12]) for elliptic boundary value problems in domains with cones in weighted Sobolev spaces.

In this paper, the existence is proved in Lemmas 8 and 17. From [12], it also follows that we can prove the existence of different solutions to problem (22) belonging to different weighted Sobolev spaces.

The difference between two such solutions equals the expression that follows from the Cauchy theorem for complex functions connected with the contour integration.

Restriction (31) means that we have to work with a very restricted class of weak solutions to (22). This also means that v_z must vanish on the axis of symmetry.

Using estimates (28) and (30) in (25) yields

$$\frac{d}{dt}|\Phi|^2_{2,\Omega} + |\nabla\Phi|^2_{2,\Omega} \le cD_2|\Gamma_{,z}|_{2,\Omega}|\nabla\Phi|_{2,\Omega} + \int_\Omega \bar{F}_r\Phi dx. \tag{32}$$

We have to emphasize that we are not able to prove estimate (24) without restriction (31).

Now, we integrate (120) with respect to time. Then, we obtain

$$\begin{aligned} |\Gamma|^2_{2,\Omega} + \|\Gamma\|^2_{1,2,\Omega^t} &\le 2\left|\int_{\Omega^t} \frac{v_\varphi}{r}\Phi\Gamma dxdt'\right| \\ &\quad + c|\bar{F}_\varphi|^2_{6/5,2,\Omega^t} + c|\Gamma(0)|^2_{2,\Omega}. \end{aligned} \tag{33}$$

Integrating (32) with respect to time and adding to (33) yields

$$\begin{aligned}\|\Phi\|^2_{V(\Omega^t)}+\|\Gamma\|^2_{V(\Omega^t)}\le c(D_2)\bigg|\int\limits_{\Omega^t}\frac{v_\varphi}{r}\Phi\Gamma dxdt'\bigg|\\ +c(D_2)(|\bar F_r|^2_{6/5,2,\Omega^t}+|\bar F_\varphi|^2_{6/5,2,\Omega^t})+c(D_2)(|\Phi(0)|^2_{2,\Omega}+|\Gamma(0)|^2_{2,\Omega}).\end{aligned}\tag{34}$$

Now, we have to estimate the first term on the r.h.s. of (34).

Introducing the quantity (see (132))

$$X(t)=\|\Phi\|_{V(\Omega^t)}+\|\Gamma\|_{V(\Omega^t)}\tag{35}$$

and recalling that constant D_8 is introduced in Assumption 1, inequality (34) takes the form

$$X^2(t)\le c(D_2)\bigg|\int\limits_{\Omega^t}\frac{v_\varphi}{r}\Phi\Gamma dxdt'\bigg|+cD_8^2,\tag{36}$$

where the first integral is called I_3.

Using estimate (123) and the estimate of L_1^2 in the proof of Lemma 11, we obtain from (36) the inequality

$$X^2(t)=c(D_2)|v_\varphi|^\varepsilon_{d,\infty,\Omega^t}|\Phi|^\theta_{2,\Omega^t}|\nabla\Phi|^{1-\theta}_{2,\Omega^t}|\nabla\Gamma|_{2,\Omega^t}+cD_8^2,\tag{37}$$

where $\theta=\left(1-\frac{3}{d}\right)\varepsilon_1-\frac{3}{d}\varepsilon_2$, $d>3$, $\varepsilon=\varepsilon_1+\varepsilon_2<1$.

To derive any estimate from (37), we use (173) in the form

$$|\Phi|^2_{2,\Omega}\le c(D_5+D_6|v_\varphi|^{\varepsilon_0}_{\infty,\Omega^t})\|\Gamma\|_{1,2,\Omega^t}+cD_7,\tag{38}$$

where ε_0 can be assumed to be an arbitrarily small positive number and D_5, D_6, D_7 are defined in Assumption 1. This is a very important estimate because the square of $|\Phi|_{2,\Omega^t}$ depends linearly on $\|\Gamma\|_{1,2,\Omega^t}$.

Using (38) in (37) yields (the estimate of I_3 is described in (122))

$$X_2(t)\le c|v_\varphi|^\varepsilon_{d,\infty,\Omega^t}[c_1(1+|v_\varphi|^{\frac{1}{2}\theta\varepsilon_0}_{\infty,\Omega^t})X^{\frac{1}{2}\theta}+c_2]X^{2-\theta}+cD_8^2,\tag{39}$$

where c_1, c_2 depend on D_5, D_6, D_7.

Since $2-\frac{1}{2}\theta$, $2-\theta$ are less than 2, Lemma 12 yields the inequality

$$X^2\le c_0|v_\varphi|^{\frac{4\varepsilon}{\theta}}_{d,\infty,\Omega^t}(1+|v_\varphi|^{2\varepsilon_0}_{\infty,\Omega^t})+c_0|v_\varphi|^{\frac{2\varepsilon}{\theta}}_{d,\infty,\Omega^t}+cD_8^2,\tag{40}$$

where $c_0=\phi(D_2,D_5,D_6,D_7)$.

Setting $d=12$ and assuming that v_φ is not small, we derive (137) in the form

$$|v_\varphi|_{12,\infty,\Omega^t}\le c|v_\varphi|^{b_0\varepsilon_0}_{\infty,\Omega^t}+\phi(D_2,D_5,D_6,D_7,D_8,D_9),\tag{41}$$

where b_0 is a positive number.

The smallness of v_φ, which must be excluded in the proof of (41), is described in Appendix A.

To prove (41), we have to pass from (140) to (141). Therefore, we need the estimate

$$\int\limits_{\Omega^t}\frac{\psi_1^2}{r^{6'}}dxdt'\le c\|\Gamma\|^2_{1,2,\Omega^t},\tag{42}$$

where $6'<6$, and we are not able to replace $6'$ by 6 (see Remark 8).

Replacing $6'$ by 6 estimate (42) takes the form

$$\int_0^t \int_\Omega \frac{\psi_1^2}{r^6} dx dt' \le c \int_0^t \|\Gamma\|^2_{H_0^1(\Omega)} dt', \tag{43}$$

where the r.h.s. can not be estimated by $\|\Gamma\|_{V(\Omega^t)}$.

Estimate (42) follows from Lemma 18 and imposes the following additional restrictions on ψ_1:

$$\psi_1|_{r=0} = 0, \quad \psi_{1,r}|_{r=0} = 0. \tag{44}$$

However, the theory developed in [13] implies that $\psi_{1,r}|_{r=0} = 0$.

Exploiting (41) in (40) yields

$$X \le c(1 + |v_\varphi|^{d_1\varepsilon_0}_{\infty,\Omega^t})|v_\varphi|^{d_2\varepsilon_0}_{\infty,\Omega^t} + \phi(D_2, D_5, D_6, D_7, D_8, D_9), \tag{45}$$

where d_1, d_2 are positive finite numbers.

Finally, we find the estimate for $|v_\varphi|_{\infty,\Omega^t}$ (see (152)). Using (152) in (45) yields (24). This ends the proof of Theorem 1. □

The problem of regularity of axially symmetric solutions to the Navier–Stokes equations has a long history. The first regularity results in the case of vanishing swirl are derived in [1,2] by O. A Ladyzhenskaya and Ukhovskii–Yudovich independently. Many references in the case of nonvanishing swirl can be found in [3].

We have to emphasize that we were able to prove Theorem 1 because the theory of weighted Sobolev spaces developed in [15] was used.

3. Notation and Auxiliary Results

First, we introduce some notations

Definition 1. *We use the following notation for Lebesque and Sobolev spaces*

$$\|u\|_{L_p(\Omega)} = |u|_{p,\Omega}, \ \ \|u\|_{L_p(\Omega^t)} = |u|_{p,\Omega^t},$$
$$\|u\|_{L_{p,q}(\Omega^t)} = \|u\|_{L_q(0,t;L_p(\Omega))} = |u|_{p,q,\Omega^t},$$

where $p, q \in [1, \infty]$. *Next,*

$$\|u\|_{H^s(\Omega)} = \|u\|_{s,\Omega}, \ \ \|u\|_{W_p^s(\Omega)} = \|u\|_{s,p,\Omega},$$
$$\|u\|_{L_q(0,t;W_p^k(\Omega))} = \|u\|_{k,p,q,\Omega^t}, \ \ \|u\|_{k,p,p,\Omega^t} = \|u\|_{k,p,\Omega^t},$$

where $s, k \in \mathbb{N} \cup \{0\}$, $H^s(\Omega) = W_2^s(\Omega)$.

We need energy-type space $V(\Omega^t)$ to be appropriate for a description of weak solutions to the Navier–Stokes equations

$$\|u\|_{V(\Omega^t)} = |u|_{2,\infty,\Omega^t} + |\nabla u|_{2,\Omega^t}.$$

We recall weighted Sobolev spaces defined by

$$\|f\|_{H^k_\mu(\mathbb{R}_+)} = \left(\int_{\mathbb{R}_+} \sum_{j=0}^k |\partial_r^j f|^2 r^{2(\mu+j-k)} r dr \right)^{1/2}$$

and

$$\|f\|_{H^k_\mu(\Omega)} = \left(\int_\Omega \sum_{|\alpha|=0}^k |D^\alpha_{r,z} f|^2 r^{2(\mu+|\alpha|-k)} r dr dz \right)^{1/2},$$

where Ω contains the axis of symmetry, $D^\alpha = \partial_r^{\alpha_1}\partial_z^{\alpha_2}$, $|\alpha| = \alpha_1 + \alpha_2$, $\alpha_i \in \mathbb{N} \cup \{0\}$, $i = 1, 2$, $k \in \mathbb{N} \cup \{0\}$, $\mu \in \mathbb{R}_+$. Moreover, we have

$$
\begin{aligned}
&H_0^0(\Omega) = L_{2,0}(\Omega) = L_2(\Omega),\\
&H_\mu^0(\Omega) = L_{2,\mu}(\Omega)
\end{aligned}
$$

and

$$
\|f\|_{L_{2,\mu}(\Omega)} = |f|_{2,\mu,\Omega}.
$$

Lemma 1. *Let $f \in L_{2,1}(\Omega^t)$, $v(0) \in L_2(\Omega)$. Then, solutions to (7) satisfy the estimate*

$$
\begin{aligned}
&\|v(t)\|_{L_2(\Omega)}^2 + \nu \int\limits_{\Omega^t} (|\nabla v_r|^2 + |\nabla v_\varphi|^2 + |\nabla v_z|^2) dx dt' \\
&+ \nu \int\limits_{\Omega^t} \left(\frac{v_r^2}{r^2} + \frac{v_\varphi^2}{r^2} \right) dx dt' \le 3\|f\|_{L_{2,1}(\Omega^t)}^2 + 2\|v(0)\|_{L_2(\Omega)}^2 \equiv D_1^2.
\end{aligned}
\tag{46}
$$

Proof. Multiplying $(7)_1$ by v_r, $(7)_2$ by v_φ, $(7)_3$ by v_z, adding the results, and integrating over Ω yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \int\limits_\Omega (v_r^2 + v_\varphi^2 + v_z^2) dx + \nu \int\limits_\Omega (|\nabla v_r|^2 + |\nabla v_\varphi|^2 + |\nabla v_z|^2) dx \\
&\quad + \nu \int\limits_\Omega \left(\frac{v_r^2}{r^2} + \frac{v_\varphi^2}{r^2} \right) dx + \int\limits_\Omega (p_{,r} v_r + p_{,z} v_z) dx \\
&= \int\limits_\Omega (f_r v_r + f_\varphi v_\varphi + f_z v_z) dx.
\end{aligned}
\tag{47}
$$

The last term on the l.h.s. of (47) vanishes in virtue of the equation of continuity $(7)_4$ and boundary conditions.

Using the fact that $v^2 = v_r^2 + v_\varphi^2 + v_z^2$, (47) takes the form

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v\|_{L_2(\Omega)}^2 + \nu \int\limits_\Omega (|\nabla v_r|^2 + |\nabla v_\varphi|^2 + |\nabla v_z|^2) dx \\
&+ \nu \int\limits_\Omega \left(\frac{v_r^2}{r^2} + \frac{v_\varphi^2}{r^2} \right) dx = \int\limits_\Omega (f_r v_r + f_\varphi v_\varphi + f_z v_z) dx.
\end{aligned}
\tag{48}
$$

Applying the Hölder inequality to the r.h.s. of (48) yields

$$
\frac{d}{dt}\|v\|_{L_2(\Omega)} \le \|f\|_{L_2(\Omega)},
\tag{49}
$$

where $f^2 = f_r^2 + f_\varphi^2 + f_z^2$.

Integrating (49) with respect to time gives

$$
\|v\|_{L_2(\Omega)} \le \|f\|_{L_{2,1}(\Omega^t)} + \|v(0)\|_{L_2(\Omega)}.
\tag{50}
$$

Integrating (48) with respect to time, using the Hölder inequality in the r.h.s. of (48) and exploiting (50), we obtain

$$\begin{aligned}&\frac{1}{2}\|v(t)\|^2_{L_2(\Omega)}+\nu\int\limits_{\Omega^t}(|\nabla v_r|^2+|\nabla v_\varphi|^2+|\nabla v_z|^2)dxdt'\\&+\nu\int\limits_{\Omega^t}\left(\frac{v_r^2}{r^2}+\frac{v_\varphi^2}{r^2}\right)dxdt'\le\|f\|_{L_{2,1}(\Omega^t)}(\|f\|_{L_{2,1}(\Omega^t)}\\&+\|v(0)\|_{L_2(\Omega)})+\frac{1}{2}\|v(0)\|^2_{L_2(\Omega)}.\end{aligned}\tag{51}$$

The above inequality implies (46). This concludes the proof. □

Lemma 2. *Consider problem (12). Assume that $f_0\in L_{\infty,1}(\Omega^t)$ and $u(0)\in L_\infty(\Omega)$. Then,*

$$\|u(t)\|_{L_\infty(\Omega)}\le\|f_0\|_{L_{\infty,1}(\Omega^t)}+\|u(0)\|_{L_\infty(\Omega)}\equiv D_2.\tag{52}$$

Proof. Multiplying $(12)_1$ by $u|u|^{s-2}$, $s>2$, integrating over Ω and by parts, we obtain

$$\begin{aligned}&\frac{1}{s}\frac{d}{dt}\|u\|^s_{L_s(\Omega)}+\frac{4\nu(s-1)}{s^2}\|\nabla|u|^{s/2}\|^2_{L_2(\Omega)}+\frac{\nu}{s}\int\limits_\Omega(|u|^s)_{,r}drdz\\&=\int\limits_\Omega f_0u|u|^{s-2}dx.\end{aligned}\tag{53}$$

From [13] it follows that $u|_{r=0}=0$. Moreover, using boundary conditions, (53) implies

$$\frac{d}{dt}\|u\|_{L_s(\Omega)}\le\|f_0\|_{L_s(\Omega)}.\tag{54}$$

Integrating (54) with respect to time and passing with $s\to\infty$, we derive (52). This ends the proof. □

Lemma 3. *Let estimates (46) and (52) hold. Then,*

$$\|v_\varphi\|_{L_4(\Omega^t)}\le D_1^{1/2}D_2^{1/2}.\tag{55}$$

Proof. We have

$$\int\limits_{\Omega^t}|v_\varphi|^4dxdt'=\int\limits_{\Omega^t}r^2v_\varphi^2\frac{v_\varphi^2}{r^2}dxdt'\le\|rv_\varphi\|^2_{L_\infty(\Omega^t)}\int\limits_{\Omega^t}\frac{v_\varphi^2}{r^2}dxdt'\le D_2^2D_1^2.$$

This implies (55) and concludes the proof. □

Lemma 4. *Consider problem (22). Assume that $\omega_1\in L_{6/5}(\Omega)$, where $\Omega=(0,R)\times(-a,a)$. Then, there a weak solution to problem (22) exists such that $\psi_1\in H^1(\Omega)$ and the estimate*

$$\|\psi_1\|_{1,\Omega}\le c|\omega_1|_{6/5,\Omega}\tag{56}$$

holds.

Proof. Multiplying $(22)_1$ by ψ_1 and using the boundary conditions, we obtain

$$\|\psi_1\|^2_{1,\Omega}+\int\limits_{-a}^a\psi_1^2|_{r=0}dz=\int\limits_\Omega\omega_1\psi_1dx.$$

Applying the Hölder and Young inequalities to the r.h.s. implies (56). The Fredholm theorem gives existence. This ends the proof. □

Remark 2. *We have to emphasize that the weak solution ψ_1 of (22) does not vanish on the axis of symmetry. It also follows from [13].*

From Lemma 2.4 in [8], we also have

Lemma 5. *Let $f \in C^\infty((0,R)\times(-a,a))$, $f|_{r\geq R} = 0$. Let $1 < r \leq 3$, $0 \leq s \leq r$, $s \leq 2$, $q \in \left[r, \frac{r(3-s)}{3-r}\right]$. Then, a positive constant $c = c(s,r)$ exists such that*

$$\left(\int\limits_\Omega \frac{|f|^q}{r^s}dx\right)^{1/q} \leq c|f|_{r,\Omega}^{\frac{3-s}{q}-\frac{3}{r}+1}|\nabla f|_{r,\Omega}^{\frac{3}{r}-\frac{3-s}{q}}, \tag{57}$$

where f does not depend on φ.

Notation 1 (see [15])**.** *First, we introduce the Fourier transform. Let $f \in S(\mathbb{R})$, where $S(\mathbb{R})$ is the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}$. Then, the Fourier transform of f and its inverse are defined by*

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}} e^{-i\lambda\tau}f(\tau)d\tau, \quad \check{\hat{f}}_{(\tau)} = \frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}} e^{i\lambda\tau}\hat{f}(\lambda)d\lambda \tag{58}$$

and $\check{\hat{f}} = \hat{\check{f}} = f$.

By $H^k_\mu(\mathbb{R}_+)$, we denote a weighted space with the norm

$$\|u\|_{H^k_\mu(\mathbb{R}_+)} = \sum_{i=0}^k \int\limits_{\mathbb{R}_+} |\partial_r^i u|^2 r^{2(\mu-k+i)}rdr.$$

In view of transformation $\tau = -\ln r$, $r = e^{-\tau}$, $dr = -e^{-\tau}d\tau$, we have the equivalence

$$\sum_{i=0}^k \int\limits_{\mathbb{R}_+} |\partial_r^i u|^2 r^{2(\mu-k+i)}rdr \sim \sum_{i=0}^k \int\limits_{\mathbb{R}} |\partial_\tau^i u'|^2 e^{2h\tau}d\tau \tag{59}$$

which holds for $u'(\tau) = u'(-\ln r) = u(r)$, $h = k-1-\mu$.

In view of the Fourier transform (58) and the Parseval identity, we have

$$\int\limits_{-\infty+ih}^{+\infty+ih} \sum_{j=0}^k |\lambda|^{2j}|\hat{u}(\lambda)|^2 d\lambda = \int\limits_{\mathbb{R}} \sum_{j=0}^k |\partial_\tau^j u|^2 e^{2h\tau}d\tau. \tag{60}$$

4. Estimates for the Stream Function ψ_1

In this Section, we derive many estimates for $\psi_1 = \psi/r$, where ψ is the stream function, in terms of $\|\Gamma\|_{1,2,\Omega^t} + |\Gamma|_{2,\infty,\Omega^t}$ (recall that $\Gamma = \omega_1$). Function ψ_1 was introduced by Thomas Hou in [16]. Lemma 6 is proved by applying the energy-type method.

We have to emphasize that the proof of Lemma 6 is much more complicated than the proof of Lemma 6 in [14] because here we need to handle the boundary terms on S_2. Inequalities (85) and (93) are proved by applying the technique of weighted Sobolev spaces developed by Kondratiev (see [12]) to problem (61). Inequalities (85) and (93) hold for ψ_1 vanishing on the axis of symmetry. The inequalities are necessary in the proof of inequality (173). Hence, to prove the global estimate (24) we require that $\psi_1|_{r=0} = 0$.

Recall that ψ_1 is a solution to the problem

$$\begin{aligned} &-\psi_{1,rr}-\psi_{1,zz}-\frac{3}{r}\psi_{1,r}=\omega_1 \quad \text{in } \Omega=(0,R)\times(-a,a), \\ &\psi_1|_S=0. \end{aligned} \tag{61}$$

Lemma 6. *For sufficiently regular solutions to (61), the following estimates hold*

$$\begin{aligned} &\int_\Omega(\psi_{1,rr}^2+\psi_{1,rz}^2+\psi_{1,zz}^2)dx+\int_\Omega\frac{1}{r^2}\psi_{1,r}^2dx+\int_{-a}^a\psi_{1,z}^2|_{r=0}dz \\ &\quad+\int_{-a}^a\psi_{1,r}^2|_{r=R}dz\le c|\omega_1|_{2,\Omega}^2 \end{aligned} \tag{62}$$

and

$$\int_\Omega(\psi_{1,zzr}^2+\psi_{1,zzz}^2)dx+\int_{-a}^a\psi_{1,zz}^2|_{r=0}dz\le c|\omega_{1,z}|_{2,\Omega}^2 \tag{63}$$

and

$$\begin{aligned} &\int_\Omega(\psi_{1,rrz}^2+\psi_{1,rzz}^2+\psi_{1,zzz}^2)dx+\int_{-a}^a\psi_{1,zz}^2|_{r=0}dz \\ &\quad+\int_{-a}^a\psi_{1,rz}^2|_{r=R}dz\le c|\omega_{1,z}|_{2,\Omega}^2. \end{aligned} \tag{64}$$

Proof. First, we prove (62). Multiplying $(61)_1$ by $\psi_{1,zz}$ and integrating over Ω yields

$$-\int_\Omega\psi_{1,rr}\psi_{1,zz}dx-\int_\Omega\psi_{1,zz}^2dx-3\int_\Omega\frac{1}{r}\psi_{1,r}\psi_{1,zz}dx=\int_\Omega\omega_1\psi_{1,zz}dx. \tag{65}$$

Integrating by parts with respect to r in the first term implies

$$\begin{aligned} &-\int_\Omega(\psi_{1,r}\psi_{1,zz}r)_{,r}drdz+\int_\Omega\psi_{1,r}\psi_{1,zzr}dx+\int_\Omega\psi_{1,r}\psi_{1,zz}drdz \\ &\quad-\int_\Omega\psi_{1,zz}^2dx-3\int_\Omega\psi_{1,r}\psi_{1,zz}drdz=\int_\Omega\omega_1\psi_{1,zz}dx. \end{aligned}$$

Continuing, we obtain

$$\begin{aligned} &-\int_{-a}^a\psi_{1,r}\psi_{1,zz}r\Big|_{r=0}^{r=R}dz+\int_\Omega\psi_{1,r}\psi_{1,zzr}dx-\int_\Omega\psi_{1,zz}^2dx \\ &\quad-2\int_\Omega\psi_{1,r}\psi_{1,zz}drdz=\int_\Omega\omega_1\psi_{1,zz}dx. \end{aligned} \tag{66}$$

The first integral in (66) vanishes because $\psi_{1,r}r|_{r=0}=0$, $\psi_{1,zz}|_{r=R}=0$. Integrating by parts with respect to z in the last term on the l.h.s. of (66) and using the fact that $\psi_{1,r}$ on S_2 vanishes, we obtain

$$\int_\Omega\psi_{1,r}\psi_{1,zzr}dx-\int_\Omega\psi_{1,zz}^2dx+2\int_\Omega\psi_{1,rz}\psi_{1,z}drdz=\int_\Omega\omega_1\psi_{1,zz}dx. \tag{67}$$

Integrating by parts with respect to z in the first term in (67), we obtain

$$-\int_0^r \psi_{1,r}\psi_{1,zr}\Big|_{z=-a}^{z=a} rdr + \int_\Omega (\psi_{1,zr}^2 + \psi_{1,zz}^2)dx - \int_\Omega (\psi_{1,z}^2)_{,r}drdz = -\int_\Omega \omega_1\psi_{1,zz}dx, \tag{68}$$

where the first integral vanishes because $\psi_{1,r}|_{S_2} = 0$ and the last term on the l.h.s. equals

$$-\int_{-a}^a \psi_{1,z}^2\Big|_{r=0}^{r=R} dz = \int_{-a}^a \psi_{1,z}^2\Big|_{r=0} dz$$

because $\psi_{1,z}|_{r=R} = 0$. Using this in (68) and applying the Hölder and Young inequalities to the r.h.s. of (68) yields

$$\int_\Omega (\psi_{1,rz}^2 + \psi_{1,zz}^2)dx + \int_{-a}^a \psi_{1,z}^2|_{r=0}dz \le c|\omega_1|_{2,\Omega}^2. \tag{69}$$

Multiply $(61)_1$ by $\frac{1}{r}\psi_{1,r}$ and integrate over Ω. Then, we have

$$3\int_\Omega \left|\frac{1}{r}\psi_{1,r}\right|^2 dx = -\int_\Omega \psi_{1,rr}\frac{1}{r}\psi_{1,r}dx - \int_\Omega \psi_{1,zz}\frac{1}{r}\psi_{1,r}dx - \int_\Omega \omega_1\frac{1}{r}\psi_{1,r}dx. \tag{70}$$

The first term on the r.h.s. of (70) equals

$$-\frac{1}{2}\int_\Omega \partial_r\psi_{1,r}^2 drdz = -\frac{1}{2}\int_{-a}^a \psi_{1,r}^2\Big|_{r=0}^{r=R} dz = -\frac{1}{2}\int_{-a}^a \psi_{1,r}^2|_{r=R}dz,$$

because $\psi_{1,r}|_{r=0} = 0$ (see [13]). Applying the Hölder and Young inequalities to the last two terms on the r.h.s. of (70) implies

$$\int_\Omega \left|\frac{1}{r}\psi_{1,r}\right|^2 dx + \frac{1}{2}\int_{-a}^a \psi_{1,r}^2\Big|_{r=R} dz \le c(|\psi_{1,zz}|_{2,\Omega}^2 + |\omega_1|_{2,\Omega}^2). \tag{71}$$

Inequalities (69) and (71) imply the estimate

$$\begin{aligned} &\int_\Omega (\psi_{1,rz}^2 + \psi_{1,zz}^2)dx + \int_\Omega \left|\frac{1}{r}\psi_{1,r}\right|^2 dx + \int_{-a}^a \psi_{1,z}^2\Big|_{r=0} dz \\ &+ \int_{-a}^a \psi_{1,r}^2\Big|_{r=R} dz \le c|\omega_1|_{2,\Omega}^2. \end{aligned} \tag{72}$$

From $(61)_1$, we have

$$|\psi_{1,rr}|_{2,\Omega}^2 \le |\psi_{1,zz}|_{2,\Omega}^2 + 3\left|\frac{1}{r}\psi_{1,r}\right|_{2,\Omega}^2 + |\omega_1|_{2,\Omega}^2. \tag{73}$$

Inequalities (72) and (73) imply (62).

Now, we show (63). Differentiate $(61)_1$ with respect to z, multiply by $-\psi_{1,zzz}$, and integrate over Ω. Then, we obtain

$$\int_\Omega \psi_{1,rrz}\psi_{1,zzz}dx + \int_\Omega \psi_{1,zzz}^2 dx + 3\int_\Omega \frac{1}{r}\psi_{1,rz}\psi_{1,zzz}dx = -\int_\Omega \omega_{1,z}\psi_{1,zzz}dx. \tag{74}$$

Integrating by parts with respect to z yields

$$\begin{aligned}\int_\Omega \psi_{1,rrz}\psi_{1,zzz}dx &= \int_\Omega (\psi_{1,rrz}\psi_{1,zz})_{,z}dx - \int_\Omega \psi_{1,rrzz}\psi_{1,zz}dx \\ &= \int_0^t \psi_{1,rrz}\psi_{1,zz}\Big|_{S_2} rdr - \int_\Omega \psi_{1,rrzz}\psi_{1,zz}dx.\end{aligned} \tag{75}$$

Projecting (61) on S_2 yields that $-\psi_{1,zz} = \omega_1$. Since $\omega_1|_{S_2} = 0$, it follows that $\psi_{1,zz}|_{S_2} = 0$ so the first term on the r.h.s. vanishes. Integrating by parts with respect to r in the second integral in (75) gives

$$-\int_\Omega (\psi_{1,rzz}\psi_{1,zz}r)_{,r}drdz + \int_\Omega \psi_{1,rzz}^2 dx + \int_\Omega \psi_{1,rzz}\psi_{1,zz}drdz,$$

where the first integral vanishes because

$$\psi_{1,rzz}r|_{r=0} = 0, \quad \psi_{1,zz}|_{r=R} = 0.$$

In view of the above considerations, (74) takes the form

$$\begin{aligned}&\int_\Omega (\psi_{1,rzz}^2 + \psi_{1,zzz}^2)dx + \int_\Omega \psi_{1,rzz}\psi_{1,zz}drdz \\ &\quad + 3\int_\Omega \psi_{1,rz}\psi_{1,zzz}drdz = -\int_\Omega \omega_{1,z}\psi_{1,zzz}dx.\end{aligned} \tag{76}$$

By integrating by parts with respect to z in the last term on the l.h.s. of (76) and using the fact that $\psi_{1,zz} = 0$ on S_2, we obtain

$$\int_\Omega (\psi_{1,rzz}^2 + \psi_{1,zzz}^2)dx - \int_\Omega \partial_r\psi_{1,zz}^2 drdz = -\int_\Omega \omega_{1,z}\psi_{1,zzz}dx. \tag{77}$$

Applying the Hölder and Young inequalities to the r.h.s. of (77) yields

$$\int_\Omega (\psi_{1,rzz}^2 + \psi_{1,zzz}^2)dx + \int_{-a}^a \psi_{1,zz}^2\Big|_{r=0} dz \le c|\omega_{1,z}|_{2,\Omega}^2,$$

where we used that $\psi_{1,zz}|_{r=R} = 0$.

The above inequality implies (63).

Finally, we show (64). Differentiate $(61)_1$ with respect to z, multiply by $\psi_{1,rrz}$, and integrate over Ω. Then, we have

$$\begin{aligned}&-\int_\Omega \psi_{1,rrz}^2 dx - \int_\Omega \psi_{1,zzz}\psi_{1,rrz}dx - 3\int_\Omega \frac{1}{r}\psi_{1,rz}\psi_{1,rrz}dx \\ &= \int_\Omega \omega_{1,z}\psi_{1,rrz}dx.\end{aligned} \tag{78}$$

Integrating by parts with respect to z in the second term in (78) and using the fact that $\psi_{1,zz}|_{S_2} = 0$ implies

$$-\int_\Omega \psi_{1,zzz}\psi_{1,rrz}dx = \int_\Omega \psi_{1,zz}\psi_{1,rrzz}dx = \int_\Omega (\psi_{1,zz}\psi_{1,rzz}r)_r drdz \\ -\int_\Omega \psi_{1,rzz}^2 dx - \int_\Omega \psi_{1,zz}\psi_{1,rzz}drdz,$$

where the first term vanishes because

$$\psi_{1,rzz}r|_{r=0} = 0, \quad \psi_{1,zz}|_{r=R} = 0.$$

Then, (78) takes the form

$$\begin{aligned}&\int_\Omega (\psi_{1,rrz}^2 + \psi_{1,rzz}^2)dx + \int_\Omega \psi_{1,zz}\psi_{1,rzz}drdz \\ &\quad + 3\int_\Omega \psi_{1,rz}\psi_{1,rrz}drdz = -\int_\Omega \omega_{1,z}\psi_{1,rrz}dx.\end{aligned} \tag{79}$$

The second term in (79) equals

$$\frac{1}{2}\int_{-a}^{a} \psi_{1,zz}^2\Big|_{r=0}^{r=R} dz = -\frac{1}{2}\int_{-a}^{a} \psi_{1,zz}^2\Big|_{r=0} dz$$

because $\psi_{1,zz}|_{r=R} = 0$, and the last term on the l.h.s. of (79) has the form

$$\frac{3}{2}\int_\Omega \partial_r\psi_{1,rz}^2 drdz = \frac{3}{2}\int_{-a}^{a} \psi_{1,rz}^2\Big|_{r=0}^{r=R} dz = \frac{3}{2}\int_{-a}^{a} \psi_{1,rz}^2\Big|_{r=R} dz$$

because $\psi_{1,rz}|_{r=0} = 0$.

Using the above expressions in (79) implies the equality

$$\begin{aligned}&\int_\Omega (\psi_{1,rrz}^2 + \psi_{1,rzz}^2)dx - \frac{1}{2}\int_{-a}^{a} \psi_{1,zz}^2\Big|_{r=0} dz + \frac{3}{2}\int_{-a}^{a} \psi_{1,rz}^2\Big|_{r=R} dz \\ &= -\int_\Omega \omega_{1,z}\psi_{1,rrz}dx.\end{aligned} \tag{80}$$

Applying the Hölder and Young inequalities in the r.h.s. of (80) gives

$$\begin{aligned}&\int_\Omega (\psi_{1,rrz}^2 + \psi_{1,rzz}^2)dx - \frac{1}{2}\int_{-a}^{a} \psi_{1,zz}^2\Big|_{r=0} dz \\ &\quad + \frac{3}{2}\int_{-a}^{a} \psi_{1,rz}^2\Big|_{r=R} dz \le c|\omega_{1,z}|_{2,\Omega}^2.\end{aligned} \tag{81}$$

Inequalities (81) and (63) imply (64). This ends the proof. □

Lemma 7. *For sufficiently regular solutions to (61) the following inequality:*

$$\left|\frac{1}{r}\psi_{1,rz}\right|_{2,\Omega} \le c|\omega_{1,z}|_{2,\Omega} \tag{82}$$

holds.

Proof. Differentiating (61) with respect to z implies

$$-\psi_{1,rrz} - \psi_{1,zzz} - \frac{3}{r}\psi_{1,rz} = \omega_{1,z}. \tag{83}$$

From (83), we have

$$\left|\frac{1}{r}\psi_{1,rz}\right|_{2,\Omega} \le c(|\psi_{1,rrz}|_{2,\Omega} + |\psi_{1,zzz}|_{2,\Omega} + |\omega_{1,z}|_{2,\Omega}). \tag{84}$$

Using (64) in (84) yields (82). This concludes the proof. □

Now, we estimate $|\frac{\psi_{1,zz}}{r}|_{2,\Omega}$.

Lemma 8. *Let ψ_1 be such a weak solution to problem (61) that it vanishes on the axis of symmetry. Then, such sufficiently regular solutions to problem (61) satisfy the estimate*

$$\int\limits_\Omega \frac{\psi_{1,zz}^2}{r^2}dx + \int\limits_\Omega \left(\psi_{1,zrr}^2 + \frac{\psi_{1,zr}^2}{r^2} + \frac{\psi_{1,z}^2}{r^4}\right)dx \le c|\omega_{1,z}|_{2,\Omega}^2. \tag{85}$$

Proof. Differentiating (61) with respect to z yields

$$\begin{gathered} -\Delta\psi_{1,z} - \frac{3}{r}\psi_{1,zr} = \omega_{1,z}, \\ \psi_{1,z}|_{S_1} = 0, \quad \psi_{1,zz}|_{S_2} = 0. \end{gathered} \tag{86}$$

Applying Lemma 17 (see also Lemma 3.1 from [15]) to problem (86) gives

$$\int\limits_\Omega \left(\psi_{1,zrr}^2 + \frac{\psi_{1,zr}^2}{r^2} + \frac{\psi_{1,z}^2}{r^4}\right)dx \le c(|\omega_{1,z}|_{2,\Omega}^2 + |\psi_{1,zzz}|_{2,\Omega}^2) \le c|\omega_{1,z}|_{2,\Omega}^2, \tag{87}$$

where (63) is used in the last inequality.

To examine the solutions to (86), we use the notation

$$u = \psi_{1,z}. \tag{88}$$

Then, (86) takes the form

$$\begin{gathered} -\Delta u - \frac{2}{r}u_{,r} = \omega_{1,z}, \\ u|_{S_1} = 0, \quad u_{,z}|_{S_2} = 0. \end{gathered} \tag{89}$$

Multiply $(88)_1$ by ur^{-2}, integrate over Ω, and express the Laplacian operator in cylindrical coordinates. Then, we have

$$-\int\limits_\Omega \left(u_{,rr} + \frac{1}{r}u_{,r} + u_{,zz}\right)ur^{-2}dx - 2\int\limits_\Omega \frac{1}{r}u_{,r}ur^{-2}dx = \int\limits_\Omega \omega_{1,z}ur^{-2}dx. \tag{90}$$

By integrating by parts with respect to z in the third term under the first integral and using the fact that $\psi_{1,zz}|_{S_2} = 0$, we obtain

$$\int\limits_\Omega \frac{u_{,z}^2}{r^2}dx = \int\limits_\Omega \left(u_{,rr} + \frac{3}{r}u_{,r}\right)ur^{-2}dx + \int\limits_\Omega \omega_{1,z}ur^{-2}dx. \tag{91}$$

Applying the Hölder and Young inequalities to the r.h.s. integrals, using the fact that $u = \psi_{1,z}$ and (87), we derive

$$\int_\Omega \frac{\psi_{1,zz}^2}{r^2} dx \le c \int_\Omega \left(\psi_{1,zrr}^2 + \frac{\psi_{1,zr}^2}{r^2} + \frac{\psi_{1,z}^2}{r^4} \right) dx + c|\omega_{1,z}|_{2,\Omega}^2. \tag{92}$$

Using (87) in (92) implies (85). This concludes the proof. □

Remark 3. *Lemma 8 is necessary in the proof of global regular axially symmetric solutions to problem (6). However, it imposes strong restrictions on solutions to (6) because the condition $\psi_1|_{r=0} = 0$ implies that $v_z|_{r=0} = 0$. We do not know how to omit the restriction in the presented proof in this paper.*

Lemma 9. *Let $\mu > 0$ and $\omega_1 \in H_\mu^1(\Omega)$. Then, for sufficiently smooth solutions to (61) the following estimate is valid:*

$$\int_\Omega \left(\psi_{1,rrr}^2 + \frac{\psi_{1,rr}^2}{r^2} + \frac{\psi_{1,r}^2}{r^4} \right) r^{2\mu} dx \le cR^{2\mu} \|\omega_1\|_{1,\Omega}^2. \tag{93}$$

Proof. To prove the lemma, we introduce a partition of unity $\{\zeta^{(i)}(r)\}_{i=1,2}$ such that

$$\sum_{i=1}^2 \zeta^{(i)}(r) = 1$$

and

$$\zeta^{(1)}(r) = \begin{cases} 1 & r \le r_0, \\ 0 & r \ge r_0 + \lambda, \end{cases}$$
$$\zeta^{(2)}(r) = \begin{cases} 0 & r \le r_0, \\ 1 & r \ge r_0 + \lambda, \end{cases}$$

where $r_0 < R$ and $\zeta^{(i)}(r)$, $i = 1,2$, are smooth functions.

Introduce the notation

$$\psi_1^{(i)} = \psi_1 \zeta^{(i)}, \quad \omega_1^{(i)} = \omega_1 \zeta^{(i)}, \quad i = 1,2. \tag{94}$$

Then, functions (94) satisfy the equations

$$\begin{aligned} &-\psi_{1,rr}^{(i)} - \psi_{1,zz}^{(i)} - \frac{3}{r}\psi_{1,r}^{(i)} = -2\psi_{1,r}\dot\zeta^{(i)} - \psi_1\ddot\zeta^{(i)} - \frac{3}{r}\psi_1\dot\zeta^{(i)} \\ &+ \omega_1^{(i)} \equiv g^{(i)}, \quad i = 1,2, \end{aligned} \tag{95}$$

where dot denotes the derivative with respect to r.

First, we consider the case $i = 1$. Differentiating (95) for $i = 1$ with respect to r yields

$$-\psi_{1,rrr}^{(1)} - \psi_{1,rzz}^{(1)} - \frac{3}{r}\psi_{1,rr}^{(1)} + \frac{3}{r^2}\psi_{1,r}^{(1)} = g_{,r}^{(1)}. \tag{96}$$

Introduce the notation

$$v = \psi_{1,r}^{(1)}, \quad f = g_{,r}^{(1)}. \tag{97}$$

Then, (96) takes the form

$$
\begin{aligned}
& -v_{,rr}-v_{,zz}-\frac{3}{r}v_{,r}+\frac{3}{r^2}v=f \quad \text{in } \Omega_{r_0+\lambda}, \\
& v|_{r=r_0}=0, \\
& v|_{S_2}=0,
\end{aligned} \tag{98}
$$

where $\Omega_{r_0+\lambda}=\{x\in\Omega\colon r\in(0,r_0+\lambda),z\in(-a,a)\}$ and $r_0+\lambda<R$.

Multiplying $(98)_1$ by r^2 yields

$$-r^2v_{,rr}-3rv_{,r}+3v=r^2(f+v_{,zz})\equiv g(r,z)$$

or equivalently

$$-r\partial_r(r\partial_r v)-2r\partial_r v+3v=g(r,z). \tag{99}$$

Introduce the new variable

$$\tau=-\ln r, \quad r=e^{-\tau}.$$

Since $r\partial_r=-\partial_\tau$, Equation (99) takes the form

$$-\partial_\tau^2 v+2\partial_\tau v+3v=g(e^{-\tau},z)\equiv g'(\tau,z). \tag{100}$$

Applying the Fourier transform (58) to (100) gives

$$\lambda^2\hat{v}+2i\lambda\hat{v}+3\hat{v}=\hat{g}'. \tag{101}$$

Looking for solutions to the algebraic equation

$$\lambda^2+2i\lambda+3=0$$

we see that it has two solutions

$$\lambda_1=-3i, \ \ \lambda_2=i.$$

For $\lambda\notin\{-3i,i\}$, we can write solutions to (101) in the form

$$\hat{v}=\frac{1}{\lambda^2+2i\lambda+3}\hat{g}'\equiv R(\lambda)\hat{g}'. \tag{102}$$

Since $R(\lambda)$ does not have poles on the line $\operatorname{Im}\lambda=1-\mu=h$, $\mu\in(0,1)$, we can use Lemma 3.1 from [15]. Then, we obtain

$$
\begin{aligned}
& \int\limits_{-\infty+ih}^{\infty+ih}\sum_{j=0}^{2}|\lambda|^{2(2-j)}|\hat{v}|^2d\lambda\le c\int\limits_{-\infty+ih}^{+\infty+ih}\sum_{j=0}^{2}|\lambda|^{2(2-j)}|R(\lambda)\hat{g}'|^2d\lambda \\
& \le c\int\limits_{-\infty+ih}^{+\infty+ih}|\hat{g}'|^2d\lambda.
\end{aligned} \tag{103}
$$

By the Parseval identity, inequality (103) becomes

$$\int\limits_{\mathbb{R}}\sum_{j=0}^{2}|\partial_\tau^j v|^2e^{2h\tau}d\tau\le c\int\limits_{\mathbb{R}}|g'|^2e^{2h\tau}d\tau.$$

Passing to variable r yields

$$\sum_{j=0}^{2}\int\limits_{\mathbb{R}_+}|\partial_r^j v|^2r^{2(\mu+j-2)}rdr\le c\int\limits_{\mathbb{R}_+}|g|^2r^{2(\mu-2)}rdr.$$

Using the fact that $g = r^2(f + v_{,zz})$, we obtain

$$\sum_{j=0}^{2} \int_{\mathbb{R}_+} |\partial_r^j v|^2 r^{2(\mu+j-2)} r dr \le c \int_{\mathbb{R}_+} |f + v_{,zz}|^2 r^{2\mu} r dr. \tag{104}$$

Recalling notation (97), we derive from (104) the inequality

$$\sum_{j=0}^{2} \int_{\Omega} |\partial_r^j \psi_{1,r}^{(1)}|^2 r^{2(\mu+j-2)} dx \le c \int_{\Omega} |g_{,r}^{(1)}|^2 r^{2\mu} dx + c \int_{\Omega} |\psi_{1,rzz}|^2 r^{2\mu} dx. \tag{105}$$

In view of (63),

$$|\psi_{1,rzz}|_{2,\Omega} \le c|\omega_{1,z}|_{2,\Omega}. \tag{106}$$

The first term on the r.h.s. of (105) can be estimated by

$$|g_{,r}^{(1)}|_{2,\mu,\Omega} \le c(|\psi_{1,rr}|_{2,\Omega} + |\psi_{1,r}|_{2,\Omega} + |\psi_1|_{2,\Omega} + |\omega_{1,r}|_{2,\Omega} + |\omega_1|_{2,\Omega}). \tag{107}$$

Lemma 6 and inequalities (105)–(107) imply

$$\begin{aligned} &\int_{\Omega} \left(|\psi_{1,rrr}^{(1)}|^2 + \frac{|\psi_{,rr}^{(1)}|^2}{r^2} + \frac{|\psi_{,r}^{(1)}|^2}{r^4} \right) r^{2\mu} r dr dz \\ &\quad + \int_{\Omega} |\psi_{1,rzz}|^2 dx \le c(|\omega_{1,r}|_{2,\Omega}^2 + |\omega_{1,z}|_{2,\Omega}^2 + |\omega_1|_{2,\Omega}^2). \end{aligned} \tag{108}$$

Function $\psi_1^{(2)}$ is a solution to the problem

$$\begin{aligned} & -\Delta\psi_1^{(2)} = -2\psi_{1,r}\dot{\zeta}^{(2)} - \psi_1\ddot{\zeta}^{(2)} + \frac{2}{r}\psi_{1,r}^{(2)} \\ & \quad -\frac{3}{r}\psi_1\dot{\zeta}^{(2)} + \omega_1^{(2)} && \text{in } \bar{\Omega}_{r_0}, \\ & \psi_1^{(2)}|_{r=R} = 0, \\ & \psi_1^{(2)} = 0 && \text{for } r \le r_0, \\ & \psi_1^{(2)} = 0 && \text{on } S_2, \end{aligned} \tag{109}$$

where $\bar{\Omega}_{r_0} = \{x \in \mathbb{R}^3 \colon r_0 \le r \le R, z \in (-a, a)\}$ and dot denotes the derivative with respect to r.

For solutions to (109), the following estimate holds:

$$\|\psi_1^{(2)}\|_{3,\Omega} \le c(\|\psi_{1,r}\|_{1,\Omega} + \|\psi_1\|_{1,\Omega} + \|\omega_1^{(2)}\|_{1,\Omega}) \le c\|\omega_1\|_{1,\Omega}. \tag{110}$$

From (56), (108), and (110), inequality (93) follows. This ends the proof. □

5. Estimates for Φ and Γ

Let $\Omega = \{(r,z) \colon r \in (0,R), z \in (-a,a)\}$. Let $\Phi = \omega_r/r$, $\Gamma = \omega_\varphi/r$, and Φ, Γ be solutions to problems (17)–(20).

Lemma 10. *Assume that $\Phi(0), \Gamma(0) \in L_2(\Omega)$, $\bar{F}_r, \bar{F}_\varphi \in L_2(0,t; L_{6/5}(\Omega))$. Let D_2 be defined by (52), and let*

$$I_3 = \int_{\Omega^t} \left| \frac{v_\varphi}{r} \Phi\Gamma \right| dx dt' < \infty.$$

Then,

$$
\begin{aligned}
&|\Phi(t)|_{2,\Omega}^2 + |\Gamma(t)|_{2,\Omega}^2 + \nu(\|\Phi\|_{1,2,\Omega^t}^2 + \|\Gamma\|_{1,2,\Omega^t}^2) \\
&\le \phi(D_2)\bigg|\int_{\Omega^t} \frac{v_\varphi}{r}\Phi\Gamma dxdt'\bigg| + \phi(D_2)(|\bar{F}_r|_{6/5,2,\Omega^t}^2 \\
&\quad + |\bar{F}_\varphi|_{6/5,2,\Omega^t}^2) + |\Phi(0)|_{2,\Omega}^2 + |\Gamma(0)|_{2,\Omega}^2 \\
&\equiv \phi(D_2)I_3 + D_8.
\end{aligned}
\tag{111}
$$

Proof. Multiplying (17) by Φ and integrating over Ω yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\Phi|_{2,\Omega}^2 + |\nabla\Phi|_{2,\Omega}^2 - \int_{-a}^{a} \Phi\bigg|_{r=0}^{r=R} dz \\
&= \int_\Omega (\omega_r\partial_r + \omega_z\partial_z)\frac{v_r}{r}\Phi dx + \int_\Omega \bar{F}_r\Phi dx,
\end{aligned}
\tag{112}
$$

where we used that $(6)_3$, $(13)_1$ implies that $\Phi|_S = 0$.

To derive the second term on the l.h.s. of (112), we consider (17) in

$$\bar{\Omega} = \{x \in \mathbb{R}^3 \colon r < R, z \in (-a,a), \varphi \in (0,2\pi)\}.$$

Then, by the Green theorem and boundary conditions we obtain the second term on the l.h.s. of (112) on $\bar{\Omega}$. Using the fact that all quantities in (112) do not depend on φ, we can drop integration with respect to φ and obtain (112).

Considering the first term on the r.h.s. of (112), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\Phi|_{2,\Omega}^2 + |\nabla\Phi|_{2,\Omega}^2 \le \int_\Omega (\omega_r\partial_r + \omega_z\partial_z)\frac{v_r}{r}\Phi dx + \int_\Omega \bar{F}_r\Phi dx \\
&\le \int_\Omega \bigg(-v_{\varphi,z}\partial_r\frac{v_r}{r} + \frac{\partial_r(rv_\varphi)}{r}\partial_z\frac{v_r}{r}\bigg)\Phi rdrdz + \int_\Omega \bar{F}_r\Phi dx \\
&= -\int_0^R v_\varphi\partial_r\frac{v_r}{r}\Phi\bigg|_{S_2} rdr + \int_\Omega v_\varphi\bigg(\bigg(\partial_z\partial_r\frac{v_r}{r}\bigg)\Phi + \partial_r\frac{v_r}{r}\partial_z\Phi\bigg)dx \\
&\quad + \int_\Omega \partial_r\bigg(rv_\varphi\partial_z\frac{v_r}{r}\Phi\bigg)drdz - \int_\Omega v_\varphi\bigg(\bigg(\partial_z\partial_r\frac{v_r}{r}\bigg)\Phi + \partial_z\frac{v_r}{r}\partial_r\Phi\bigg)dx \\
&\quad + \int_\Omega \bar{F}_r\Phi dx = -\int_0^R v_\varphi\partial_r\frac{v_r}{r}\Phi\bigg|_{S_2} rdr + \int_{-a}^{a} rv_\varphi\partial_z\frac{v_r}{r}\Phi\bigg|_{r=0}^{r=R} dz \\
&\quad + \int_\Omega v_\varphi\bigg(\partial_r\frac{v_r}{r}\partial_z\Phi - \partial_z\frac{v_r}{r}\partial_r\Phi\bigg)dx + \int_\Omega \bar{F}_r\Phi dx \\
&\equiv -\int_0^R v_\varphi\partial_r\frac{v_r}{r}\Phi\bigg|_{S_2} rdr + \int_{-a}^{a} rv_\varphi\partial_z\frac{v_r}{r}\Phi\bigg|_{r=0}^{r=R} dz + I + \int_\Omega \bar{F}_r\Phi dx,
\end{aligned}
\tag{113}
$$

where the boundary terms on S_1 and S_2 vanish because $\Phi|_{S_2} = 0$, $v_\varphi|_{r=R} = 0$, $v_r|_{r=R} = 0$, $\Phi|_{r=R} = 0$ and

$$\int_{-a}^{a} rv_\varphi\partial_z\frac{v_r}{r}\Phi\bigg|_{r=0} dz = 0$$

because [13] implies the following expansions near the axis of symmetry

$$v_\varphi = a_1(z,t)r + a_2(z,t)r^3 + \cdots,$$
$$v_r = \bar{a}_1(z,t)r + \bar{a}_2(z,t)r^3 + \cdots$$

and $\Phi = -\frac{v_{\varphi,z}}{r}$.

Finally, $I \le I_1 + I_2$, where

$$\begin{aligned} I_1 &\le \int_\Omega \left| v_\varphi \partial_r \frac{v_r}{r} \Phi_{,z} \right| dx, \\ I_2 &\le \int_\Omega \left| v_\varphi \partial_z \frac{v_r}{r} \Phi_{,r} \right| dx. \end{aligned} \tag{114}$$

Now, we estimate I_1 and I_2. Recall that $\frac{v_r}{r} = -\psi_{1,z}$. Then,

$$\begin{aligned} I_1 &\le \int_\Omega |v_\varphi \psi_{1,rz} \Phi_{,z}| dx = \int_\Omega \left| r v_\varphi \frac{\psi_{1,rz}}{r} \Phi_{,z} \right| dx \\ &\le |r v_\varphi|_{\infty,\Omega} \left| \frac{\psi_{1,rz}}{r} \right|_{2,\Omega} |\Phi_{,z}|_{2,\Omega} \equiv I_1^1. \end{aligned}$$

From (52) and (82), we have (recall that $\Gamma = \omega_1$)

$$I_1^1 \le c D_2 |\Gamma_{,z}|_{2,\Omega} |\Phi_{,z}|_{2,\Omega}. \tag{115}$$

Similarly, we calculate

$$\begin{aligned} I_2 &\le \int_\Omega |v_\varphi \psi_{1,zz} \Phi_{,r}| dx \le |r v_\varphi|_{\infty,\Omega} \left| \frac{\psi_{1,zz}}{r} \right|_{2,\Omega} |\Phi_{,r}|_{2,\Omega} \\ &\le c D_2 |\Gamma_{,z}|_{2,\Omega} |\Phi_{,r}|_{2,\Omega}, \end{aligned} \tag{116}$$

where (85) is used.

Finally, the last term on the r.h.s. of (113) is bounded by

$$\varepsilon |\Phi|_{6,\Omega}^2 + c(1/\varepsilon) |\bar{F}_r|_{6/5,\Omega}^2. \tag{117}$$

Using estimates (115)–(117) in (113), assuming that ε is sufficiently small and applying the Poincaré inequality we obtain

$$\frac{d}{dt} |\Phi|_{2,\Omega}^2 + \|\Phi\|_{1,\Omega}^2 \le c D_2 |\Gamma_{,z}|_{2,\Omega} |\nabla \Phi|_{2,\Omega} + c |\bar{F}_r|_{6/5,\Omega}^2. \tag{118}$$

Multiplying (18) by Γ, integrating over Ω, and using the boundary conditions and explanation about applying the Green theorem below (112), we obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} |\Gamma|_{2,\Omega}^2 + |\nabla \Gamma|_{2,\Omega}^2 - \int_{-a}^{a} \Gamma^2 \bigg|_{r=0}^{r=R} dz \\ &\le 2 \left| \int_\Omega \frac{v_\varphi}{r} \Phi \Gamma dx \right| + \int_\Omega \bar{F}_\varphi \Gamma dx. \end{aligned} \tag{119}$$

Using the fact that $\Gamma|_{r=R} = 0$, by applying the Hölder and Young inequalities to the last term on the r.h.s. of (119) and using the Poincaré inequality we derive

$$\frac{d}{dt}|\Gamma|_{2,\Omega}^2 + \|\Gamma\|_{1,\Omega}^2 \le 2\int_\Omega \frac{v_\varphi}{r}\Phi\Gamma dx + c|\bar{F}_\varphi|_{6/5,\Omega}^2. \tag{120}$$

From (118) and (120), we have

$$\begin{aligned}&\frac{d}{dt}(|\Phi|_{2,\Omega}^2 + |\Gamma|_{2,\Omega}^2) + \|\Phi\|_{1,\Omega}^2 + \|\Gamma\|_{1,\Omega}^2 \le \phi(D_2)\bigg|\int_\Omega \frac{v_\varphi}{r}\Phi\Gamma dxdt'\bigg| \\ &+ \phi(D_2)(|\bar{F}_r|_{6/5,\Omega}^2 + |\bar{F}_\varphi|_{6/5,\Omega}^2),\end{aligned} \tag{121}$$

where ϕ is an increasing positive function. Integrating (121) with respect to time yields (111). This ends the proof. □

Lemma 11. *Let the assumptions of Lemma 16 hold.*

Let $v_\varphi \in L_\infty(0,t;L_d(\Omega))$, $d > 3$. Let $\theta = \left(1 - \frac{3}{d}\right)\varepsilon_1 - \frac{3}{d}\varepsilon_2 > 0$, $\varepsilon = \varepsilon_1 + \varepsilon_2$. Let $\varepsilon_0 > 0$ be arbitrary small.

Then,

$$\begin{aligned}I_3 \le\ & c|v_\varphi|_{d,\infty,\Omega^t}^\varepsilon [c_1(1 + |v_\varphi|_{\infty,\Omega^t}^{\frac{1}{2}\theta\varepsilon_0})\|\Gamma\|_{1,2,\Omega^t}^{\frac{1}{2}\theta} \\ &+ c_2]|\nabla\Phi|_{2,\Omega^t}^{1-\theta}|\nabla\Gamma|_{2,\Omega^t},\end{aligned} \tag{122}$$

where c_1, c_2 depending on D_5, D_6, D_7 are introduced in L_1^4 below.

Proof. We examine

$$\begin{aligned}I_3 &= \int_{\Omega^t} \left|rv_\varphi \frac{\Phi}{r}\frac{\Gamma}{r}\right| dxdt' \\ &\le \int_{\Omega^t} |rv_\varphi|^{1-\varepsilon}|v_\varphi|^\varepsilon \left|\frac{\Phi}{r^{1-\varepsilon_1}}\right|\left|\frac{\Gamma}{r^{1-\varepsilon_2}}\right| dxdt' = I_3^1,\end{aligned}$$

where $\varepsilon = \varepsilon_1 + \varepsilon_2$ and ε_i, $i = 1,2$, are positive numbers.

Using (52) and applying the Hölder inequality in I_3^1 yields

$$\begin{aligned}I_3^1 &\le D_2^{1-\varepsilon}\left(\int_{\Omega^t} |v_\varphi|^{2\varepsilon}\left|\frac{\Phi}{r^{1-\varepsilon_1}}\right|^2 dxdt'\right)^{1/2}\left|\frac{\Gamma}{r^{1-\varepsilon_2}}\right|_{2,\Omega^t} \\ &\equiv D_2^{1-\varepsilon} L|\Gamma/r^{1-\varepsilon_2}|_{2,\Omega^t} \equiv I_3^2.\end{aligned}$$

By the Hardy inequality, we obtain

$$\left\|\frac{\Gamma}{r}\right\|_{L_{2,\varepsilon_2}(\Omega^t)} \le c\|\nabla\Gamma\|_{L_{2,\varepsilon_2}(\Omega^t)} \le cR^{\varepsilon_2}|\nabla\Gamma|_{2,\Omega^t}. \tag{123}$$

Now, we estimate L,

$$\begin{aligned}L &= \left(\int_0^t\int_\Omega |v_\varphi|^{2\varepsilon}\left|\frac{\Phi}{r^{1-\varepsilon_1}}\right|^2 dxdt'\right)^{1/2} \\ &\le \left[\int_0^t |v_\varphi|_{2\varepsilon\sigma,\Omega}^{2\varepsilon}\left(\int_\Omega \left|\frac{\Phi}{r^{1-\varepsilon_1}}\right|^q dx\right)^{2/q} dt'\right]^{1/2} \equiv L_1,\end{aligned}$$

where $1/\sigma + 1/\sigma' = 1$, $q = 2\sigma'$. Let $d = 2\varepsilon\sigma$. Then,

$$\sigma' = \frac{d}{d-2\varepsilon} \quad \text{so} \quad q = \frac{2d}{d-2\varepsilon}.$$

Continuing,

$$L_1 \le \sup_t |v_\varphi|^{\varepsilon}_{d,\Omega} \bigg(\int_0^t \bigg| \frac{\Phi}{r^{1-\varepsilon_1}} \bigg|^2_{q,\Omega} dt' \bigg)^{1/2} \equiv L_1^1 L_1^2.$$

Now, we estimate the second factor L_1^2.

For this purpose, we use Lemma 5 for $r = 2$. Let $\frac{s}{q} = 1 - \varepsilon_1$. Then $q \in [2, 2(3-s)]$. Since $s = (1-\varepsilon_1)q$ we have the restriction $2 \le q \le 6 - 2s = 6 - 2(1-\varepsilon_1)q$. Then,

$$2 \le q \le \frac{6}{3 - 2\varepsilon_1} \tag{124}$$

and $\frac{6}{3-2\varepsilon_1} > 2$ for any $\varepsilon_1 \in (0,1)$.

Hence, Lemma 5 implies

$$\begin{aligned} L_1^2 &= \bigg(\int_0^t \bigg| \frac{\Phi}{r^{1-\varepsilon_1}} \bigg|^2_{q,\Omega} dt' \bigg)^{1/2} \\ &\le c \bigg(\int_0^t |\Phi|^{2(\frac{3-s}{q} - \frac{1}{2})}_{2,\Omega} |\nabla\Phi|^{2(\frac{3}{2} - \frac{3-s}{q})}_{2,\Omega} dt' \bigg)^{1/2} \\ &\le c |\Phi|^{\frac{3-s}{q} - \frac{1}{2}}_{2,\Omega^t} |\nabla\Phi|^{\frac{3}{2} - \frac{3-s}{q}}_{2,\Omega^t} \equiv L_1^3, \end{aligned}$$

where we used that for $\theta = \frac{3-s}{q} - \frac{1}{2}$, $1 - \theta = \frac{3}{2} - \frac{3-s}{q}$ so the Hölder inequality can be applied.

Using (173) in L_1^3, we have

$$\begin{aligned} L_1^3 &\le c(D_5^{\frac{1}{2}\theta} |\nabla\Gamma|^{\frac{1}{2}\theta}_{2,\Omega^t} + D_6^{\frac{1}{2}\theta} |v_\varphi|^{\frac{1}{2}\theta\varepsilon_0}_{\infty,\Omega^t} \|\Gamma\|^{\frac{1}{2}\theta}_{1,2,\Omega^t} + D_7^{\frac{1}{2}\theta}) \cdot |\nabla\Phi|^{1-\theta}_{2,\Omega^t} \\ &\equiv [c_1(1 + |v_\varphi|^{\frac{1}{2}\theta\varepsilon_0}_{\infty,\Omega^t}) \|\Gamma\|^{\frac{1}{2}\theta}_{1,2,\Omega^t} + c_2] |\nabla\Phi|^{1-\theta}_{2,\Omega^t} \equiv L_1^4, \end{aligned}$$

where c_1, c_2 depend on D_5, D_6, D_7.

To justify the above inequality, we have to know that the following inequalities hold:

$$\theta = \frac{3-s}{q} - \frac{1}{2} > 0 \tag{125}$$

and

$$1 - \theta = \frac{3}{2} - \frac{3-s}{q} > 0. \tag{126}$$

Consider (125). Using the form of q and $\frac{s}{q}$ we have

$$\frac{3}{q} - \frac{s}{q} - \frac{1}{2} > 0 \quad \text{so} \quad \frac{3(d-2\varepsilon)}{2d} - (1-\varepsilon_1) - \frac{1}{2} > 0.$$

Hence,

$$\frac{3}{2} - \frac{3}{d}\varepsilon - 1 + \varepsilon_1 - \frac{1}{2} > 0 \quad \text{so} \quad \varepsilon_1 - \frac{3}{d}(\varepsilon_1 + \varepsilon_2) > 0.$$

Therefore, the following inequality

$$\left(1-\frac{3}{d}\right)\varepsilon_1-\frac{3}{d}\varepsilon_2>0 \tag{127}$$

holds for $d>3$ and ε_2 small. Moreover, (127) implies

$$\varepsilon_1>\frac{3}{d}\frac{d}{d-3}\varepsilon_2=\frac{3}{d-3}\varepsilon_2. \tag{128}$$

To exmine (126), we calculate

$$\frac{3}{2}-\frac{3(d-2\varepsilon)}{2d}+1-\varepsilon_1=1+\frac{3}{d}\varepsilon-\varepsilon_1=1-\left(1-\frac{3}{d}\right)\varepsilon_1+\frac{3}{d}\varepsilon_2. \tag{129}$$

Since (129) must be positive, we have the restriction

$$1+\frac{3}{d}\varepsilon_2>\left(1-\frac{3}{d}\right)\varepsilon_1. \tag{130}$$

Using (128) in (130) implies

$$1+\frac{3}{d}\varepsilon_2>\frac{3}{d}\varepsilon_2$$

so there is no contradiction.

Hence, we have

$$\begin{aligned}\theta&=\left(1-\frac{3}{d}\right)\varepsilon_1-\frac{3}{d}\varepsilon_2,\\ 1-\theta&=1-\left(1-\frac{3}{d}\right)\varepsilon_1+\frac{3}{d}\varepsilon_2,\end{aligned} \tag{131}$$

where $d>3$.

Finally,

$$I_3\le c|v_\varphi|^{\varepsilon}_{d,\infty,\Omega^t}[c_1(1+|v_\varphi|^{\frac{1}{2}\theta\varepsilon_0}_{\infty,\Omega^t})\|\Gamma\|^{\frac{1}{2}\theta}_{1,2,\Omega^t}+c_2]|\nabla\Phi|^{1-\theta}_{2,\Omega^t}\cdot|\nabla\Gamma|_{2,\Omega^t}.$$

This implies (122) and ends the proof. □

Introduce the quantity

$$X(t)=\|\Phi\|_{V(\Omega^t)}+\|\Gamma\|_{V(\Omega^t)}. \tag{132}$$

Lemma 12. *Let the assumptions of Lemmas 10 and 11 hold. Let* $\theta=\left(1-\frac{3}{d}\right)\varepsilon_1-\frac{3}{d}\varepsilon_2$, $\varepsilon=\varepsilon_1+\varepsilon_2$. *Then,*

$$X^2\le c_0|v_\varphi|^{\frac{4\varepsilon}{\theta}}_{d,\infty,\Omega^t}(1+|v_\varphi|^{2\varepsilon_0}_{\infty,\Omega^t})+c_0|v_\varphi|^{\frac{2\varepsilon}{\theta}}_{d,\infty,\Omega^t}+D_8^2, \tag{133}$$

where $c_0=\phi(D_5,D_6,D_7)$.

Proof. In view of notation (132), inequalities (111) and (122) imply

$$\begin{aligned}X^2&\le c|v_\varphi|^{\varepsilon}_{d,\infty,\Omega^t}[c_1(1+|v_\varphi|^{\frac{1}{2}\theta\varepsilon_0}_{\infty,\Omega^t})X^{1-\frac{1}{2}\theta}\\ &\quad+c_2X^{1-\theta}]X+D_8\equiv\alpha_1X^{2-\frac{1}{2}\theta}+\alpha_2X^{2-\theta}+D_8^2.\end{aligned} \tag{134}$$

Applying the Young inequality in (134) implies

$$X^2\le c\alpha_1^{\frac{4}{\theta}}+c\alpha_2^{\frac{2}{\theta}}+D_8^2.$$

This yields (133) and concludes the proof. □

Remark 4. *Consider exponents in (133). Then,*

$$\delta = \frac{4\varepsilon}{\theta} = \frac{4\varepsilon}{(1-\frac{3}{d})\varepsilon_1 - \frac{3}{d}\varepsilon_2}, \quad \delta_0 = \frac{2\varepsilon}{(1-\frac{3}{d})\varepsilon_1 - \frac{3}{d}\varepsilon_2}. \tag{135}$$

For ε_2 small, we have

$$\delta = \frac{4}{1-\frac{3}{d}} + \varepsilon_*, \quad \delta_0 = \frac{2}{1-\frac{3}{d}} + \varepsilon_{0*},$$

where ε_*, ε_{0*} are positive numbers that can be chosen very small.

For $d = 12$ it follows that

$$\delta = \frac{16}{3} + \varepsilon_*, \quad \delta_0 = \frac{8}{3} + \varepsilon_{0*}. \tag{136}$$

This ends the remark.

Lemma 13. *Assume that $\varepsilon_1 > a\varepsilon_2$, $s > 1$, $a = \frac{16+6'}{3\cdot 6'-16}$, $b = \frac{2\cdot 6'(3\varepsilon_1-\varepsilon_2)}{(6'\cdot 3-16)(\varepsilon_1-a\varepsilon_2)}$, and we choose $6'$ as arbitrarily close to 6 and*

$$D_9^s(s) = s^2|f_\varphi|^s_{\frac{3s}{2s+1},s,\Omega^t} + |v_\varphi(0)|^s_{s,\Omega} < \infty.$$

Then, excluding cases in which either $v_\varphi = 0$ or v_φ is small, we have

$$|v_\varphi|^{6'}_{12,\infty,\Omega^t} \le c|v_\varphi|^{b\varepsilon_0}_{\infty,\Omega^t} + \phi(D_5, D_6, D_7) + c(D_8 + D_9^{12}). \tag{137}$$

Proof. Multiply $(7)_2$ by $v_\varphi|v_\varphi|^{s-2}$, integrate over Ω, and exploit the relation $\frac{v_r}{r} = -\psi_{1,z}$. Then, we obtain

$$\begin{aligned} \frac{1}{s}\frac{d}{dt}|v_\varphi|^s_{s,\Omega} + \frac{4\nu(s-1)}{s^2}|\nabla|v_\varphi|^{s/2}|^2_{2,\Omega} &= \int\limits_\Omega \psi_{1,z}|v_\varphi|^s dx \\ &+ \int\limits_\Omega f_\varphi v_\varphi |v_\varphi|^{s-2} dx. \end{aligned} \tag{138}$$

Integrating by parts in the first term on the r.h.s. of (138) and applying the Hölder and Young inequalities yields

$$\left|\int\limits_\Omega \psi_{1,z}|v_\varphi|^s dx\right| \le \int\limits_0^R \psi_1|v_\varphi|^s|_{S_2} r dr + \varepsilon|\partial_z|v_\varphi|^{s/2}|^2_{2,\Omega} + c(1/\varepsilon)\int\limits_\Omega \psi_1^2|v_\varphi|^s dx,$$

where the boundary term vanishes because $\psi_1|_{S_2} = 0$.

By the Poincaré inequality,

$$|\nabla|v_\varphi|^{s/2}|^2_{2,\Omega} \ge c|v_\varphi|^s_{3s,\Omega}$$

so we can estimate the second term on the r.h.s. of (138) by

$$|f_\varphi|_{\frac{3s}{2s+1},\Omega}|v_\varphi|^{s-1}_{3s,\Omega} \le \varepsilon_1|v_\varphi|^s_{3s,\Omega} + c(1/\varepsilon_1)|f_\varphi|^s_{\frac{3s}{2s+1},\Omega}.$$

By using the above estimates with sufficiently small ε, ε_1 in (138), we derive the inequality

$$\begin{aligned}&\frac{1}{s}\frac{d}{dt}|v_\varphi|^s_{s,\Omega}+\frac{1}{s}|\nabla|v_\varphi|^{s/2}|^2_{2,\Omega}+\frac{1}{s}|v_\varphi|^s_{3s,\Omega}\\&\le cs\int\limits_\Omega\psi_1^2|v_\varphi|^s dx+cs|f_\varphi|^s_{\frac{3s}{2s+1},\Omega}.\end{aligned}\tag{139}$$

In view of Lemma 2, the first term on the r.h.s. of (139) is bounded by

$$cs|u|^{6'}_{\infty,\Omega^t}\int\limits_\Omega\frac{\psi_1^2}{r^{6'}}|v_\varphi|^{s-6'}dx\le csD_2^{6'}|v_\varphi|^{s-6'}_{\infty,\Omega}\int\limits_\Omega\frac{\psi_1^2}{r^{6'}}dx,$$

where $6'<6$, but $6'$ may be assumed to be arbitrarily close to 6.

Using the estimate in (139) yields

$$\frac{1}{s}\frac{d}{dt}|v_\varphi|^s_{s,\Omega}\le csD_2^{6'}|v_\varphi|^{s-6'}_{\infty,\Omega}\int\limits_\Omega\frac{\psi_1^2}{r^{6'}}dx+cs|f_\varphi|^s_{\frac{3s}{2s+1},\Omega}.\tag{140}$$

By integrating (140) with respect to time and using Lemma 18, we obtain

$$\begin{aligned}|v_\varphi|^s_{s,\Omega}&\le c_1s^2D_2^{6'}|v_\varphi|^{s-6'}_{\infty,\Omega^t}\|\Gamma\|^2_{1,2,\Omega^t}\\&\quad+cs^2|f_\varphi|^s_{\frac{3s}{2s+1},s,\Omega^t}+|v_\varphi(0)|^s_{s,\Omega}\\&\equiv c_1s^2D_2^{6'}|v_\varphi|^{s-6'}_{\infty,\Omega^t}\|\Gamma\|^2_{1,2,\Omega^t}+cD_9^s(s),\end{aligned}\tag{141}$$

$c_1=cR^{2\mu}\big(1+\frac{4}{(6-6')^2}\big)$.

Dividing (141) by $|v_\varphi|^{s-6'}_{\infty,\Omega^t}$ implies

$$\left|\frac{|v_\varphi|_{s,\infty,\Omega^t}}{|v_\varphi|_{\infty,\Omega^t}}\right|^{s-6'}|v_\varphi|^{6'}_{s,\Omega}\le c_1s^2D_2^{6'}\|\Gamma\|^2_{1,\Omega^t}+\frac{c}{|v_\varphi|^{s-6'}_{\infty,\Omega^t}}D_9^s(s).\tag{142}$$

The division by $|v_\varphi|_{\infty,\Omega^t}$ is justified because the following two cases are excluded from this paper:

(1) In the case in which $v_\varphi=0$, the existence of global regular solutions to problem (6) is proved in [1,2,17].
(2) The existence of global regular solutions to problem (6) for v_φ sufficiently small is proved in Appendix A.

Since cases (1) and (2) are not considered in this paper, we can show the existence of positive constants c_0 and c_1 such that

$$\left|\frac{|v_\varphi|_{s,\infty,\Omega^t}}{|v_\varphi|_{\infty,\Omega^t}}\right|^{s-6'}\ge\bar c_0\tag{143}$$

and

$$\frac{1}{|v_\varphi|^{s-6'}_{\infty,\Omega^t}}\le\bar c_1.\tag{144}$$

In view of (143) and (144), inequality (142) takes the form

$$\bar c_0|v_\varphi|^{6'}_{s,\infty,\Omega^t}\le c_1s^2\|\Gamma\|^2_{1,2,\Omega^t}+c\bar c_1D_9^s(s).\tag{145}$$

Let $d = 12$. Then, $\theta = \frac{1}{4}(3\varepsilon_1 - \varepsilon_2)$ and (133) for $d = 12$ takes the form

$$X^2 \le c_0 |v_\varphi|_{12,\infty,\Omega^t}^{\frac{16\varepsilon}{3\varepsilon_1-\varepsilon_2}} (1 + |v_\varphi|_{\infty,\Omega^t}^{2\varepsilon_0}) + c_0 |v_\varphi|_{12,\infty,\Omega^t}^{\frac{8\varepsilon}{3\varepsilon_1-\varepsilon_2}} + D_8. \tag{146}$$

Taking (145) for $s = 12$ and using (146) yields

$$\begin{aligned} |v_\varphi|_{12,\infty,\Omega^t}^{6'} &\le c_2 |v_\varphi|_{12,\infty,\Omega^t}^{\frac{16\varepsilon}{3\varepsilon_1-\varepsilon_2}} (1 + |v_\varphi|_{\infty,\Omega^t}^{2\varepsilon_0}) \\ &\quad + c_2 |v_\varphi|_{12,\infty,\Omega^t}^{\frac{8\varepsilon}{3\varepsilon_1-\varepsilon_2}} + cD_8 + cD_9^{12}, \end{aligned} \tag{147}$$

where $C_2 = \frac{144 c_1 c_0}{\bar{c}_0}$.

To derive any estimate from (147), we need

$$\frac{16\varepsilon}{3\varepsilon_1 - \varepsilon_2} < 6'. \tag{148}$$

We see that (148) holds for

$$\varepsilon_1 > \frac{16 + 6'}{3 \times 6' - 16} \varepsilon_2 \equiv a\varepsilon_2, \tag{149}$$

where $a > 11$.

In view of the Young inequality, (147) implies

$$|v_\varphi|_{12,\infty,\Omega^t}^{6'} \le c |v_\varphi|_{\infty,\Omega^t}^{b\varepsilon_0} + c + c(D_8 + D_9^{12}), \tag{150}$$

where $b = \frac{2 \times 6'(3\varepsilon_1 - \varepsilon_2)}{(6' \times 3 - 16)(\varepsilon_1 - a\varepsilon_2)}$. The above inequality implies (137) and concludes the proof. □

Remark 5. *Exploiting (150) in (146) implies the inequality*

$$X^2 \le c(1 + |v_\varphi|_{\infty,\Omega^t}^{2\varepsilon_0}) |v_\varphi|_{\infty,\Omega^t}^{d\varepsilon_0} + \phi(D_5, D_7, D_8, D_9), \tag{151}$$

where $d = \frac{16 b\varepsilon}{3\varepsilon_1 - \varepsilon_2}$ and X is introduced in (132).

To prove Theorem 1, we need an estimate for $|v_\varphi|_{\infty,\Omega^t}$. For this purpose, we need the result.

Lemma 14. *Assume that quantities D_2, D_5, D_7, D_8, and D_9 are bounded. Assume that $f_\varphi/r \in L_1(0,t;L_\infty(\Omega))$, $v_\varphi(0) \in L_\infty(\Omega)$.*

Then, an increasing positive function ϕ exists such that

$$|v_\varphi|_{\infty,\Omega^t} \le \phi(D_2, D_5, D_7, D_8, D_9, \|f_\varphi/r\|_{L_1(0,t;L_\infty(\Omega))}, |v_\varphi(0)|_{\infty,\Omega}). \tag{152}$$

Proof. Recall Equation $(7)_2$ for v_φ

$$v_{\varphi,t} + v \cdot \nabla v_\varphi - \nu \left(\Delta v_\varphi - \frac{1}{r^2} v_\varphi \right) = \psi_{1,z} v_\varphi + f_\varphi, \tag{153}$$

where $\frac{v_r}{r} = -\psi_{1,z}$.

Multiplying (153) by $v_\varphi |v_\varphi|^{s-2}$ and integrating over Ω yields

$$\begin{aligned} &\frac{1}{s} \frac{d}{dt} |v_\varphi|_{s,\Omega}^s + \frac{4\nu(s-1)}{s^2} |\nabla |v_\varphi|^{s/2}|_{2,\Omega}^2 + \nu \int_\Omega \frac{|v_\varphi|^s}{r^2} dx \\ &= \int_\Omega \psi_{1,z} v_\varphi^2 |v_\varphi|^{s-2} dx + \int_\Omega f_\varphi v_\varphi |v_\varphi|^{s-2} dx, \end{aligned} \tag{154}$$

where we used that $v_\varphi|_{S_1} = 0$, $v_{\varphi,z}|_{S_2} = 0$.

The first term on the r.h.s. of (154) is bounded by

$$\int_\Omega |\psi_{,z}|\,|v_\varphi|^{s/2} \frac{|v_\varphi|^{s/2}}{r} dx \le \varepsilon \int_\Omega \frac{|v_\varphi|^s}{r^2} dx + c(1/\varepsilon) \int_\Omega \psi_{,z}^2 |v_\varphi|^s dx,$$

where the second integral is bounded by

$$|rv_\varphi|_{\infty,\Omega}^2 \int_\Omega |\psi_{1,z}|^2 |v_\varphi|^{s-2} dx \le D_2^2 |\psi_{1,z}|_{s,\Omega}^2 |v_\varphi|_{s,\Omega}^{s-2}.$$

The second term on the r.h.s. of (154) is estimated by

$$\begin{aligned}
&\int_\Omega |f_\varphi|\,|v_\varphi|^{s-1} dx = \int_\Omega \left|\frac{f_\varphi}{r}\right| r |v_\varphi|^{s-1} dx \\
&\le |rv_\varphi|_{\infty,\Omega} \int_\Omega \left|\frac{f_\varphi}{r}\right| |v_\varphi|^{s-2} dx \le D_2 \left|\frac{f_\varphi}{r}\right|_{s/2,\Omega} |v_\varphi|_{s,\Omega}^{s-2}.
\end{aligned}$$

Using the above estimates in (154) and assuming that ε is sufficiently small, we obtain the inequality

$$\frac{1}{s}\frac{d}{dt}|v_\varphi|_{s,\Omega}^s \le D_2^2 \left(|\psi_{1,z}|_{s,\Omega}^2 |v_\varphi|_{s,\Omega}^{s-2} + \left|\frac{f_\varphi}{r}\right|_{s/2,\Omega} |v_\varphi|_{s,\Omega}^{s-2} \right).$$

Simplifying, we obtain

$$\frac{1}{2}\frac{d}{dt}|v_\varphi|_{s,\Omega}^2 \le D_2^2 \left(|\psi_{1,z}|_{s,\Omega}^2 + \left|\frac{f_\varphi}{r}\right|_{s/2,\Omega} \right).$$

Integrating with respect to time and passing with $s \to \infty$, we derive

$$|v_\varphi(t)|_{\infty,\Omega}^2 \le D_2^2 \left(\int_0^t |\psi_{1,z}|_{\infty,\Omega}^2 dt' + \int_0^t \left|\frac{f_\varphi}{r}\right|_{\infty,\Omega} dt' \right) + |v_\varphi(0)|_{\infty,\Omega}^2. \tag{155}$$

Since $\int_0^t |\psi_{1,z}|_{\infty,\Omega}^2 dt' \le X^2$, we can apply (151). Then, (155) takes the form

$$\begin{aligned}
|v_\varphi|_{\infty,\Omega^t}^2 &\le D_2^2 (1 + |v_\varphi|_{\infty,\Omega^t}^{2\varepsilon_0}) |v_\varphi|_{\infty,\Omega^t}^{\frac{96\varepsilon}{\varepsilon_1 - 11\varepsilon_2}\varepsilon_0} + D_2 \phi(D_5, D_7, D_8, D_9) \\
&\quad + D_2^2 \int_0^t \left|\frac{f_\varphi}{r}\right|_{\infty,\Omega} dt' + |v_\varphi(0)|_{\infty,\Omega}^2.
\end{aligned} \tag{156}$$

Hence, for ε_0 sufficiently small we derive (152). This ends the proof. $\square$

Remark 6. *Inequalities (151) and (152) imply*

$$X \le \phi(D_2, D_5, D_7, D_8, D_9, |f_\varphi/r|_{\infty,1,\Omega^t}, |v_\varphi(0)|_{\infty,\Omega}). \tag{157}$$

The above inequality proves Theorem 1.

6. Estimates for the Swirl

Applying the energy method and using the estimate for the weak solution (see Lemma 1) and L_∞-estimate for swirl (see Lemma 2), we derive the estimate

$$\|u\|_{L_\infty(0,t;H^1(\Omega))} + \|u\|_{L_2(0,t;H^2(\Omega))} \le \phi(\text{data}).$$

This is a new result, and it is necessary in the proof of (173).

In this Section, we find estimates for solutions to the problem

$$\begin{aligned} &u_{,t} + v\cdot\nabla u - \nu\Delta u + 2\nu\frac{u_{,r}}{r} = rf_\varphi \equiv f_0 && \text{in } \Omega^t,\\ &u|_{S_1} = 0, && \text{in } S_1^t,\\ &u_{,z} = 0 && \text{on } S_2^t,\\ &u|_{t=0} = u(0) && \text{in } \Omega. \end{aligned} \tag{158}$$

Lemma 15. *Assume that D_1, D_2 are described by (46) and (52), respectively. Let $u_{,z}(0)$, $u_{,r}(0) \in L_2(\Omega)$, $f_0 \in L_2(\Omega^t)$.*

Then, the solutions to (158) satisfy the estimates

$$|u_{,z}(t)|_{2,\Omega}^2 + \nu|\nabla u_{,z}|_{2,\Omega^t}^2 \le c(D_1^2D_2^2 + |u_{,z}(0)|_{2,\Omega}^2 + |f_0|_{2,\Omega^t}^2) \equiv cD_3^2, \tag{159}$$

$$\begin{aligned} &|u_{,r}(t)|_{2,\Omega}^2 + \nu(|u_{,rr}|_{2,\Omega^t}^2 + |u_{,rz}|_{2,\Omega^t}^2) \le cD_1^2(1+D_2^2)\\ &\quad + |u_{,r}(0)|_{2,\Omega}^2 + |f_0|_{2,\Omega^t}^2 + |f_0|_{4/3,2,S_1^t}^2 \equiv cD_4^2. \end{aligned} \tag{160}$$

Proof. Differentiate (158) with respect to z, multiply by $u_{,z}$, and integrate over Ω. To apply the Green theorem, we have to consider problem (158) in domain $\bar\Omega = \{x \in \mathbb{R}^3 : r < R, z \in (-a,a), \varphi \in (0,2\pi)\}$. Then, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|u_{,z}|_{2,\bar\Omega}^2 - \nu\int_{\bar\Omega} \operatorname{div}(\nabla u_{,z}u_{,z})d\bar x + \nu\int_{\bar\Omega}|\nabla u_{,z}|^2 d\bar x\\ &\quad + 2\nu\int_{\bar\Omega} u_{,zr}u_{,z}drdzd\varphi\\ &= -\int_{\bar\Omega} v_{,z}\cdot\nabla u\cdot u_{,z}d\bar x - \int_{\bar\Omega} v\cdot\nabla u_{,z}u_{,z}d\bar x + \int_{\bar\Omega} f_{0,z}u_{,z}d\bar x, \end{aligned} \tag{161}$$

where $d\bar x = dxd\varphi$.

The second term on the l.h.s. of (161) equals $-\nu\int_{\bar S}\bar n\cdot\nabla u_{,z}u_{,z}dSd\varphi = 0$ because $u_{,z}|_S = 0$.

The last term on the l.h.s. of (161) takes the form

$$+\nu\int_{\bar\Omega}\partial_r u_{,z}^2 drdzd\varphi = \nu\int_0^{2\pi}\int_{-a}^{a} u_{,z}^2\Big|_{r=0}^{r=R} dzd\varphi = 0$$

because $u_{.z}|_{r=R} = 0$, and [13] implies that $u_{,z}|_{r=0} = 0$.

Integrating by parts in the first term on the r.h.s. of (161) gives

$$-\int_{\bar\Omega} v_{,z}\cdot\nabla(u\cdot u_{,z})d\bar x + \int_{\bar\Omega} v_{,z}\cdot\nabla u_{,z}ud\bar x \equiv I_1 + I_2,$$

where

$$I_1 = -\int_0^{2\pi}\int_S v_{,z}\cdot\bar{n}uu_{,z}dSd\varphi = 0$$

because $u_{,z}|_S = 0$. Applying the Hölder and Young inequalities to I_2 yields

$$|I_2| \le \varepsilon\int_{\bar{\Omega}}|\nabla u_{,z}|^2 d\bar{x} + c(1/\varepsilon)|u|^2_{\infty,\Omega}\int_{\bar{\Omega}} v^2_{,z}d\bar{x}.$$

The second term on the r.h.s. of (161) takes the form

$$-\frac{1}{2}\int_{\bar{\Omega}} v\cdot\nabla u^2_{,z}d\bar{x} = -\frac{1}{2}\int_{\bar{S}} v\cdot\bar{n}u^2_{,z}d\bar{S}$$

which vanishes because $v\cdot\bar{n}|_S = 0$.

Integrating by parts in the last term on the r.h.s. of (161) yields

$$\int_{\bar{\Omega}} f_{0,z}u_{,z}d\bar{x} = \int_{\bar{\Omega}}(f_0u_{,z})_{,z}d\bar{x} - \int_{\bar{\Omega}} f_0u_{,zz}d\bar{x} \equiv J_1 + J_2,$$

where

$$J_1 = \int_0^{2\pi}\int_0^R f_0u_{,z}\Big|_{S_2} rdrd\varphi = 0$$

because $u_{,z}|_{S_2} = 0$ and

$$|J_2| \le \varepsilon|u_{,zz}|^2_{2,\bar{\Omega}} + c(1/\varepsilon)|f_0|^2_{2,\bar{\Omega}}.$$

Using the above results in (161), assuming that ε is sufficiently small and performing integration with respect to φ, we obtain

$$\frac{d}{dt}|u_{,z}|^2_{2,\Omega} + \nu|\nabla u_{,z}|^2_{2,\Omega} \le c|u|^2_{\infty,\Omega}|u_{,z}|^2_{2,\Omega} + c|f_0|^2_{2,\Omega}. \tag{162}$$

Integrating (162) with respect to time gives

$$\begin{aligned}|u_{,z}(t)|^2_{2,\Omega} + \nu|\nabla u_{,z}|^2_{2,\Omega^t} &\le c|u|^2_{\infty,\Omega^t}|v_{,z}|^2_{2,\Omega^t} + |u_{,z}(0)|^2_{2,\Omega} + c|f_0|^2_{2,\Omega^t}\\ &\le cD_1^2D_2^2 + |u_{,z}(0)|^2_{2,\Omega} + c|f_0|^2_{2,\Omega^t}.\end{aligned} \tag{163}$$

Using Lemmas 1 and 2, we have

$$\begin{aligned}&|u_{,z}(t)|^2_{2,\Omega} + \nu|\nabla u_{,z}|^2_{2,\Omega^t} \le cD_1^2D_2^2\\ &\quad + c|f_0|^2_{2,\Omega^t} + |u_{,z}(0)|^2_{2,\Omega}.\end{aligned} \tag{164}$$

The above inequality implies (159).

Differentiating (158) with respect to r gives

$$u_{,rt} + v\cdot\nabla u_{,r} + v_{,r}\cdot\nabla u - \nu(\Delta u)_{,r} + \frac{2\nu}{r}u_{,rr} - \frac{2\nu}{r^2}u_{,r} = f_{0,r}. \tag{165}$$

Multiplying (165) by $u_{,r}$ and integrating over Ω yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|u_{,r}|^2_{2,\Omega} + \int_\Omega v_{,r}\cdot\nabla uu_{,r}dx + \int_\Omega v\cdot\nabla u_{,r}u_{,r}dx - \nu\int_\Omega(\Delta u)_{,r}u_{,r}dx\\ &\quad + 2\nu\int_\Omega\frac{1}{r}u_{,rr}u_{,r}dx - 2\nu\int_\Omega\frac{u^2_{,r}}{r^2}dx = \int_\Omega f_{0,r}u_{,r}dx.\end{aligned} \tag{166}$$

Now, we examine the particular terms in (166). The second term equals

$$\begin{aligned}
&\int_\Omega v_{,r}\cdot\nabla u u_{,r} r dr dz = \int_\Omega (v_{r,r}\partial_r u + v_{z,r}\partial_z u)u_{,r} r dr dz\\
&= \int_\Omega (r v_{r,r}u_{,r} + r v_{z,r}u_{,z})u_{,r} dr dz = \int_\Omega (r v_{r,r}u_{,r}u_{,r} + r v_{z,r}u_{,r}u_{,z}) dr dz\\
&= \int_\Omega (r v_{r,r}u_{,r}u)_{,r} dr dz + \int_\Omega (r v_{z,r}u_{,r}u)_{,z} dr dz\\
&\quad - \int_\Omega [(r v_{r,r}u_{,r})_{,r} + (r v_{z,r}u_{,r})_{,z}] u dr dz \equiv J_1 + J_2\\
&\quad - \int_\Omega [(r v_{r,r}u_{,r})_{,r} + (r v_{z,r}u_{,r})_{,z}] u dr dz \equiv J_1 + J_2 + I,
\end{aligned}$$

where we used that

$$J_1 = \int_{-a}^{a} r v_{r,r} u_{,r} u\Big|_{r=0}^{r=R} dz = 0$$

because $u|_{S_1} = 0$ and

$$J_2 = \int_0^R r v_{z,r} u_{,r} u\Big|_{z=-a}^{z=a} dr = 0$$

because $v_{z,r}|_{S_2} = 0$ (see [13]). Continuing, we write I in the form

$$\begin{aligned}
I = &-\int_\Omega [(r v_{r,r})_{,r} + (r v_{z,r})_{,z}] u_{,r} u dr dz\\
&-\int_\Omega [r v_{r,r} u_{,rr} + r v_{z,r} u_{,rz}] u dr dz \equiv I_1 + I_2.
\end{aligned}$$

To estimate I_1, we calculate

$$I_1^1 = (r v_{r,r})_{,r} + (r v_{z,r})_{,z} = r v_{r,rr} + v_{r,r} + r v_{z,rz}.$$

Since $v = v_r\bar{e}_r + v_z\bar{e}_z$ is divergence-free, we have

$$v_{r,r} + v_{z,z} + \frac{v_r}{r} = 0. \tag{167}$$

Since Equation (167) is satisfied identically in Ω, we can differentiate (167) with respect to r. Then, we obtain

$$v_{r,rr} + v_{z,zr} + \frac{v_{r,r}}{r} - \frac{v_r}{r^2} = 0.$$

Hence

$$I_1^1 = \frac{v_r}{r}.$$

Then, I_1 equals

$$I_1 = -\int_\Omega \frac{v_r}{r} u_{,r} u dr dz.$$

Therefore,

$$\left|\int_0^t I_1 dt'\right| \le \left|\frac{v_r}{r}\right|_{2,\Omega^t} \left|\frac{u_{,r}}{r}\right|_{2,\Omega^t} |u|_{\infty,\Omega^t}. \tag{168}$$

Next,
$$|I_2| \le \varepsilon(|u_{,rr}|_{2,\Omega}^2 + |u_{,rz}|_{2,\Omega}^2) + c(1/\varepsilon)|u|_{\infty,\Omega}^2(|v_{r,r}|_{2,\Omega}^2 + |v_{z,r}|_{2,\Omega}^2).$$
The third integral in (166) equals
$$\begin{aligned}
J &= -\nu\int_\Omega (\Delta u)_{,r}u_{,r}dx = -\nu\int_\Omega \left(u_{,rrr} + \left(\frac{1}{r}u_{,r}\right)_{,r} + u_{,rzz}\right)u_{,r}rdrdz \\
&= -\nu\int_\Omega \left[\left(u_{,rr} + \frac{1}{r}u_{,r}\right)u_{,r}r\right]_{,r}drdz + \nu\int_\Omega u_{,rr}(u_{,r}r)_{,r}drdz \\
&\quad + \nu\int_\Omega \frac{1}{r}u_{,r}(u_{,r}r)_{,r}drdz + \int_\Omega u_{,rz}^2dx = -\nu\int_{-a}^a \left(u_{,rr} + \frac{1}{r}u_{,r}\right)u_{,r}r\Big|_{r=0}^{r=R}dz \\
&\quad + \nu\int_\Omega (u_{,rr}^2 + u_{,rz}^2)dx + \nu\int_\Omega \frac{u_{,r}^2}{r^2}dx + 2\nu\int_\Omega u_{,rr}u_{,r}drdz,
\end{aligned}$$
where the last term equals
$$\nu\int_\Omega (u_{,r}^2)_{,r}drdz = \nu\int_{-a}^a u_{,r}^2\Big|_{r=0}^{r=R}dz = \nu\int_{-a}^a u_{,r}^2\Big|_{r=R}dz \tag{169}$$
because $u_{,r}|_{r=0} = (v_\varphi + v_{\varphi,r}r)|_{r=0} = 0$.

To examine the boundary term in J, we recall the expansion of v_φ near the axis of symmetry (see [13])
$$v_\varphi = a_1(z,t)r + a_2(z,t)r^3 + \cdots,$$
so
$$u = a_1(z,t)r^2 + a_2(z,t)r^4 + \cdots$$
Then, $(u_{,rr} + \frac{1}{r}u_{,r})u_{,r}r|_{r=0} = 0$, and we have to emphasize that all calculations in this paper are performed for sufficiently regular solutions.

Therefore, the boundary term in J equals
$$J_1 = -\nu\int_{-a}^a \left(u_{,rr} + \frac{1}{r}u_{,r}\right)u_{,r}r\Big|_{r=R}dz.$$
Projecting $(158)_1$ on S_1 yields
$$-\nu\left(u_{,rr} + \frac{1}{r}u_{,r}\right) + 2\nu\frac{u_{,r}}{r} = f_0 \quad \text{on } S_1.$$
Hence,
$$u_{,rr}|_{S_1} = \left(\frac{u_{,r}}{r} - \frac{1}{\nu}f_0\right)\Big|_{S_1}.$$
Using the expression in J_1 gives
$$J_1 = -2\nu\int_{-a}^a u_{,r}^2\Big|_{r=R}dz + \int_{-a}^a f_0u_{,r}r\Big|_{r=R}dz.$$
The fourth term in (166) equals (169).

Using the above estimates and expressions in (166) yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u_{,r}|_{2,\Omega}^2+\nu\int_\Omega(u_{,rr}^2+u_{,rz}^2)dx+\nu\int_\Omega\frac{u_{,r}^2}{r^2}dx\\
&-2\nu\int_\Omega\frac{u_{,r}^2}{r^2}dx\le\int_\Omega\left|\frac{v_r}{r}u_{,r}u\right|drdz\\
&+\varepsilon(|u_{,rr}|_{2,\Omega}^2+|u_{,rz}|_{2,\Omega}^2)+c(1/\varepsilon)|u|_{\infty,\Omega}^2(|v_{r,r}|_{2,\Omega}^2+|v_{z,r}|_{2,\Omega}^2)\\
&+c(1/\varepsilon)|f_0|_{2,\Omega}^2+\left|\int_{-a}^a f_0u_{,r}r\right|_{r=R}dz.
\end{aligned}\tag{170}$$

Integrating (170) with respect to time and assuming that ε is sufficiently small, we obtain

$$\begin{aligned}
&|u_{,r}(t)|_{2,\Omega}^2+\nu(|u_{,rr}|_{2,\Omega^t}^2+|u_{,rz}|_{2,\Omega^t}^2)\le\nu\left|\frac{u_{,r}}{r}\right|_{2,\Omega^t}^2\\
&+c\left|\frac{v_r}{r}\right|_{2,\Omega^t}\left|\frac{u_{,r}}{r}\right|_{2,\Omega^t}|u|_{\infty,\Omega^t}+c|u|_{\infty,\Omega^t}^2(|v_{r,r}|_{2,\Omega^t}^2+|v_{z,r}|_{2,\Omega^t}^2)\\
&+c|f_0|_{2,\Omega^t}^2+|u_{,r}(0)|_{2,\Omega}^2+\nu\int_0^t\int_{-a}^a u_{,r}^2\bigg|_{r=R}dxdt'\\
&+\left|\int_0^t\int_{-a}^a f_0u_{,r}r\bigg|_{r=R}dxdt'\right|.
\end{aligned}\tag{171}$$

Using

$$\int_{\Omega^t}\left|\frac{u_{,r}}{r}\right|^2dxdt'\le\int_{\Omega^t}\left(|v_{\varphi,r}|^2+\frac{v_\varphi^2}{r^2}\right)dxdt'\le cD_1^2$$

and

$$\int_0^t\int_{-a}^a u_{,r}^2\bigg|_{r=R}dxdt'\le\varepsilon|\nabla u_{,r}|_{2,\Omega^t}^2+c(1/\varepsilon)|u_{,r}|_{2,\Omega^t}^2,$$

$$\begin{aligned}
&\left|\int_0^t\int_{-a}^a f_0u_{,r}\bigg|_{r=R}dxdt'\right.\le\varepsilon_1|u_{,r}|_{4,2,S_1^t}^2+c(1/\varepsilon_1)|f_0|_{4/3,2,S_1^t}^2\\
&\le\varepsilon_1(|u_{,rr}|_{2,\Omega^t}^2+|u_{,rz}|_{2,\Omega^t}^2)+c(1/\varepsilon_1)|f_0|_{4/3,2,S_1^t}^2
\end{aligned}$$

and Lemmas 1 and 2 we have

$$\begin{aligned}
&|u_{,r}(t)|_{2,\Omega}^2+\nu(|u_{,rr}|_{2,\Omega^t}^2+|u_{,rz}|_{2,\Omega^t}^2)\le c(D_1^2+D_1^2D_2+D_1^2D_2^2)\\
&+c|f_0|_{2,\Omega^t}^2+c|f_0|_{4/3,2,S_1^t}^2+|u_{,r}(0)|_{2,\Omega}^2.
\end{aligned}\tag{172}$$

This inequality implies (160) and concludes the proof. □

7. Estimates for ω_r, ω_z

Inequality (173) is the most important inequality in this paper. To prove it, we need results from Sections 4 and 6 and from Lemma 2. By the energy method, we derive (174), where the first term on the r.h.s., denoted by J, is nonlinear. The aim of the proof of Lemma 16 is to show that

$$(*)\qquad J\le c|u|_{\infty,\Omega^t}(D_1+\|u\|_{L_2(0,t;H^2(\Omega))})\|\Gamma\|_{L_2(0,t;H^1(\Omega))}\equiv J_1^1.$$

Using (46), (52), (160), and (161), we obtain

$$(**) \qquad J_1^1 \le \phi(\text{data})\|\Gamma\|_{L_2(0,t;H^1(\Omega))}$$

so it is linear with respect to the norm of Γ.

To show $(*)$, we replace ω_r, ω_z in J by derivatives of u described by (13) and express components of velocity v_r, v_z by derivatives of ψ using (15). Performing appropriate integration by parts in J, we are able to extract the norm $|u|_{\infty,\Omega^t}$.

Then, J becomes bilinear. Then, estimates (46), (159), and (160) imply $(*)$ by the Hölder inequality.

Lemma 16. *Assume that $D_5 = D_2(D_1 + D_3 + D_4)$, $D_6 = D_2^{1-\varepsilon_0} D_3$, where D_1, D_2 are introduced in (46) and (52) and D_3, D_4 are introduced in (159) (160), respectively. Let*

$$\begin{aligned} D_7 = &|F_r|^2_{6/5,2,\Omega^t} + |F_z|^2_{6/5,2,\Omega^t} + |\omega_r(0)|^2_{2,\Omega} + |\omega_z(0)|^2_{2,\Omega} \\ &+ |f_\varphi|_{2,S_1^t}(D_3 + D_4) < \infty. \end{aligned}$$

Let ε_0 be an arbitrary small positive number, and let $v_\varphi \in L_\infty(\Omega^t)$.

Let $\Gamma \in L_2(0,t;H^1(\Omega))$.

Then,

$$\begin{aligned} &\|\omega_r\|^2_{V(\Omega^t)} + \|\omega_z\|^2_{V(\Omega^t)} + |\Phi|^2_{2,\Omega^t} \le cD_5|\Gamma_{,z}|_{2,\Omega^t} \\ &\quad + cD_6|v_\varphi|^{\varepsilon_0}_{\infty,\Omega^t}\|\Gamma\|_{1,2,\Omega^t} + cD_7. \end{aligned} \tag{173}$$

Proof. Multiplying $(9)_1$ by ω_r, $(9)_3$ by ω_z, integrating over Ω^t, and adding yield

$$\begin{aligned} &\frac{1}{2}(|\omega_r(t)|^2_{2,\Omega} + |\omega_z(t)|^2_{2,\Omega}) + \nu(|\nabla\omega_r|^2_{2,\Omega^t} + |\nabla\omega_z|^2_{2,\Omega^t}) \\ &\quad + \nu\left|\frac{\omega_r}{r}\right|^2_{2,\Omega^t} - \nu\int_{S^t} \bar{n}\cdot\nabla\omega_r\omega_r dS_1dt' - \nu\int_{S^t} \bar{n}\cdot\nabla\omega_z\omega_z dS_1dt' \\ &= \int_{\Omega^t}[v_{r,r}\omega_r^2 + v_{z,z}\omega_z^2 + (v_{r,z} + v_{z,r})\omega_r\omega_z]dxdt' \\ &\quad + \int_{\Omega^t}(F_r\omega_r + F_z\omega_z)dxdt' + \frac{1}{2}(|\omega_r(0)|^2_{2,\Omega} + |\omega_z(0)|^2_{2,\Omega}) \\ &\equiv J + \int_{\Omega^t}(F_r\omega_r + F_z\omega_z)dxdt' + \frac{1}{2}(|\omega_r(0)|^2_{2,\Omega} + |\omega_z(0)|^2_{2,\Omega}). \end{aligned} \tag{174}$$

Now, we examine the boundary terms from the l.h.s.

Since $\omega_r = -v_{\varphi,z}$ and $v_\varphi|_{r=R} = 0$, we obtain

$$-\int_{S_1} \bar{n}\cdot\nabla\omega_r\omega_r dS_1 = 0$$

and

$$-\int_{S_2} \bar{n}\cdot\nabla\omega_r\omega_r dS_2 = 0$$

because $v_{\varphi,z}|_{S_2} = 0$.

Using $\omega_z = v_{\varphi,r} + \frac{v_\varphi}{r}$, we derive

$$-\nu\int_{S_1^t} \bar{n}\cdot\nabla\omega_z\omega_z dS_1dt' = -\nu\int_0^t\int_{-a}^a \partial_r\left(v_{\varphi,r} + \frac{v_\varphi}{r}\right)\left(v_{\varphi,r} + \frac{v_\varphi}{r}\right)\bigg|_{r=R} Rdzdt' \equiv I_1.$$

Since $v_\varphi|_{r=R}=0$ I_1 takes the form

$$I_1=-\nu\int_0^t\int_{-a}^a\left(v_{\varphi,rr}+\frac{v_{\varphi,r}}{r}\right)v_{\varphi,r}\bigg|_{r=R}Rdzdt'.$$

Projecting (7)$_2$ on S_1 yields

$$-\nu\left(v_{\varphi,rr}+\frac{1}{r}v_{\varphi,r}\right)=f_\varphi\quad\text{on}\ \ S_1.$$

Hence,

$$\begin{aligned}I_1&=R\int_0^t\int_{-a}^a f_\varphi v_{\varphi,r}\bigg|_{r=R}dzdt'=\int_0^t\int_{-a}^a f_\varphi\left(u_{,r}-\frac{1}{R}u\right)\bigg|_{r=R}dzdt'\\&=\int_0^t\int_{-a}^a f_\varphi u_{,r}\bigg|_{r=R}dxdt'.\end{aligned}\tag{175}$$

Finally,

$$-\nu\int_{S_2^t}\bar{n}\cdot\nabla\omega_z\omega_z dS_2dt'=-\nu\int_{S_2^t}\frac{1}{r}u_{,zr}\frac{1}{r}u_{,r}dS_2dt'=0$$

because $v_{\varphi,z}|_{S_2}=0$.

Using (13) and (21) in J implies

$$\begin{aligned}J=\int_{\Omega^t}\bigg[&-\frac{1}{r^2}u_{,z}^2(\psi_{1,z}+r\psi_{1,rz})+\left(\frac{1}{r}u_{,r}\right)^2(r\psi_{1,zr}+2\psi_{1,z})\\&-\frac{1}{r^2}u_{,r}u_{,z}(-r\psi_{1,zz}+3\psi_{1,r}+r\psi_{1,rr})\bigg]dxdt'\equiv J_1+J_2+J_3.\end{aligned}$$

We integrate by parts in J_1 and use the boundary conditions on S_2. Then, we have

$$\begin{aligned}J_1=&-\int_{\Omega^t}\left[u\frac{u_{,z}}{r^2}(\psi_{1,z}+r\psi_{1,rz})\right]_{,z}rdrdzdt'+\int_{\Omega^t}\frac{1}{r^2}uu_{,zz}(\psi_{1,z}+r\psi_{1,rz})dxdt'\\&+\int_{\Omega^t}\frac{1}{r^2}uu_{,z}(\psi_{1,zz}+r\psi_{1,rzz})dxdt'.\end{aligned}$$

Since $u_{,z}|_{S_2}=0$ the boundary term vanishes.

Now, we estimate the particular terms in J_1,

$$\begin{aligned}J_{11}&=\left|\int_{\Omega^t}uu_{,zz}\frac{1}{r}\psi_{1,rz}dxdt'\right|\le|u|_{\infty,\Omega^t}|u_{,zz}|_{2,\Omega^t}\left|\frac{1}{r}\psi_{1,rz}\right|_{2,\Omega^t},\\J_{12}&=\left|\int_{\Omega^t}u\frac{u_{,z}}{r}\psi_{1,rzz}dxdt'\right|\le|u|_{\infty,\Omega^t}\left|\frac{u_{,z}}{r}\right|_{2,\Omega^t}|\psi_{1,rzz}|_{2,\Omega^t},\\J_{13}&=\left|\int_{\Omega^t}u\frac{u_{,z}}{r}\frac{\psi_{1,zz}}{r}dxdt'\right|\le|u|_{\infty,\Omega^t}\left|\frac{u_{,z}}{r}\right|_{2,\Omega^t}\left|\frac{\psi_{1,zz}}{r}\right|_{2,\Omega^t},\\J_{14}&=\left|\int_{\Omega^t}\frac{1}{r^2}uu_{,zz}\psi_{1,z}dxdt'\right|=\left|\int_{\Omega^t}uu_{,zz}\frac{\psi_{1,z}}{r^2}dxdt'\right|\\&\le|u|_{\infty,\Omega^t}|u_{,zz}|_{2,\Omega^t}\left|\frac{\psi_{1,z}}{r^2}\right|_{2,\Omega^t}.\end{aligned}$$

Next, we consider J_2,

$$
\begin{aligned}
J_2 &= \int_{\Omega^t} \frac{1}{r^2} u_{,r}^2 (r\psi_{1,zr} + 2\psi_{1,z}) r dr dz dt' = \int_{\Omega^t} \frac{1}{r} u_{,r}^2 (r\psi_{1,zr} + 2\psi_{1,z}) dr dz dt' \\
&= \int_0^t \int_{-a}^{a} \left[\frac{1}{r} u u_{,r} (r\psi_{1,zr} + 2\psi_{1,z}) \right] \bigg|_{r=0}^{r=R} dz dt' \\
&\quad - \int_{\Omega^t} u u_{,rr} \left(\frac{1}{r} \psi_{1,zr} + \frac{2}{r^2} \psi_{1,z} \right) dx dt' \\
&\quad - \int_{\Omega^t} u u_{,r} \left(\psi_{1,zrr} - \frac{2}{r^2} \psi_{1,z} + \frac{2}{r} \psi_{1,zr} \right) dr dz dt',
\end{aligned}
$$

where the boundary term for $r = R$ vanishes because $u|_{r=R} = 0$. To examine the boundary term at $r = 0$, we recall from [13] the expressions near the axis of symmetry

$$
u = a_1(z,t) r^2 + a_2(z,t) r^4 + \cdots,
$$

so

$$
u_{,r} = 2a_1(z,t) r + 4a_2(z,t) r^3 + \cdots
$$

Then,

$$
\frac{1}{r} u u_{,r} (r\psi_{1,zr} + 2\psi_{1,z}) \sim c r^2 (r\psi_{1,zr} + 2\psi_{1,z}).
$$

The above expression vanishes for $r = 0$ because $\psi_{1,z}$ is bounded near the axis of symmetry.

Now, we estimate the particular terms in J_2,

$$
\begin{aligned}
J_{21} &= \left| \int_{\Omega^t} u u_{,rr} \frac{1}{r} \psi_{1,rz} dx dt' \right| \le |u|_{\infty,\Omega^t} |u_{,rr}|_{2,\Omega^t} \left| \frac{1}{r} \psi_{1,zr} \right|_{2,\Omega^t}, \\
J_{22} &= \left| \int_{\Omega^t} u u_{,rr} \frac{1}{r^2} \psi_{1,z} dx dt' \right| \le |u|_{\infty,\Omega^t} |u_{,rr}|_{2,\Omega^t} \left| \frac{1}{r^2} \psi_{1,z} \right|_{2,\Omega^t}, \\
J_{23} &= \left| \int_{\Omega^t} u \frac{u_{,r}}{r} \psi_{1,zrr} dx dt' \right| \le |u|_{\infty,\Omega^t} \left| \frac{u_{,r}}{r} \right|_{2,\Omega^t} |\psi_{1,zrr}|_{2,\Omega^t}, \\
J_{24} &= \left| \int_{\Omega^t} u \frac{u_{,r}}{r} \frac{1}{r^2} \psi_{1,z} dx dt' \right| \le |u|_{\infty,\Omega^t} \left| \frac{u_{,r}}{r} \right|_{2,\Omega^t} \left| \frac{1}{r^2} \psi_{1,z} \right|_{2,\Omega^t}, \\
J_{25} &= \left| \int_{\Omega^t} u \frac{u_{,r}}{r} \frac{1}{r} \psi_{1,zr} dx dt' \right| \le |u|_{\infty,\Omega^t} \left| \frac{u_{,r}}{r} \right|_{2,\Omega^t} \left| \frac{1}{r} \psi_{1,zr} \right|_{2,\Omega^t}.
\end{aligned}
$$

Finally, we examine J_3. Integrating by parts with respect to z, we have

$$
\begin{aligned}
J_3 &= -\int_{\Omega^t} \left[\frac{1}{r^2} u_{,r} (-r\psi_{1,zz} + 3\psi_{1,r} + r\psi_{1,rr}) u \right]_{,z} dx dt' \\
&\quad + \int_{\Omega^t} u \frac{1}{r^2} u_{,rz} (-r\psi_{1,zz} + 3\psi_{1,r} + r\psi_{1,rr}) dx dt' \\
&\quad + \int_{\Omega^t} u \frac{1}{r^2} u_{,r} (-r\psi_{1,zzz} + 3\psi_{1,rz} + r\psi_{1,rrz}) dx dt',
\end{aligned}
$$

where the boundary term vanishes because

$$
\psi_{1,r}|_{S_2} = 0, \quad \psi_{1,rr}|_{S_2} = 0 \quad \text{and} \quad \psi_{1,zz}|_{S_2} = -\omega_1|_{S_2} = 0.
$$

Now, we estimate the particular terms in J_3,

$$J_{31} = \left|\int_{\Omega^t} uu_{,rz}\frac{1}{r}\psi_{1,zz}dxdt'\right| \le |u|_{\infty,\Omega^t}|u_{,rz}|_{2,\Omega^t}\left|\frac{\psi_{1,zz}}{r}\right|_{2,\Omega^t},$$

$$\begin{aligned}J_{32} &= \left|\int_{\Omega^t} u\frac{1}{r^2}u_{,rz}\psi_{1,r}dxdt'\right| = \left|\int_{\Omega^t}\frac{u}{r^{\varepsilon_0}}u_{,rz}\frac{\psi_{1,r}}{r^{2-\varepsilon_0}}dxdt'\right| \\ &\le |u|^{1-\varepsilon_0}_{\infty,\Omega^t}|v_\varphi|^{\varepsilon_0}_{\infty,\Omega^t}|u_{,rz}|_{2,\Omega^t}\left|\frac{\psi_{1,r}}{r^2}\right|_{L_2(0,t;L_{2,\varepsilon_0}(\Omega))},\end{aligned}$$

where $\varepsilon_0 > 0$ can be chosen to be as small as we want. Thus,

$$\begin{aligned}J_{33} &= \left|\int_{\Omega^t}\frac{u}{r^{\varepsilon_0}}u_{,rz}\frac{1}{r^{1-\varepsilon_0}}\psi_{1,rr}dxdt'\right| \le |u|^{1-\varepsilon_0}_{\infty,\Omega^t}|v_\varphi|^{\varepsilon_0}_{\infty,\Omega^t}|u_{,rz}|_{2,\Omega^t}\left|\frac{\psi_{1,rr}}{r^{1-\varepsilon_0}}\right|_{2,\Omega^t},\\ J_{34} &= \left|\int_{\Omega^t} u\frac{u_{,r}}{r}\psi_{1,zzz}dxdt'\right| \le |u|_{\infty,\Omega^t}\left|\frac{u_{,r}}{r}\right|_{2,\Omega^t}|\psi_{1,zzz}|_{2,\Omega^t},\\ J_{35} &= \left|\int_{\Omega^t} u\frac{u_{,r}}{r}\frac{1}{r}\psi_{1,rz}dxdt'\right| \le |u|_{\infty,\Omega^t}\left|\frac{u_{,r}}{r}\right|_{2,\Omega^t}\left|\frac{\psi_{1,rz}}{r}\right|_{2,\Omega^t},\\ J_{36} &= \left|\int_{\Omega^t} u\frac{u_{,r}}{r}\psi_{1,rrz}dxdt'\right| \le |u|_{\infty,\Omega^t}\left|\frac{u_{,r}}{r}\right|_{2,\Omega^t}|\psi_{1,rrz}|_{2,\Omega^t}.\end{aligned}$$

Summarizing the above estimates, we obtain

$$\begin{aligned}|J| \le{}& c|u|_{\infty,\Omega^t}\Big[(|u_{,zz}|_{2,\Omega^t} + |u_{,zr}|_{2,\Omega^t} + |u_{,rr}|_{2,\Omega^t})\cdot\\ &\cdot\left(\left|\frac{1}{r}\psi_{1,rz}\right|_{2,\Omega^t} + \left|\frac{1}{r}\psi_{1,zz}\right|_{2,\Omega^t} + \left|\frac{1}{r^2}\psi_{1,z}\right|_{2,\Omega^t}\right)\\ &+\left(\left|\frac{u_{,r}}{r}\right|_{2,\Omega^t} + \left|\frac{u_{,z}}{r}\right|_{2,\Omega^t}\right)\Big(|\psi_{1,rzz}|_{2,\Omega^t} + |\psi_{1,zrr}|_{2,\Omega^t}\\ &+|\psi_{1,zzz}|_{2,\Omega^t} + \left|\frac{1}{r}\psi_{1,zz}\right|_{2,\Omega^t} + \left|\frac{1}{r}\psi_{1,zr}\right|_{2,\Omega^t} + \left|\frac{1}{r^2}\psi_{1,z}\right|_{2,\Omega^t}\Big)\Big]\\ &+c|u|^{1-\varepsilon_0}_{\infty,\Omega^t}|v_\varphi|^{\varepsilon_0}_{\infty,\Omega^t}|u_{,rz}|_{2,\Omega^t}\left(\left|\frac{\psi_{1,rr}}{r}\right|_{L_2(0,t;L_{2,\varepsilon_0}(\Omega))} + \left|\frac{\psi_{1,r}}{r^2}\right|_{L_2(0,t;L_{2,\varepsilon_0}(\Omega))}\right).\end{aligned}$$

Using (52), (159), (160), and the estimates from (46)

$$\begin{aligned}\left|\frac{u_{,r}}{r}\right|_{2,\Omega^t} &\le \left|\frac{v_\varphi}{r}\right|_{2,\Omega^t} + |v_{\varphi,r}|_{2,\Omega^t} \le cD_1,\\ \left|\frac{u_{,z}}{r}\right|_{2,\Omega^t} &\le |v_{\varphi,z}|_{2,\Omega^t} \le cD_1\end{aligned}$$

we obtain the following estimate for J,

$$\begin{aligned}|J| \le{}& c[D_2(D_3+D_4) + D_1D_2]\Big(|\psi_{1,rrz}|_{2,\Omega^t} + |\psi_{1,rzz}|_{2,\Omega^t}\\ &+|\psi_{1,zzz}|_{2,\Omega^t} + \left|\frac{1}{r}\psi_{1,rz}\right|_{2,\Omega^t} + \left|\frac{1}{r}\psi_{1,zz}\right|_{2,\Omega^t} + \left|\frac{1}{r^2}\psi_{1,z}\right|_{2,\Omega^t}\Big)\\ &+cD_2^{1-\varepsilon_0}D_3|v_\varphi|^{\varepsilon_0}_{\infty,\Omega^t}\left(\left|\frac{1}{r}\psi_{1,rr}\right|_{L_2(0,t;L_{2,\varepsilon_0}(\Omega))} + \left|\frac{1}{r^2}\psi_{1,r}\right|_{L_2(0,t;L_{2,\varepsilon_0}(\Omega))}\right) \equiv J'.\end{aligned}$$

From (64), we have (recall that $\omega_1 = \Gamma$)

$$|\psi_{1,rrz}|_{2,\Omega^t} + |\psi_{1,rzz}|_{2,\Omega^t} + |\psi_{1,zzz}|_{2,\Omega^t} \le c|\Gamma_{,z}|_{2,\Omega^t}. \tag{176}$$

Estimates (82) and (85) imply

$$\left|\frac{\psi_{1,rz}}{r}\right|_{2,\Omega^t} + \left|\frac{\psi_{1,zz}}{r}\right|_{2,\Omega^t} + \left|\frac{\psi_{1,z}}{r^2}\right|_{2,\Omega^t} \le c|\Gamma_{,z}|_{2,\Omega^t}. \tag{177}$$

Finally, (93) yields

$$\left|\frac{1}{r}\psi_{1,rr}\right|_{L_2(0,t;L_{2,\varepsilon_0}(\Omega))} + \left|\frac{1}{r^2}\psi_{1,r}\right|_{L_2(0,t;L_{2,\varepsilon_0}(\Omega))} \le cR^{\varepsilon_0}\|\Gamma\|_{1,2,\Omega^t}. \tag{178}$$

Recall that (177) is valid for $\psi_1|_{r=0} = 0$.

This restriction implies that $v_z|_{r=0} = 0$, so it is a strong restriction on the solutions proved in this paper.

Using (176)–(178) in J' yields

$$J' \le cD_2(D_1 + D_3 + D_4)|\Gamma_{,z}|_{2,\Omega^t} + cD_2^{1-\varepsilon_0}D_3|v_\varphi|^{\varepsilon_0}_{\infty,\Omega^t}\|\Gamma\|_{1,2,\Omega^t}.$$

In view of Lemma 15, the term I_1 introduced in (175) is bounded by

$$I_1 \le c|f_\varphi|_{2,S_1^t}\|u\|_{2,2,\Omega^t} \le c|f_\varphi|_{2,S_1^t}(D_3 + D_4).$$

Using the estimates in (174), we obtain

$$\begin{aligned}
&\|\omega_r\|^2_{V(\Omega^t)} + \|\omega_z\|^2_{V(\Omega^t)} + |\Phi|^2_{2,\Omega^t} \\
&\le cD_2(D_1 + D_3 + D_4)|\Gamma_{,z}|_{2,\Omega^t} + cD_2^{1-\varepsilon_0}D_3|v_\varphi|^{\varepsilon_0}_{\infty,\Omega^t}\|\Gamma\|_{1,2,\Omega^t} \\
&\quad + c(|F_r|^2_{6/5,2,\Omega^t} + |F_z|^2_{6/5,2,\Omega^t}) + c(|\omega_r(0)|^2_{2,\Omega} \\
&\quad + |\omega_z(0)|^2_{2,\Omega}) + c|f_\varphi|_{2,S_1^t}(D_3 + D_4),
\end{aligned} \tag{179}$$

where we used

$$\begin{aligned}
\left|\int_\Omega (F_r\omega_r + F_z\omega_z)dxdt'\right| &\le \varepsilon(|\omega_r|^2_{6,\Omega} + |\omega_z|^2_{6,\Omega}) \\
&\quad + c(1/\varepsilon)(|F_r|^2_{6/5,\Omega} + |F_z|^2_{6/5,\Omega}).
\end{aligned}$$

Hence, (179) implies (173) and concludes the proof. □

8. Estimates for the Stream Function in Weighted Sobolev Spaces

Recall that the stream function ψ_1 is a solution to problem (22). To increase the regularity of the weak solutions to (22), we need appropriate estimates for ψ_1 assuming the sufficient regularity of vorticity w_1.

Remark 7. *In Lemma 4, the existence of weak solutions to problem (22) satisfying estimate (56) is proved. Inequality (62) implies that the weak solution belongs to $H^2(\Omega)$ and the estimate holds*

$$\|\psi_1\|_{2,\Omega} \le c|w_1|_{2,\Omega}. \tag{180}$$

Assuming that $w_{1,z} \in L_2(\Omega)$ estimates (63) and (64) increase the regularity of ψ_1 such that $\psi_{1,z} \in H^2(\Omega)$ and the estimate holds

$$\|\psi_{1,z}\|_{H^2(\Omega)} \le c(|w_{1,z}|_{2,\Omega} + |w_1|_{2,\Omega}). \tag{181}$$

Estimate (181) is derived by the technique of the energy method. The method is not sufficiently strong to derive an estimate for $|\psi_{1,rrr}|_{2,\Omega}$.

Moreover, estimate (181) is not sufficient to prove estimate (24) of Theorem 1. To prove Theorem 1, we need estimated (85) and (93). To prove the estimates, we need the theory of weighted Sobolev spaces developed by Kondratiev [12] that are used to examine elliptic boundary value problems in domains with cones.

Unfortunately, estimates (85) and (93) hold for such weak solutions that ψ_1 vanishes on the axis of symmetry. This implies that the v_z coordinate of velocity must also vanish on the axis of symmetry. Therefore, Theorem 1 holds for a smaller class than the class of weak solutions. This means that the regularity problem for axially symmetric solutions to the Navier–Stokes equations is solved only partially.

Now, we show the existence of solutions to problem (22) in weighted Sobolev spaces.

Lemma 17. *Assume that* ψ_1 *is a solution to (61). Assume that* $\omega_{1,z}, \omega_1 \in L_2(\Omega)$. *Then,*

$$\begin{aligned} &\int_\Omega \left(\psi_{1,zrr}^2 + \frac{1}{r^2}\psi_{1,zr}^2 + \frac{1}{r^4}\psi_{1,z}^2\right)dx + \int_\Omega \psi_{1,zzz}^2 dx \\ &\le c\int_\Omega (|\omega_{1,z}|^2 + |\omega_1|^2)dx. \end{aligned} \tag{182}$$

Proof. To prove the lemma, we need weighted Sobolev spaces defined by Fourier transform (58) and introduced in (59) and (60). Therefore, to examine problem (22) in weighted Sobolev spaces we have to derive estimates with respect to r and z, separately. To derive an estimate with respect to r, we have to examine solutions to (22) independently as well in a neighborhood of the axis of symmetry as in a neighborhood located in a positive distance from it. To perform such considerations, we treat z as a parameter and we introduce a partition of unity $\{\zeta^{(1)}(r), \zeta^{(2)}(r)\}$ such that

$$\sum_{i=1}^2 \zeta^{(i)}(r) = 1$$

and

$$\zeta^{(1)}(r) = \begin{cases} 1 & \text{for } r \le r_0 \\ 0 & \text{for } r \ge 2r_0 \end{cases}, \quad \zeta^{2)}(r) = \begin{cases} 0 & \text{for } r \le r_0 \\ 1 & \text{for } r \ge 2r_0, \end{cases}$$

where $0 < r_0$ is fixed in such a way that $2r_0 < R$.

Let $\psi_1^{(i)} = \psi_1\zeta^{(i)}$, $\omega_1^{(i)} = \omega_1\zeta^{(i)}$ and $\dot\zeta^{(i)} = \frac{d}{dr}\zeta^{(i)}$, $\ddot\zeta^{(i)} = \frac{d^2}{dr^2}\zeta^{(i)}$, $i = 1, 2$. Moreover, functions $\zeta^{(1)}, \zeta^{(2)}$ are smooth.

Then, we obtain from (22) the following two problems:

$$\begin{cases} -\Delta\psi_1^{(1)} - \dfrac{2}{r}\psi_{1,r}^{(1)} = \omega_1^{(1)} - 2\psi_{1,r}\dot\zeta^{(1)} & \\ \qquad - \psi_1\ddot\zeta^{(1)} - \dfrac{2}{r}\psi_1\dot\zeta^{(1)} & \text{in } \Omega^{(1)}, \\ \psi_1^{(1)} = 0 & \text{on } S_2^{(1)}, \end{cases} \tag{183}$$

where

$$\Omega^{(1)} = \{(r,z)\colon r > 0, z \in (-a,a)\},$$
$$S_2^{(1)} = \{(r,z)\colon r > 0, z \in \{-a,a\}\}$$

and

$$\begin{cases} -\Delta\psi_1^{(2)} - \frac{2}{r}\psi_{1,r}^{(2)} = \omega_1^{(2)} - 2\psi_{1,r}\dot{\zeta}^{(2)} & \\ \quad - \psi_1\ddot{\zeta}^{(2)} - \frac{2}{r}\psi_1\dot{\zeta}^{(2)} & \text{in } \Omega^{(2)}, \\ \psi_1^{(2)} = 0 & \text{on } S_1, \\ \psi_1^{(2)} = 0 & \text{on } S_2^{(2)}, \end{cases} \tag{184}$$

where

$$\Omega^{(2)} = \{(r,z) \colon r_0 < r < R, z \in (-a,a)\},$$
$$S_2^{(2)} = \{(r,z) \colon r_0 < r < R, z \in \{-a,a\}\}.$$

We temporarily simplify the notation using

$$\begin{aligned} &u = \psi_1^{(1)}, \ \ w = \psi_1^{(2)}, \\ &f = \omega_1^{(1)} - 2\psi_{1,r}\dot{\zeta}^{(1)} - \psi_1\ddot{\zeta}^{(1)} - \frac{2}{r}\psi_1\dot{\zeta}^{(1)}, \\ &b = \omega_1^{(2)} - 2\psi_{1,r}\dot{\zeta}^{(2)} - \psi_1\ddot{\zeta}^{(2)} - \frac{2}{r}\psi_1\dot{\zeta}^{(2)}. \end{aligned} \tag{185}$$

Then, (183) and (184) become

$$\begin{aligned} -\Delta u - \frac{2}{r}u_{,r} = f \quad & \text{in } \Omega^{(1)}, \\ u = 0 \quad & \text{on } S_2^{(1)} \end{aligned} \tag{186}$$

and

$$\begin{aligned} -\Delta w - \frac{2}{r}w_{,r} = b \quad & \text{in } \Omega^{(2)}, \\ w = 0 \quad & \text{on } S_1, \\ w = 0 \quad & \text{on } S_2^{(2)}. \end{aligned} \tag{187}$$

First, we consider problem (186). We rewrite it in the form

$$\begin{aligned} -u_{,rr} - \frac{3}{r}u_{,r} = f + u_{,zz} \quad & \text{in } \Omega^{(1)}, \\ u = 0 \qquad\qquad & \text{on } S_2^1. \end{aligned} \tag{188}$$

For a fixed $z \in (-a,a)$ and given r.h.s. of (188), we obtain the ordinary differential equation

$$-u_{,rr} - \frac{3}{r}u_{,r} = f + u_{,zz} \quad \text{in } \mathbb{R}_+. \tag{189}$$

Properties of solutions to (189) depend on the behavior of u at $r = 0$.

Multiplying (189) by r^2 yields

$$-r^2 u_{,rr} - 3ru_{,r} = r^2(f + u_{,zz}) \equiv g(r,z) \tag{190}$$

or equivalently

$$-r\partial_r(r\partial_r u) - 2r\partial_r u = g(r,z). \tag{191}$$

Introduce the new variable

$$\tau = -\ln r, \quad r = e^{-\tau}. \tag{192}$$

Since $r\partial_r = -\partial_\tau$, we see that (191) takes the form

$$-\partial_\tau^2 u + 2\partial_\tau u = g(e^{-\tau}, z) = g'(\tau, z). \tag{193}$$

Utilizing the Fourier transform (58) to (193), we obtain

$$\lambda^2\hat{u} + 2i\lambda\hat{u} = \hat{g}'.$$

For $\lambda \notin \{0, -2i\}$, we have

$$\hat{u} = \frac{1}{\lambda(\lambda+2i)}\hat{g}' \equiv R(\lambda)\hat{g}'. \tag{194}$$

Introduce the quantity

$$h(k,\mu) = 1 + k - \mu. \tag{195}$$

Consider the case $k = 0$, $\mu = 0$. Then, $h(0,0) = 1$. Theorem 1.1 from Section 1 in [12] (see also Lemma 3.1 from [15]) yields.

Let $f + u_{,zz} \in L_2(\mathbb{R}_+)$, and $R(\lambda)$ does not have poles on the line $Im\ \lambda = 1$.

Then, we have

$$\int\limits_{-\infty+ih(0,0)}^{+\infty+ih(0,0)} \sum_{j=0}^{2} |\lambda|^{2(2-j)}|\hat{u}|^2 d\lambda \le c \int\limits_{-\infty+ih(0,0)}^{+\infty+ih(0,0)} |\hat{g}'|^2 d\lambda. \tag{196}$$

Using (60) and that $h(0,0) = 1$, we obtain

$$\int\limits_{\mathbb{R}} \sum_{j=0}^{2} |\partial_\tau^j u|^2 e^{2\tau} d\tau \le c \int\limits_{\mathbb{R}} |g'|^2 e^{2\tau} d\tau.$$

Passing to variables r and using the definition of g in (190) yields

$$\int\limits_{\mathbb{R}_+} \left(|u_{,rr}|^2 + \frac{1}{r^2}|u_{,r}|^2 + \frac{1}{r^4}|u|^2\right) rdr \le c \int\limits_{\mathbb{R}_+} |f + u_{,zz}|^2 rdr. \tag{197}$$

Using notation (185) and the estimate for the weak solutions, we obtain from (197) the inequality

$$\begin{aligned} &\int\limits_{\mathbb{R}_+\cap \operatorname{supp}\zeta^{(1)}} |\psi_{1,rr}^{(1)}|^2 + \frac{1}{r^2}|\psi_{1,r}^{(1)}|^2 + \frac{1}{r^4}|\psi_1^{(1)}|^2 rdr \\ &\le c \int\limits_{\mathbb{R}_+\cap \operatorname{supp}\zeta^{(1)}} |\omega_1|^2 rdr + c \int\limits_{\mathbb{R}_+\cap \operatorname{supp}\zeta^{(1)}} (|\psi_{1,r}|^2 + |\psi_1|^2) rdr \\ &\quad + c \int\limits_{\mathbb{R}_+\cap \operatorname{supp}\zeta^{(1)}} |\psi_{1,zz}|^2 rdr. \end{aligned} \tag{198}$$

For solutions to (187), we have the estimate

$$\begin{aligned} &\|w\|_{H^2((0,R)\cap \operatorname{supp}\zeta^{(2)})} \le c\|b\|_{L_2((0,R)\cap \operatorname{supp}\zeta^{(2)})} \\ &\quad + c\|w_{,r}\|_{L_2((0,R)\cap \operatorname{supp}\zeta^{(2)})}. \end{aligned} \tag{199}$$

In view of notation (185), we obtain

$$\begin{aligned} &\int\limits_{(0,R)\cap \operatorname{supp}\zeta^{(2)}} (|\psi_{1,rr}^{(2)}|^2 + |\psi_{1,r}^{(2)}|^2 + |\psi_1^{(2)}|^2) rdr \\ &\le c \int\limits_{(0,R)\operatorname{supp}\zeta^{(2)}} (|\omega_1|^2 + |\psi_{1,zz}|^2 + |\psi_{1,r}|^2 + |\psi_1|^2) rdr. \end{aligned} \tag{200}$$

Adding (198) and (200) and integrating the result with respect to z and using (56) yields

$$\begin{aligned}&\int\limits_{\Omega}\left(\psi_{1,rr}^2+\frac{1}{r^2}\psi_{1,r}^2+\frac{1}{r^4}\psi_1^2\right)dx\\&\leq c\int\limits_{\Omega}(|\omega_1|^2+|\psi_{1,zz}|^2)dx.\end{aligned}\tag{201}$$

Replacing ψ_1 by $\psi_{1,z}$ and ω_1 by $\omega_{1,z}$ we obtain, from (201) and (63), estimate (182). This ends the proof. □

Lemma 18. *Assume that ψ_1 is a solution to (61). Assume that $\mu \in (0,1)$, $\omega_1 \in H^1(\Omega)$, and $\Omega = (0,R) \times (-a,a)$.*
Then,

$$\begin{aligned}&\int\limits_{\Omega}\left(\psi_{1,rrr}^2+\frac{1}{r^2}\psi_{1,rr}^2+\frac{1}{r^4}\psi_{1,r}^2+\frac{1}{r^6}\psi_1^2\right)r^{2\mu}dx+\|\psi_1\|_{H^2(\Omega)}^2\\&+\int\limits_{\Omega}(\psi_{1,zrr}^2+\psi_{1,zzr}^2+\psi_{1,zzz}^2)dx\leq c\left(1+\frac{1}{\mu^2}\right)\|\omega_1\|_{H^1(\Omega)}^2.\end{aligned}\tag{202}$$

Proof. Recall the partition of unity introduced in the proof of Lemma 17. Recall also the local problems (183), (184), and notation (185). Then, we can examine problems (186) and (187). First, we examine problem (186).

Applying the Mellin transform any solution to (189) reads in the form (194).

In this case, we introduce the quantity

$$h(1,\mu)=2-\mu.\tag{203}$$

Since operator $R(\lambda)$ does not have poles on the line $\mathrm{Im}\,\lambda = h(1,\mu)$ we have (see Theorem 1.1 from Section 1 in [12]) (see also Lemma 3.1 from [15])

$$\int\limits_{-\infty+ih(1,\mu)}^{+\infty+ih(1,\mu)}\sum_{j=0}^{3}|\lambda|^{2(3-j)}|\hat{u}|^2d\lambda\leq c\int\limits_{-\infty+ih(1,\mu)}^{+\infty+ih(1,\mu)}\sum_{j=0}^{1}|\lambda|^{2(1-j)}|\hat{g}'|^2d\lambda.\tag{204}$$

Using (60) for $h(1,\mu)=2-\mu$, we obtain

$$\int\limits_{\mathbb{R}}\sum_{j=0}^{3}|\partial_\tau^j u|^2e^{2(3-j)\tau}d\tau\leq c\int\limits_{\mathbb{R}}\sum_{j=0}^{1}|\partial_\tau^j g'|^2e^{2(1-j)\tau}d\tau.\tag{205}$$

In view of equivalence (59), inequality (205) takes the form

$$\begin{aligned}&\int\limits_{\mathbb{R}_+}\left(|u_{rrr}|^2+\frac{1}{r^2}|u_{rr}|^2+\frac{1}{r^4}|u_r|^2+\frac{1}{r^6}|u|^2\right)r^{2\mu}rdr\\&\leq c\int\limits_{\mathbb{R}_+}|(f+u_{zz})_{,r}|^2r^{2\mu}rdr+c\int\limits_{\mathbb{R}_+}|f+u_{zz}|^2r^{2\mu-2}rdr,\end{aligned}\tag{206}$$

where $z \in (-a,a)$ and $\mu \in (0,1)$.

Integrating (206) with respect to z and exploiting notation (185) yields

$$
\begin{aligned}
&\int_{-a}^{a} dz \int_{\mathbb{R}_+\cap\operatorname{supp}\zeta^{(1)}} \left(|\psi_{1,rrr}^{(1)}|^2 + \frac{1}{r^2}|\psi_{1,rr}^{(1)}|^2 + \frac{1}{r^4}|\psi_{1,r}^{(1)}|^2 + \frac{1}{r^6}|\psi_1^{(1)}|^2\right) r^{2\mu} r dr \\
&\le c\int_{-a}^{a} dz \int_{\mathbb{R}_+\cap\operatorname{supp}\zeta^{(1)}} (|\partial_r(\omega_1+\psi_{1,zz})|^2 + |\omega_1+\psi_{1,zz}|r^{-2}) r^{2\mu} r dr.
\end{aligned} \tag{207}
$$

For solutions to problem (187) and notation (185), we obtain

$$
\begin{aligned}
&\int_{-a}^{a} dz \|\psi_1^{(2)}\|^2_{H^3_\mu(\mathbb{R}_+\cap\operatorname{supp}\zeta^{(2)})} \\
&\le c\int_{-a}^{a} dz (\|\omega_1\|^2_{H^1(\mathbb{R}_+\cap\operatorname{supp}\zeta^{(2)})} + \|\psi_{1,zz}\|^2_{H^1(\mathbb{R}_+\cap\operatorname{supp}\zeta^{(2)})}).
\end{aligned} \tag{208}
$$

From (207), (208), and the Hardy inequality (see [18] (Ch. 1, Sect. 2.16))

$$
\int_{\mathbb{R}_+} |\omega_1+\psi_{1,zz}|^2 r^{2\mu-2} r dr \le \frac{1}{\mu^2}\int_{\mathbb{R}_+} |(\omega_1+\psi_{1,zz})_{,r}|^2 r^{2\mu} r dr \tag{209}
$$

we obtain

$$
\begin{aligned}
&\int_\Omega \left(\psi_{1,rrr}^2 + \frac{1}{r^2}\psi_{1,rr}^2 + \frac{1}{r^4}\psi_{1,r}^2 + \frac{1}{r^6}\psi_1^2\right) r^{2\mu} dx \\
&\le c\left(1+\frac{1}{\mu^2}\right)\left[\|\omega_1\|^2_{H^1(\Omega)} + \int_\Omega (\psi_{1,zzr}^2 + \psi_{1,zzz}^2) dx\right].
\end{aligned} \tag{210}
$$

Using estimates (56), (62), and (63) in (210) implies (202) and ends the proof. □

Remark 8. *Since* $\mu > 0$, *the Hardy inequality (209) does not require that* $\omega_1 + \psi_{1,zz}|_{r=0} = 0$.

9. Conclusions

The main result of this paper is the proof of (24). Since $\Gamma = \omega_\varphi/r$, we obtain from (24) the estimate

$$
\|\omega_\varphi\|_{L_\infty(0,t;L_2(\Omega))} \le \phi(\text{data}), \tag{211}
$$

where we used that $r < R$ and R is finite. This means that (211) does not hold for the Cauchy problem.

Using problem (14) and relations (15), we obtain

$$
\begin{aligned}
&\|v'\|_{L_\infty(0,t;L_6(\Omega))} \le c\|\psi\|_{L_\infty(0,t;H^2(\Omega))} \\
&\le c\|\omega_\varphi\|_{L_\infty(0,t;L_2(\Omega))} \le \phi(\text{data}),
\end{aligned} \tag{212}
$$

where $v' = (v_r, v_z)$.

Consider the Stokes problem implied by (6)

$$
\begin{aligned}
&v_t - \nu\Delta v + \nabla p = -v'\cdot\nabla v + f && \text{in } \Omega^T, \\
&\operatorname{div} v = 0 && \text{in } \Omega^T, \\
&v\cdot\bar{n}|_S = 0,\ \ \omega_\varphi|_S = 0,\ \ v_\varphi|_{S_1} = 0,\ \ v_{\varphi,z}|_{S_2} = 0 && \text{on } S^T, \\
&v|_{t=0} = v(0) && \text{in } \Omega.
\end{aligned} \tag{213}
$$

Using (212) and the energy estimate (46), we have

$$
\|v'\cdot\nabla v\|_{L_2(0,t;L_{3/2}(\Omega))} \le \phi(\text{data}). \tag{214}
$$

Assuming more regularity on data that was needed in the proof of Theorem 1 and using [19] (see also [20]), we obtain the following estimate for solutions to problem (213):

$$\|v\|_{W^{2,1}_{3/2,2}(\Omega^t)} \le \phi(\text{data}). \tag{215}$$

By the imbedding and (215), we have

$$|\nabla v|_{5/2,\Omega^t} \le c\|v\|_{W^{2,1}_{3/2,2}(\Omega^t)} \le \phi(\text{data}). \tag{216}$$

The above inequality and (212) imply

$$|v' \cdot \nabla v|_{L_{5/2}(0,t;L_{30/17}(\Omega))} \le \phi(\text{data}). \tag{217}$$

Applying [19] (see also [20]), we obtain

$$\|v\|_{W^{2,1}_{30/17,5/2}(\Omega^t)} + |\nabla p|_{\frac{30}{17},\frac{5}{2},\Omega^t} \le \phi(\text{data}), \tag{218}$$

where an additional regularity on data is imposed.

This means that (24) implies any regularity of solutions to problem (6) assuming the appropriate regularity of data.

The existence of solutions can be proved by appropriately choosing a fixed-point theorem.

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Appendix A. Existence of Regular Local Solutions to (1)

Recall the quantities

$$u_1 = \frac{v_\varphi}{r}, \ \omega_1 = \frac{\omega_\varphi}{r}, \ \psi_1 = \frac{\psi}{r}, \ f_1 = \frac{f_\varphi}{r}, \ F_1 = \frac{F_\varphi}{r}. \tag{A1}$$

In view of the fact that [16] system (6) is equivalent to the following one:

$$\begin{aligned}
&u_{1,t} + v\cdot\nabla u_1 - \nu\left(\Delta u_1 + \frac{2}{r}u_{1,r}\right) = 2u_1\psi_{1,z} + f_1,\\
&\omega_{1,t} + v\cdot\nabla\omega_1 - \nu\left(\Delta\omega_1 + \frac{2}{r}\omega_{1,r}\right) = 2u_1u_{1,z} + F_1,\\
&-\Delta\psi_1 - \frac{2}{r}\psi_{1,r} = \omega_1,\\
&u_1|_{S_1} = 0, \ u_{1,z}|_{S_2} = 0, \ \psi_1|_S = 0, \ \omega_1|_S = 0,\\
&u_1|_{t=0} = u_1(0),\\
&\omega_1|_{t=0} = \omega_1(0).
\end{aligned} \tag{A2}$$

Multiplying $(A2)_1$ by $u_1|u_1|^2$, integrating over Ω, and using boundary conditions yields

$$\frac{d}{dt}|u_1|^4_{4,\Omega} + \nu|u_1|^4_{4,\Omega} \le c|\omega_1|^2_{2,\Omega}|u_1|^4_{4,\Omega} + c|f_1|^4_{4,\Omega}. \tag{A3}$$

Multiply $(A2)_2$ by ω_1, integrate over Ω, and exploit the boundary conditions. Then, we have

$$\frac{d}{dt}|\omega_1|^2_{2,\Omega} + \nu|\omega_1|^2_{2,\Omega} \le c|u_1|^4_{4,\Omega} + c|F_1|^2_{2,\Omega}. \tag{A4}$$

Introduce the quantity

$$\bar{X}(t) = |u_1(t)|^4_{4,\Omega} + |\omega_1(t)|^2_{2,\Omega}. \tag{A5}$$

Then, (A3) and (A4) imply

$$\frac{d}{dt}\bar{X} + \nu\bar{X} \le c_0\bar{X}^2 + G, \tag{A6}$$

where

$$G(t) = c(|f_1(t)|^4_{4,\Omega} + |F_1(t)|^2_{2,\Omega}). \tag{A7}$$

Lemma A1. *Assume that $f_1 \in L_4(\Omega^t)$, $F_1 \in L_2(\Omega^t)$, $u_1(0) \in L_4(\Omega)$, $\omega_1 \in L_2(\Omega)$, and $t \le T$. Let $H(t) = |f_1|^4_{4,\Omega^t} + |F_1|^2_{2,\Omega^t} + |u_1(0)|^4_{4,\Omega} + |\omega_1(0)|^2_{2,\Omega}$. Assume that T is so small that*

$$T < \frac{1}{4cc_0H(T)},$$

where c appears in (A3), (A4), and c_0 in (A6).

Then, for $t \le T$ a local solution to the problem (A2) exists such that $u_1 \in L_\infty(0,t;L_4(\Omega))$, $\omega_1 \in L_\infty(0,t;L_4(\Omega))$, $t \le T$, and

$$\sup_t(|u_1(t)|^4_{4,\Omega} + |\omega_1(t)|^2_{2,\Omega}) \le 2cH \equiv \phi_1. \tag{A8}$$

Proof. Integrating (A6) with respect to time and introducing the quantity

$$X(t) = \sup_{t' \le t}\bar{X}(t') \tag{A9}$$

we obtain

$$X(t) + \nu\int_0^t \bar{X}(t')dt' \le c_0t^2X^2(t) + cH(t). \tag{A10}$$

Dropping the second term on the l.h.s. yields

$$X(t) \le c_0t^2X^2(t) + cH(t). \tag{A11}$$

Let X' be such that

$$X' = c_0t^2(X')^2 + cH(t) \tag{A12}$$

and $X \le X'$. To show the existence of solutions to (A12), we use the following method of successive approximations:

$$X'_{n+1} = c_0t^2X'^2_n + cH$$

Assume that $X'_n \le 2cH$ and t is so small that $4cc_0t^2H \le 1$. Assuming that $X'_0 = 0$, we obtain

$$X'_n \le 2cH \quad \text{for all } n \in \mathbb{N}. \tag{A13}$$

To show convergence, we introduce the differences $Y'_n = X'_n - X'_{n-1}$, which are solutions to the problem

$$Y'_{n+1} \le c_0t(X'_n + X'_{n-1})Y'_n. \tag{A14}$$

Assuming that t is so small that $4cc_0tH < 1$, we have the convergence of the sequence $\{X'_n\}$.

Hence, a solution to problem (A2) exists such that $u_1 \in L_\infty(0,t;L_4(\Omega))$, $\omega_1 \in L_\infty(0,t;L_2(\Omega))$ and (A8) holds. This ends the proof. □

Lemma A2. *Let the assumptions of Lemma A1 hold. Let* $f \in W_2^{1,1/2}(\Omega^t)$, $v(0) \in W_2^2(\Omega)$. *Then, there exists a solution to problem (6) such that* $v \in W_2^{3,3/2}(\Omega^t)$, $\nabla p \in W_2^{1,1/2}(\Omega^t)$ *and the estimate holds*

$$\begin{aligned} &\|v\|_{W_2^{3,3/2}(\Omega^t)} + \|\nabla p\|_{W_2^{1,1/2}(\Omega^t)} \le c(\|f\|_{W_2^{1,1/2}(\Omega^t)} \\ &\quad + \|v(0)\|_{W_2^2(\Omega)} + \phi(\|f\|_{W_2^{1,1/2}(\Omega^t)}, \|v(0)\|_{W_2^2(\Omega)}, H(t)), \end{aligned} \tag{A15}$$

where $t \le T$, *and where* ϕ *is an increasing positive function.*

Proof. From (A8), we have

$$|v'|_{6,\infty,\Omega^t} \le c\|\psi_1\|_{2,\infty,\Omega^t} \le c|\omega_1|_{2,\infty,\Omega^t} \le \phi_1, \tag{A16}$$

where $v' = (v_r, v_z)$. From (46), we have

$$|\nabla v|_{2,\Omega^t} \le D_1. \tag{A17}$$

Estimates (A16) and (A17) imply

$$|v' \cdot \nabla v|_{\frac{3}{2},2,\Omega^t} \le \phi_1 D_1. \tag{A18}$$

Now, we consider the Stokes problem

$$\begin{aligned} &v_{,t} - \nu\Delta v + \nabla p = -v' \cdot \nabla v + f, \\ &\operatorname{div} v = 0, \\ &v \cdot \bar{n}|_S = 0, \ \ (v_{r,z} - v_{z,r})|_S = 0, \\ &v_\varphi|_{S_1} = 0, \ \ v_{\varphi,z}|_{S_2} = 0, \\ &v|_{t=0} = v(0). \end{aligned} \tag{A19}$$

Applying the theory of Sobolev spaces with mixed norm developed in [19], we have the existence of solutions to (A19) and the estimate

$$\begin{aligned} &\|v\|_{W^{2,1}_{\frac{3}{2},2}(\Omega^t)} + |\nabla p|_{\frac{3}{2},2,\Omega^t} \le c(|f|_{\frac{3}{2},2,\Omega^t} \\ &\quad + \|v(0)\|_{B^1_{\frac{3}{2},2}(\Omega)} + \phi_1 D_1) \equiv D_2. \end{aligned} \tag{A20}$$

In view of the imbedding

$$|\nabla v|_{\frac{5}{2},\Omega^t} \le c\|v\|_{W^{2,1}_{\frac{3}{2},2}(\Omega^t)}$$

we obtain

$$|v' \cdot \nabla v|_{\frac{30}{17},\frac{5}{2},\Omega^t} \le \phi_1 D_2.$$

Since

$$L_{\frac{5}{2}}(0,t;L_{\frac{30}{17}}(\Omega)) \subset L_2(0,t;L_{\frac{3}{2}}(\Omega))$$

we have an increase in the regularity of solutions to (A19). Continuing the considerations, we obtain (A15). This ends the proof. □

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