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VARMA Models with Single- or Mixed-Frequency Data: New Conditions for Extended Yule–Walker Identification

Celina Pestano-Gabino *, Concepción González-Concepción and María Candelaria Gil-Fariña

Department of Applied Economics and Quantitative Methods, Universidad de La Laguna (ULL), 38200 San Cristóbal de La Laguna, Spain; cogonzal@ull.edu.es (C.G.-C.); mgil@ull.edu.es (M.C.G.-F.)
* Correspondence: cpestano@ull.edu.es

Abstract: This paper deals with the identifiability of VARMA models with VAR order greater than or equal to the MA order, in the context of mixed-frequency data (MFD) using extended Yule–Walker equations. The main contribution is that necessary and sufficient conditions for identifiability in the single-frequency data case are expressed in an original way and yield new results in the MFD case. We also provide two counterexamples that answer an open question in this topic about whether certain sufficient conditions are necessary for identifiability. Therefore, this paper expands the set of models that can be identified with MFD using extended Yule–Walker equations. The main idea is that with MFD, some autocovariance blocks are not available from observed variables and, in some cases, the new conditions in this paper can be used to reconstruct all the non-available covariance blocks from available covariance blocks.

Keywords: VARMA model; mixed-frequency data; identification; extended Yule–Walker method; Hankel matrix; autocovariance

MSC: 37M10; 62H12; 93B30; 62M10; 15B99

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1. Introduction

In time series models, developing statistically efficient and computationally quick methods for identifying and estimating VAR or VARMA models with single-frequency data (SFD) or mixed-frequency data (MFD) is an important task. In this paper, "SFD means that all the variables of a model are observed at the same discrete-time frequency at which the model operates, and MFD means that some of the variables are observed at the same discrete-time frequency at which the model operates, and others are observed at one or more lower frequencies" [1].

Linear algebra tools are extensively used to study transfer functions, Yule–Walker equations and Hankel matrices associated with identifying and estimating VAR or VARMA models (see, for instance, [1–13]). In particular, the extended Yule–Walker method (XYW) was proposed by [5], and it is considered in several later papers, in particular in [2,3,8] for estimating VAR models from available covariance matrices of MFD. Two of the principal and parallel strands of this literature which extend the method to the case of VARMA models are [1,2]. The first considers both exact and generic identification and the second considers only exact identification. They make both common and differing assumptions involving the parameters of a VARMA model and prove that their assumptions are, as a whole, sufficient to identify all of the parameters of a VARMA model with MFD. In particular, [1] questions whether its "conditions as a whole are necessary for identification". This question has motivated our work, and our aim is to expand the methodology to more identifiable models.

This paper is focused on identification. In econometrics and statistics, identification means the coefficients (parameters) of a model are determined uniquely from data covariances (and higher moments, depending on the data distribution) under assumed conditions for the coefficients of a model.

Our aim is to provide conditions for the identifiability of VARMA models that cannot be identified by following the procedure in [1] and, in this regard, to complement it. We use a subset of conditions in [1] and some more, and we prove with two counterexamples that the whole set of conditions in [1] is not necessary. The main results are obtained from subsystems of extended Yule–Walker equations. Therefore, we expand the set of models identified with extended Yule–Walker methods.

Because with MFD some autocovariance blocks are not available from observed variables, the main idea of our work is to provide conditions that, in addition to ensuring that the model with SFD is identifiable, allow rebuilding the unknown blocks from available covariance blocks. This thus yields all the complete autocovariances. As is well known, there is a bijection between the covariance and the corresponding spectral density of the process (see, for instance, [6]), meaning we can ensure the identifiability of the model.

Section 2 summarizes and comments on the sufficient conditions used in [1] to identify VARMA models with SFD and MFD. In Section 3, we prove our main theoretical results. With a suitable change, our paper expands the applicability of XYW methods to more identifiable models. Section 4 tests and illustrates our main insight with two counterexamples. We close this paper with Conclusions, References and an Appendix with a MATLAB subroutine used in a counterexample.

2. The Six Sufficient Conditions for Identification in [1]

Zadrozny, in [1], considers the VARMA(r, q) model

$$y_t = A_1 y_{t-1} + \ldots + A_r y_{t-r} + B_0 \varepsilon_t + B_1 \varepsilon_{t-1} + \ldots + B_q \varepsilon_{t-q} \tag{1}$$

where y_t denotes an $n \times 1$ vector of observed variables; $p = \max\{r, q+1\}$; A_i for $i = 1, \ldots, p$ and B_j for $j = 0, 1, \ldots, p-1$ are $n \times n$ matrices, $A_r \neq 0$, $B_q \neq 0$, $A_i = 0$ if $i = r+1, \ldots, p$; $B_i = 0$ if $i = q+1, \ldots, p-1$; and ε_t denotes an $n \times 1$ white noise vector.

In order to express the conditions in [1], and others in this paper, we need the following notation:

$$\boldsymbol{a(z)} = I - A_1 z - \ldots - A_p z^p,\ \boldsymbol{b(z)} = B_0 + B_1 z + \ldots + B_q z^q, \text{ where } z \in \mathbb{C},$$

$$\boldsymbol{F} = \begin{pmatrix} A_1 & I_n \cdots & 0 \\ \vdots & 0 \ddots & \vdots \\ \vdots & \vdots \ddots & I_n \\ A_p & 0 \cdots & 0 \end{pmatrix} \text{is } np \times np, \boldsymbol{G} = \begin{pmatrix} B_0 \\ \vdots \\ B_{p-1} \end{pmatrix} \text{is } np\ n, \boldsymbol{H} = (I_n, 0_{n\times n}, \ldots, 0_{n\times n}) \text{is } n \times np,$$

$$\boldsymbol{C_L(F, G)} = [G\ FG\ F^2G \ldots F^{L-1}G] \text{ is } np \times nL \text{ and}$$

$$\boldsymbol{O_L(F, H)} = [H^t\ (HF)^t (HF^2)^t \ldots (HF^{L-1})^t]^{\mathrm{t}} \text{ is } nL \times np, \text{ for } L = 1, 2, \ldots$$

Assuming that the VARMA(r, q) model is stationary and that $\sum = \mathrm{E}(\varepsilon_t \varepsilon_t^t)$ is positive definite:

$\boldsymbol{K(z)} = a^{-1}(z) b(z) = \sum_{i=0}^{\infty} K_i z^i$ is the transfer function,

$\boldsymbol{C_i} = \mathrm{E}(y_t y_{t-i}^T) = \sum_{j=0}^{\infty} K_{i+j} \sum K_j^t$ for $i \in \mathbb{Z}$, is the i-th population covariance matrix.

To study mixed-frequency data cases (with stock variables), we consider $n = n_1 + n_2$ variables; the first $\boldsymbol{n_1}$ variables are high-frequency variables observed in every period and, given $N \in \mathbb{N}$, the last $\boldsymbol{n_2}$ variables are low-frequency variables observed only for $t \in \{0, N, 2N, 3N, \ldots\}$. Furthermore, we consider the following partition:

$$\boldsymbol{C_i} = \begin{pmatrix} \boldsymbol{C_i^{ff}} & \boldsymbol{C_i^{fs}} \\ \boldsymbol{C_i^{sf}} & \boldsymbol{C_i^{ss}} \end{pmatrix},$$

where C_i^{ff}, C_i^{fs}, C_i^{sf} *and* C_i^{ss} are $n_1 \times n_1$, $n_1 \times n_2$, $n_2 \times n_1$ and $n_2 \times n_2$ blocks, respectively.

Note that all the covariance blocks are available from variables observed with MFD, except C_i^{ss} if $I \neq kN$, for any integer k. However, the XYW method in [1] only considers the first n_1 columns in each C_i. Therefore, let us denote:

$\widetilde{\boldsymbol{C}}_{\boldsymbol{i}}$ as the first n_1 columns of C_i, for $i \in \mathbb{Z}$;

$\boldsymbol{H_1}$ as the matrix with the first n_1 rows of H;

$\boldsymbol{H_2}$ as the matrix with the last n_2 rows of H.

Zadrozny, in [1], proves that under the following six sufficient conditions, a VARMA model (1) is identified by the population covariance of its variables observed with MFD.

Condition I: VARMA model (1) is stationary, i.e., det $a(z) \neq 0$ if $|z| \leq 1$.

Condition II: VARMA model (1) is regular with B_0 lower triangular and non-singular and $\sum = I_n$.

Condition III: VARMA model (1) is miniphase, i.e., det $b(z) \neq 0$ if $|z| < 1$.

Condition IV: rank $C_{np}(F, G) = np$.

Condition V: rank $C_L(F, V H_1^t) = np$ and rank $O_L(F, H_1) = np$, for sufficiently large L, where $V = \sum_{k=0}^{\infty} F^k G \sum G^t \left(F^t\right)^k = [C_{np}(F, G)\ldots]\,[C_{np}(F, G)\ldots]^t$ which exists because the model is stationary.

Note on Condition V: Condition V in [1] (p. 441) reads: "VARMA model (2.1) is observable for sufficiently large L, for the MFD being considered". Condition V written above can be read in [1] (p. 445): "Sections 3 and 4 proved that parameters of VARMA model (2.1) are identified (…) for MFD if $C_L(F, V H_1^t)$ and $O_L(F, H_1)$ have full rank, (…) for sufficiently large L", because to identify VAR parameters with the specific procedure in [1] (pp. 441–442), rank $D_1 = np$ is necessary, and therefore the full rank of $C_L(F, V H_1^t)$. (We have taken into account that matrices D_1 and E_1 in [1] (p. 442) must be written without a subscript in the second $\widetilde{H}$). Moreover, to identify VMA parameters with the specific procedure in [1] (pp. 442–444), the full rank of $O_L(F, H_1)$ is necessary.

Condition VI: The $nq \times nq$ matrix $\begin{pmatrix} -B_1 B_0^{-1} & I_n & & 0 \\ \vdots & 0 & \ddots & \\ & \vdots & & I_n \\ -B_q B_0^{-1} & 0 & \cdots & 0 \end{pmatrix}$ is diagonalizable, i.e., it has a linearly independent set of eigenvectors.

Remark 1. *Zadrozny, in [1], proves that Conditions I, II, III and IV are sufficient for identifiability with SFD. However, if Conditions I, II and III hold, Condition IV is sufficient but not necessary in the SFD case (see Counterexamples 1 and 2).*

Deistler et al. in [6] prove that if $q > r$, a VARMA(r, q) model with MFD is not identifiable. Next, we will show that rank O_L (F, H_1) = np in Condition V excludes not only the case $q > r$, but also the case $q = r$.

Lemma 1. *If $q \geq r$, then rank $O_L(F, H_1) < np$.*

Proof. Considering that if $q \geq r$ then $p = q + 1$ and $A_{r+1} = \ldots = A_p = 0$.

On the one hand, the first $n(p-r)$ columns in the $n(p-r) \times np$ matrix

$$(H^t (HF)^t (HF^2)^t \cdots\ (HF^{p-r-1})^t)^t$$

form a lower triangular matrix with ones on the diagonal.

Therefore, its rank is exactly $n(p-r)$. As a result

$$\mathrm{rank}(H_1^t (H_1 F)^t (H_1 F^2)^t \cdots (H_1 F^{p-r-1})^t)^t = n_1(p-r). \tag{2}$$

On the other hand, considering the $nr \times np$ matrix$((HF^{p-r})^t \cdots (HF^{p-1})^t)^t$, it is easy to see that the submatrix formed by its last nr columns is lower triangular with ones on the diagonal, and therefore,

$$\mathrm{rank}((HF^{p-r})^t \cdots (HF^{p-1})^t)^t = nr. \tag{3}$$

Moreover, if $i \geq p$, $HF^i = A_1HF^{i-1} + \ldots + A_rHF^{i-r}$ and, from (3), rank $((HF^{p-r})^t \cdots (HF^{L-1})^t)^t = nr$.

As a consequence:

$$\mathrm{rank}((H_1F^{p-r})^t \cdots (H_1F^{L-1})^t)^t \leq nr \qquad \text{if } L > p \tag{4}$$

From (2) and (4), rank $O_L(F, H_1) \leq n_1(p-r) + nr = n_1p - n_1r + (n_1 + n_2)r = n_1p + n_2r < np$ and Lemma 1 has been proven. □

Note that if we do not have the hypothesis from Lemma 1, i.e., if $q < r$, then $p = r$ and we do not have (2). Therefore, rank $O_L(F, H_1)$ could be equal to np.

3. Reconstructing Missing Blocks in Autocovariance Matrices

Our goal in this work is to extend the set of models that can be identified with Yule–Walker methods in the MFD case. We provide new conditions to identify the VARMA model in the MFD case from an original form of rewriting necessary and sufficient conditions in the SFD case. These conditions are expressed based on the parameters of the model. We will demonstrate with two counterexamples that the sufficient conditions in [1] are not necessary to identify MFD models. Thus, we consider one of the questions opened in [1] to be resolved.

In our proposal, we treat the cases $r > q$ and $r = q$ separately, giving new conditions to replace the corresponding ones in [1]. In addition, the fourth and fifth conditions are different in each of the two cases.

3.1. Case I: $r > q$

In this case, our main result will be Theorem 1. We previously introduced the necessary notation, the new conditions and some previous results (Hanzon's Theorem and Lemma 2).

Let us denote the following matrices:

$$\boldsymbol{G^*} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ B_0 \\ \vdots \\ B_q \end{pmatrix} \text{ and } \boldsymbol{X} = \begin{pmatrix} C_q \\ C_{q-1} \\ \vdots \\ C_{q-r+1} \end{pmatrix} \text{ are } nr \times n, \boldsymbol{F^*} = \begin{pmatrix} A_1 & A_2 & & \cdots & A_r \\ I_n & 0 & & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & 0 & 0 \\ 0 & \cdots & 0 & I_n & 0 \end{pmatrix} \text{is } nr \times nr,$$

$$\boldsymbol{\theta} = \begin{pmatrix} H_1F^* \\ \vdots \\ H_1(F^*)^{nr} \\ H_2(F^*)^{kN-q} \\ H_2(F^*)^{kN-q}(F^*)^N \\ \vdots \\ H_2(F^*)^{kN-q}(F^*)^{(nr-1)N} \end{pmatrix} \text{is } n^2r \times nr, J = \begin{pmatrix} C^{ff}_{q+1} & C^{fs}_{q+1} \\ & \vdots \\ C^{ff}_{q+nr} & C^{fs}_{q+nr} \\ C^{sf}_{kN} & C^{ss}_{kN} \\ C^{sf}_{(k+1)N} & C^{ss}_{(k+1)N} \\ & \vdots \\ C^{sf}_{(k+nr-1)N} & C^{ss}_{(k+nr-1)N} \end{pmatrix} \text{is } n^2r \times n,$$

where k is an integer such that $kN > q$.

$\boldsymbol{\theta_b}$ is the submatrix formed with columns of θ, such that the ith column of θ is in$\boldsymbol{\theta_b}$, if $I \neq 1, \ldots, n_1, n+1, \ldots, n+n_1, 2n+1, \ldots, 2n+n_1, \ldots, (r-1)n+1, \ldots, (r-1)n+n_1$ and also the ith row of $X = \begin{pmatrix} C_q \\ C_{q-1} \\ \vdots \\ C_{q-r+1} \end{pmatrix}$ is not a row of C_j or C_{-j} with $j \in \{0, N, 2N, 3N \ldots\}$.

$\boldsymbol{\theta_a}$ is the submatrix of θ with the columns that are not in θ_b.

Below we set out the new conditions. Note that although [1] refers to a state space model associated with the VARMA model, we only need to note the algebraic properties that hold in certain matrices constructed from the parameters of the VARMA model.

Condition ii: The VARMA model is regular with B_0 = I and $\sum$ positive definite.

Condition iv.1: rank$C_{nr}(F, G^*) = nr$.

Condition v.1: rank $C_{nr}(F, V^*(F^t)^{r-q-1}H_1^t)$ = nr where

$V^* = \sum_{k=0}^{\infty} F^k G^* \sum (G^*)^t (F^t)^k = [C_{np}(F, G^* \sum^{1/2}) \ldots] [C_{np}(F, G^* \sum^{1/2}) \ldots]^t$, which exists because the model is stationary.

Condition vi:θ_b is full column rank.

We will also make use of [11] (Theorem 3.1.3.2–1 (iii), Theorem 3.1.2.3–29 (iii) and Corollary 2.4.3–25), which we summarize in the following theorem that we call Hanzon's Theorem.

Hanzon's Theorem: Considering $K_{-i} = 0$ if $i > 0$, for $i, j, h \in \mathbb{N}$, we denote

$$\boldsymbol{M_{i,j,h}} = \begin{pmatrix} K_i & K_{i+1} \cdots & K_{i+h-1} \\ K_{i+1} & K_{i+2} \ddots & K_{i+h} \\ \vdots & \vdots \ddots & \vdots \\ K_{i+j-1} & \cdots & K_{i+j+h-2} \end{pmatrix} \text{and}$$

$$\boldsymbol{Q_{i,j,h}} = \begin{pmatrix} C_i & C_{i+1} \cdots & C_{i+h-1} \\ C_{i+1} & C_{i+2} \ddots & C_{i+h} \\ \vdots & \vdots \ddots & \vdots \\ C_{i+j-1} & \cdots & C_{j+j+h-2} \end{pmatrix}$$

If model (1) is stationary, regular and miniphase, the following conditions are satisfied for any orders r and q:

1. rank $M_{q-r+1,\infty,\infty}$ = rank $M_{q-r+1,r,nr}$
2. rank $Q_{q-r+1,r,\infty}$ = rank $Q_{q-r+1,r,nr}$
3. rank $M_{q-r+1,r,nr}$ = rank $Q_{q-r+1,r,nr}$

As a consequence, we can deduce the following Lemma.

Lemma 2. *Suppose $r > q$ and Conditions I, II or ii, and III hold. Therefore, Condition IV implies Condition iv.1.*

Proof. We can easily see that

$$K_i = HF^i G \text{ for } i \geq 0$$

and $M_{0,r,nr} = O_r(F, H)\, C_{nr}(F, G)$. Taking into account that $O_r(F, H)$ is full column rank nr and, from Condition IV, $C_{nr}(F, G)$ is full row rank nr, then rank $M_{0,r,nr} = nr$.

By Hanzon's Theorem, rank $M_{0,r,nr}$ = rank $M_{0,r,\infty} = nr$, rank $M_{q-r+1,r,nr}$ = rank $M_{q-r+1,r,\infty}$. Since all the columns of $M_{0,r,\infty}$ are columns of the matrix $M_{q-r+1,r,\infty}$, rank $M_{q-r+1,r,\infty} \geq$ rank $M_{0,r,\infty}$. Since the matrix has nr rows, rank $M_{q-r+1,r,\infty} = nr$. By Hanzon's Theorem, rank $M_{q-r+1,r,nr}$ = rank $M_{q-r+1,r,\infty} = nr$.

We can easily see that

$$K_j = HF^{r-q-1+j}G^* \text{ for } j \geq q-r+1, \text{ (note that } q-r+1 \leq 0)$$

and $M_{q-r+1,r,nr} = O_r(F, H)\, C_{nr}(F, G^*)$. Considering that $M_{q-r+1,r,nr}$ is full rank and $O_r(F, H)$ is full column rank nr, then $C_{nr}(F, G^*)$ is full row rank nr, i.e., Condition iv.1 holds. □

In light of the above, we can now state the following Theorem.

Theorem 1. *If $r > q$ and Conditions I, II or ii, and III hold:*

(a) *Condition iv.1 is necessary and sufficient for identifiability of the VARMA(r, q) model (1) in the SFD case.*

(b) *Conditions iv.1, v.1 and vi are sufficient for identifiability of the VARMA(r, q) model (1) in the MFD case.*

Proof. For $j \geq 0$, $C_{q-r+1+j} = \sum_{i=0}^{\infty} \boldsymbol{K}_{i+q-r+1+j} \sum \boldsymbol{K}_i^t = \sum_{i=0}^{\infty} \boldsymbol{H}\boldsymbol{F}^{i+j}\boldsymbol{G}^* \sum \boldsymbol{G}^{*t}(\boldsymbol{F}^{i-q+r-1})^t\boldsymbol{H}^t = HF^j(\sum_{i=0}^{\infty} \boldsymbol{F}^i\boldsymbol{G}^*\sum \boldsymbol{G}^{*t}(\boldsymbol{F}^i)^t)(\boldsymbol{F}^{-q+r-1})^t\boldsymbol{H}^t$.

Therefore, denoting $V^* = \sum_{i=0}^{\infty} F^iG^* \sum G^{*t}\left(F^i\right)^t$,

$$C_{q-r+1+j} = H\, F^j V^*(F^{-q+r-1})^t H^t \text{ for } j \geq 0.$$

Taking into account that $M_{q-r+1,r,nr} = O_r(F, H)\, C_{nr}(F, G^*)$, $O_r(F, H)$ is full column rank nr and, from Condition iv.1, we have that $C_{nr}(F, G^*)$ is full row rank nr, then rank $M_{q-r+1,r,nr} = nr$.

From Hanzon's Theorem, rank $M_{q-r+1,r,nr}$ = rank $Q_{q-r+1,r,nr} = nr$, and therefore, under Conditions I, II or ii, III and iv.1, a VARMA model is identified with population covariance of its variables observed with SFD. Note that $(A_1, A_2, \ldots, A_r)$ can be uniquely identified from the autocovariance matrices of the process, solving the following system:

$$(A_r \ldots\ A_1)Q_{q-r+1,r,nr} = \left(C_{q+1}C_{q+2}\ldots C_{q+nr}\right).$$

Taking into account that $Q_{q-r+1,r,nr} = O_r(F, H)\, C_{nr}(F, V^*(F^t)^{r-q+1}H^t)$, rank $Q_{q-r+1,r,nr} = nr$ and $O_r(F, H)$ is full column rank nr then

$$C_{nr}(F, V^*(F^t)^{r-q+1}H^t) \text{ is full row rank } nr.$$

If we substitute H for H_1 in the autocovariances

$$\widetilde{C}_{j+q-r+1} = HF^jV^*(F^{-q+r-1})^tH_1^t \text{ for } j\ \geq 0.$$

Denoting

$$\widetilde{Q}_{q-r+1,r,L} = \begin{pmatrix} \widetilde{C}_{q-r+1} & \widetilde{C}_{q-r+2}\cdots & \widetilde{C}_{q-r+L} \\ \widetilde{C}_{q-r+2} & \widetilde{C}_{q-r+3}\iddots & \widetilde{C}_{q-r+L+1} \\ \vdots & \vdots\ \ddots & \vdots \\ \widetilde{C}_q & \widetilde{C}_{q+1}\cdots & \widetilde{C}_{q+L-1} \end{pmatrix}$$

we have

$$\widetilde{Q}_{q-r+1,r,L} = O_r(F, H)\; C_L(F, V * (F^t)^{r-q+1}H_1^t).$$

Keeping Hamilton–Cayley in mind, rank $C_L(F, V^*(F^t)^{r-q+1}H_1^t)$ does not change when $L > nr$.

Condition v.1 implies rank$\widetilde{Q}_{q-r+1,r,nr} = nr$, and therefore $(A_1, A_2, \ldots, A_r)$ can be uniquely identified from available autocovariance matrices of the process in the MFD case, solving the system $(A_r \ldots\ A_1)\widetilde{Q}_{q-r+1,r,nr} = (\widetilde{C}_{q+1}\widetilde{C}_{q+2}\ldots\widetilde{C}_{q+nr})$ as follows:

$$(A_r \ldots\ A_1) = (\widetilde{C}_{q+1}\widetilde{C}_{q+2}\ldots\widetilde{C}_{q+nr})\widetilde{Q}^T_{q-r+1,r,nr}[\widetilde{Q}_{q-r+1,r,nr}\widetilde{Q}^T_{q-r+1,r,nr}]^{-1} \tag{5}$$

The last part of the proof of this theorem aims to show that, once $(A_1, A_2, ..., A_r)$ are calculated, if Condition vi holds, we can uniquely reconstruct the unknown blocks of the autocovariance matrices.

Note that $H(F^*)^i X = C_{q+i}$ for $i > 0$, i.e., $H_1(F^*)^i X = \begin{pmatrix} C_{q+i}^{ff} & C_{q+i}^{fs} \end{pmatrix}$ and $H_2(F^*)^i X = \begin{pmatrix} C_{q+i}^{sf} & C_{q+i}^{ss} \end{pmatrix}$. Therefore, $J = \theta X$ and, in particular, $J_2 = \theta X_2$, where J_2 and X_2 are the submatrices formed by the last n_2 columns of J and X, respectively.

Note that, keeping Hamilton–Cayley in mind, θ has been defined such that its rank does not change when some rows of $H(F^*)^i$are added to θ, for some $i > nr$. Moreover, J has been defined only with available autocovariance blocks. If we rearrange the rows of X_2 such that $\theta X_2 = (\theta_a\ \theta_b) \begin{pmatrix} X_{2a} \\ X_{2b} \end{pmatrix}$, we make it such that X_{2b} is the submatrix containing the unknown blocks. To calculate X_{2b}, we solve the following system of linear equations:

$$J_2 - \theta_a X_{2a} = \theta_b X_{2b}, \tag{6}$$

for Condition vi, (6) has as a unique solution

$$X_{2b} = \left(\theta_b^t \theta_b\right)^{-1} \theta_b^t (J_2 - \theta_a X_{2a}).$$

As a consequence, we have C_i^{ss} for $i = q - r + 1, \ldots, q$ and we can calculate C_i^{ss} for $i > q$, considering that $C_i = A_1 C_{i-1} + \ldots + A_r C_{i-r}$ for $i > q$.

Therefore, Theorem 1 has been proven. □

The following Corollary 1 is a consequence of section (a) of the previous Theorem and of the Theorem in [9].

Corollary 1. *Suppose $r > q$ and that Conditions I, ii and III hold. In this case: rank$(A_r \vdots B_q) = n$ and $(a(z), b(z))$ is left coprime iff rank$C_{nr}(F, G^*) = nr$.*

Note that, unlike [1], we can consider rank $A_r < n$ or rank $B_q < n$.

3.2. Case 2: $r = q$

In this section, our main result will be Theorem 2.

For this case, neither G nor G^* allow us to state sufficient conditions similar to those in Theorem 1. Therefore, we consider the following $nr \times n$ matrix:

$$\boldsymbol{G^{**}} = \begin{pmatrix} A_1 + B_1 \\ \vdots \\ A_r + B_r \end{pmatrix},$$

and, considering $p = r$, we state the following conditions, which change with respect to Case 1.

Condition iv.2: rank $C_{nr}(F, G^{**}) = nr$.

Condition v.2: rank $C_{nr}(F, \left(G^{**} \sum \widetilde{B_0^t} + \mathrm{F}V^{**} H_1^t\right)) = nr$ where $\widetilde{B_0^t}$ denotes the first n_1 columns of B_0^t and $V^{**} = \sum_{i=0}^{\infty} F^i G^{**} \sum G^{**t} \left(F^i\right)^t$, which exists because the model is stationary.

We are in a position to state the following theorem.

Theorem 2. *If $r = q$ and Conditions I, II or ii, and III hold:*

(a) *Condition iv.2 is necessary and sufficient for identifiability of the VARMA(r, r) model (1) in the SFD case.*

(b) *Conditions iv.2, v.2 and vi are sufficient for identifiability of the VARMA(r, r) model (1) in the MFD case.*

Proof. The proof of Theorem 2 is similar to that of Theorem 1, except for some specific details, because with G^{**}, we have that

$$K_{j+1} = HF^{j}G^{**} \text{ if } j \geq 0,$$

$$C_j = HF^{j-1}G^{**}\sum B_0^t + HF^j\sum_{i=0}^{\infty} F^iG^{**}\sum G^{**t}\left(F^i\right)^t H^t = HF^{j-1}\left(G^{**}\sum B_0^t + FV^{**}H^t\right)$$
$$\text{for } j \geq 1 \text{ with } V^{**} = \sum_{i=0}^{\infty} F^iG^{**}\sum G^{**t}\left(F^i\right)^t.$$

Taking into account that

$$M_{1,r,nr} = O_r(F, H)\, C_{nr}(F, G^{**}) \text{ and } Q_{1,r,nr} = O_r(F, H)\, C_{nr}(F, (G^{**}\textstyle\sum B_0{}^t + FV^{**}H^t)),$$

Conditions I, II or ii, III and iv.2 imply the full rank nr of the matrices $M_{1,r,nr}$, $Q_{1,r,nr}$ and $C_{nr}(F, (G^{**}\sum B_0^t + FV^{**}H^t))$.

If in the autocovariances we substitute H for H_1 and B_0^t for $\widetilde{B}_0^t$:
$\widetilde{C}_{j+q-r+1} = HF^{j-1}(G^{**}\sum \widetilde{B}_0^t + FV^{**}H_1^t)$ for $j \geq 1$

$$\widetilde{Q}_{1,r,nr} = \begin{pmatrix} \widetilde{C}_1 & \widetilde{C}_2\cdots & \widetilde{C}_{nr} \\ \widetilde{C}_2 & \widetilde{C}_3\ddots & \widetilde{C}_{nr+1} \\ \vdots & \vdots\ \ddots & \vdots \\ \widetilde{C}_r & \widetilde{C}_{r+1}\cdots & \widetilde{C}_{nr+r-1} \end{pmatrix} \text{and then}$$

$$\widetilde{Q}_{1,r,nr} = O_r(F, H)\, C_{nr}(F, (G^{**}\textstyle\sum \widetilde{B}_0^t + FV^{**}H_1^t)$$

Condition v.2 implies rank$\widetilde{Q}_{1,r,nr} = nr$, and therefore $(A_1, A_2, \ldots, A_r)$ can be uniquely identified from the autocovariance matrices of the process, solving (5).

The last part of the proof of this theorem is identical to that of Theorem 1. Therefore, Theorem 2 has been proven. □

As a consequence of section (a) of the previous Theorem and of the Theorem in [9], we give the following Corollary 2.

Corollary 2. *Suppose r = q and that Conditions I, ii and III hold. In this case,* rank$(A_r \vdots B_r) = n$ *and (a(z), b(z)) is left coprime iff rank* $C_{nr}(F, G^{**}) = nr$.

Note that the results in this section consider certain blocks available in the autocovariance matrices that [1] ignores. In particular, if $r > q + 1$, we use C_i^{ff} and C_i^{sf} for $i = q - r + 1$, …, 1, in $\widetilde{Q}_{q-r+1,r,L}$ solving (5) and C_i^{fs} for $i = q + 1, \ldots, q + nr$ and $i = \{kN, (k+1)N, \ldots (k + nr - 1)N\}$ in X_{2a} solving (6).

4. Counterexamples

In Counterexample 1, the conditions in Theorem 1 hold, and thus the VARMA model is identified with MFD. However, Condition IV in [1] does not hold. Therefore, it is not necessary for identifiability in the SFD case. We remark that rank $A_r < n$, but rank$(A_r \vdots B_q) = n$.

Counterexample 1. *Consider the VARMA(3, 1) model with $A_0 = B_0 = I$,*

$$A_1 = \begin{pmatrix} 0 & -1/2 \\ -1/2 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} -1/4 & 0 \\ 0 & -1/4 \end{pmatrix}, A_3 = \begin{pmatrix} -1/2 & -1/4 \\ -1/4 & -1/8 \end{pmatrix}, B_1 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 0 \end{pmatrix}$$

and $E(\varepsilon_{t}{}_t^t) = I_n$.

We have $q = 1$, $r = 3$, $nr = 6$ and we consider $n_1 = 1$.
The model is identifiable in the SFD case because

$$\text{rank}\begin{pmatrix} K_{q-r+1} & K_{q-r+2} & \cdots K_{q+(n-1)r} \\ \vdots & \vdots & \vdots \\ K_q & K_{q+1} & \cdots\ K_{q+nr-1} \end{pmatrix} = \text{rank}\begin{pmatrix} K_{-1} & K_0 & \cdots K_4 \\ K_0 & K_1 & \ldots K_5 \\ K_1 & K_2 & \cdots K_6 \end{pmatrix} = 6.$$

Note that Conditions I, ii, III, iv.1 hold.

However, Condition IV is not satisfied because rank $[G \ldots F^{np}G] = 5 \neq 6$ and, as a consequence, rank $C_L(F, V H_1^t) < np$ and Condition V do not hold. Therefore, A_1, A_2 and A_3 could not be uniquely calculated using the procedure in [1].

We prove that Condition v.1 holds (with MATLAB, see Appendix A) as follows:

(i) First, we computed C_i (i = 0, 1, 2, 3) by solving the Yule–Walker equations:

$$C_0 - A_1C_{-1} - A_2C_{-2} - A_3C_{-3} = I + B_1{K_1}^t$$

$$C_1 - A_1C_0 - A_2C_{-1} - A_3C_{-2} = B_1$$

$$C_2 - A_1C_1 - A_2C_0 - A_3C_{-1} = 0$$

$$C_3 - A_1C_2 - A_2C_1 - A_3C_0 = 0$$

where $K_1 = B_1 + A_1$. We then computed $C_i = A_1C_{i-1} + A_2C_{i-2} + A_3C_{i-3}$. for $i > 4$.

(ii) Second, we obtained that rank $\widetilde{Q}_{q-r+1,r,nr} = 6$.

(iii) Taking into account that $\widetilde{Q}_{q-r+1,r,L} = O_r(F, H)\ C_L(F, V^*(F^t)^{r-q+1}H_1^t)$, $O_r(F, H)$ is full column rank nr, $C_L(F, V^*(F^t)^{r-q+1}H_1^t)$ has nr rows and rank$\widetilde{Q}_{q-r+1,r,nr} = 6 = nr$, and we can affirm that rank $C_L(F, V^*(F^t)^{r-q+1}H_1^t) = nr = 6$; i.e., Condition v.1 holds.

Therefore, A_1, A_2 and A_3 can be uniquely determined by solving (5).

Regarding Condition vi, in this example, θ has 6 columns, where θ_b is the submatrix with the 2nd and 6th columns of θ and rank $\theta_b = 2$, i.e., Condition vi holds.

Taking into account that $X = \begin{pmatrix} C_1 \\ C_0 \\ C_{-1} \end{pmatrix}$, from (6), we can identify C_1^{22}andC_{-1}^{22}. As a consequence, C_0 and C_1 are complete. Finally, the unknown C_i^{22} for $i > 1$ can be identified considering that $C_i = A_1C_{i-1} + A_2C_{i-2} + A_3C_{i-3}$ for $i > 1$. Therefore, this model is identified in the MFD case.

In the following example, Condition V does not hold because $r = q$ (Lemma 1). However, the conditions in Theorem 2 hold and therefore the VARMA model is identified with MFD.

Counterexample 2. *Consider the VARMA(1, 1) model where $A_0 = B_0 = I$, $A_1 = \begin{pmatrix} -1/2 & -1/4 \\ 1 & 1/2 \end{pmatrix}$, $B_1 = \begin{pmatrix} 1 & 4 \\ -1/4 & 1 \end{pmatrix}$ and $E(\varepsilon_t\varepsilon_t^t) = I_n$.*

We have $r = q = 1$, $nr = 2$ and we consider $n_1 = 1$. The autocovariance matrices are

$$C_0 = \begin{pmatrix} 4753/256 & -1025/128 \\ -1025/128 & 949/64 \end{pmatrix}, C_1 = \begin{pmatrix} -201/32 & 275/64 \\ 229/16 & -51/32 \end{pmatrix}, C_2 = \begin{pmatrix} -7/16 & -7/4 \\ 7/8 & 7/2 \end{pmatrix},$$
$$C_i = 0$$

for $i > 2$.

Note that Conditions I, ii, III and iv.2 hold.

Regarding Condition v.2, note that $\widetilde{Q}_{1,r,nr} = O_r(F, H))C_{nr}(F, (G^{**}\sum \widetilde{B}_0^t + FV^{**}H_1^t) = \begin{pmatrix} \widetilde{C}_1 & \widetilde{C}_2 \end{pmatrix} = \begin{pmatrix} -201/32 & -7/16 \\ 229/16 & 7/8 \end{pmatrix}$. Due to the fact that rank $\begin{pmatrix} \widetilde{C}_1 & \widetilde{C}_2 \end{pmatrix} = nr = 2$, $O_r(F, H)$ has nr columns and $C_{nr}(F, (G^{**}\sum \widetilde{B}_0^t + FV^{**}H_1^t))$ has nr rows, then rank $C_{nr}(F, (G^{**}\sum \widetilde{B}_0^t + FV^{**}H_1^t)) = nr = 2$; i.e., Condition v.2 holds.

Therefore, A_1 is uniquely determined by solving (5).

Regarding Condition vi, taking into account that $HF^* = A_1$ is a submatrix of θ and θ_b is the second column of θ, then rank $\theta_b = 1$ and Condition vi holds. Taking into account that $X = C_1$, from (6), we can identify C_1^{22}. Since C_0 and C_1 are complete, the unknown C_i^{22} for i > 1 can be identified considering that $C_i = A_1 C_{i-1}$ for $i > 1$. Therefore, this model is identified in the MFD case.

5. Conclusions

In this work, we have helped to expand the set of VARMA models identified by extended Yule–Walker methods. It provides new necessary and sufficient conditions in the simple-frequency data case, and sufficient conditions in the mixed-frequency data case. The main results are embodied in two theorems, two corollaries and two counterexamples. The two counterexamples allow us to affirm that models are identifiable for which the sufficient conditions for identifiability in [1] do not hold.

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Appendix A

% Numerical evidence: Condition v.1 holds in Counterexample 1

% Software: MATLAB R2022b. Free version in https://es.mathworks.com/products/matlab.html (accessed on 5 December 2023)

% We write the following linear system, $AX = B$, to calculate C_0, C_1, C_2 and C_3

A = [1 0 0 0 0 0.50 0 0 0.25 0 0 0 0.50 0.25 0 0
0 1 0 0 0 0 0 0.50 0 0 0.25 0 0 0 0.50 0.25
0 0 1 0 0.50 0 0 0 0 0.25 0 0 0.25 0.1250 0 0
0 0 0 1 0 0 0.50 0 0 0 0 0.25 0 0 0.25 0.125

```
0 0 0.50 0 1.25 0 0 0 0.50 0.25 0 0 0 0 0 0
0 0 0 0.50 0 1 0.25 0 0 0 0.50 0.25 0 0 0 0
0.50 0 0 0 0 0.25 1 0 0.25 0.125 0 0 0 0 0 0
0 0.50 0 0 0 0 0 1.25 0 0 0.25 0.125 0 0 0 0
0.25 0 0 0 0.50 0.25 0.50 0 1 0 0 0 0 0 0 0
0 0.25 0 0 0 0 0.50 0.75 0 1 0 0 0 0 0 0
0 0 0.25 0 0.75 0.125 0 0 0 0 1 0 0 0 0 0
0 0 0 0.25 0 0.50 0.25 0.125 0 0 0 1 0 0 0 0
0.50 0 0.25 0 0.25 0 0 0 0.50 0 1 0 0 0
0 0.50 0 0.25 0 0.25 0 0 0 0 0 0.50 0 1 0 0
0.25 0 0.1250 0 0 0 0.25 0 0.50 0 0 0 1 0
0 0.25 0 0.125 0 0 0 0.25 0 0.50 0 0 0 0 0 1]

B = [1.25 0 0.25 1 0.5 0.5 0.5 0 0 0 0 0 0 0 0 0]
X = inv(A)*B′

C0 = [X(1) X(2);X(3) X(4)]
C1 = [X(5) X(6);X(7) X(8)]
C2 = [X(9) X(10);X(11) X(12)]
C3 = [X(13) X(14);X(15) X(16)]

% Calculations to obtain the autocovariance C4, C5 and C6
A3 = [1/2 1/4
1/4 1/8]

A2 = [1/4 0
0 1/4]

A1 = [0 1/2
1/2 0]

C4 = −A3*C1−A2*C2−A1*C3
C5 = −A3*C2−A2*C3−A1*C4
C6 = −A3*C3−A2*C4−A1*C5

% Rank of Qs

Qs = [C1′ C0 C1 C2 C3 C4
C0 C1 C2 C3 C4 C5
C1 C2 C3 C4 C5 C6 ]
RangoQs = rank(Qs)
singularvaluesQs = svd(Qs)

% Considering only the odd columns in Qs
QsOdd = [Qs(:, 1) Qs(:, 3) Qs(:, 5) Qs(:, 7) Qs(:, 9) Qs(:, 11)]
RangoQsOdd = rank(QsOdd)
singularvaluesQsOdd = svd(QsOdd)

% Considering only the even columns in Qs
QsEven = [Qs(:, 2) Qs(:, 4) Qs(:, 6) Qs(:, 8) Qs(:, 10) Qs(:, 12)]
RangoQsEven = rank(QsEven)
singularvaluesQsEven = svd(QsEven)
```

```
%%%%%%
A =
1.0000 0 0 0 0 0.5000 0 0 0.2500 0 0 0 0.5000 0.2500 0 0
0 1.0000 0 0 0 0 0 0.5000 0 0 0.2500 0 0 0 0.5000 0.2500
0 0 1.0000 0 0.5000 0 0 0 0 0.2500 0 0.2500 0.1250 0 0 0
0 0 0 1.0000 0 0 0.5000 0 0 0 0 0.2500 0 0 0.2500 0.1250
0 0 0.5000 0 1.2500 0 0 0 0.5000 0.2500 0 0 0 0 0 0
0 0 0 0.5000 0 1.0000 0.2500 0 0 0 0.5000 0.2500 0 0 0 0
0.5000 0 0 0 0 0.2500 1.0000 0 0.2500 0.1250 0 0 0 0 0 0
0 0.5000 0 0 0 0 0 1.2500 0 0 0.2500 0.1250 0 0 0 0
0.2500 0 0 0 0.5000 0.2500 0.5000 0 1.0000 0 0 0 0 0 0 0
0 0.2500 0 0 0 0 0.5000 0.7500 0 1.0000 0 0 0 0 0 0
0 0 0.2500 0 0.7500 0.1250 0 0 0 0 1.0000 0 0 0 0 0
0 0 0 0.2500 0 0.5000 0.2500 0.1250 0 0 0 1.0000 0 0 0 0
0.5000 0 0.2500 0 0.2500 0 0 0 0 0 0.5000 0 1.0000 0 0 0
0 0.5000 0 0.2500 0 0.2500 0 0 0 0 0 0.5000 0 1.0000 0 0
0.2500 0 0.1250 0 0 0 0.2500 0 0.5000 0 0 0 0 0 1.0000 0
0 0.2500 0 0.1250 0 0 0 0.2500 0 0.5000 0 0 0 0 0 1.0000

B =
1.2500 0 0.2500 1.0000 0.5000 0.5000 0.5000 0 0 0 0 0 0 0 0 0

X =
1.7457
0.1876
0.1876
1.2901
0.5771
0.2847
−0.2865
0.0682
−0.6529
0.0452
−0.5153
−0.4018
−0.8064
−0.2866
−0.0618
−0.2478

C0 =
1.7457 0.1876
0.1876 1.2901

C1 =
0.5771 0.2847
−0.2865 0.0682

C2 =
−0.6529 0.0452
−0.5153 −0.4018

C3 =
−0.8064 −0.2866
−0.0618 −0.2478
```

A3 =
0.5000 0.2500
0.2500 0.1250

A2 =
0.2500 0
0 0.2500

A1 =
0 0.5000
0.5000 0

C4 =
−0.0228 −0.0468
0.4236 0.1640

C5 =
0.4451 0.0675
0.2545 0.1243

C6 =
0.2971 0.1548
−0.1191 0.0279

Qs =
0.5771 −0.2865 1.7457 0.1876 0.5771 0.2847 −0.6529 0.0452 −0.8064 −0.2866 −0.0228 −0.0468
0.2847 0.0682 0.1876 1.2901 −0.2865 0.0682 −0.5153 −0.4018 −0.0618 −0.2478 0.4236 0.1640
1.7457 0.1876 0.5771 0.2847 −0.6529 0.0452 −0.8064 −0.2866 −0.0228 −0.0468 0.4451 0.0675
0.1876 1.2901 −0.2865 0.0682 −0.5153 −0.4018 −0.0618 −0.2478 0.4236 0.1640 0.2545 0.1243
0.5771 0.2847 −0.6529 0.0452 −0.8064 −0.2866 −0.0228 −0.0468 0.4451 0.0675 0.2971 0.1548
−0.2865 0.0682 −0.5153 −0.4018 −0.0618 −0.2478 0.4236 0.1640 0.2545 0.1243 −0.1191
0.0279

RangoQs =
6

singularvaluesQs =
2.9541
2.5268
1.2708
1.0259
0.2446
0.0917

QsOdd =
0.5771 1.7457 0.5771 −0.6529 −0.8064 −0.0228
0.2847 0.1876 −0.2865 −0.5153 −0.0618 0.4236
1.7457 0.5771 −0.6529 −0.8064 −0.0228 0.4451
0.1876 −0.2865 −0.5153 −0.0618 0.4236 0.2545
0.5771 −0.6529 −0.8064 −0.0228 0.4451 0.2971
−0.2865 −0.5153 −0.0618 0.4236 0.2545 −0.1191

RangoQsOdd =
6

singularvaluesQsOdd =
2.7937
2.2169
0.5019
0.2229
0.0897
0.0383

QsEven =
-0.2865 0.1876 0.2847 0.0452 -0.2866 -0.0468
0.0682 1.2901 0.0682 -0.4018 -0.2478 0.1640
0.1876 0.2847 0.0452 -0.2866 -0.0468 0.0675
1.2901 0.0682 -0.4018 -0.2478 0.1640 0.1243
0.2847 0.0452 -0.2866 -0.0468 0.0675 0.1548
0.0682 -0.4018 -0.2478 0.1640 0.1243 0.0279

RangoQsEven =
6

singularvaluesQsEven =
1.5738
1.4737
0.3418
0.1908
0.1172
0.0195

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